A shorter proof of Thomassen's theorem on Tutte paths in plane graphs

Kenta Ozeki*

National Institute of Informatics,
2-1-2 Hitotsubashi, Chiyoda-ku,
Tokyo 101-8430, Japan
and
JST, ERATO, Kawarabayashi Large Graph Project
e-mail: ozeki@nii.ac.jp

Abstract

A graph is said to be *Hamiltonian-connected* if there exists a Hamiltonian path between any given pair of distinct vertices. In 1983, Thomassen proved that every 4-connected plane graph is Hamiltonian-connected, using the concept of *Tutte subgraph*. In this paper, we give a new proof to Thomassen's theorem.

1 Introduction

In 1956, Tutte [9] proved that every 4-connected plane graph contains a Hamiltonian cycle. Extending the technique of Tutte, Thomassen [8] proved that every 4-connected plane graph is *Hamiltonian-connected*, i.e. there is a Hamiltonian path between any given pair of distinct vertices. For the proof of these two results, they considered a stronger concept, called a *Tutte* subgraph.

Let T be a subgraph of a graph G. A T*-bridge* of G is either (i) an edge of $G - E(T)$ with both end vertices on T or (ii) a subgraph of G induced by the edges in a component of $G - V(T)$ and all edges from that component to T. A T-bridge with the former type is said to be *trivial*, while the latter is *non-trivial*. For a T-bridge B of G, the vertices in $B \cap T$ are the *attachments* of B (on T). We say that T is a *Tutte subgraph* in G if every T-bridge of G has at most three attachments on T. For another subgraph C of G, the subgraph T is a C*-Tutte subgraph* in G if T is a Tutte subgraph in G and every T-bridge of G containing an edge of C has at most two attachments on T. A *Tutte path* (respectively, a *Tutte cycle*) in a graph is a path (respectively, a cycle) which is a Tutte

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subgraph. For a connected plane graph G, the boundary walk of the outer face is called the *outer walk* of G. Furthermore, if it is a cycle, then it is the *outer cycle* of G.

Thomassen [8] proved the following result. (Although Thomassen's proof in [8] contained a small omission, it was corrected by Chiba and Nishizeki [1].)

Theorem 1 (Thomassen [8]) *Let G be a 2-connected plane graph, let C be the outer cycle of G, let $x \in V(C)$, let $y \in V(G) - \{x\}$, and let $e \in E(C)$. Then G has a C-Tutte path from x to y through e.*

It is easy to see that Theorem 1 implies the result stating that every 4-connected plane graph is Hamiltonian-connected. In fact, for a given pair of distinct vertices x and y in a given 4-connected plane graph G, we first specify a facial cycle C incident with x and an edge e in C. Actually, we can specify the edge e (and the cycle C) so that neither x nor y is an end vertex of e. Then we consider the graph G so that C is the outer cycle, and it follows from Theorem 1 that G has a C-Tutte path T from x to y through e. By the choice of the edge e, we have $|T| \geq 4$. If there exists a non-trivial T-bridge B of G, then the attachments of B form a cut set of order at most 3 that separates $T - (B \cap T)$ from $B - (B \cap T)$, which contradicts that G is 4-connected. Therefore, there exists no non-trivial T-bridge of G, which means that T is a Hamiltonian path. This shows that G is Hamiltonian-connected, and we are done.

Note that in several papers, finding Tutte subgraphs is a crucial method to show Hamiltonicity or other related properties of graphs on surfaces. See for example, [2, 3, 4, 5, 6, 7, 10]. The purpose of this note is to give a simpler proof to Theorem 1. Indeed, we show the following statement.

Theorem 2 *Let G be a connected plane graph and let C be the outer walk of G. Then both of the following hold:*

(I) *Let $x \in V(C)$, $y \in V(G) - \{x\}$ and $e \in E(C)$. If G has a path from x to y through e, then G has a C-Tutte path from x to y through e.*

(II) *Let $x, y \in V(C)$ with $x \neq y$, and $S \subset V(C) - \{x, y\}$ with $|S| \leq 2$.*

(a) *Suppose that $|S| = 1$ and C has a subpath Q_1 from z to y with $x \in V(Q_1)$, where $\{z\} = S$. Then G has a path T from x to y such that $V(T) \cap S = \emptyset$ and $T \cup S$ is a Q_1-Tutte subgraph in G.*

(b) *Suppose that $|S| = 2$ and C has a subpath Q_2 from x to y with $V(Q_2) \cap S = \emptyset$. Then G has a path T from x to y such that $V(T) \cap S = \emptyset$ and $T \cup S$ is a Q_2-Tutte subgraph in G.*

