2-Factors of cubic bipartite graphs

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Abstract

For a positive integer $k \geq 3$, a $\{C_n|n \geq k\}$-factor in a graph $G$ is a spanning subgraph of $G$ in which each component is a cycle of length at least $k$. Kano, Lee and Suzuki proved that every 2-connected bipartite cubic graph contains a $\{C_n|n \geq 6\}$-factor. In this paper, we prove that every bipartite cubic graph $G$ contains a $\{C_n|n \geq 8\}$-factor, provided that $G$ satisfies some conditions related to 6-cycles.

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1 Introduction

In this paper, we consider only finite and simple graphs, except for otherwise stated.

A 2-factor of a graph $G$ is a spanning subgraph of $G$ in which all vertices have degree exactly 2. Note that a 2-factor consists of vertex-disjoint cycles such that every vertex in $G$ is contained in some of them. It is easy to see that any two cycles in a cubic graph either share an edge or are vertex-disjoint. Thus, in the remaining of this paper, we simply use the term “disjoint” for two cycles in cubic graphs regardless the meanings of vertex-disjoint and edge-disjoint.

Petersen [8] proved that every 2-connected cubic graph contains a 2-factor. For a set $\mathcal{F}$ of connected graphs, an $\mathcal{F}$-factor of a graph $G$ is a spanning subgraph in which every component is a member of $\mathcal{F}$. Note that a 2-factor is nothing but a $\{C_n:n \geq 3\}$-factor, where we denote by $C_n$ an $n$-cycle, that is, a cycle of length exactly $n$. Thus, Petersen’s theorem can be re-stated as follows.

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Theorem 1. [Petersen [8]] Every 2-connected cubic graph has a \{C_n|n \geq 3\}-factor.

Because of the relation to several applications, such as Travelling Salesman Problem (see [1]), many researchers have been interested in the length of each cycle in a 2-factor, that is, a \{C_n|n \geq k\}-factor for a large integer k. There are several known theorems.

Theorem 2. Let \(G\) be a 2-connected cubic graph. Then the following holds.

1. \(G\) has a \{C_n|n \geq 4\}-factor. (Kawarabayashi, Matsuda, Oda and Ota [6])
2. If \(G\) is 3-connected, then \(G\) has a \{C_n|n \geq 5\}-factor. (Jackson and Yoshimoto [4])
3. If \(G\) is a bipartite graph, then \(G\) has a \{C_n|n \geq 6\}-factor. (Kano, Lee and Suzuki [5])

Note that any connected bipartite cubic graph is 2-connected, see [5, Lemma 8]. So in Theorem 2(3), 2-connectedness in the assumption is not necessary.

In this paper, we focus on bipartite cubic graphs and attempt to find a \{C_n|n \geq 8\}-factor. As in Figure 1, which is same as [5, Figure 1(c)], there are 2-connected bipartite cubic graphs in which every 2-factor contains a 6-cycle. Thus, it is impossible to guarantee for every 2-connected bipartite cubic graph to have a \{C_n|n \geq 8\}-factor, but we show that with additional conditions.

Figure 1: A bipartite cubic graph in which every 2-factor contains a 6-cycle.

An edge \(e\) in a graph \(G\) is called a chord of a cycle \(C\) in \(G\) if \(e\) is not an edge in \(C\), but both end vertices of \(e\) are contained in \(C\). We are now ready to state our main theorem.

Theorem 3. Let \(G\) be a cubic bipartite graph satisfying the following conditions.

\(G1\) Each 6-cycle of \(G\) has at most one chord, and
\(G2\) for each 6-cycle \(D\), if \(D\) shares exactly one edge with another 6-cycle, then \(D\) does not share an edge with any other 6-cycles.
Then $G$ has a $\{C_n \mid n \geq 8\}$-factor.

Note that considering the example in Figure 1, it seems that the condition (G1) is essential while we need condition (G2) only for technical reason for our proofs. So, we expect the following conjecture:

**Conjecture 1** Let $G$ be a cubic bipartite graph such that each 6-cycle of $G$ has at most one chord. Then $G$ has a $\{C_n \mid n \geq 8\}$-factor.

Tutte [9] suggested the statement that every 3-connected cubic bipartite graph contains a Hamiltonian cycle. There are several known counterexamples to this statement, but the following weaker version might hold.

**Conjecture 2** Let $G$ be a 3-connected cubic bipartite graph of order at least 8. Then $G$ has a $\{C_n \mid n \geq 8\}$-factor.

If there exists a 6-cycle $D$ of a cubic graph $G$ with at least two chords, then either $G$ is $K_{3,3}$ or the two vertices in $D$ that are not incident with the chords of $D$ form a cut set of order 2. This shows that if Conjecture 1 is true, then Conjecture 2 is also true. We hope that Theorem 3 would be a step toward to solve those Conjectures.

Note that in several literatures, the reduction of short cycles is a powerful method to show the existence of a $\{C_n \mid n \geq k\}$-factor for some $k$. For example, the reduction of 3-cycles and 4-cycles is a key idea to prove Theorem 2(1)–(3) in [6], and in [4, 5], respectively. In fact, this paper follows the latter reduction to reduce 4-cycles, and in addition, we also reduce 6-cycles, depending on the neighbouring situations. (In order to restrict the neighbouring situations of 6-cycles, we need the condition (G2).) Note that reduction to other structures can be found in [2, 3], but their reductions are more restricted. As long as the authors know, this paper is the first to attempt to reduce 6-cycles to find a 2-factor in which each cycle has length longer than 6.

