A complete bipartite graph without properly colored cycles of length four

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Abstract

A subgraph of an edge-colored graph is said to be properly colored, or shortly PC, if any two adjacent edges have different colors. Fujita, Li and Zhang gave a decomposition theorem for edge-colorings of complete bipartite graphs without PC $C_4$. However, their decomposition just focus on only three colors, and does not deal with all of the colors. In this paper, we give a new and detailed decomposition theorem for edge-colorings of complete bipartite graphs without PC $C_4$. Our decomposition gives a corollary on the existence of a monochromatic star with almost sharp bound.

Keywords: Properly colored cycle, complete bipartite graph, minimum color degree,

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This work was partially supported by the project P202/12/G061 of the Czech Science Foundation and by the project LO1506 of the Czech Ministry of Education, Youth and Sports.

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This work was partially supported by JST ERATO Grant Number JPMJER1201, Japan, and JSPS KAKENHI Grant Number 18K03391.

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This work was supported by JSPS KAKENHI Grant Number 18K03402.
1 Introduction

In this paper, we consider edge-colorings of complete graphs and complete bipartite graphs. Note that such an edge-coloring is not necessarily proper. See [16] for any notation not defined here. The complete graph of order $n$ is denoted by $K_n$, and the complete bipartite graph with bipartitions of size $n$ and $m$, respectively, is denoted by $K_{n,m}$. We identify an edge-coloring of a graph with an edge-colored graph. A subgraph of an edge-coloring is said to be properly colored, or shortly PC, if any two adjacent edges have different colors.

The contribution of this paper is (1) to give a new and detailed decomposition theorem for edge-colorings of complete bipartite graphs without PC $C_4$, and then (2) to show some application for the existence of a large monochromatic star in edge-colorings of complete bipartite graphs without PC $C_4$.

1.1 Complete or complete bipartite graphs without certain PC subgraphs

Complete graphs or complete bipartite graphs without certain PC subgraphs have been shown to have particular properties. The most famous one is the characterization due to Gallai [11] for the case of no PC $C_3$ (or equivalently rainbow $C_3$, see Section 1.3). Indeed, the characterization gives a special partition of the vertices, which is called a Gallai partition.

**Theorem 1 (Gallai [11])** For $n \geq 2$, if $G$ is an edge-coloring of $K_n$ without PC $C_3$, then there exist at most two colors $i$ and $j$ and a partition of $V(G)$ into at least two parts such that for any two different parts, all edges between them are colored by the same color that is $i$ or $j$.

Recently Fujita, Li and Zhang gave a characterization of edge-colorings of complete bipartite graphs without PC $C_4$. (Because of presentation reason, we change the indices from those in [10].) Since $C_4$ is a shortest possible cycle in a bipartite graph, in some sense, this is a bipartite analogous to Theorem 1. Before stating their result, we define some terminology.

The color degree of a vertex $x$ in an edge-coloring $G$, denoted by $d^c_G(x)$, is the number of distinct colors appearing in edges incident with $x$. The minimum color degree of $G$ is the minimum of $d^c_G(x)$ over all vertices $x$ in $G$. For a disjoint vertex subset $X$ and $Y$ of an edge-coloring $G$ of a graph, we denote by $C(X,Y)$ the set of colors that are used for some edge between $X$ and $Y$. If $X$ consists of only one vertex $x$, then we abbreviate $C(x,Y)$ instead of $C(\{x\},Y)$. Similarly, we write $C(X,y)$ when $Y = \{y\}$.

**Theorem 2 (Fujita, Li and Zhang [10])** Let $G$ be an edge-coloring of a complete bipartite graph without PC $C_4$, and let $X$ and $Y$ be the bipartitions of $G$. Suppose that the minimum color degree is at least two. Then, for some set of three colors,
say 1, 2, 3 by symmetry, $X$ and $Y$ can be partitioned into $\{A_1, A_2, A_3, X_1, X_2, X_3, \tilde{X}\}$ and $\{B_1, B_2, B_3, Y_1, Y_2, Y_3, \tilde{Y}\}$, respectively, satisfying the following conditions for any $i \in \{1, 2, 3\}$. (The indices are taken modulo 3.)

(P1) $A_i, B_i \neq \emptyset$;

(P2) $C(A_i, B_{i-1} \cup B_i) = \{i\}$ and $C(A_i, B_{i+1}) = \{i+1\}$;

(P3) $C(A_i, Y_{i-1} \cup Y_i \cup \tilde{Y}) \subseteq \{i\}$ and $C(A_i, y_{i+1}) = \{i, i+1\}$ for each vertex $y_{i+1} \in Y_{i+1}$;

(P4) $C(X_{i-1} \cup X_i \cup \tilde{X}, B_i) \subseteq \{i\}$ and $C(x_{i+1}, B_i) = \{i, i+1\}$ for each vertex $x_{i+1} \in X_{i+1}$.

However, they focus on only three colors and do not discuss colors on the edges between $X_1 \cup X_2 \cup X_3 \cup \tilde{X}$ and $Y_1 \cup Y_2 \cup Y_3 \cup \tilde{Y}$. In order to deal with them, one might think that after getting a partition as in Theorem 2, we can apply Theorem 2 again to $G - (A_1 \cup A_2 \cup A_3 \cup B_1 \cup B_2 \cup B_3)$. This gives us a recursive structure, which expresses the colors of all edges. However, the same color may appear several times, and the structure would be complicated. In fact, while Fujita et al. [10] obtained a corollary on a large monochromatic star, their bound was not best possible when there are more than three colors. We will discuss this in Section 5.

Considering the above situation, we give a new and simpler partition considering all colors in this paper. Before stating our main theorem, we give some definition.