Note that statement (I) implies Theorem 1 as an immediate corollary and statement (II-b) is exactly Theorem (2.4) in [5]. Our new ideas to prove Theorem 2 (I) are the following;

- Not assuming 2-connectedness.
  Since Theorem 1 deals with 2-connected graphs only, in order to use the induction hypothesis, we have to use some tricks to make reduced graphs 2-connected. In fact, the proof in [8] first consider 2-cuts with certain conditions, and reduce a given graph, if such a 2-cut exists, with several cases depending on the properties of the 2-cut. On the other hand, since we only assume the connectedness in Theorem 2, we do not need to consider the step. Instead, we have to consider whether there exists a cut vertex (see Claim 1), but the proof of it is much easier than to deal with 2-cuts with certain conditions.

- A shorter proof to statement (II).
  In order to show statement (I), we need a statement like (II). Thomassen [8] (and other researchers, for example in [1, 5]) has considered a block decomposition to show statement (II), together with the induction hypothesis for statement (I). In this paper, we actually give a shorter proof to statement (II), directly using the induction hypothesis for statement (I). Because of that, we could reduce the length of the proof.

Notice also that statement (I) guarantees the existence of a C-Tutte path connecting two given vertices through a given edge, which easily implies the existence of a C-Tutte path connecting two given vertices through a given vertex. In Section 2, we will often use the latter statement when we consider the induction hypothesis.

Let P be a path or a cycle with fixed direction. For two vertices x and y in P, $P[x, y]$ denotes the subpath of P from x to y (along the direction).

2 Proof of Theorem 2

We prove Theorem 2 by using simultaneous induction on the order of G. Actually, we will show the following two statements.

(i) Theorem 2 (I) holds for a graph G if both Theorems 2 (I) and (II) hold for all graphs G' with $|G'| < |G|$.

(ii) Theorem 2 (II) holds for a graph G if Theorem 2 (I) holds for all graphs G' with $|G'| \leq |G|$.

Note that both Theorems 2 (I) and (II) clearly hold for all graphs of order at most 3. Hence proving the above two statements completes the proof of Theorem 2.

Proof of statement (i).

Let G be a plane graph, let C be the outer walk of G, let $x \in V(C)$, let $y \in V(G) - \{x\}$, and let $e \in E(C)$. Suppose that G has a path from x to y through e. We first show the following claim.

Claim 1 *If G has a cut vertex, then G has a C-Tutte path from x to y through e.*

Proof. Suppose contrary that G has a cut vertex v. Let G_1' be a component of $G - v$ with $V(G_1') \cap V(C) \neq \emptyset$, let G_1 be the subgraph of G induced by $V(G_1') \cup \{v\}$, and let $G_2 = G - V(G_1')$. Note that $|G_1|, |G_2| < |G|$. If $V\big(G_2 - v\big) \cap V(C) \neq \emptyset$, then we can use the symmetry between G_1 and G_2, and hence by symmetry, we may assume that $x \in V(G_1)$; Otherwise, that is, if $V\big(G_2 - v\big) \cap V(C) = \emptyset$, then $x \in V(G_1)$ since $x \in V(C)$. In either case, we have $x \in V(G_1)$. Let C_1 be the restriction of C into G_1.

Assume that y is also contained in G_1. Since G has a path from x to y through e, the edge e is also contained in G_1 and G_1 has a path from x to y through e. Since we assumed that Theorem 2 (I) holds for all graphs G' with $|G'| < |G|$, G_1 has a C_1-Tutte path T from x to y through e. Then we can easily see that T is a C-Tutte path in G from x to y through e, and we are done.

Hence we may assume that y is contained in $G_2 - v$. Let C_2 be the restriction of C into G_2. Here we assume that e is contained in G_1, but we can show the other case (when e is contained in G_2) by the almost same arguments. Since G has a path from x to y through e, G_1 has a path from x to v through e and G_2 has a path from v to y. Since we assumed that Theorem 2 (I) holds for all graphs G' with $|G'| < |G|$, G_1 has a C_1-Tutte path T_1 from x to v through e and G_2 has a C_2-Tutte path T_2 from v to y. Then letting $T = T_1 \cup T_2$, we see that T is a C-Tutte path in G from x to y through e, and we are done. This completes the proof of Claim 1. $\square$

Throughout the rest of the proof of statement (i), it follows from Claim 1 that we may assume that G is 2-connected. Thus, C is a cycle. We fix the direction of C in the clockwise order.