## 2 Preliminary

In this section, we define some terminology. Let $G$ be a cubic graph. For two 6-cycles $D$ and $D'$ in $G$ sharing at least one edge, their union $\bar{D}$ is defined as $V(\bar{D}) = V(D) \cup V(D')$ and $E(\bar{D}) = E(D) \cup E(D')$. We call $\bar{D}$ a 6-cycle pair of Type-$i$, where $i$ is the number of edges shared by $D$ and $D'$.

For an edge subset $T$ in $G$, we say that a subgraph $H$ in $G$ is $T$-intersecting if $H$ contains at least one edge in $T$. If $H$ is not $T$-intersecting, we say that $H$ is $T$-disjoint. Furthermore, a $\{C_n \mid n \geq 6\}$-factor $F$ in $G$ is $T$-good if each 6-cycle in $F$ either is $T$-intersecting or shares exactly one edge with some 4-cycle. We show the following lemma.
Lemma 1. Let $G$ be a cubic bipartite graph, and let $T = \emptyset$. If $G$ contains a $T$-good $\{C_n | n \geq 6\}$-factor in $G$, then $G$ contains a $\{C_n | n \geq 8\}$-factor.

Proof. Let $F$ be a $T$-good $\{C_n | n \geq 6\}$-factor in $G$. We choose such $F$ so that the number of components in $F$ is as few as possible. Suppose that $F$ contains a 6-cycle $D = abcedfa$. Since $F$ is $T$-good and $T = \emptyset$, $D$ shares exactly one edge with a 4-cycle. By symmetry, we may assume that the edge $ab$ is shared by a 4-cycle, say $D' = abbl'a'$. Since $F$ is a 2-factor in the cubic graph $G$, the edge $a'b'$ is contained in another cycle in $F$. Then, by switching the two edges $ab$ and $a'b'$ in $F$ with the two edges $aa'$ and $bb'$, we obtain a $T$-good $\{C_n | n \geq 6\}$-factor with fewer number of components, contradicting the choice of $F$. Thus, $F$ contains no 6-cycle, and hence $F$ is a $\{C_n | n \geq 8\}$-factor in $G$. □

The following lemma is almost trivial, but useful. We will use it implicitly several times in our proof.

Lemma 2. Every minimal edge cut of a 2-factor contains even number of edges.

We briefly explain an outline of our proof. From a cubic bipartite graph $G$ satisfying the conditions (G1) and (G2), we recursively reduce 6-cycle pairs of Type-1 (Section 3.1), those of Type-3 (Section 3.2), 6-cycles (Section 3.3) and then 4-cycles (Section 3.4), if they satisfy certain conditions. To explain those reduction, we let $G_0 = G$ and $T_0 = \emptyset$. For $i \geq 1$, let $G_i$ be the graph obtained from $G_{i-1}$ by the $i$th reduction, and $T_i$ be the set of edges in $G_i$ created by the reductions we have done before obtaining $G_i$ from $G_0$. For each $i \geq 1$, we also show the following in each section:

- The new graph $G_i$ is a simple cubic bipartite graph.
- If the new graph $G_i$ contains a $T_i$-good $\{C_n | n \geq 6\}$-factor, then $G_{i-1}$ contains a $T_{i-1}$-good $\{C_n | n \geq 6\}$-factor.

By Lemma 1, those claims enable us to focus on the simple cubic bipartite graph $G_s$ obtained finally by the all reductions. By using Theorem 2(3), $G_s$ contains a $\{C_n | n \geq 6\}$-factor $F_s$. In the last section (Section 3.5), we show that $G_s$ contains a $T_s$-good $\{C_n | n \geq 6\}$-factor, which completes the proof.

3 Proof of Theorem 3

Proof. Let $G$ be a cubic bipartite graph satisfying the conditions (G1) and (G2). As explained in the above outline of the proof, we recursively reduce some configurations in $G$ one by one. Let $G_0 = G$ and $T_0 = \emptyset$. 

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3.1 Reduction for 6-cycle pairs of Type-1 with particular properties

Let $i \geq 1$, and suppose that the graph $G_{i-1}$ contains a $T_{i-1}$-disjoint 6-cycle pair $\tilde{D}_i$ of Type-1 that is disjoint from any 4-cycles in $G_{i-1}$. Then we reduce $\tilde{D}_i$ and update $T_{i-1}$ as follows. Let $\tilde{D}_i$ consist of the 10-cycle $abcdefgkl$ with a chord $bg$. Suppose that two vertices in $\{a, c, d, e, f, h, k, l\}$ are adjacent in $G_{i-1}$ but not in $\tilde{D}_i$. Since $\tilde{D}_i$ is disjoint from any 4-cycles and $G_{i-1}$ is bipartite, either $dk \in E(G_{i-1})$ or $el \in E(G_{i-1})$. Note that one of the edges $dk$ and $el$ do not exist, since otherwise $delkd$ is a 4-cycle in $G_{i-1}$, contradicting the condition of $\tilde{D}_i$. This arguments distinguish the $T_{i-1}$-disjoint 6-cycle pair $\tilde{D}_i$ of Type-1 into two types, namely, Type-1(i): neither $dk$ nor $el$ exist in $G_{i-1}$ and Type-1(ii): exactly one of $dk$ and $el$ exist in $G_{i-1}$. We reduce $\tilde{D}_i$ of each type as follows.

