Color number of cubic graphs having spanning tree with bounded number of leaves

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Abstract

The color number $c(G)$ of a cubic graph $G$ is the minimum cardinality of a color class of a proper 4-edge-coloring of $G$. It is well-known that every cubic graph $G$ satisfies $c(G) = 0$ if $G$ has a Hamiltonian cycle and $c(G) \leq 2$ if $G$ has a Hamiltonian path, respectively. In this paper, we extend those results by obtaining a bound for the color number of cubic graphs having a spanning tree with bounded number of leaves.

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1 Introduction

Edge-coloring of cubic graphs are extensively studied in the past few decades. Several theorems and conjectures on edge-coloring of cubic graphs were formulated.

In this paper, we deal with only simple graphs, that is, graphs without multiple edges nor loops. Let $G$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. A $k$-edge-coloring of $G$ is a mapping $f : E(G) \rightarrow \{1, 2, \ldots, k\}$. If $f(e) \neq f(e')$ for any two adjacent edges $e$ and $e'$, then $f$ is a proper $k$-edge-coloring of $G$. A graph is $k$-edge-colorable if it has

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a proper $k$-edge-coloring. The *edge-chromatic number* $\chi'(G)$, also known as the *chromatic index*, of a graph $G$ is the least $k$ such that $G$ is $k$-edge-colorable. Let $\Delta = \Delta(G)$ be the maximum degree of $G$. Vizing’s theorem [12] states that each graph has the edge-chromatic number $\Delta(G)$ or $\Delta(G) + 1$. By Vizing’s theorem, the set of cubic graphs is divided into two classes; A cubic graph is said to be class 1 if $\chi'(G) = 3$ and class 2 if $\chi'(G) = 4$, respectively.

Since class 2 cubic graphs, in particular *snarks*, which is defined to be a connected class 2 cubic graph with certain connectivity condition, are closely related to several famous conjectures, such as Cycle Double Cover Conjecture [11] and Berge-Fulkerson Conjecture [4], those graphs have attracted by many researchers. To study such conjectures, it is important to measure how far apart a given class 2 cubic graph is from being class 1. The natural measure is the *color number* or sometimes called the *resistance* of a cubic graph $G$, denoted by $c(G)$, which is the minimum number of edges whose removal from $G$ yields a 3-edge-colorable graph. The color number is equivalent with the minimum cardinality of a color class of a proper 4-edge-coloring of $G$. Note that a cubic graph $G$ is 3-edge-colorable if and only if $c(G) = 0$. See [3] for more measures.

It is well-known that every cubic graph with Hamiltonian cycle is 3-edge-colorable, that is, $c(G) = 0$. This fact can be extended to the following, see [13, p. 37].

**Theorem 1** Let $G$ be a cubic graph. If $G$ has a Hamiltonian path, then $c(G) \leq 2$.

Note that the bound $c(G) \leq 2$ is best possible: For example, the Petersen graph has a Hamiltonian path and the color number exactly two. With this relation in mind, we are interested in a property related to the Hamiltonicity such that all cubic graphs satisfying that property has a small color number. In this paper, we show that the existence of a spanning tree with bounded number of leaves is in fact such a property. Actually, the following is the main theorem, which is an extension of Theorem 1.

**Theorem 2** Let $k$ be an integer at least 2, and let $G$ be a cubic graph. If $G$ has a spanning tree with at most $k$ leaves, then $c(G) \leq 2k - 2$.

The proof of Theorem 1 is easy; We first color all edges in a Hamiltonian path alternately by blue and green. For each end vertex $v$ of the Hamiltonian path, choose one edge incident to $v$, and color the two chosen edges by violet. All other edges form a matching, so we can color them by red. This gives a proper 4-edge-coloring with only two edges of violet color. However, the proof of Theorem 2 is not so simple since the more violet edges required, the more careful we are so that the chosen violet edges actually form a matching. In our proof, we solve this issue by taking a suitable partition of a spanning tree, see Lemma 4.

In the rest of this paper, we prove Theorem 2 in the next section. In Section 3, we discuss the best possibility of the bound in Theorem 2 and the relation to a well-known parameter called *oddness* of a cubic graph.

