[\(a, b\)]-Factors of Graphs on Surfaces

Ryota Matsubara\(^1\), Haruhide Matsuda\(^1\), Nana Matsuo\(^1\), Kenta Noguchi\(^2\), and Kenta Ozeki\(^3\)

\(^1\) Department of Mathematics, Shibaura Institute of Technology
307 Fukasaku, Saitama, 337-8577 Japan
\{ryota, hmatsuda, mf15069\}@sic.shibaura-it.ac.jp
\(^2\) Department of Information Sciences, Tokyo University of Science
2641 Yamazaki, Noda, Chiba, 278-8510 Japan
noguchi@rs.tus.ac.jp
\(^3\) Faculty of Environment and Information Sciences,
Yokohama National University
79-7 Tokiwadai, Hodogaya-ku, Yokohama, 240-8501 Japan
ozeki-kenta-xr@ynu.ac.jp

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Abstract

A well-known conjecture of Grünbaum [2] and Nash-Williams [6] asserts that every 4-connected toroidal graph has a Hamiltonian cycle. Related to this conjecture, Kawarabayashi and Ozeki [4] proved two results on a 2-factor and a 3-factor. In this paper, motivated by these results, we give several sufficient conditions for a graph embedded in a surface to have an \([a, b]\)-factor. We also show that several conditions are best possible.

Keywords: graph; surface; factor; Hamiltonian cycle

1 Introduction and main theorems

All graphs considered in the paper are simple. Tutte [9] proved that every 4-connected planar graph has a Hamiltonian cycle. Thomas and Yu [7] extended the Tutte’s theorem to that for projective planar graphs. These results are sharp in the sense that we cannot replace the condition “4-connected” by “3-connected”. For 4-connected...
graphs embedded in a torus, a well-known conjecture of Grünbaum [2] and Nash-Williams [6] asserts that every 4-connected toroidal graph has a Hamiltonian cycle. While this conjecture is still open, Thomas and Yu [8] proved that every 5-connected toroidal graph has a Hamiltonian cycle. Since a Hamiltonian cycle is a special kind of a 2-factor, which is a two-regular spanning subgraph, it is quite natural to consider the existence of a 2-factor. In fact, Dean and Ota [1] showed that every 4-connected graph on a torus has a 2-factor. This is a fundamental research for the above-mentioned conjecture by Grünbaum and Nash-Williams. Note that the connectivity condition for the existence of a 2-factor is also best possible.

A surface is a compact connected 2-manifold without boundary. We define the Euler genus \( g \) of a surface \( \Sigma \) as \( 2 - \chi(\Sigma) \), where \( \chi(\Sigma) \) is the Euler characteristic of \( \Sigma \). A closed curve \( \gamma \) on a surface \( \Sigma \) is contractible if \( \gamma \) bounds a disc. A non-contractible curve is a closed curve that is not contractible. A graph \( G \) is embedded in a surface \( \Sigma \) if the vertices of \( G \) are distinct points of \( \Sigma \) and every edge of \( G \) is a simple arc such that it connects two points that are its end vertices and the interior is disjoint from other edges and vertices. In particular, if each face is homeomorphic to a disk, then \( G \) is 2-cell embedded in \( \Sigma \). The face-width of a graph \( G \) embedded in a non-spherical surface is the smallest possible cardinality of the intersection of \( G \) with a non-contractible curve on the surface. In the case that \( G \) is a plane graph, we define its face-width as \( +\infty \). The face-width is sometimes called the representativity.

Concerning the condition that face-width is sufficiently large, Kawarabayashi and Ozeki [4] proved two results: One is that every 4-connected graph embedded in a surface of Euler genus \( g \) with minimum degree at least 5 and face-width at least \( 4g - 12 \) has a 2-factor. The other is that every 5-connected graph of even order embedded in a surface of Euler genus \( g \) with face-width at least \( \max\{44g - 117, 5\} \) has a 3-factor, which is a three-regular spanning subgraph. The connectivity conditions of these results are best possible.

Let \( G \) be a graph and let \( a \) and \( b \) be two integers with \( 1 \leq a < b \). A spanning subgraph \( F \) of \( G \) is called an \([a, b]\)-factor if each vertex of \( F \) has the degree between \( a \) and \( b \). In particular, if \( a = b = k \), then an \([a, b]\)-factor is a \( k \)-factor, which is a \( k \)-regular spanning subgraph.

Motivated by above results, we obtain the following result on an \([a, b]\)-factor.

**Theorem 1.1.** Let \( 1 \leq a < b \) and \( b \geq 3 \) be integers and let \( G \) be a graph embedded in a surface of Euler genus \( g \). Suppose that \( \delta(G) \geq a + 2 \) and face-width of \( G \) is at least \( \frac{(b + 1)(2g - 4) - 2}{b - 2} \). Then \( G \) has an \([a, b]\)-factor.