**Type-1(i): Neither $dk$ nor $el$ exist in $G_{i-1}$.**

In this type, for $x \in \{a, c, d, e, f, h, k, l\}$, let $x'$ be the vertex in $G_{i-1}$ such that $xx'$ is an edge in $G_{i-1}$ but not in $\tilde{D}_i$, see the left of Figure 2. Note that some vertices of $a', c', d', e', f', h', k', l'$ might possibly coincide. Then remove $V(\tilde{D}_i)$ and add four new edges $e'd', e'f', h'k'$ and $l'a'$, see the right of Figure 2. Let $G_i$ be the obtained graph, and let $T_i$ be the set of edges in $G_i$ that are not in $G$. In other words, $T_i$ is obtained from $T_{i-1}$ by deleting all edges $xx'$ for $x \in \{a, c, d, e, f, h, k, l\}$ (if contained in $T_{i-1}$) and adding the edges $e'd', e'f', h'k'$ and $l'a'$. As we will see in Figure 2, $G_i$ is still bipartite (and hence it does not have any loops), and since the degree of any vertices in $G_i$ is same as that in $G_{i-1}$, $G_i$ is still cubic. Since $\tilde{D}_i$ is disjoint from any 4-cycles in $G_{i-1}$, $G_i$ is still a simple graph.

![Figure 2: Reduction of the 6-cycle pair $\tilde{D}_i$ of Type-1(i).](image)

**Type-1(ii): Exactly one of $dk$ and $el$ exist in $G_{i-1}$.**

In this type, we may assume $dk \in E(G_{i-1})$ by symmetry. Note that $abcdkla$ is a 6-cycle sharing an edge with the 6-cycle $abghkla$ in $\tilde{D}_i$. Since no edge in $\tilde{D}_i$ is contained in $T_{i-1}$, it follows from the condition (G2) that $abcdkla$ is not a 6-cycle in $G$. This implies that $dk \in T_{i-1}$.
For $x \in \{a, c, e, f, h, l\}$, let $x'$ be the vertex in $G_{i-1}$ such that $xx'$ is an edge in $G_{i-1}$ but not in $\tilde{D}_i$, see the left of Figure 3. Then remove $V(\tilde{D}_i)$ and add six new vertices $u_a, u_c, u_e, u_f, u_h$ and $u_l$ together with 12 new edges $a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l, u_au_i, u_cu_h, u_hue, u_euf$ and $ufua$, see the right of Figure 3. Let $G_i$ be the obtained graph, and let $T_i$ be the set of edges in $G_i$ that are not in $G$. In other words, $T_i$ is obtained from $T_{i-1}$ by deleting $dk$ and all edges $xx'$ for $x \in \{a, c, e, f, h, l\}$ (if contained in $T_{i-1}$) and adding the 12 new edges. As we will see in Figure 3, $G_i$ is still bipartite and cubic. It is clear from the construction that $G_i$ is still a simple graph.

Let $G_p$ be the graph obtained by the above reduction as long as we can. So, any $T_p$-disjoint 6-cycle pair $\tilde{D}$ of Type-1 in $G_p$ shares an edge with a 4-cycle. Note that $G_p$ is a simple, cubic and bipartite graph. In order to show our main theorem, we first prove the following claim.

Claim 1. Let $1 \leq i \leq p$. If $G_i$ contains a $T_i$-good $\{C_n|n \geq 6\}$-factor, then $G_{i-1}$ contains a $T_{i-1}$-good $\{C_n|n \geq 6\}$-factor.

To show this claim, suppose that $G_i$ has a $T_i$-good $\{C_n|n \geq 6\}$-factor $F_i$. Depending on types of $\tilde{D}_i$, we modify each edge in $T_i - T_{i-1}$ contained in $F_i$ into some suitable paths in $G_{i-1}$ as follows. This modification converts $F_i$ in $G_i$ into a $T_{i-1}$-good $\{C_n|n \geq 6\}$-factor $F_{i-1}$ in $G_{i-1}$. By the condition of $\tilde{D}_i$, no 4-cycle in $G_{i-1}$ shares an edge with $\tilde{D}_i$. If there is a 6-cycle in $G_{i-1}$ sharing an edge with $\tilde{D}_i$, except for $abghkla$ and $bcdefgb$, then it is $T_{i-1}$-intersecting, since otherwise, the 6-cycle $abghkla$ or $bcdefgb$ contradicts the condition (G2). Therefore, no 4-cycle nor $T_{i-1}$-disjoint 6-cycle in $G$, except for $abghkla$ and $bcdefgb$, shares an edge with $\tilde{D}_i$. In the following, this property ensures that the obtained $\{C_n|n \geq 6\}$-factor $F_{i-1}$ in $G_{i-1}$ is in fact $T_{i-1}$-good.

**Type-1(i):** Recall that $G_i$ is obtained from $G_{i-1}$ by reducing $\tilde{D}_i$ as in Figure 2. In this

![Figure 3: Reduction of the 6-cycle pair $\tilde{D}_i$ of Type-1(ii).](image-url)
case, we divide the proof into five cases, depending on how many edges in \{c'd', e'f', h'k', l'a'\} are used in \(F_i\).

**Case 1(i)-1:** No edges in \{c'd', e'f', h'k', l'a'\} are used in \(F_i\).

In this case, we add the 10-cycle abcdefghkla as a component of \(F_i\).