## 2 Proof of the main theorem

The main idea for the proof of Theorem 2 is to use a *path-factor* of a graph, which is a spanning subgraph $F$ such that every component in $F$ is a path of order at least two. Note that a path-factor can be also regarded as a set of pairwise vertex-disjoint paths. A path-factor in a graph has been studied in some literature (e.g. [6]). We use the following lemma to find a path-factor with few paths in a tree having a bounded number of leaves.
To state our lemma, we need the following definition. A tree $T$ is said to be *special* if it is recursively obtained by the following operations.

- The trivial tree, that is the tree consisting of only one vertex, is a special tree.
- Let $T$ be a special tree and let $v$ be a vertex in $T$ of degree at most two. If we add three new vertices $u, \ell_1, \ell_2$ together with three new edges $uv, u\ell_1, u\ell_2$, then the obtained tree is also special.

Some examples of special trees are shown in Figure 1. The graph $K_{1,3}$ in Figure 1 (a) is the second smallest special tree. The trees $T_1$ and $T_2$ in Figure 1 (b) and (c) are special trees obtained from the trees in (a) and (b), respectively, by applying the above operation along the corresponding vertices. The tree $T$ in Figure 1 (d) is a special tree obtained by recursively applying the above operation. Note that in the operation to construct special trees, the vertices $v$ and $u$ have degree at most three and exactly three in the obtained tree, respectively. In addition, it is easy to see the following fact, which will be used later.

**Fact 3** In any special tree $T$, no vertex of degree at most two is adjacent with a vertex of degree at most two.

![Figure 1: Examples of special trees](image)

Lemma 4 Let $T$ be a tree of maximum degree at most three. If $T$ is not special, then $T$ has a path-factor with at most $k$ paths, where $k$ is the number of leaves in $T$. 
Proof of Lemma 4. Let $T$ be a tree of maximum degree at most three that is not special. We prove this lemma by the induction on $k$, where $k$ is the number of leaves in $T$. If $k = 2$, the statement trivially holds. Thus, we may assume that $k \geq 3$.

For each leaf $\ell$, we take a path $Q_\ell$ in $T$ from $\ell$ to the vertex of degree three such that $Q_\ell$ contains no other vertex of degree three. Such a path is uniquely determined. Since $k \geq 3$, there are two leaves $\ell_1$ and $\ell_2$ such that $Q_{\ell_1}$ and $Q_{\ell_2}$ share an end vertex, say $u$.

Suppose first that at least one of $Q_{\ell_1}$ and $Q_{\ell_2}$, say $Q_{\ell_1}$, has three or more vertices. Let $T' = T - V(Q_{\ell_1} - u)$. Note that $u$ has degree two in $T'$, and hence $T'$ has at most $k - 1$ leaves. Furthermore, since the neighbor of $\ell_2$ in $T'$ is contained in $Q_{\ell_2}$, $\ell_2$ is adjacent with a vertex of degree two in $T'$. It follows from Fact 3 that $T'$ is not a special tree. Thus, by induction hypothesis, $T'$ has a path-factor $F'$ with at most $k - 1$ paths. Then, by adding $Q_{\ell_1} - u$ to $F'$ we obtain a path-factor of $G$ with at most $k$ paths.

Therefore, we may assume that both of $Q_{\ell_1}$ and $Q_{\ell_2}$ have two vertices. Let $P$ be the path obtained by the concatenation of $Q_{\ell_1}$ and $Q_{\ell_2}$ at $u$. Note that $P$ consists of the three vertices $\ell_1, u$, and $\ell_2$. Let $T' = T - V(P)$. Note that $T$ is obtained from $T'$ by adding three vertices $u, \ell_1, \ell_2$ together with three new edges $uv, u\ell_1, \ell_2u$, where $v$ is the neighbor of $u$ in $T$ other than $\ell_1, \ell_2$. Thus, if $T'$ is special, then it follows from the definition of special trees that $T$ is also special, a contradiction. Thus we may assume that $T'$ is not special. By induction hypothesis, $T'$ has a path-factor $F'$ with at most $k - 1$ paths. Similarly to the previous case, we obtain a path-factor with at most $k$ paths. This completes the proof of Lemma 4. □

The converse of Lemma 4 can be proved by the standard induction on the number of steps we performed when we construct special trees. Since we do not use this statement for the proof of our main theorem, we leave its proof for the reader.

Now, we are ready to prove Theorem 2.