Let $G$ be a graph and let $k$ be the number of colors used in $G$. A mapping $\varphi : V(G) \to \{1, \ldots, k\}$ is said to be good if for $i, j \in \{1, \ldots, k\}$, each edge between $\varphi^{-1}(i)$ and $\varphi^{-1}(j)$ is colored by either $i$ or $j$. Notice that when $G$ is an edge-coloring of a complete bipartite graph, then $\varphi^{-1}(i)$ induces a monochromatic complete bipartite graph of color $i$. Those sets $\varphi^{-1}(i)$ for $i \in \{1, \ldots, k\}$ form indeed a partition of the vertices.

**Theorem 3** Let $G$ be an edge-coloring of a complete bipartite graph without PC $C_4$, and let $X$ and $Y$ be the bipartitions of $G$. Then $G$ admits a good mapping from $V(G)$ to $\{1, \ldots, k\}$. Furthermore, if the minimum color degree is at least two, then the following holds. For some set of three colors, say 1, 2, 3 by symmetry, there are six vertices $\tilde{x}_i \in \varphi^{-1}(i) \cap X$ and $\tilde{y}_i \in \varphi^{-1}(i) \cap Y$ satisfying the following condition for $i \in \{1, 2, 3\}$; (The indices are taken modulo 3.)

$$C(\tilde{x}_i, Y - \varphi^{-1}(i+1)) = \{i\} \quad \text{and} \quad C(X - \varphi^{-1}(i+1), \tilde{y}_i) = \{i\}.$$ 

After preliminary sections, in Section 3, we prove that Theorem 3 implies Theorem 2. The proof of Theorem 3 will appear in Section 4. Then we will discuss a corollary on a monochromatic star in Section 5.

1.2 Color degree condition

Erdős and Tuza [7] showed that using a main idea of Theorem 1, an edge-coloring of $K_n$ admits a PC $C_3$ if the minimum color degree is at least $\log_2 n + O(1)$. As pointed
out by Axenovich, Jiang and Tuza in [4, Proposition 4.3], the bound is best possible, except for the $O(1)$ part. They also proved the following for the existence of a PC $C_4$.

**Theorem 4 (Axenovich, Jiang and Tuza [4, Theorem 4.10])** Let $G$ be an edge-coloring of the complete graph $K_n$. If the minimum color degree is at least three, then $G$ contains a PC $C_4$.

As in [10], Theorem 2 (and also Theorem 3) directly gives a bipartite analogue of Theorem 4.

**Theorem 5 (Fujita, Li and Zhang, [10])** Let $G$ be an edge-coloring of a complete bipartite graph. If the minimum color degree of $G$ is at least three, then $G$ admits a PC $C_4$.

It was mentioned in [10] that the condition “the minimum color degree of $G$ is at least three” in Theorem 5 is best possible. However, they did not explicitly mention that they need at least three colors in their example. In fact, we can improve this when there are only two colors, see Lemma 7 in Section 2.

We introduce more results on PC cycles. A PC cycle in an edge-coloring of (non-complete) graphs with color degree conditions were considered in several papers [1, 14, 15]. Chen and Daykin [5] showed that for each integers $t, n, \ell$ with $t \geq 2$, $n \geq 25t$ and $n \geq \ell \geq 2$, an edge-coloring of the complete bipartite graph $K_{n,n}$ admits a PC $C_{2\ell}$ if no vertex is incident to $t$ edges of the same color. More related results can be found in surveys [8, 9].

### 1.3 Rainbow cycles

A subgraph of an edge-coloring is said to be *rainbow* if no two edges in the subgraph have the same color. Note that a rainbow $C_3$ is nothing but a PC $C_3$, but for many other graphs, such as $C_4$, those two concepts do not coincide. In fact, the properties of graphs without rainbow $C_4$ are very different from those without PC $C_4$.

For example, in contrast to Theorem 4, Axenovich, Jiang and Tuza proved that an edge-coloring of the complete graph $K_n$ contains a rainbow $C_4$ if the minimum color degree is at least $4n^{2/3}$ ([4, Theorem 4.7]), and there is an edge-coloring of the complete graph $K_n$ of minimum color degree $\log_2 n - 1$, but without rainbow $C_4$ ([4, Proposition 4.3]). In some sense, this gap shows the difficulty of dealing with a rainbow $C_4$, compared with a PC $C_4$. It might be interesting to consider the counterpart of Theorem 5 as follows.

**Problem 6** Determine the sharp function $f(n)$ such that an edge-coloring of the complete bipartite graph $K_{n,n}$ contains a rainbow $C_4$ if the minimum color degree at least $f(n)$.

The proof of [4, Theorem 4.7] can be adapted to show that an edge-coloring of the complete bipartite graph $K_{n,m}$ with $n \geq m$ contains a rainbow $C_4$ if the minimum color
degree is at least $4(nm)^{1/3}$. This directly gives $f(n) = O(n^{2/3})$. On the other hand, the same construction in the proof of [4, Proposition 4.3] shows $f(n) = \Omega(\log_2 n)$. There is still a huge gap between known upper bound and lower bound of $f(n)$, similarly to the case of the complete graph.

Several researchers have been interested in the condition on the number of colors that forces the existence of a rainbow $C_4$, which is called an anti-Ramsey theory. Alon [2] proved that “an edge-coloring of $K_n$ with at least $\lceil 4n/3 \rceil$ colors admits a rainbow $C_4$”, which was conjectured by Erdős, Simonovits and Sós [6]. (Note that their conjecture is also for rainbow $C_k$ ($k \geq 4$), which was proved by Montellano-Ballesteros and Neumann-Lara [12].) Axenovich, Jiang, and Kündgen [3], and independently Mubayi and West [13] considered the bipartite analogous, showing that “an edge-coloring of $K_{n,m}$ with at least $m+n$ colors admits a rainbow $C_4$”.