**Case 1(i)-2:** Exactly one edge in \{c'd', e'f', h'k', l'a'\} is used in \(F_i\).

By symmetry, we may assume that the edge c'd' is used in \(F_i\). Then we replace the edge c'd' with the path c'balkhgfedd', see Figure 4.

![Figure 4: Conversion of the 6-cycle pair \(\widetilde{D_i}\) of Type-1(i) in Case 1(i)-2.](image)

**Case 1(i)-3:** Exactly two edges in \{c'd', e'f', h'k', l'a'\} are used in \(F_i\).

In this subcase, we have three possibilities up to symmetry.

1. \(\{c'd', e'f', h'k', l'a'\} \cap E(F_i) = \{c'd', e'f'\}\).
   In this case, we replace the edges c'd' and e'f' with the paths c'balkhgf'f' and d'de'e', respectively, see Figure 5.

![Figure 5: Conversion of the 6-cycle pair \(\widetilde{D_i}\) of Type-1(i) in Case 1(i)-3 (1).](image)

2. \(\{c'd', e'f', h'k', l'a'\} \cap E(F_i) = \{c'd', h'k'\}\).
   In this case, we replace the edges c'd' and h'k' with the paths c'balakk' and d'defghh',

3. \(\{c'd', e'f', h'k', l'a'\} \cap E(F_i) = \{e'f', h'k'\}\).
   In this case, we replace the edges e'f' and h'k' with the paths c'alghfgh and d'efghh',


respectively, see Figure 6.

Figure 6: Conversion of the 6-cycle pair \( \tilde{D}_i \) of Type-1(i) in Case 1(i)-3 (2).

\[
(3) \{c'd', e'f', h'k', a'l'\} \cap E(F_i) = \{c'd', a'l'\}.
\]

In this case, we replace the edges \( c'd' \) and \( a'l' \) with the paths \( d'abc'e' \), and \( d'de'fghkl'l' \), respectively, see Figure 7.

Figure 7: Conversion of the 6-cycle pair \( \tilde{D}_i \) of Type-1(i) in Case 1(i)-3 (3).

Case 1(i)-4: Exactly three edges in \{\( c'd', e'f', h'k', l'a' \)\} are used in \( F_i \).

By symmetry, we may assume that \( \{c'd', e'f', h'k', a'l'\} \cap E(F_i) = \{c'd', e'f', a'l'\} \). Then we replace the edges \( c'd', e'f' \) and \( a'l' \) with the paths \( d'abc'e' \), \( d'de'fghkl'l' \), respectively, see Figure 8.
Case 1(i)-5: All edges in \{c'd', e'f', h'k', l'a'\} are used in \(F_i\).

In this case, we replace the edges c'd', e'f', h'k' and a'l' with the paths a'abcc', d'dee', f'fghh' and k'kll', respectively, see Figure 9.

Type-1(ii): Recall that \(G_i\) is obtained from \(G_{i-1}\) by reducing \(\tilde{D}_i\) as in Figure 3. In this case, we divide the proof into four cases, depending on how many edges in \{a'ua, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} are used in \(F_i\). By Lemma 2, its number must be even, so 0, 2, 4 or 6. Note that the vertices \(u_a\) and \(u_f\), the vertices \(u_c\) and \(u_h\), and the vertices \(u_e\) and \(u_l\) are symmetric, respectively.

Case 1(ii)-1: No edges in \{a'ua, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} are used in \(F_i\).

In this case, we add the 10-cycle abcdefghkla as a component of \(F_i\).

Case 1(ii)-2: Exactly two edges in \{a'ua, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} are used in \(F_i\).

In this subcase, we have four possibilities up to symmetry. We will show each possibility one by one.

1. \{a'ua, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \(\cap E(F_i) = \{a'ua, f'u_f\}\).

In this case, we replace the path \(a'uaualhahu_euf'\) with the path \(a'alkghbcdefff'\), see
Figure 10: Conversion of the 6-cycle pair $\widetilde{D}_i$ of Type-1(ii) in Case 1(ii)-2 (1).

\[ \{a'ua, e'u_c, e'u_e, f'u_fh'h'u_l, l'ui\} \cap E(F_i) = \{f'u_fh'h'u_l, l'ui\}. \]

In this case, we replace the path $f'u_fh'h'u_l$ with the path $f'fhklabcdee'$, see Figure 11.

Figure 11: Conversion of the 6-cycle pair $\widetilde{D}_i$ of Type-1(ii) in Case 1(ii)-2 (2).

\[ \{a'ua, e'u_c, e'u_e, f'u_fh'h'u_l, l'ui\} \cap E(F_i) = \{e'u_e, h'u_h\}. \]

In this case, we replace the path $e'u_eu_fh'h'u_l$ with the path $e'fhhlkabcde'$ and then add the 6-cycle $abcdkla$, see Figure 12. Note that the 6-cycle $abcdkla$ contains the edge $dk$ that belongs to $T_{i-1}$, and hence it is $T_{i-1}$-intersecting.
Figure 12: Conversion of the 6-cycle pair $\tilde{D}_i$ of Type-1(ii) in Case 1(ii)-2 (3).

(4) $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{h'u_h, c'u_c\}$.

In this case, we replace the path $h'u_h u_a u_f a'u_a e'c'$ with the path $h'khlgf'edc'c'$, see Figure 13.

Figure 13: Conversion of the 6-cycle pair $\tilde{D}_i$ of Type-1(ii) in Case 1(ii)-2 (4).