Proof of Theorem 2. By Theorem 1, we may assume that $k \geq 3$. Let $G$ be a cubic graph. Assume that $G$ has no spanning tree with $k - 1$ leaves, and let $T$ be a spanning tree of $G$ with $k$ leaves. We first prove the following claims.

Claim 1 The leaves in $T$ are pairwise nonadjacent.

Proof: If two leaves $\ell_1$ and $\ell_2$ are adjacent in $G$, then by adding the edge $\ell_1\ell_2$ into the tree $T$ and removing a suitable edge, we obtain a spanning tree of $G$ with $k - 1$ leaves, a contradiction. Thus, any two leaves are nonadjacent. ■

Claim 2 $T$ is not a special tree.

Proof: Suppose on the contrary that $T$ is a special tree. Note that $T$ has $k$ leaves, which implies that the number of vertices of degree three is $k - 2$. By Fact 3, the number of vertices of degree two in $T$ is at most $k - 3$.

Exactly $2k$ edges in $G \setminus E(T)$ in total are incident with the leaves of $T$. Since there are at most $k - 3$ vertices of degree two in $T$, there are at most $k - 3$ edges in $G \setminus E(T)$ that are incident with the vertices of degree two. Hence, at least $\frac{1}{2}(k + 3)$ edges in $G \setminus E(T)$ connect two leaves of $T$, which contradicts Claim 1. Thus, $T$ is not a special tree. ■

By Lemma 4, $T$ has a path-factor $F$ with at most $k$ paths. Note that $F$ is a path-factor also in $G$. We take such a path-factor in $G$ with as few paths as possible.
Let \( S \) be the set of end vertices of paths in \( F \). Note that \(|S| \leq 2k\). By the choice of \( F \), no two vertices in \( S \) are adjacent in \( G \) except for the case that they are end vertices of the same path in \( F \).

First, color the edges of the paths in \( F \) with blue and green alternately. Let \( H = G - E(F) \), and we will color the edges in \( H \) by red and violet appropriately. Note that each vertex in \( S \) (resp. not in \( S \)) has degree exactly two (resp. one) in \( H \). Since no two vertices in \( S \) can be adjacent except for the case that they are end vertices of the same path in \( F \), \( H \) is a vertex-disjoint union of paths each of which has at most four vertices. Then we can color all edges in \( H \) with red and violet alternately, and in particular, if we use violet on as few edges as possible, every violet edge is incident to a vertex in \( S \). Thus, there are at most \( 2k \) violet edges, and this completes the proof of Theorem 2.

\[ \square \]

3 Conclusion

In this paper, we show Theorem 2 which states that any cubic graph containing a spanning tree with at most \( k \) leaves has the color number at most \( 2k - 2 \), which is an extension of Theorem 1. The bound \( "2k - 2" \) in Theorem 2 is best possible, in the sense of the next theorem.

Theorem 5 For each \( k \geq 2 \), there is a 3-edge-connected cubic graph \( G \) such that \( G \) has a spanning tree with \( k \) leaves and \( c(G) = 2k - 2 \).

Proof: Let \( P^* \) be the Petersen graph with one vertex removed. It is easy to see that \( P^* \) does not admit a 3-edge-coloring. Let \( n = k - 1 \). For \( i \in \{1, 2, \ldots, n\} \), let \( P_i \) and \( P'_i \) be the copies of \( P^* \), and let \( u_i, v_i, w_i \) (resp. \( u'_i, v'_i, w'_i \)) be the vertices of degree two in \( P_i \) (resp. \( P'_i \)). For any \( i \in \{1, 2, \ldots, n\} \), add edges \( w_iu_{i+1}, v_iv'_i, u'_iw'_{i+1} \), where the subscript is taken modulo \( n \), to obtain a 3-edge-connected cubic graph \( G \) as shown in Figure 2.

Consider a 4-edge-coloring of \( G \). Since each copy of \( P^* \) must contain all of the four colors, we see that \( c(G) \geq 2n \). On the other hand, it is straightforward to check that \( G \) admits a 4-edge-coloring for which there are only \( 2n \) edges of the fourth color (so that each copy of \( P^* \) contains one edge of the fourth color). This implies that \( c(G) = 2n = 2k - 2 \).