In particular, Theorem 1.1 with \( a = 2 \) and \( b = 3 \) gives the following corollary.

**Corollary 1.2.** Let \( G \) be a graph embedded in a surface \( \Sigma \) of Euler genus \( g \). Suppose that \( \delta(G) \geq 4 \) and face-width of \( G \) is at least \( 8g - 18 \). Then \( G \) has a \([2, 3]\)-factor.

We can regard a 2-factor and a 3-factor as an extremal structure of a \([2, 3]\)-factor, and hence Corollary 1.2 lies between the two results in [4]. Note that we do not need
any connectivity condition for the existence of a $[2,3]$-factor, while it is required for the existence of a 2-factor or a 3-factor.

Although Theorem 1.1 does not contain the case $a = 1$ and $b = 2$, we also show the following result.

**Theorem 1.3.** Let $G$ be a graph embedded in a surface $\Sigma$ of Euler genus $g \leq 2$. Suppose that $\delta(G) \geq 3$. Then $G$ has a $[1,2]$-factor.

As we will show in Section 4.1, the condition of Euler genus $g \leq 2$ is necessary in Theorem 1.3. In fact, for any surface $\Sigma$ of Euler genus $g \geq 3$, there exist infinitely many graphs $G$ embedded in $\Sigma$ such that $\delta(G) \geq 3$ and the face-width of $G$ is sufficiently large, but $G$ has no $[1,2]$-factor.

We now discuss how sharp the lower bound of $\delta(G)$ in Theorems 1.1 and 1.3 is. Note that for integers $a$ and $b$ with $1 \leq a < b$, if a graph $G$ with $\delta(G) \geq a$ contains a vertex $v$ of degree at least $b + 1$ and all of the neighbors of $v$ have degree exactly $a$, then $G$ clearly has no $[a,b]$-factor. This suggests that the minimum degree condition $\delta(G) \geq a$ cannot guarantee the existence of an $[a,b]$-factor for any $1 \leq a < b$. Therefore we need the condition $\delta(G) \geq a + 1$.

Furthermore, for the cases $a \in \{1,2,3,4\}$, the condition $\delta(G) \geq a + 2$ in Theorems 1.1 and 1.3 is best possible. We shall show this in Section 4.2. On the other hand, we can lower the minimum degree condition of Theorem 1.1 by one for the case of $a = 5$.

**Theorem 1.4.** Let $G$ be a graph embedded in a surface of Euler genus $g$. Suppose that $\delta(G) \geq 6$ and face-width of $G$ is at least $12g - 24$. Then $G$ has a $[5,6]$-factor.

It is well known that any graph $G$ embedded in a surface $\Sigma$ of Euler genus $g$ with $|V(G)| \geq 3$ satisfies

$$|E(G)| \leq 3|V(G)| + 3g - 6,$$

which is obtained by Lemma 2.1 in Section 2 and hence the average degree is at most $6 + \frac{6g - 12}{|V(G)|}$. This implies that for any fixed surface $\Sigma$, there are only finitely many graphs embedded in $\Sigma$ with $\delta(G) \geq 7$. Therefore, it is less motivated to focus on the existence of an $[a,b]$-factor for $6 \leq a < b$. Theorems 1.1–1.4 give all the interesting cases of sufficient conditions for the existence of an $[a,b]$-factor with $1 \leq a < b$.

The remaining part of this paper is organized as follows: In the next section, we prepare some notation and preliminary results that will be used in our proofs. Section 3 deals with proofs of Theorems 1.1 and 1.3. After showing sharpness of some conditions of Theorems 1.1 and 1.3 in Section 4, we prove Theorem 1.4 in Section 5.

## 2 Notation and preliminary results

For a vertex $v$ of a graph $G$, we denote by $N_G(v)$ the *neighborhood* of $v$ in $G$, and we denote by $\deg_G(v)$ the *degree* of $v$ in $G$. For a subset $S$ of $V(G)$, let $N_G(S) = \bigcup_{v \in S} N_G(v)$ and let $\deg_G(S) = \sum_{v \in S} \deg_G(v)$. We denote by $\delta(G)$ the *minimum degree* of $G$. For two disjoint sets $S$ and $T$ of $V(G)$, we denote by $E_G(S,T)$ the set of edges in $G$ between $S$ and $T$, and let $e_G(S,T) := |E_G(S,T)|$. 


To show our theorems, we need the following results.