2 Preliminaries

Let $G$ be an edge-coloring of a graph. Recall that for a disjoint vertex subset $X$ and $Y$ of $G$, we denote by $C(X,Y)$ the set of colors that are used for some edge between $X$ and $Y$. With abuse of notation, for an edge $e$ of $G$, we denote the color of $e$ by $c(e)$. The color sequence of a cycle or a path in an edge-coloring is a sequence of colors of the edges along its order.

The following lemma is used several times in our proofs. It gives a counterpart of the ”best possibility” of Theorem 5 for the case where only two colors appear. (Note that the contraposition of Lemma 7 states that “every edge-coloring of a complete bipartite graph with two colors admits a PC $C_4$ if the minimum color degree is two”.)

**Lemma 7** Let $G$ be an edge-coloring of a complete bipartite graph with at most two colors, say colors 1 and 2, and let $X$ and $Y$ be bipartitions of $G$. If $G$ admits no PC $C_4$, then either there exists a vertex $x$ in $X$ such that $c(x,Y) = \{1\}$ or there exists a vertex $y$ in $Y$ such that $c(X,y) = \{2\}$.

**Proof of Lemma 7.** Suppose that every vertex in $Y$ is incident with an edge of color 1. Let $x \in X$ be such that the number of edges incident with $x$ of color 1 is maximum over all vertices in $X$. Suppose further that there exists an edge $xy$ in $G$ with $c(xy) = 2$. By the assumption, $y$ is incident with an edge of color 1, say $x'y$. Then if $c(xy') = 1$ for $y' \in Y$, then $c(x'y') = 1$. Otherwise the cycle $xyx'y'x$ has the color sequence 2121, a contradiction. However, this implies that $x'$ is incident with more edges of color 1 than $x$, contradicting the choice of $x$. Therefore, all edges incident with $x$ are of color 1, and we are done. \[\square\]
3 Proof of Theorem 2 using Theorem 3

Let $G$ be an edge-coloring of a complete bipartite graph without PC $C_4$, and let $X$ and $Y$ be the bipartitions of $G$. Suppose that the minimum color degree is at least two. Then by Theorem 3, $G$ admits a good mapping $\varphi$ from $V(G)$ to $\{1, \ldots, k\}$, and for some set of three colors, say $1, 2, 3$ by symmetry, there are six vertices $\tilde{x}_i \in \varphi^{-1}(i) \cap X$ and $\tilde{y}_i \in \varphi^{-1}(i) \cap Y$ for $i \in \{1, 2, 3\}$ as in Theorem 3. Let $W$ be the set of vertices of color degree exactly two, that is, $W = \{x \in V(G) : d_G^c(x) = 2\}$, and for $i \in \{1, 2, \ldots, k\}$, let

$$A_i = \varphi^{-1}(i) \cap W \cap X \quad \text{and} \quad B_i = \varphi^{-1}(i) \cap W \cap Y.$$

By the second part of Theorem 3, each vertex $x_j$ in $X_j$ for $j \in \{4, \ldots, k\}$ is incident to $\tilde{y}_i$ for $i \in \{1, 2, 3\}$ with $c(x_j, \tilde{y}_i) = i$, and hence $x_j \notin W$. This implies that $A_j = \emptyset$ for $j \in \{4, \ldots, k\}$, and by symmetry, $B_j = \emptyset$. We remark that $\{A_1, A_2, A_3, B_1, B_2, B_3\}$ is a partition of $W$ and all other vertices have color degree at least three.

We take $\hat{X}$ and $\hat{Y}$ as large as possible subject to the property that

- any vertex $x \in \hat{X}$ satisfies $C(x, B_i) = \{i\}$ for $i \in \{1, 2, 3\}$, and
- any vertex $y \in \hat{Y}$ satisfies $C(y, A_i) = \{i\}$ for $i \in \{1, 2, 3\}$.

For any $i \in \{1, 2, 3\}$, let

$$X_i = \varphi^{-1}(i) \cap X - A_i - \hat{X} \quad \text{and} \quad Y_i = \varphi^{-1}(i) \cap Y - B_i - \hat{Y}.$$

Then we will show that those sets satisfy conditions (P1)–(P4). By symmetry, it suffices to check only the case $i = 1$. Furthermore, by the symmetry between $X$ and $Y$, conditions (P3) and (P4) are symmetric, and hence we will omit to check condition (P4).

Since $\varphi$ is a good mapping, $C(\tilde{x}_1, \varphi^{-1}(2) \cap Y) \subseteq \{1, 2\}$. Thus, the property $C(\tilde{x}_1, Y - \varphi^{-1}(2)) = \{1\}$ implies that $\tilde{x}_1 \in A_1$. Similarly, we have $\tilde{y}_1 \in B_1$, and hence condition (P1) is satisfied.

Since $A_1, B_1, Y_1 \subseteq \varphi^{-1}(1)$ and $\varphi$ is a good mapping, we have $C(A_1, B_1 \cup Y_1) = \{1\}$. By the property $C(X - \varphi^{-1}(3), \tilde{y}_2) = \{2\}$ and $A_1 \subseteq X - \varphi^{-1}(3)$, we have $C(A_1, \tilde{y}_2) = \{2\}$. Thus, since each vertex $a_1 \in A_1$ has the color degree exactly two, we see $C(A_1, Y) = \{1, 2\}$ and $C(A_1, B_3 \cup Y_3) = \{1\}$. Note that $C(A_1, \hat{Y}) \subseteq \{1\}$ by the definition of $\hat{Y}$. Thus, the first part of conditions (P2) and (P3) holds.