Thus, it remains only to show that \( G \) has a spanning tree with \( k \) leaves. It is easy to see the following:

- \( P_1 \) contains a spanning subgraph \( Q_1 \) such that \( u_1 \) is a vertex of degree 0 in \( Q_1 \) and \( Q_1 - u_1 \) is a Hamiltonian path in \( P_1 - u_1 \) connecting \( v_1 \) and \( w_1 \).
- \( P'_1 \) contains a spanning subgraph \( Q'_1 \) corresponding to \( Q_1 \).
- For \( 2 \leq i \leq n \), \( P_i \) has two vertex-disjoint paths \( R_i \) and \( S_i \) such that \( R_i \) connects \( u_i \) and \( w_i \), \( S_i \) connects \( v_i \) and some vertex and paths \( R_i \) and \( S_i \) contain 4 and 5 vertices, respectively.
- For \( 2 \leq i \leq n \), \( P'_i \) has a spanning tree \( T'_i \) such that its leaves are precisely \( u'_i, v'_i \) and \( w'_i \).

Then from \( Q_1 \cup Q'_1 \cup \bigcup_{i=2}^{n} (R_i \cup S_i \cup T'_i) \) by adding the edges \( w_iu_{i+1}, v_iv'_i, u'_iw'_{i+1} \) for \( i \in \{1, 2, \ldots, n\} \), where the subscript is taken modulo \( n \), we obtain a spanning tree \( T \) as shown in Figure 3. Note that each \( P_i \) with \( 2 \leq i \leq n \) contains exactly one leaf that is an end vertex of \( S_i \) other than \( v_i \), and there are no other leaves except for \( u_1 \) and \( u'_1 \). This implies that \( T \)
Figure 2: A 3-edge-connected cubic graph $G$ has $n + 1 = k$ leaves. This completes the proof.

Before closing this paper, we discuss the relation to the oddness of a cubic graph. The oddness of a cubic graph $G$, denoted by $\omega(G)$, is the smallest number of odd cycles in a 2-factor, where a 2-factor is a spanning subgraph in which every vertex has degree 2. This is a measure how far apart a given class 2 cubic graph is from being class 1, other than the color number. Since it has been shown that cubic graphs with small oddness have several good properties (see [5] for example), the oddness has been widely studied.

For a cubic graph $G$, it is clear that $\omega(G) \geq c(G)$ from the definition, and shown that $\omega(G) = 2$ if and only if $c(G) = 2$ [9, Lemma 2.5]. Therefore, it is natural to think whether we obtain the bound on $\omega(G)$, instead of the bound on $c(G)$ in Theorem 2, if $G$ has a spanning tree with at most $k$ leaves. We leave this as an open problem.

**Problem 6** Find an upper bound on $\omega(G)$ for a cubic graph $G$ having a spanning tree with at most $k$ leaves.

Note that the gap between $\omega(G)$ and $c(G)$ can be arbitrary large even for snarks (that is, for cubic graphs assuming certain connectivity conditions). In fact, Allie [1] proved that there is no constant $k$ such that $\omega(G) \leq kr(G)$ holds for every cubic graph $G$.

On the other hand, the cubic graph $G$ constructed in the proof of Theorem 5 satisfies $\omega(G) = 2k - 2$. This can be seen as follows (we use the same notation as in the proof of Theorem 5): Each $P_i$ contains vertex-disjoint subgraphs $C_i$ and $Q_i$ such that $C_i$ is a cycle of
order 5 and $Q_i$ is a path connecting $u_i$ and $w_i$ of order 4. Then we have the cycle of order 4$n$ that is obtained by the union of $Q_i$ for $1 \leq i \leq n$ together with the edges $w_1u_i+1$ taken module $n$. This cycle and the $n$ cycles $C_i$’s contain all vertices in $\bigcup_{i=1}^{n} P_i$. Similarly, we can take the cycle of order 4$n$ and the $n$ cycles of order 5 symmetrically in $\bigcup_{i=1}^{n} P_i'$, and hence $G$ has a 2-factor with $2n$ odd cycles. This shows that $\omega(G) \leq 2n$. Since any 2-factor of $G$ must contain a cycle of order 5 in each $P_i$ and $P_i'$, we also have $\omega(G) = 2n$.

Therefore, as a possible answer to Problem 6, it might be true that $\omega(G) \leq 2k - 2$ for any cubic graph containing a spanning tree with at most $k$ leaves.

References