Note that the case $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{c'u_c, l'u_l\}$ is symmetric to (3), and the case $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{l'u_l, a'u_a\}$ is symmetric to (2), respectively. This completes the proof for Case 1(ii)-2.

**Case 1(ii)-3: Exactly four edges in $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\}$ are used in $F_i$.**

In this subcase, we have six possibilities up to symmetry. We will show each possibility one by one.

(1) $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{l'u_l, a'u_a, f'u_f, e'u_e\}$.

In this case, we replace the paths $l'u_l u_h e'c'$ and $a'u_a u_f f'$ with the paths $l'khgf'f'$ and
e'edcba', see Figure 14.

Figure 14: Conversion of the 6-cycle pair $\tilde{D}_i$ of Type-1(ii) in Case 1(ii)-3 (1).

(2) \(\{a'u_a, e'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{a'u_a, f'u_f, e'u_e, h'u_h\}\).
In this case, we replace the paths $a'u_au_au_ah'$ and $f'u_fu_ee'$ with the paths $a'alkhh'$ and $f'fgbcdee'$, see Figure 15.

Figure 15: Conversion of the 6-cycle pair $\tilde{D}_i$ of Type-1(ii) in Case 1(ii)-3 (2).

(3) \(\{a'u_a, e'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{f'u_f, e'u_e, h'u_h, e'u_c\}\).
In this case, we replace the paths $f'u_fu_au_ue'c'$ and $e'u_euh'c'$ with the paths $f'fgbalkhh'$ and $ecdee'$, see Figure 16.
Figure 16: Conversion of the 6-cycle pair $\tilde{D}_i$ of Type-1(ii) in Case 1(ii)-3 (3).

(4) $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{e'u_e, h'u_h, c'u_c, l'u_l\}$.

In this case, we replace the paths $e'u_eu_fu_{a}u_{l}l'$ and $h'u_hu_ce'$ with the paths $e'efgball'$ and $h'hkdec'$, see Figure 17.

Figure 17: Conversion of the 6-cycle pair $\tilde{D}_i$ of Type-1(ii) in Case 1(ii)-3 (4).

Note that the case $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{h'u_h, c'u_c, l'u_l, a'u_a\}$ is symmetric to (3), and the case $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{e'u_e, l'u_l, a'u_a, f'u_f\}$ is symmetric to (2), respectively.

(5) $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{f'u_f, e'u_e, c'u_c, l'u_l\}$.

In this case, we replace the paths $f'u_fu_{a}u_{l}l'$ and $c'u_cuhue'$ with the paths $f'fghkdec'$ and $l'labcc'$, see Figure 18.
Figure 18: Conversion of the 6-cycle pair $\tilde{D}_i$ of Type-1(ii) in Case 1(ii)-3 (5).

(6) $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{a'u_a, f'u_f, h'u_h, c'u_c\}$.

In this case, we replace the paths $a'u_au_uf$ and $h'u_heuf'$ with the paths $a'alkdef'$ and $c'cbghh'$, see Figure 19.

Figure 19: Conversion of the 6-cycle pair $\tilde{D}_i$ of Type-1(ii) in Case 1(ii)-3 (6).

Note that the case $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{e'u_e, h'u_h, l'u_l, a'u_a\}$ is symmetric to (5). This completes the proof for Case 1(ii)-3.

**Case 1(ii)-4:** All edges in $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\}$ are used in $F_i$.

In this case, we replace the paths $a'u_au_uf'$, $c'u_ueuf'$ and $c'u_ufhl'$ (or the paths $l'u_au_ah'$ and $f'u_ufue'$ with the paths $l'lkhh'$, $a'abgff'$ and $c'dce'$, see Figure 20.
Recall that no 4-cycle nor $T_{i-1}$-disjoint 6-cycle in $G$, except for $abghkla$ and $bcdefgb$, shares an edge with $\tilde{D}_i$. Thus, in either case, the obtained $\{C_n|n \geq 6\}$-factor $F_{i-1}$ in $G_{i-1}$ is $T_{i-1}$-good. This completes the proof of Claim 1.

By Lemma 1, it suffices to find a $T_0$-good $\{C_n|n \geq 6\}$-factor in $G_0$. Thus, by recursively applying Claim 1, it suffices to show that $G_p$ has a $T_p$-good $\{C_n|n \geq 6\}$-factor.