Let $e = u_1u_2$. If $\{x, y\} = \{u_1, u_2\}$, then the edge xy itself is a C-Tutte path in G, and we are done. Thus, we may assume that $\{x, y\} \neq \{u_1, u_2\}$. This and symmetry imply that $y \neq u_1, u_2$, since otherwise, we can change the roles of x and y. Furthermore, it follows from the symmetry between u_1 and u_2 that we may assume that G has a path from x to y through u_1 and then u_2. Then the subpath $C[x, u_1]$ of C, which is directed in the clockwise order, satisfies that $G - V\big(C[x, u_1]\big)$ has a path from u_2 to y.

Let H be the component of $G - V\big(C[x, u_1]\big)$ containing u_2 and y, and let C_H be the outer walk of H. Take a vertex w in $V(C_H) \cap V(C)$ so that there exists a path in H from u_2 to y through w, and w is as close to x on $C[u_2, x]$ as possible under the condition. Since u_2 can play the role of w except for the last condition, such a vertex w exists. Since we assumed that Theorem 2 (I) holds for all graphs G' with $|G'| < |G|$, H has a C_H-Tutte path T_H from u_2 to y through w. See Figure 1.

We claim that we can take such a path T_H so that $|T_H| \geq 3$, unless u_2 and y are connected by an edge that is a cut-edge of H. Suppose that $|T_H| \leq 2$. Since $y \neq u_2$, we have $|T_H| = 2$, T_H consists of only u_2 and y, and $w = u_2$ or $w = y$. If there exists a vertex w' such that $w' \neq u_2, y$ and H has a path from u_2 to y through w', then instead of the path T_H, we can take a C_H-Tutte path T_H' from u_2 to y through w'. Note that T_H' has the properties same as T_H together with the condition $|T_H'| \geq 3$, and the claim holds.

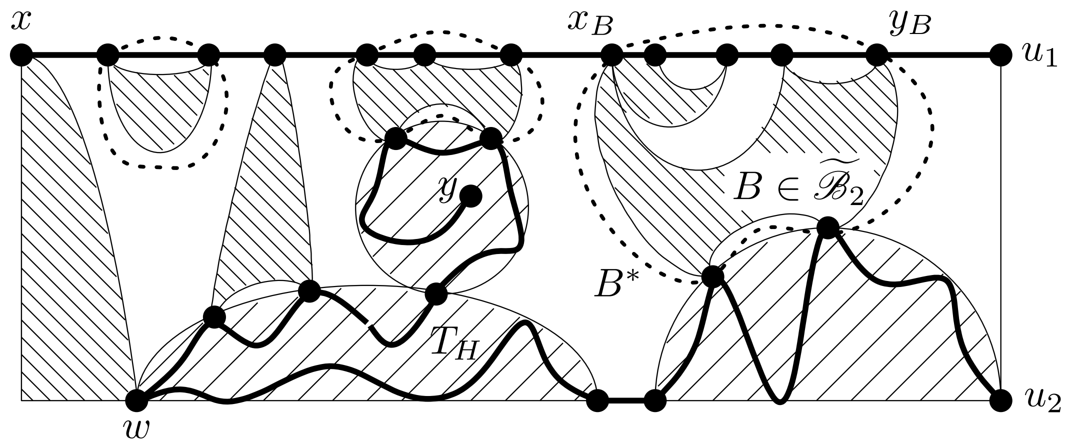

Figure 1: The C_H-Tutte path T_H and non-trivial $\big(C[x,u_1] \cup T_H\big)$-bridges of G. (The outer rectangle represents the outer cycle C of G. The regions with falling diagonals stroke from top left to bottom right represent non-trivial $\big(C[x,u_1] \cup T_H\big)$-bridges of G having an attachment on $C[x,u_1]$. The regions bounded by dotted curves are graphs B^* for $B \in \widetilde{\mathscr{B}_2}$.)

Therefore, there does not exist such a vertex w'. We can easily see that these properties imply that u_2 and y are connected by an edge that is a cut-edge of H. Then the claim holds.

Let $\mathscr{B}_1$ (and resp. $\mathscr{B}_2$) be the set of non-trivial $\big(C[x,u_1] \cup T_H\big)$-bridges B of G such that B has exactly one (resp. at least two) attachments on $C[x,u_1]$. For any $B \in \mathscr{B}_2$, let x_B and y_B be the attachments of B on $C[x,u_1]$ such that x_B is as close to x on $C[x,u_1]$ as possible and y_B is as close to u_1 on $C[x,u_1]$ as possible. Note that $x_B \neq y_B$ and $C[x_B,y_B]$ is contained in $C[x,u_1]$ for any $B \in \mathscr{B}_2$. For $B, B' \in \mathscr{B}_2$, we write $B' \preceq B$ if either (i) $B = B'$, or (ii) B' is contained in the disk bounded by $Q \cup C[x_B,y_B]$, where Q is a path in B connecting x_B and y_B. Since G is a plane graph and $C[x,u_1]$ is a subpath of the outer cycle C of G, the binary relation $\preceq$ is a partial order on $\mathscr{B}_2$. Let $\widetilde{\mathscr{B}_2}$ be the set of maximal elements of $\mathscr{B}_2$ with respect to the partial order $\preceq$. By the planarity, we have the following claim.