**Theorem 2.1 (Lovász [5])**. Let \( a \) and \( b \) be two integers with \( 1 \leq a < b \). Then a graph \( G \) has an \([a,b]\)-factor if and only if for any two disjoint subsets \( S,T \subset V(G) \),

\[
\lambda_G(S, T) := b|S| + \deg_{G-S}(T) - a|T| \geq 0.
\]

For a graph \( G \) embedded in \( \Sigma \), Euler’s formula states that \(|V(G)| - |E(G)| + |F(G)| \geq 2 - g\) holds, where \( F(G) \) be the set of faces in \( G \). Furthermore, if \( G \) is connected and 2-cell embedded in \( \Sigma \), the equality holds. (However, the above inequality suffices for our proofs.)

A cycle \( C \) of a graph \( G \) embedded in a surface \( \Sigma \) is called *non-contractible* if \( C \) is non-contractible as a curve on \( \Sigma \). The *edge-width* of \( G \) is defined as the length of a shortest non-contractible cycle in \( G \); if \( G \) contains no non-contractible cycle, then we define its edge-width as \( +\infty \). Note that if \( G \) is 2-cell embedded in a non-spherical surface, then \( G \) always contains a non-contractible cycle. If there is a non-contractible cycle \( C \), then it can be considered as a non-contractible curve in \( \Sigma \). In particular, such a non-contractible curve can be deformed by homotopic shift so that it hits only \( V(C) \). Therefore, the edge-width of any graph is greater than or equal to its face-width. (Note that if \( G \) is a triangulation, then they coincide.)

We say that a subgraph \( H \) of \( G \) is *flat* if there exists a disk \( \Delta \) in \( \Sigma \) that bounds \( H \), that is, \( \Delta \) contains all vertices and edges in \( H \). It is easy to see that if \( H \) is not flat, then \( H \) contains a non-contractible cycle.

For a face \( f \) of a graph \( G \) embedded in a surface, we denote by \( \deg_G(f) \) the length of the boundary walk of \( f \). Note that any bridge of \( G \) contained in \( f \) is counted twice in \( \deg_G(f) \). This and Euler’s formula give the next lemmas as easy consequences.

**Lemma 2.1.** Let \( G \) be a connected graph embedded in a surface of Euler genus \( g \). Then the following holds;

\[
\sum_{v \in V(G)} \left( \deg_G(v) - 6 \right) + \sum_{f \in F(G)} \left( 2\deg_G(f) - 6 \right) \leq 6g - 12.
\]

**Lemma 2.2.** Let \( H \) be a bipartite subgraph of a graph embedded in a surface of Euler genus \( g \). Then the following holds;

\[
|E(H)| \leq \begin{cases} 2|V(H)| - 2 & \text{if } H \text{ is flat;} \\ 2|V(H)| + 2g - 4 & \text{if } H \text{ is not flat.} \end{cases}
\]

Since Lemma 2.2 plays an important role for our theorems, we give the proof.

**Proof of Lemma 2.2** Euler’s formula implies

\[
|V(H)| - |E(H)| + |F(H)| \geq 2 - g.
\]

Suppose first that \( H \) is not flat. In this case, it is clear that \(|V(H)| \geq 3\). Thus, since \( H \) is bipartite, every face of \( H \) is bounded by a closed walk of length at least four, and
hence $2|E(H)| \geq 4|F(H)|$, or $2|F(H)| \leq |E(H)|$. Substituting this into the above Euler’s formula, we obtain $|E(H)| \leq 2|V(H)| + 2g - 4$, and we are done.

When $H$ is flat, we can use Euler’s formula as $g = 0$. If $|V(H)| \geq 3$, then the same argument as above implies $|E(H)| \leq 2|V(H)| - 4$; otherwise, either $|V(H)| = 1$ and $|E(H)| = 0$, or $|V(H)| = 2$ and $|E(H)| \leq 1$. In either case, we have $|E(H)| \leq 2|V(H)| - 2$. ■

3 Proofs of Theorems 1.1 and 1.3

3.1 Proof of Theorem 1.1

Suppose that $G$ satisfies all the conditions of Theorem 1.1 but has no $[a, b]$-factor. By Theorem 2.1, there exist two disjoint subsets $S, T \subseteq V(G)$, which satisfy the following inequality;

$$\lambda_G(S, T) := b|S| + \deg_{G-S}(T) - a|T| \leq -1.$$  \hspace{1cm} (1)

Choose such subsets $S$ and $T$ so that $T$ is minimal. If $T = \emptyset$, then (1) implies $0 \leq \lambda_G(S, \emptyset) = b|S| \leq -1$, a contradiction. Therefore, we may assume $T \neq \emptyset$. The minimality of $T$ implies the following claim.

Claim 3.1. For any $x \in T$, $\deg_{G-S}(x) \leq a - 1$.