Suppose that $C(A_1, B_2) \neq \{2\}$. Then there exists an edge $a_1b_2$ with $a_1 \in A_1$, $b_2 \in B_2$ and $c(a_1, b_2) \neq 2$. Since $A_1 \subseteq \varphi^{-1}(1)$, we have $c(a_1b_2) = 1$. Since the first part of condition (P2) for $i = 2$ and $3$ implies $C(A_2, B_2) = \{2\}$ and $C(A_3, B_2) = \{3\}$, the vertex $b_2$ has color degree at least three, contradicting $b_2 \in B_2 \subseteq W$. Therefore, condition (P2) is also satisfied.

Suppose that the second part of condition (P3) does not hold, that is, there is a vertex $y_2 \in Y_2$ such that $C(A_1, y_2) \neq \{1, 2\}$. Since $A_1 \subseteq \varphi^{-1}(1)$ and $y_2 \in Y_2 \subseteq \varphi^{-1}(2)$, either $C(A_1, y_2) = \{1\}$ or $C(A_1, y_2) = \{2\}$. Suppose $C(A_1, y_2) = \{1\}$. By the first part of condition (P3) for $i = 2$ and $3$, we have $C(A_2, y_2) = \{2\}$ and $C(A_3, y_2) = \{3\}$.
This shows that moving $y_2$ from $Y_2$ to $\hat{Y}$ leads to a contradiction with the maximality of $\hat{Y}$. Therefore, we may assume $C(A_1, y_2) = \{2\}$. Recall that $C(A_2, y_2) = \{2\}$ and $C(A_3, y_2) = \{3\}$. Since the color degree of $y_2$ is at least three, there is a vertex $x$ in $X - (A_1 \cup A_2 \cup A_3)$ with $c(xy_2) \notin \{2, 3\}$. Since $y_2 \in Y_2 \subseteq \varphi^{-1}(2)$, $X_2 \subseteq \varphi^{-1}(2)$ and $X_3 \subseteq \varphi^{-1}(3)$, we see $x \notin X_2 \cup X_3$. If $x \notin \hat{X}$, then the definition of $\hat{X}$ implies that the cycle $a_1y_2xb_1a_1$ is a PC $C_4$ with color sequence $2j31$, where $a_1 \in A_1$, $b_1 \in B_3$ and $j = c(xy_2)$, a contradiction. Thus, we further have $x \notin \hat{X}$. Therefore, it remains only the case $x \in X_1$. In this case, $c(xy_2) = 1$.

If $c(xb_3) = 3$ for some $b_3 \in B_3$, then the cycle $a_1y_2xb_3a_1$ is a PC $C_4$ with color sequence $2131$, where $a_1 \in A_1$, a contradiction. Thus, we have $C(x, B_3) = \{1\}$. Note that $C(x, B_1) = \{1\}$ and $C(x, B_2) = \{2\}$ by the first part of condition (P3) for $i = 1$ and 2. Since the color degree of $x$ is at least three, there is a vertex $y$ in $Y - (B_1 \cup B_2 \cup B_3)$ with $c(xy) \notin \{1, 2\}$. However, we obtain a contradiction, as follows:

- Since $x \in X_1$ and $\varphi$ is a good mapping, we have $y \notin Y_1 \cup Y_2$.
- If $y \in Y_3 \cup \hat{Y}$, then the cycle $a_1y_2xya_1$ is a PC $C_4$ with color sequence $21j1$, where $a_1 \in A_1$ and $j = c(xy) \notin \{1, 2\}$.

This contradiction shows the second part of condition (P3), which completes the proof of Theorem 2. □

4 Proof of Theorem 3

4.1 Proof of the first part

We first prove the existence of a good mapping by induction on the number of vertices. If one of the bipartitions of $G$ is empty, then the assertion trivially holds. Thus, we may assume that both bipartitions have at least one vertex.

Suppose that $G$ does not admit a good mapping. Let $X$ and $Y$ be the bipartitions of $G$, let $x_0 \in X$, let $G' = G - x_0$ and let $X' = X - \{x_0\}$. By the induction hypothesis, $G'$ admits a good mapping $\varphi : V(G') \to \{1, \ldots, k\}$. We call a vertex $y$ in $Y$ irregular if $c(x_0y) \neq \varphi(y)$. We take a good mapping $\varphi$ of $G'$ so that

(X1) the number of irregular vertices is as small as possible.

We now focus on the properties of irregular vertices, showing the following three claims.

Claim 1 For any irregular vertex $y$, there exists a vertex $x \in X'$ such that $\varphi(x) \neq \varphi(y) = c(xy)$.

Proof Suppose, to the contrary, that there exists an irregular vertex $y_0$ such that for every $x \in X'$ with $\varphi(x) \neq \varphi(y_0)$, we have $c(xy_0) \neq \varphi(y_0)$. Since $\varphi$ is a good mapping, we have $c(xy_0) = \varphi(x)$ for such a vertex $x \in X'$. In this case, let $\varphi'(y_0) = c(x_0y_0)$ and

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\( \varphi'(v) = \varphi(v) \) for all \( v \in X' \cup Y - \{y_0\} \). Then we see that \( \varphi' \) is a good mapping of \( G' \) for which \( y_0 \) is not irregular, which contradicts condition (X1). \( \blacksquare \)

**Claim 2** There are two irregular vertices \( y_1 \) and \( y_2 \) such that \( c(x_0y_1) \neq c(x_0y_2) \).

Suppose that there is a color \( i \) such that all irregular vertices \( y \) satisfy \( c(x_0y) = i \). Then setting \( \varphi(x_0) = i \), we can obtain a good mapping of \( G \), a contradiction. Therefore, there are two irregular vertices \( y_1 \) and \( y_2 \) such that \( c(x_0y_1) \neq c(x_0y_2) \). \( \blacksquare \)