### 3.2 Reduction for 6-cycle pairs of Type-3 with particular properties

Let $i \geq p + 1$, and suppose that the graph $G_{i-1}$ contains a $T_{i-1}$-disjoint 6-cycle pair $\tilde{D}_i$ of Type-3 that is disjoint from any 4-cycles in $G_{i-1}$. Then we reduce $\tilde{D}_i$ in $G_{i-1}$ and update $T_{i-1}$ as follows. Since $\tilde{D}_i$ is disjoint from any 4-cycles, the three edges shared by two 6-cycles in $\tilde{D}_i$ must be consecutive and we name each vertex in $\tilde{D}_i$ as in the left of Figure 21. For $x \in \{b, c, e, f, g, h\}$, let $x'$ be the vertex such that $xx'$ is an edge in $G_{i-1}$ but not in $\tilde{D}_i$. If $x' = y$ or $x' = y'$ for some vertices $x, y \in \{b, c, e, f, g, h\}$, then we can find a 4-cycle sharing an edge with $\tilde{D}_i$, contradicting the condition of $\tilde{D}_i$. Thus, all vertices in the left of Figure 21 are pairwise distinct. This implies that the following reduction does not create multiple edges: Remove $V(\tilde{D}_i)$ and add two new vertices $u$ and $v$ together with six new edges $ub', ug', uf', vc', ve'$ and $vh'$, see the right of Figure 21. Let $G_i$ be the obtained graph, and let $T_i$ be the set of edges in $G_i$ that are not in $G$. In other words, $T_i$ is obtained from $T_{i-1}$ by deleting all edges $xx'$ for $x \in \{b, c, e, f, g, h\}$ (if contained in $T_{i-1}$) and adding the six new edges. As in Figure 21, the obtained graph is still cubic and bipartite. As explained in the above, $G_i$ is still a simple graph.
Let \( G_q \) be the graph obtained by the above reduction as long as we can. So, any \( T_q \)-disjoint 6-cycle pair \( \tilde{D} \) of Type-3 in \( G_q \) shares an edge with a 4-cycle. Note that \( G_q \) is a simple, cubic and bipartite graph. We next show the following claim.

\[ \text{Claim 2. Let } p + 1 \leq i \leq q. \text{ If } G_i \text{ contains a } T_i \text{-good } \{ C_n \mid n \geq 6 \} \text{-factor, then } G_{i-1} \text{ contains a } T_{i-1} \text{-good } \{ C_n \mid n \geq 6 \} \text{-factor.} \]

To show this claim, suppose that \( G_i \) has a \( T_i \)-good \( \{ C_n \mid n \geq 6 \} \)-factor \( F_i \). Similarly to the proof of Claim 1, we will convert \( F_i \) in \( G_i \) into a \( T_{i-1} \)-good \( \{ C_n \mid n \geq 6 \} \)-factor \( F_{i-1} \) in \( G_{i-1} \).

Recall that \( G_i \) is obtained from \( G_{i-1} \) by reducing \( \tilde{D}_i \) as in Figure 21. By the condition of \( \tilde{D}_i \), no 4-cycle in \( G_{i-1} \) shares an edge with \( \tilde{D}_i \). Since \( F_i \) is a 2-factor of \( G_i \), \( F_i \) contains exactly two edges incident with \( u \) and exactly two edges incident with \( v \). Note that the pairs \( \{ bb', cc' \}, \{ ee', ff' \} \) and \( \{ hh', gg' \} \) are pairwise symmetric, and hence we may assume that the edges \( ub' \) and \( vc' \) are both used in \( F_i \). Then, depending on how other edges are used in \( F_i \), we have two subcases, namely, Case 2-1: \( \{ ub', ug', vc', vh' \} \subseteq E(F_i) \) and Case 2-2: \( \{ ub', ug', vc', ve' \} \subseteq E(F_i) \).

**Case 2-1: The paths \( b'ug' \) and \( c'vh' \) are used in \( F_i \).**

In this case we replace these paths with the paths \( b'bec' \) and \( g'gafedhh' \), respectively, see Figure 22. Suppose that the new cycle using the path \( b'bec' \), say \( D \), is a \( T_{i-1} \)-disjoint 6-cycle. Then the 6-cycle \( abcedf \)a shares exactly one edge with \( D \) and three edges with \( afedg \). Since those three 6-cycles are all \( T_{i-1} \)-disjoint, they are also contained in \( G \), which contradicts the condition (G2). Thus the new cycle using the path \( b'bec' \) is either of length at least 8 or \( T_{i-1} \)-intersecting. On the other hand, the new cycle using the path \( g'gafedhh' \) contains the vertices \( g', g, a, f, e, d, h, h' \), and hence its length is at least 8.
Case 2-2: The paths $b'ug'$ and $c've'$ are used in $F_i$.

In this case, we replace these paths with $b'bafee'$ and $c'cdhgg'$, respectively, see Figure 23. If the new cycle using the path $b'bafee'$, say $D_i$, is a $T_{i-1}$-disjoint 6-cycle, then by the same argument as in Case 2-1, the 6-cycle $abcdhga$ shares exactly one edge with $D_i$ and three edges with $abcdefa$, which contradicts the condition (G2). Thus, the new cycle using the path $b'bafee'$ either has length at least 8 or is $T_{i-1}$-intersecting. Similarly, the new cycle using the path $c'cdhgg'$ either has length at least 8 or is $T_{i-1}$-intersecting.

This completes the proof of Claim 2.

As in the end of Section 3.1, it suffices to show that $G_p$ has a $T_p$-good $\{C_n|n \geq 6\}$-factor. Thus, by recursively applying Claim 2, it also suffices to show that $G_q$ has a $T_q$-good $\{C_n|n \geq 6\}$-factor.