**Proof.** Let $x \in T$ and put $T' = T \setminus \{x\}$. By the choice of $T$, we have $\lambda_G(S, T') \geq 0$ and $\lambda_G(S, T) \leq -1$. Thus, $1 \leq \lambda_G(S, T') - \lambda_G(S, T) \leq a - \deg_{G-S}(x)$, implying $\deg_{G-S}(x) \leq a - 1$. ■

We construct a new bipartite graph $H$ from $G$ by letting $V(H) = S \cup T$ and $E(H) = E_G(S, T)$. Then

$$|V(H)| = |S| + |T| \quad \text{and} \quad |E(H)| = e_G(S, T).$$

By Lemma 2.2, we have

$$e_G(S, T) \leq \begin{cases} 2|S| + 2|T| - 2 & \text{if } H \text{ is flat;} \\ 2|S| + 2|T| + 2g - 4 & \text{if } H \text{ is not flat.} \end{cases} \hspace{1cm} (2)$$

Since $\delta(G) \geq a + 2$, we obtain

$$\deg_{G-S}(T) \geq (a + 2)|T| - e_G(S, T).$$ \hspace{1cm} (3)

It follows from (1) and (3) that

$$e_G(S, T) \geq b|S| + 2|T| + 1.$$

By the above inequality and (2), we obtain

$$(b - 2)|S| \leq \begin{cases} -3 & \text{if } H \text{ is flat;} \\ 2g - 5 & \text{if } H \text{ is not flat.} \end{cases} \hspace{1cm} (4)$$
In particular, if $H$ is flat or $g \leq 2$, then it deduces a contradiction. Thus, we may assume that $H$ is not flat and $g \geq 3$. By Claim 3.1 and (3), we obtain

$$(a - 1)|T| \geq \deg_{G-S}(T) \geq (a + 2)|T| - e_G(S, T)$$

and thus

$$e_G(S, T) \geq 3|T|.$$ 

Since $H$ is not flat, substituting this inequality into (2) yields

$$|T| \leq 2|S| + 2g - 4.$$ 

Thus, by (4) and the assumption $b \geq 3$,

$$|S| + |T| \leq 3|S| + 2g - 4 \leq \frac{(b + 1)(2g - 4) - 3}{b - 2}. \quad (5)$$

Since $H$ is not flat, there exists a non-contractible cycle $C$ in $H$. Note that $C$ is also non-contractible in $G$, and we can find a non-contractible curve $\gamma$ such that $\gamma \cap G \subseteq V(C) \subseteq V(H) = S \cup T$. Using $S \cap T = \emptyset$ and (5), we have

$$|\gamma \cap G| \leq |S \cup T| = |S| + |T| \leq \frac{(b + 1)(2g - 4) - 3}{b - 2}.$$ 

However, this contradicts that the face-width of $G$ is at least $\frac{(b + 1)(2g - 4) - 2}{b - 2}$. This proves Theorem 1.1. \hfill $\Box$

### 3.2 Proof of Theorem 1.3

The proof is quite similar to that of Theorem 1.1. Assume that $G$ has no $[1, 2]$-factor. By (4) in the proof of Theorem 1.1, we obtain $0 = (b - 2)|S| \leq 2g - 5$. This contradicts that $g \leq 2$ and proves Theorem 1.3. \hfill $\Box$

### 4 Sharpness

In this section, we discuss the sharpness of our results.

#### 4.1 The condition on Euler genus in Theorem 1.3

Let $\Sigma$ be a surface with Euler genus $g \geq 3$, let $H$ be a triangulation of $\Sigma$ with sufficiently large face-width and let $G$ be the face subdivision of $H$, which is obtained by adding a single vertex into each face of $H$ and joining it to all vertices on the corresponding boundary. It is easy to see that the face-width of $G$ is equal to that of $H$ (see Figure 1).

Since $H$ is a triangulation of $\Sigma$, Euler's formula gives

$$|F(H)| = 2|V(H)| + 2g - 4 > 2|V(H)|,$$
Figure 1: Each bold curve represents a non-contractible curve in $\Sigma$.

because $g \geq 3$. Suppose that $G$ contains a $[1, 2]$-factor $R$. Since $\deg_{R}(x) \geq 1$ for any $x \in V(G) - V(H)$, $R$ must have at least $|F(H)|$ edges between $V(G) - V(H)$ and $V(H)$. On the other hand, $R$ can have at most $2|V(H)|$ edges between $V(G) - V(H)$ and $V(H)$ because any vertex in $R$ has degree at most 2. Thus, we obtain $|F(H)| \leq 2|V(H)|$, which contradicts the above inequality. Therefore, $G$ contains no $[1, 2]$-factor.

4.2 The minimum degree condition in Theorem 1.1

We show that the condition $\delta(G) \geq a + 2$ is sharp for the case of $a = 1, 2, 3, 4$. Actually, we prove that for any integer $a \in \{1, 2, 3, 4\}$, any integer $b > a$ and any surface $\Sigma$, there exist infinitely many graphs $G$ embedded in $\Sigma$ such that $\delta(G) = a + 1$ and the face-width of $G$ is sufficiently large, but $G$ has no $[a, b]$-factor.