**Claim 3** For each pair of two irregular vertices \( y \) and \( y' \), we have \( \varphi(y) = \varphi(y') \).

**Proof.** Suppose that there are two irregular vertices \( y \) and \( y' \) with \( \varphi(y) \neq \varphi(y') \).

If \( c(x_0y) = c(x_0y') \), then for irregular vertices \( y_1 \) and \( y_2 \) as in Claim 2, we have \( c(x_0y) = c(x_0y') \neq c(x_0y_1) \) for some \( i \in \{1, 2\} \). Furthermore, either \( \varphi(y) \neq \varphi(y_1) \) or \( \varphi(y') \neq \varphi(y_1) \). Thus, by changing the name of \( y \), with \( y' \) or \( y \), we may assume that \( c(x_0y) \neq c(x_0y'). \)

By symmetry, we may also assume that \( \varphi(y) = 1 \) and \( \varphi(y') = 2 \). Let \( i \) and \( j \) be the colors with \( c(x_0y) = i \) and \( c(x_0y') = j \). Note that \( 1 \neq i \neq j \neq 2 \). By Claim 1, there exist vertices \( x, x' \in X' \) such that \( \varphi(x) \neq \varphi(y) = 1 = c(xy) \) and \( \varphi(x') \neq \varphi(y') = 1 = c(x'y'). \)

If \( x = x' \), then the cycle \( x_0yxy'y_0 \) is a PC \( C_4 \) with color sequence \( i12j \), a contradiction. Thus, we have \( x \neq x' \).

Let \( p = \varphi(x) \). Since \( \varphi \) is a good mapping and \( \varphi(y') = 2 \), the color \( c(xy) \) is either \( p \) or \( 2 \). Then consider the cycle \( x_0yxy'y_0 \) whose color sequence is either \( i1pj \) or \( i12j \). Since it is not a PC \( C_4 \), we must have \( c(xy') = p = \varphi(x) = j \). By the same argument, we also obtain \( c(x'y') = \varphi(x') = i \). Therefore, the cycle \( yxy'y'x \) is a PC \( C_4 \) with color sequence \( 1i2j \), a contradiction. \( \blacksquare \)

By Claim 3 and symmetry, we may assume that \( \varphi(y) = 1 \) for each irregular vertex \( y \). Let \( X'_1 = \varphi^{-1}(1) \cap X' \). We now take a good mapping \( \varphi \) of \( G' \) so that

(X2) \( X'_1 \) is as large as possible, subject to condition (X1).

Consider the complete bipartite subgraph induced by \( X' - X'_1 \) and \( S \), where \( S \) is the set of all irregular vertices. By Lemma 7 with regarding the colors \( 2, \ldots, k \) as one color, either there exists a vertex \( \tilde{x} \) in \( X' - X'_1 \) such that \( C(\tilde{x}, S) = \{1\} \) or there exists a vertex \( \tilde{y} \) in \( S \) such that \( 1 \notin C(X' - X'_1, \tilde{y}) \). By Claim 1, the latter cannot occur. Therefore, there exists a vertex \( \tilde{x} \) in \( X' - X'_1 \) such that \( c(\tilde{x}y) = 1 \) for all irregular vertices \( y \).

If \( c(\tilde{x}y) = \varphi(y) \) for all vertices \( y \in Y - S \), then the mapping \( \varphi' \) is also good, where \( \varphi'(\tilde{x}) = 1 \) and \( \varphi'(v) = \varphi(v) \) for every \( v \in X' \cup Y - \{\tilde{x}\} \), which contradicts condition (X2). Therefore, there exists a vertex \( y_0 \in Y - S \) such that \( c(x_0y_0) \neq \varphi(y_0) \). Since \( \varphi \) is a good mapping, we have \( c(x_0y_0) = \varphi(\tilde{x}) \neq 1 \). Since \( y_0 \) is not irregular, we have \( c(x_0y_0) = \varphi(y_0) \neq \varphi(\tilde{x}) \). By Claim 2, there are two irregular vertices \( y_1 \) and \( y_2 \) such that \( c(x_0y_1) \neq c(x_0y_2) \). By symmetry, we may assume that \( c(x_0y_1) \neq \varphi(y_0) \). Since
$y_1$ is irregular, we have $c(x_0y_1) \neq 1$. Then the cycle $x_0y_1\tilde{x}y_0x_0$ has the color sequence $p_1\varphi(\tilde{x})\varphi(y_0)$, where $p = c(x_0y_1)$, a contradiction. This proves the first part of the statement.

### 4.2 Proof of the second part

We now have that $G$ admits a good mapping from $V(G)$ to $\{1, \ldots, k\}$, where $k$ is the number of colors. We may assume that all of the $k$ colors appear in $G$. For $i \in \{1, \ldots, k\}$, let $X_i = X \cap \varphi^{-1}(i)$ and $Y_i = Y \cap \varphi^{-1}(i)$. Notice that $X_i$ or $Y_i$ may possibly be an empty set, but at least one of them is non-empty. Assume that all vertices have the color degree at least two.