### 3.3 Reduction for 6-cycles with particular properties

We next reduce 6-cycles with particular properties. Let $i \geq q + 1$, and suppose that $G_{i-1}$ contains a $T_{i-1}$-disjoint 6-cycle $D_i$ that is disjoint from any 4-cycles in $G_{i-1}$. Then we reduce $D_i$ from $G_{i-1}$ and update $T_{i-1}$ as follows. Let $D_i = abcdefa$, and for
Let $G_i$ be the vertex such that $xx'$ is an edge in $G_{i-1}$ but not in $D_i$, see the left of Figure 24. If $x' = y$ or $x' = y'$ for some vertices $x, y \in \{a, b, c, d, e, f\}$, then we can find a 4-cycle sharing an edge with $D_i$, contradicting the condition of $D_i$. Thus, all vertices in the left of Figure 24 are pairwise distinct. This implies that the following reduction does not create multiple edges: Remove $V(D_i)$ and add two new vertices $u$ and $v$ together with six new edges $ua', uc', ue', vb', vd'$ and $vf'$, see the right of Figure 24. Let $G_i$ be the obtained graph, and let $T_i$ be the set of edges in $G_i$ that are not in $G$. In other words, $T_i$ is obtained from $T_{i-1}$ by deleting all edges $xx'$ for $x \in \{a, b, c, d, e, f\}$ (if contained in $T_{i-1}$) and adding the six new edges. As in Figure 24, the obtained graph is still cubic and bipartite. As explained in the above, $G_i$ is still a simple graph.

![Figure 24: Reduction of the 6-cycle $D_i$.](image)

Let $G_r$ be the graph obtained by the above reduction as long as we can. So, any $T_r$-disjoint 6-cycle $D$ in $G_r$ shares an edge with a 4-cycle. Note that $G_r$ is a simple, cubic and bipartite graph. We next show the following claim.

**Claim 3.** Let $q + 1 \leq i \leq r$. If $G_i$ contains a $T_i$-good $\{C_n|n \geq 6\}$-factor, then $G_{i-1}$ contains a $T_{i-1}$-good $\{C_n|n \geq 6\}$-factor.

Similarly to the proofs of Claims 1 and 2, suppose that $G_i$ has a $T_i$-good $\{C_n|n \geq 6\}$-factor $F_i$, and we will convert $F_i$ into a $T_i$-good $\{C_n|n \geq 6\}$-factor $F_{i-1}$ in $G_{i-1}$.

Recall that $G_i$ is obtained from $G_{i-1}$ by reducing $D_i$ as in Figure 24. By the condition of $D_i$, there is no 4-cycle sharing an edge with $D_i$ in $G_{i-1}$. In this case, by symmetry, we have two subcases, namely, Case 3-1: $\{ua', ue', vb', vf'\} \subseteq E(F_i)$ and Case 3-2: $\{ua', ue', vb', vd'\} \subseteq E(F_i)$.

**Case 3-1: The paths $a'ue'$ and $b'vf'$ are used in $F_i$.**

In this case, we can replace these paths with the paths $a'aff'$ and $b'bcdee'$, respectively, see Figure 25.

Let $D$ be the new cycle in $F_{i-1}$ using the path $a'aff'$, and suppose that $D$ is a $T_{i-1}$-disjoint 6-cycle. Since both $D_i$ and $D$ are $T_{i-1}$-disjoint, $D_i$ and $D$ form a 6-cycle pair of
Type-1 in $G$, and also in $G_p$. By the condition of $G_p$, the 6-cycle pair of Type-1 must share an edge with some 4-cycle in $G_p$. Since $D_i$ is disjoint from any 4-cycles, $D$ shares exactly one edge with the 4-cycle. This shows that $D$ is either of length at least 8 or a $T_{i-1}$-intersecting or shares exactly one edge with a 4-cycle.

Let $D'$ be the new cycle in $F_{i-1}$ using the path $b'bcdee'$ and suppose that $D'$ is a $T_{i-1}$-disjoint 6-cycle. Since both $D_i$ and $D'$ are $T_{i-1}$-disjoint, $D_i$ and $D'$ form a 6-cycle pair of Type-3 in $G$, and hence in $G_q$. By the condition of $G_q$, the 6-cycle pair of Type-3 must share an edge with some 4-cycle in $G_q$. Thus, by the same argument as above, $D'$ is either of length at least 8 or a $T_{i-1}$-intersecting or shares exactly one edge with a 4-cycle.

![Figure 25: Conversion of the 6-cycle $D_i$ in Case 3-1.](image)

**Case 3-2: The paths $a'ue'$ and $b'vd'$ are used in $F_i$.**

In this case, we can replace these paths with the paths $a'afee'$ and $b'bcdd'$, respectively, see Figure 26. Since $F_i$ is a $\{C_n|n \geq 6\}$-factor of $G_i$, the cycle using the path $a'ue'$ in $F_i$ has length at least 6. Since the length of the new cycle using the path $a'afee'$ is increased by two, its length is at least 8. In the same manner, we can show that the new cycle using the path $b'bcdd'$ also has length at least 8, as desired.
This completes the proof of Claim 3.

Thus, again by recursively applying Claim 3, it also suffices to show that $G_r$ has a $T_r$-good $\{C_n | n \geq 6\}$-factor.

### 3.4 Reduction for 4-cycles with particular properties

Finally, we reduce 4-cycles with particular properties. Let $i \geq r + 1$, and suppose that $G_{i-1}$ contains a $T_{i-1}$-disjoint 6-cycle $D$ with exactly one chord. Let $D = abcdefa$ with a chord $ad \in E(G_{i-1})$. Then $abcda$ is a 4-cycle, and let $D_i = abcda$. We reduce $D_i$ from $G_{i-1}$ and update $T_{i-1}$ as follows. For $x \in \{b, c\}$, let $x'$ be the vertex such that $xx'$ is an edge in $G_{i-1}$ but not in $D_i$. Since the 6-cycle $D$ has exactly one chord in $G_{i-1}$, $b' \neq e$ and $c' \neq f$. Then, we remove the four vertices $a, b, c, d$ and add two new vertices $u$ and $v$ together with five new edges $fu, c'u, b'v, ev$ and $uv$, see Figure 27. Let $G_i$ be the obtained graph, and let $T_i$ be the set of edges in $G_i$ that are not in $G$. In other words, $T_i$ is obtained from $T_{i-1}$ by deleting the edges $ad, bb', cc'$ (if contained in $T_{i-1}$) and adding the five new edges. As in Figure 27, the obtained graph is still cubic and bipartite. As explained in the above, $G_i$ is still a simple graph.