Let $H$ be any graph embedded in $\Sigma$ such that $\delta(H) \geq a + 1$ and the face-width is sufficiently large, and let $x$ and $y$ be any pair of vertices that are incident with the same face. We divide this section into four cases depending on $a$.

**Case 1.** $a = 1$.

Let $t$ be an integer with $t > 2b$. Construct a graph $G$ from $H$ and $t$ isolated vertices by joining each of $x$ and $y$ to all $t$ isolated vertices. See Figure 2. Note that $\delta(G) = 2$ and the face-width of $G$ is one more than or equal to that of $H$. Let $S_1 = \{x, y\}$ and $T_1$ be the set of $t$ isolated vertices. By $t > 2b$, we obtain

$$\lambda_G(S_1, T_1) = b|S_1| + \deg_{G - S_1}(T_1) - |T_1| = 2b + 0 - t = 2b - t < 0.$$ 

Hence by Theorem 2.1, $G$ has no $[1, b]$-factor.
Figure 2: The graph $G$ in the case of $a = 1$.

Case 2. $a = 2$.

Let $t$ be an integer with $t > b$. Denote a path of order two by $P_2$ and construct a graph $G$ from $H$ and $t$ copies of $P_2$ by joining each of $x$ and $y$ to all vertices in $t$ copies of $P_2$. See Figure 3. Note that $\delta(G) = 3$. Let $S_2 = \{x, y\}$ and $T_2$ be the set of all vertices in $t$ copies of $P_2$. Since $t > b$ and $\deg_{G - S_2}(T_2) = |T_2| = 2t$, we obtain

$$\lambda_G(S_2, T_2) = b|S_2| + \deg_{G - S_2}(T_2) - 2|T_2| = 2b - 2t < 0.$$ 

Hence by Theorem 2.1, $G$ has no $[2, b]$-factor.

Figure 3: The graph $G$ in the case of $a = 2$.

Case 3. $a = 3$.

Let $t$ be an integer with $t > b$ and let $O^-$ be the graph obtained by removing an edge from an octahedral graph. Construct a graph $G$ from $H$ and $t$ copies of $O^-$ by identifying each of $x$ and $y$ with the two vertices whose degrees are three in each copy of $O^-$, respectively. See Figure 4. Denote $S_3 = \{x, y\}$ and $T_3$ to be the set of all vertices of degree four in the $t$ copies of $O^-$'s.

Note that $\delta(G) = 4$ and $\deg_{G - S_3}(T_3) = (2 + 3 + 2 + 3)t = 10t$. By $t > b$ and $|T_3| = 4t$, we obtain

$$\lambda_G(S_3, T_3) = b|S_3| + \deg_{G - S_3}(T_3) - 3|T_3| = 2b + 10t - 12t = 2b - 2t < 0.$$
Hence by Theorem 2.1, $G$ has no $[3, b]$-factor.

$$\begin{array}{c}
\text{Figure 4: The graph } G \text{ in the case of } a = 3.
\end{array}$$

**Case 4.** $a = 4$.

Let $t$ be an integer with $t > b$ and let $I^-$ be the graph obtained by removing an edge from an icosahedral graph. Construct a graph $G$ from $H$ and $t$ copies of $I^-$ by identifying each of $x$ and $y$ with the two vertices of degree four in each copy of $I^-$, respectively. See Figure 5. Let $S_4 = \{x, y\}$ and $T_4 = \{v \in V(I^-) \mid \deg_{I^-}(v) = 5 \text{ and } N_G(v) \cap \{x, y\} \neq \emptyset\}$. Note that $\delta(G) = 5$. By $t > b$, $|T_4| = 6t$ and $e_G(S_4, T_4) = 8t$, we obtain

$$\begin{align*}
\lambda_G(S_4, T_4) &= b|S_4| + \deg_{G-S_4}(T_4) - 4|T_4| \\
&= 2b + (5|T_4| - e_G(S_4, T_4)) - 4|T_4| \\
&= 2b - 2t < 0.
\end{align*}$$

Hence by Theorem 2.1, $G$ has no $[4, b]$-factor.

$$\begin{array}{c}
\text{Figure 5: The graph } G \text{ in the case of } a = 4.
\end{array}$$
5 Proof of Theorem 1.4

To prove Theorem 1.4, we consider the edge-width condition instead of the face-width one. Actually, we prove the following theorem.

Theorem 5.1. Let $G$ be a graph embedded in a surface $\Sigma$ of Euler genus $g$. Suppose that $\delta(G) \geq 6$ and edge-width of $G$ is at least $12g - 24$. Then $G$ has a $[5, 6]$-factor.