Now we construct a directed bipartite graph $H$ as follows. The bipartition of $H$, denoted by $C_x$ and $C_y$, are copies of the color set $\{1, \ldots, k\}$, where we let $C_x = \{x_1, \ldots, x_k\}$, $C_y = \{y_1, \ldots, y_k\}$ and $i_x$ and $i_y$ both correspond to the color $i$. Let $i_x \in C_x$ and $j_y \in C_y$. If either $i = j$ or $X_i = Y_j = \emptyset$, then we do not put an edge between $i_x$ and $j_y$ in $H$. Suppose that $i \neq j$ and $X_i \cup Y_j \neq \emptyset$. Since $\varphi$ is a good mapping, the subgraph of $G$ induced by $X_i \cup Y_j$ is a complete bipartite graph with at most two colors, which are $i$ and $j$. Then it follows from Lemma 7 that either

(i) there exists a vertex $x$ in $X_i$ such that $C(x, Y_j) = \{i\}$ or

(ii) there exists a vertex $y$ in $Y_j$ such that $C(X_i, y) = \{j\}$.

Since $i \neq j$, (i) and (ii) do not occur at the same time. Note that when $X_i \neq \emptyset$ and $Y_j = \emptyset$, then (i) must occur, and when $X_i = \emptyset$ and $Y_j \neq \emptyset$, then (ii) must occur. If (i) occurs, then we put a directed edge in $H$ from $i_x$ to $j_y$; otherwise, we put a directed edge with opposite direction. By the definition, if there is an edge out-going from the vertex $i_x$ (resp. $j_y$) in $H$, then $X_i \neq \emptyset$ (resp. $Y_j \neq \emptyset$).

We give the following two claims.

**Claim 4** The directed bipartite graph $H$ does not contain a directed $C_4$.

**Proof.** Suppose that $H$ contains a directed $C_4$, say $i_xj_yp_xq_yi_x$, where $i_x, p_x \in C_x$ and $j_y, q_y \in C_y$. By the definition of $H$, we have $i \neq j \neq p \neq q \neq i$. Since $H$ contains the directed edge from $i_x$ to $j_y$, there exists a vertex $x$ in $X_i$ such that $C(x, Y_j) = \{i\}$. Similarly, there exist vertices $y \in Y_j$, $x' \in X_p$ and $y' \in Y_q$ such that $C(X_p, y) = \{j\}$, $C(x', Y_q) = \{p\}$ and $C(X_i, y') = \{q\}$. Then the cycle $xyx'y'x$ in $G$ is a PC $C_4$ with color sequence $ijpq$, a contradiction.

**Claim 5** We may assume that each $i_x$ in $C_x$ with $X_i \neq \emptyset$ has the in-degree at least one in $H$, and so is each $j_y$ in $C_y$ with $Y_j \neq \emptyset$.

**Proof.** Suppose that the first statement does not hold. By symmetry, we may assume that $X_1 \neq \emptyset$ and the vertex $1_x$ in $C_x$ has the in-degree 0 in $H$. Since $X_1 \neq \emptyset$, all edges in $E_H(1_x, C_y - \{1_y\})$ exist and are outgoing from $1_x$.  

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Consider the complete bipartite subgraph of \( G \) induced by \( X_1 \cup \bar{Y} \), where \( \bar{Y} = \bigcup_{j=2}^{k} Y_j = Y - Y_1 \). By Lemma 7 with regarding the colors \( 2, \ldots, k \) as one color, either there exists a vertex \( \bar{x} \) in \( X_1 \) such that \( c(\bar{x}, \bar{Y}) = \{1\} \), or there exists a vertex \( \bar{y} \) in \( \bar{Y} \) such that \( 1 \notin c(X_1, \bar{y}) \). Since the edge \( 1_x j_y \) in \( H \) is directed from \( 1_x \) to \( j_y \) for all \( j \) with \( 2 \leq j \leq k \), the latter cannot occur, and hence the former holds. Since \( \varphi \) is a good mapping, \( c(\bar{x}, Y_1) = \{1\} \). Therefore, the vertex \( \bar{x} \) has color degree one in \( G \). contradicting the condition that minimum color degree is at least two.

The second statement can be shown in a symmetrical way. This completes the proof of Claim 5.

Using the claims obtained, we also give the next claim.

Claim 6 There exists a color \( i \in \{1, \ldots, k\} \) such that the in-degree of \( i_x \) or \( i_y \) is exactly one.

Proof. Suppose that any vertex in \( H \) has the in-degree at least two. Within the proof of this claim, we call this assumption condition (*) . Now take a vertex in \( H \) with maximum out-degree. By symmetry we may assume that it is \( i_x \) in \( C_x \). We denote the set of all out-neighbors of \( i_x \) by \( N_i^+(i_x) \). Then by condition (*), there exist two vertices \( i_y \) and \( j_y \) in \( C_y \) with directed edges from \( i_y \) to \( 1_x \) and from \( j_y \) to \( 1_x \), respectively. Note that \( 1 \neq i \neq j \neq 1 \), \( Y_1 \neq \emptyset \) and \( Y_j \neq \emptyset \).

By condition (*) for \( i_y \), there exists a vertex \( p_x \) with \( p \neq j \) and with a directed edge from \( p_x \) to \( i_y \). Note again that \( p \neq i \) and \( X_p \neq \emptyset \). By Claim 4 and by the directed path \( p_x i_y 1_x \), for all \( r_y \in N_i^+(i_x) \) with \( r \neq p \), the edge \( p_x r_y \) exists and is directed from \( p_x \) to \( r_y \). Thus, \( N_i^+(1_x) - \{p_y\} \subseteq N_i^+(p_x) \). Since \( i_y \in N_i^+(p_x) - N_i^+(1_x) \), it follows from the choice of \( i_x \) that \( p_y \in N_i^+(1_x) \) and \( N_i^+(1_x) - \{p_y\} = N_i^+(p_x) - \{i_y\} \). In particular, we have \( j_y \notin N_i^+(p_x) \), which implies that the edge \( j_y p_x \) is directed from \( j_y \) to \( p_x \).