Let $G_s$ be the graph obtained by the above reduction as long as we can. Note that
Claim 4. Let $r + 1 \leq i \leq s$. If $G_i$ contains a $T_i$-good $\{C_n | n \geq 6\}$-factor, then $G_{i-1}$ contains a $T_{i-1}$-good $\{C_n | n \geq 6\}$-factor.

Similarly to the proofs of Claims 1–3, suppose that $G_i$ has a $T_i$-good $\{C_n | n \geq 6\}$-factor $F_i$, and we will convert $F_i$ into a $T_{i-1}$-good $\{C_n | n \geq 6\}$-factor $F_{i-1}$ in $G_{i-1}$. Recall that $G_i$ is obtained from $G_{i-1}$ by reducing $D_i$ as in Figure 27. In this case, we have two subcases, depending on whether $uv \in E(F_i)$ or not.

Case 4-1: The edge $uv$ is used in $F_i$.
Suppose that $F_i$ passes through $uv$ with the path $b'vuc'$. Then we replace the path $b'vuc'$ in $F_i$ with the path $b'badcc'$, see Figure 28. Note that the length of the cycle using $uv$ in $F_i$ is increased by two, and hence the new cycle has length at least 8, as desired. In the case when $F_i$ passes through $uv$ with the other ways, we can similarly convert the cycle using $uv$ in $F_i$ to a cycle of length at least 8.

Case 4-2: The edge $uv$ is not used in $F_i$.
In this case, $F_i$ contains the two paths $fuc'$ and $b've$. Then we first replace them with the paths $fabb'$ and $c'cde$, respectively, see Figure 29. If the new cycle using the path $fabb'$ does not contain the path $c'cde$, then we further change the edges $ab$ and $cd$ with $ad$ and $bc$. In either case, the new cycle using the edge $fa$ contains all the vertices $a, b, c, d, e, f, b'$ and $c'$, and hence the new cycle has length at least 8, as desired.
This completes the proof of Claim 4.

Thus, again by recursively applying Claim 4, it also suffices to show that $G_s$ has a $T_s$-good $\{C_n|n \geq 6\}$-factor.

### 3.5 Finding a $T_s$-good $\{C_n|n \geq 6\}$-factor in $G_s$

Then we proceed to the final step of the proof. Since $G_s$ is a simple cubic bipartite graph, we can apply Theorem 2(3) to $G_s$ and obtain a $\{C_n|n \geq 6\}$-factor $F_s$ in $G_s$. We take such a $\{C_n|n \geq 6\}$-factor $F_s$ so that the number of $T_s$-disjoint 6-cycles is as small as possible.

Now we prove that every 6-cycle in $F_s$ either is $T_s$-intersecting or shares exactly one edge with some 4-cycle. Suppose that there is a $T_s$-disjoint 6-cycle $D = abedefa$ in $F_s$. This means that $D$ is also a 6-cycle in $G$. Thus, if $D$ shares no edges with any 4-cycles in $G_s$, then we must reduce $D$ in $G_r$ (or $G_p$ or $G_q$ if $D$ forms a 6-cycle pair of Type-1 or -3 that is disjoint from any 4-cycles, respectively), a contradiction. Therefore, $D$ shares an edge with a 4-cycle in $G_s$. If $D$ shares exactly one edge with a 4-cycle, then there is nothing to prove. Thus, we may assume that such a 4-cycle does not exist.

Suppose that $D$ shares exactly three edges with a 4-cycle $D'$. Since $G_s$ is a cubic graph, those three edges must be consecutive on $D$. So, we may assume by symmetry that $ab, bc$ and $cd$ are the three edges, which means that $ad$ is a chord of $D$. By the condition of $G_s$, $D$ has another chord, say $be$ by symmetry. By the condition (G1), at least one of $ad$ and $be$ is not an edge in $G$, that is, it is contained in $T_s$. Thus, we can replace the 6-cycle $D$ in $F_s$ with the $T_s$-intersecting 6-cycle $adbecfa$, which contradicts the choice of $F_s$. This implies that $D$ shares exactly three edges with no 4-cycles. Since $D$ shares an edge with a 4-cycle in $G_s$, there is a 4-cycle $D'$ that shares exactly two edges with $D$. If such two edges are not consecutive in $D$, then $D$ has two chords and there is a 4-cycle sharing three edges with $D$, a contradiction. Thus, $D$ shares two consecutive edges, say $ab$ and $bc$ by symmetry, with a 4-cycle $D'$. Let $D' = abca'$. Since $G_s$ is a cubic graph and the two neighbors $a$ and $c$ of $a'$ are both contained in $D$, $a'$ cannot be contained in any cycles in the 2-factor $F_s$, a contradiction. This shows that $F_s$ is a $T_s$-good $\{C_n|n \geq 6\}$-factor in $G_s$. This completes the proof of Theorem 3.

\[\square\]

**References**