As we have explained in Section 2, the edge-width of any graph is greater than or equal to its face-width. Therefore, Theorem 5.1 directly implies Theorem 1.4.

The usefulness of the edge-width relies on the property that for a graph $G$ embedded in a surface and an edge $e$ in $G$, the edge-width of $G - e$ is greater than or equal to that of $G$, while this does not hold for the face-width. This property allows us to restrict ourselves to only “edge minimal graphs,” which shows that $V_{\geq 7}$ is independent, see the first paragraph of the proof of Theorem 5.1.

The proof of Theorem 5.1 depends on the following theorem.

Theorem 5.2 (Generalized Marriage Theorem [3]). Let $H$ be a bipartite graph with two partite sets $X$ and $Y$, and let $f : X \to \mathbb{N}$ be a function. Then $H$ has a spanning subgraph $R$ such that

$$\deg_R(x) = f(x) \quad \text{for all } x \in X$$

and

$$\deg_R(y) \leq 1 \quad \text{for all } y \in Y$$

if and only if

$$|N_H(S)| \geq \sum_{x \in S} f(x) \quad \text{for all } S \subseteq X.$$

Proof of Theorem 5.1. Suppose that $G$ satisfies all the conditions of Theorem 5.1, but has no $[5, 6]$-factor. We choose such a graph $G$ so that $E(G)$ is minimal. This choice gives the property that $V_{\geq 7}$ is an independent set of $G$, where $V_{\geq 7} = \{x \in V(G) \mid \deg_G(x) \geq 7\}$; if there exists an edge $x_1x_2$ in $G$ with $x_1, x_2 \in V_{\geq 7}$, then $G - x_1x_2$ also satisfies all the conditions of Theorem 5.1 and has no desired factor. However, this contradicts the minimality of $E(G)$.

Since $\delta(G) \geq 6$, any vertex in $V(G) \setminus V_{\geq 7}$ has degree exactly six in $G$. By Lemma 2.1 and the fact $\deg_G(f) \geq 3$ for any face $f$, we obtain

$$|V_{\geq 7}| \leq \sum_{x \in V_{\geq 7}} \left( \deg_G(x) - 6 \right) \leq 6g - 12. \quad (6)$$

Let $H$ be a bipartite graph with two partite sets $V_{\geq 7}$ and $N_G(V_{\geq 7})$ such that $E(H)$ is the set of all edges in $G$ joining $V_{\geq 7}$ and $N_G(V_{\geq 7})$, that is, $E(H) = E_G(V_{\geq 7}, N_G(V_{\geq 7}))$. In other words, $H$ is obtained from $G$ by deleting all vertices in $G - V_{\geq 7} - N_G(V_{\geq 7})$ and all edges connecting two vertices in $N_G(V_{\geq 7})$. Since $V_{\geq 7}$ is independent in $G$, we have $\deg_H(x) = \deg_G(x) \geq 7$ for any $x \in V_{\geq 7}$.
Suppose that $H$ has a spanning subgraph $R$ satisfying
\[
\deg_R(x) = \deg_H(x) - 6 \quad \text{for all } x \in V_{\geq 7}
\]
and
\[
\deg_R(y) \leq 1 \quad \text{for all } y \in V(G) \setminus V_{\geq 7}.
\]

Then $\overline{R} := G - E(R)$ is a $[5, 6]$-factor of $G$ since for $x \in V_{\geq 7}$ and $y \in V(G) \setminus V_{\geq 7},$
\[
\deg_{\overline{R}}(x) = \deg_G(x) - \deg_R(x) = \deg_G(x) - (\deg_H(x) - 6) = 6,
\]
and
\[
6 = \deg_G(y) \geq \deg_{\overline{R}}(y) = \deg_G(y) - \deg_R(y) \geq 6 - 1 = 5.
\]
This contradicts that $G$ has no $[5, 6]$-factor and thus such a spanning subgraph $R$ does not exist in $H$. By Theorem 5.2 and (7), there exists $S \subseteq V_{\geq 7}$ satisfying the inequality
\[
|N_H(S)| < \sum_{x \in S} \left( \deg_H(x) - 6 \right). \tag{8}
\]

Let $G_S$ be the subgraph of $G$ induced by $S \cup N_H(S)$. So, $G_S$ is obtained from $H \cap (S \cup N_H(S))$ by taking back all edges connecting two vertices in $N_H(S)$. Then we may assume that $G_S$ is connected; otherwise, suppose that $G_S$ is not connected. Let $G_1, G_2, \ldots, G_t$ be the components of $G_S$, and let $S_i = S \cap V(G_i)$ for $1 \leq i \leq t$. Since $|N_H(S)| = \sum_{i=1}^t |N_H(S_i)|$ and $\sum_{x \in S} \left( \deg_H(x) - 6 \right) = \sum_{i=1}^t \sum_{x \in S_i} \left( \deg_H(x) - 6 \right); \tag{8}$ implies that there exists an integer $i$ with $1 \leq i \leq t$ such that $|N_H(S_i)| < \sum_{x \in S_i} \left( \deg_H(x) - 6 \right).$ Thus, $S_i$ also satisfies the condition corresponding to (8), but $G_i$ is connected. So we can use $S_i$ instead of $S$.