Again by condition (*) for \( j_y \), there exists a vertex \( q_x \) with \( q \neq i \) and with a directed edge from \( q_x \) to \( j_y \). Since the edge \( p_x j_y \) is directed from \( j_y \) to \( p_x \), we have \( q \neq p \). By the same argument as in the previous paragraph, we see that the edge \( i_y q_x \) is directed from \( i_y \) to \( q_x \). However, this implies that \( p_x i_y q_x j_y p_x \) is a directed \( C_4 \), contradicting Claim 4.

By Claim 6 and symmetry, we may assume that the in-degree of \( 1_x \) is exactly one. Again by symmetry, we may also assume that

- there are a directed edge from \( 2_y \) to \( 1_x \) and directed edges from \( 1_x \) to \( j_y \) in \( H \) for any \( j \in \{3, \ldots, k\} \).

By Claim 5 to \( 2_y \), there is a directed edge from \( i_x \) to \( 2_y \) for some \( i \in \{3, \ldots, k\} \), say \( i = 3 \) by symmetry. If there is a directed edge from \( j_y \) to \( 3_x \) for some \( j \) with \( j \neq 1, 2, 3 \), then \( 1_x j_y 2_y 3_x \) is a directed \( C_4 \), a contradiction. Thus, by Claim 5 to \( 3_x \),

- there are a directed edge from \( 1_y \) to \( 3_x \) and directed edges from \( 3_x \) to \( j_y \) in \( H \) for any \( j \in \{2, \ldots, k\} \) with \( j \neq 3 \).
Suppose that there is a directed edge from $i_x$ to $1_y$ for some $i \neq 2$. Because of the directed path $i_x1_y3_x2_y$, there is not a directed edge from $2_y$ to $i_x$, and hence there is a directed edge from $i_x$ to $2_y$. Then for any $j \in \{3, \ldots, k\}$, because of the directed path $i_x2_y1_xj_y$, there is a directed edge from $i_x$ to $j_y$. However, these imply that the in-degree of $i_x$ is 0, contradicting Claim 5. Therefore, by Claim 5 to $1_y$,

- there are a directed edge from $2_x$ to $1_y$ and directed edges from $1_y$ to $i_x$ in $H$ for any $i \in \{3, \ldots, k\}$.

Suppose that there is a directed edge from $j_y$ to $2_x$ for some $j \in \{4, \ldots, k\}$. Then because of the directed path $j_y2_x1_x3_x$, there is a directed edge from $j_y$ to $3_x$. However, $1_xj_y3_x2_y1_x$ is a directed $C_4$, a contradiction. Therefore, by Claim 5 to $2_x$,

- there are a directed edge from $3_y$ to $2_x$ and directed edges from $2_x$ to $j_y$ in $H$ for any $j \in \{1, \ldots, k\}$ with $j \neq 2, 3$.

Now, by changing the role of $1_x$ with $2_x$ and $3_x$, we also have

- there are a directed edge from $3_x$ to $2_y$ and directed edges from $2_y$ to $i_x$ in $H$ for any $i \in \{1, \ldots, k\}$ with $i \neq 2, 3$, and

- there are a directed edge from $1_x$ to $3_y$ and directed edges from $3_y$ to $i_x$ in $H$ for any $i \in \{2, \ldots, k\}$ with $i \neq 3$.

Consider the complete bipartite subgraph of $G$ induced by $X_1 \cup \overline{Y_1}$, where $\overline{Y_1} = Y - Y_2$. By Lemma 7 with regarding the colors $3, \ldots, k$ as one color, either there exists a vertex $\tilde{x}_1$ in $X_1$ such that $c(\tilde{x}_1, \overline{Y_1}) = \{1\}$, or there exists a vertex $\overline{y}$ in $\overline{Y_1}$ such that $1 \notin c(X_1, \overline{y})$. If the latter occurs, then it contradicts that the edge $1_xi_y$ in $H$ is directed from $1_x$ to $i_y$, where $i$ is a color satisfying $\overline{y} \in Y_i$. Thus, the former holds. By the same way, we can check all required conditions, and we are done. \qed

## 5 Monochromatic large star

In this section, we discuss the existence of a large monochromatic star in an edge-coloring of a complete bipartite graph without PC $C_4$. Fujita, Li and Zhang [10, Theorem 4 (ii) and (iii)] proved the following, using Theorem 2.

**Theorem 8** Let $G$ be an edge-coloring of a complete bipartite graph without a PC $C_4$. Then $G$ contains a monochromatic star $K_{1,t}$ with $t = \frac{2n}{3}$ where $n$ is the order of the smaller bipartition of $G$.

Furthermore, they constructed an edge-coloring of a complete bipartite graph without a PC $C_4$ nor a monochromatic star $K_{1,t}$ with $t > \frac{2n}{3}$ [10, Remark 2]. However, their example uses only three colors, and it is natural to ask what happens if there are more than three colors. Theorem 3 can answer this question implying the following theorem. As we will show in Proposition 10 later, the conclusion is best possible, except for the constant term, and the constant term cannot be improved by more than $\frac{1}{6}$.
Theorem 9 Let $G$ be an edge-coloring of a complete bipartite graph without PC $C_4$, and let $k$ be the number of colors used in $G$. Then $G$ contains a monochromatic star $K_{1,t}$ with $t = \min \left\{ n, \frac{2n}{3} + \frac{k-3}{6} \right\}$ where $n$ is the order of the smaller bipartition of $G$.

Proof. Suppose that an edge-coloring $G$ of a complete bipartite graph admits no PC $C_4$. Let $X$ and $Y$ be the bipartition of $G$. By symmetry, we may assume that $|X| \geq |Y| = n$. For $i \in \{1, \ldots, k\}$, let $X_i = X \cap \varphi^{-1}(i)$ and $Y_i = Y \cap \varphi^{-1}(i)$. When $k \leq 2$, then trivially or by Lemma 7 we are done. Thus, we may assume that $k \geq 3$. Furthermore, we may also assume that the minimum color degree is at least two.

By Theorem 3, $G$ admits a good mapping from $V(G)$ to $\{1, \ldots, k\}$ satisfying the conditions in Theorem 3. Let 
\[
\overline{X}_i = X - \varphi^{-1}(i+1) \cap X \quad \text{and} \quad \overline{Y}_i = Y - \varphi^{-1}(i+1) \cap Y
\]
for $i \in \{1, 2, 3\}$ with indices taken module 3. Since for $i \in \{1, 2, 3\}$, we have a monochromatic star $K_{1,t_i}$ of color $i$ with center $\overline{x}_i$ or $\overline{y}_i$ where $t_i = |\overline{Y}_i| \text{ or } |\overline{X}_i|$, respectively. Note that 
\[
|\overline{X}_1| + |\overline{X}_2| + |\overline{X}_3| = 3|X| - \left( |X_1| + |X_2| + |X_3| \right) = 2|X| + \sum_{i=4}^{k} |X_i|.
\]
Similarly, we also obtain 
\[
|\overline{Y}_1| + |\overline{Y}_2| + |\overline{Y}_3| = 2|Y| + \sum_{i=4}^{k} |Y_i|.
\]
Since all of the $k$ colors appear in $G$, $X_i \neq \emptyset$ or $Y_i \neq \emptyset$ for any $i \in \{4, \ldots, k\}$, and hence $\sum_{i=4}^{k} |X_i| + \sum_{i=4}^{k} |Y_i| \geq k - 3$. This implies that 
\[
|\overline{X}_1| + |\overline{X}_2| + |\overline{X}_3| + |\overline{Y}_1| + |\overline{Y}_2| + |\overline{Y}_3| \geq 2|X| + 2|Y| + k - 3 \geq 4n + (k - 3).
\]
Therefore, we have either 
\[
|\overline{X}_i| \geq \frac{2n}{3} + \frac{k-3}{6} \quad \text{or} \quad |\overline{Y}_i| \geq \frac{2n}{3} + \frac{k-3}{6}
\]
for some $i \in \{1, 2, 3\}$. So, there exists a monochromatic star of color $i$ with center $\overline{x}_i$ or $\overline{y}_i$ and with desired size. \qed

Proposition 10 There exists an edge-coloring $G$ of a complete bipartite graph with exactly $k$ colors such that $G$ admits neither a PC $C_4$ nor a monochromatic star $K_{1,t}$ with 
\[
t = \begin{cases} 
\frac{2n}{3} + \frac{k-1}{6} & \text{if } k \text{ is even}, \\
\frac{2n}{3} + \frac{k+4}{6} & \text{if } k \text{ is odd},
\end{cases}
\]
where $n$ is the order of the smaller bipartition of $G$. 12
Proof. We here prove the case \( k \) is an even integer, but the case \( k \) is odd can be shown by suitable modification.

We set pairwise disjoint sets \( X_i \) and \( Y_i \) for \( i \in \{1, \ldots, k\} \) satisfying the following conditions for a large integer \( r \);

\[
\begin{align*}
|X_1| &= |X_2| = |X_3| = |Y_1| = |Y_2| = |Y_3| = r, \\
|X_i| &= 1 \quad \text{and} \quad |Y_i| = 0 \quad \text{if} \ i \ \text{is an even integer with} \ 4 \leq i \leq k-2, \\
|X_i| &= 0 \quad \text{and} \quad |Y_i| = 1 \quad \text{if} \ i \ \text{is an odd integer with} \ 5 \leq i \leq k-1, \\
|X_k| &= 1 \quad \text{and} \quad |Y_k| = 1.
\end{align*}
\]

Now we construct an edge-coloring of the complete bipartite graph with bipartition \( \bigcup_{i=1}^{k} X_i \) and \( \bigcup_{i=1}^{k} Y_i \). We first put an edge of color \( k \) between \( X_k \) and \( Y_k \). Then for \( i \in \{4, \ldots, k-1\} \), we play the following operation iteratively with the reverse order;

- If \( i \) is an even integer, then put edges between \( X_i \) and \( \bigcup_{j=i+1}^{k} Y_j \) of color \( i \);
- If \( i \) is an odd integer, then put edges between \( Y_i \) and \( \bigcup_{j=i+1}^{k} X_j \) of color \( i \).

For \( i \in \{1, 2, 3\} \), we connect \( X_i \) to \( Y_i \) and \( Y_i \) to \( X_i \) by edges of color \( i \) respectively, where

\[
X_i = X - \varphi^{-1}(i+1) \cap X \quad \text{and} \quad Y_i = Y - \varphi^{-1}(i+1) \cap Y
\]

for \( i \in \{1, 2, 3\} \) with indices taken module 3. Then we obtain an edge-coloring, say \( G \), of the complete bipartite graph with bipartition \( X = \bigcup_{i=1}^{k} X_i \) and \( Y = \bigcup_{i=1}^{k} Y_i \), and it uses exactly \( k \) colors. Let \( n = |X| = |Y| = 3r + \frac{k-2}{2} \).

It is easy to see that \( G \) contains a monochromatic star of center in \( X_1 \) and the leaves \( Y_1 \), and its size is maximum in \( G \). Since \( r = \frac{n}{3} - \frac{k-2}{6} \), note that

\[
|Y_1| = 2r + \frac{k-2}{2} = \frac{2n}{3} + \frac{k-2}{6},
\]

which completes the proof. \( \square \)

References