We have the following claim.

**Claim 5.1.** $G_S$ is flat.

**Proof.** If $G_S$ is not flat, then $G_S$ contains a non-contractible cycle $C$. Thus, by (6) and (8), we have
\[
|C| \leq |S| + |N_H(S)| \leq |V_{\geq 7}| + |N_H(S)| < 2(6g - 12) = 12g - 24.
\]
This contradicts that the edge-width of $G$ is at least $12g - 24$. Hence the claim holds. 

Since $G_S$ is flat, there exists a disk $\Delta$ on $\Sigma$ containing all vertices and edges in $G_S$, and let $\widetilde{G_S}$ be the subgraph of $G$ consisting of all vertices and edges contained in $\Delta$. In other words, $G_S$ is obtained from $\widetilde{G_S}$ by deleting all vertices not in $S \cup N_G(S)$. Take such a disk $\Delta$ (and hence such a subgraph $\widetilde{G_S}$) as small as possible. Note that planar embedding of $\widetilde{G_S}$ is directly obtained from $\Delta$. So, from now on, we regard $\widetilde{G_S}$ as a plane graph, and let $f_0$ be the outer face of $\widetilde{G_S}$. Let $B$ be the set of vertices $y$ in the boundary of $f_0$ with $y \not\in S$. Since $N_H(S) \subseteq V(G_S) \subseteq V(\widetilde{G_S})$ and $\widetilde{G_S}$ consists of all vertices and edges contained in $\Delta$, all neighbors of any vertex in $\widetilde{G_S} - B$ are contained in $\widetilde{G_S}$. This implies that
\[
\deg_{\widetilde{G_S}}(x) = \deg_G(x) \geq 6 \quad \text{for all } x \in V(\widetilde{G_S}) \setminus B. \tag{9}
\]
Conversely, \( \deg_{G_S}(y) \leq \deg_G(y) \) for \( y \in B \) and the inequality can be strict, since such a vertex \( y \) may have a neighbor outside of \( \Delta \).

Suppose that there exists a vertex \( y \in B \setminus N_H(S) \). Since \( G_S \) is connected, there exists a component of \( G_S - \{ y \} \) that contains all vertices in \( S \cup N_H(S) \). However, such a component is induced by the vertices contained in some disk on \( \Sigma \), contradicting the minimality of \( \Delta \). Therefore, we have \( B \subseteq N_H(S) \).

By Lemma 2.1 for the plane graph \( \tilde{G}_S \), we obtain

\[
\sum_{v \in V(\tilde{G}_S)} \left( \deg_{\tilde{G}_S}(v) - 6 \right) + \sum_{f \in F(\tilde{G}_S)} \left( 2 \deg_{\tilde{G}_S}(f) - 6 \right) \leq -12.
\]

By (9) and \( S \cap B = \emptyset \), we obtain

\[
\sum_{v \in V(\tilde{G}_S)} \left( \deg_{\tilde{G}_S}(v) - 6 \right) = \sum_{x \in S} \left( \deg_{\tilde{G}_S}(x) - 6 \right) + \sum_{y \in B} \left( \deg_{\tilde{G}_S}(y) - 6 \right) + \sum_{v \in V(\tilde{G}_S) - S - B} \left( \deg_{\tilde{G}_S}(v) - 6 \right) \geq \sum_{x \in S} \left( \deg_{\tilde{G}_S}(x) - 6 \right) + \sum_{y \in B} \left( \deg_{\tilde{G}_S}(y) - 6 \right).
\]

Since \( \deg_{\tilde{G}_S}(f) \geq 3 \) for all \( f \in F(\tilde{G}_S) \), we obtain

\[
\sum_{f \in F(\tilde{G}_S)} \left( 2 \deg_{\tilde{G}_S}(f) - 6 \right) \geq 2 \deg_{\tilde{G}_S}(f_0) - 6.
\]

Hence we have

\[
\sum_{x \in S} \left( \deg_{\tilde{G}_S}(x) - 6 \right) + \sum_{y \in B} \left( \deg_{\tilde{G}_S}(y) - 6 \right) + \left( 2 \deg_{\tilde{G}_S}(f_0) - 6 \right) \leq -12.
\]

This, together with (8) and (9), implies that

\[
|N_H(S)| < \sum_{x \in S} \left( \deg_H(x) - 6 \right) = \sum_{x \in S} \left( \deg_{\tilde{G}_S}(x) - 6 \right) \leq \sum_{y \in B} \left( 6 - \deg_{\tilde{G}_S}(y) \right) - 2 \deg_{\tilde{G}_S}(f_0) - 6. \tag{10}
\]

For a positive integer \( i \), denote \( B_i = \{ y \in B \mid \deg_{G_S}(y) = i \} \) and \( B_{\geq i} = \{ y \in B \mid \deg_{G_S}(y) \geq i \} \). We denote the boundary walk of \( f_0 \) by \( \partial f_0 \), and give a fixed (clockwise) direction to it. We obtain the following claim.

**Claim 5.2.** \( \deg_{G_S}(f_0) \geq 2|B_1| + \frac{3}{2}|B_2| + |B_{\geq 3}| \).

**Proof.** For \( y \in B \), let \( \text{pre}(y) \) (resp. \( \text{suc}(y) \)) be the edge incoming to (resp. outgoing from) \( y \) along \( \partial f_0 \). (If a vertex \( y \in B \) appears twice or more in \( \partial f_0 \), then we choose arbitrary one as \( \text{pre}(y) \) and \( \text{suc}(y) \).) Let

\[
\mathcal{P}_1 = \left\{ (y, \text{pre}(y)) \mid y \in B \right\} \quad \text{and} \quad \mathcal{P}_2 = \left\{ (y, \text{suc}(y)) \mid y \in B \right\}.
\]
We count $|\mathcal{P}_1| + |\mathcal{P}_2|$ in two different ways: It is equal to $2|B|$ and at most $2|\partial f_0| = 2\deg_{\widetilde{G}_S}(f_0)$. Furthermore, the latter can be improved by the following reason. For an edge $e$ with $e = \text{pre}(y)$ or $e = \text{suc}(y)$ with $y \in B$, if the end vertex $x$ of $e$ other than $y$ is contained in $S$, then $e$ is contained in $\partial f_0$, but $(x, e) \notin \mathcal{P}_1 \cup \mathcal{P}_2$. Now, we count the number of such edges $e$.

Figure 6: The vertices in $B_1, B_2$ and $B_{\geq 3}$ in the graph $\widetilde{G}_S$. The shaded area represents the disk $\Delta$. Black circles and white squares are vertices in $S$ and those in $V(\widetilde{G}_S) - S$, respectively. In particular, $y_1 \in B_1$, $y_2 \in B_2$ and all other vertices in $B$ (white squares) belong to $B_{\geq 3}$, respectively.

Let $y \in B$. Since $B \subseteq N_H(S)$, there exists a vertex $x \in S$ with $xy \in E(\widetilde{G}_S)$. If $y \in B_1$, then the unique neighbor $x$ of $y$ in $\widetilde{G}_S$ is contained in $S$, and hence $\text{pre}(y)$ and $\text{suc}(y)$ coincide and both are incident with a vertex in $S$. Thus, both $\text{pre}(y)$ and $\text{suc}(y)$ are contained in $\partial f_0$, but $(x, \text{pre}(y)) \notin \mathcal{P}_1$ and $(x, \text{suc}(y)) \notin \mathcal{P}_2$. On the other hand, if $y \in B_2$, then both edges incident with $y$ appear in $\partial f_0$, and hence at least one of $\text{pre}(y)$ and $\text{suc}(y)$ is incident with a vertex in $S$. These imply

$$2|B| = |\mathcal{P}_1| + |\mathcal{P}_2| \leq 2\deg_{\widetilde{G}_S}(f_0) - 2|B_1| - |B_2|.$$ 

Since $|B| = |B_1| + |B_2| + |B_{\geq 3}|$, we obtain the desired inequality.

By (10) and Claim 5.2, we obtain

$$|N_H(S)| < \sum_{y \in B} \left( 6 - \deg_{\widetilde{G}_S}(y) \right) - 2\deg_{\widetilde{G}_S}(f_0) - 6$$

$$\leq \sum_{y \in B} \left( 6 - \deg_{\widetilde{G}_S}(y) \right) - 2\left( 2|B_1| + \frac{3}{2}|B_2| + |B_{\geq 3}| \right) - 6$$

$$= \sum_{y \in B_1} \left( 2 - \deg_{\widetilde{G}_S}(y) \right) + \sum_{y \in B_2} \left( 3 - \deg_{\widetilde{G}_S}(y) \right) + \sum_{y \in B_{\geq 3}} \left( 4 - \deg_{\widetilde{G}_S}(y) \right) - 6$$

$$\leq |B_1| + |B_2| + |B_{\geq 3}| - 6 \leq |N_H(S)| - 6.$$ 

This is a contradiction. This proves Theorem 5.1.
References