Claim 2 *For any $B, B' \in \widetilde{\mathscr{B}_2}$ with $B \neq B'$, $C[x_B,y_B]$ and $C[x_{B'},y_{B'}]$ are edge-disjoint.*

For $B \in \mathscr{B}_1 \cup \widetilde{\mathscr{B}_2}$, let $S_B = V(B) \cap V(T_H)$. If $B \in \mathscr{B}_1$, then let $B^* = B$, and if $B \in \widetilde{\mathscr{B}_2}$, then let B^* be the subgraph of G induced by the union of all elements $B' \in \widetilde{\mathscr{B}_2}$ such that $B' \preceq B$, together with $C[x_B,y_B]$. Let C_B be the outer walk of B^*. Note that if $B \in \widetilde{\mathscr{B}_2}$, then $C[x_B,y_B]$ is a subpath of C_B with $V(C[x_B,y_B]) \cap S_B = \emptyset$. We need the following claim.

Claim 3 *For any $B \in \mathscr{B}_1 \cup \widetilde{\mathscr{B}_2}$, we have $|S_B| \leq 2$. Furthermore, if B^* contains an edge of $C[u_2,x] - \{x\}$, then $|S_B| \leq 1$*

Proof. Let $B \in \mathscr{B}_1 \cup \widetilde{\mathscr{B}_2}$. If $S_B = \emptyset$, then the claim holds. Hence suppose that $S_B \neq \emptyset$. Since B has an attachment on $C[x, u_1]$, $B - V\big(C[x, u_1]\big)$ is a T_H-bridge of H containing an edge of C_H. Since T_H is a C_H-Tutte path in H, we have $|S_B| \leq 2$. So, the first statement holds.

Suppose that B^* contains an edge of $C[u_2, x] - \{x\}$ and $|S_B| = 2$ for some $B \in \mathscr{B}_1 \cup \widetilde{\mathscr{B}_2}$. Since B has an attachment on $C[x, u_1]$ and B^* contains an edge of $C[u_2, x] - \{x\}$, it follows from the definition of B^* that B contains an edge of $C[w, x] - \{x\}$. Let $\{z_1, z_2\} = S_B$, and by symmetry, we may assume that T_H passes through z_1 first and then z_2. Since B is a $\big(C[x, u_1] \cup T_H\big)$-bridge, $B - V\big(C[x, u_1]\big)$ is connected, and hence there exists a path a contradiction. Thus, there exists a path Q in $B - V\big(C[x, u_1]\big)$ from z_1 to z_2 through a vertex w' for some $w' \in \big(V(B) - \{x, z_1, z_2\}\big) \cap V\big(C[u_2, x]\big)$.

Since $w \in V(T_H)$, it follows from the choice of w' that $w' \in V\big(C[w, x]\big) - \{w, x\}$. Then connecting $T_H[u_2, z_1]$, Q and $T_H[z_2, y]$, we can find a path in H from u_2 to y through w', which contradicts the choice of w. $\square$

Let $B \in \widetilde{\mathscr{B}_2}$. If $|T_H| \geq 3$, then $V(T_H) - S_B \neq \emptyset$ since $|S_B| \leq 2$ by Claim 3. Suppose that $|T_H| = 2$. In this case, u_2 and y are connected by an edge that is a cut-edge of H. This implies that at least one of u_2 and y is not an attachment of B^*, and hence $|S_B| \leq 1$. In either case, $V(T_H) - S_B \neq \emptyset$. Since $V(T_H) - S_B \subset V(G) - V(B^*)$, we obtain $|B^*| < |G|$. Since we assumed that Theorems 2 (I) and (II) hold for all graphs G' with $|G'| < |G|$, B^* has a path T_B from x_B to y_B with $V(T_B) \cap S_B = \emptyset$ such that $T_B \cup S_B$ is a C_B-Tutte subgraph in B^* if $S_B = \emptyset$ (by Theorem (I)), or such that $T_B \cup S_B$ is a Q_B-Tutte subgraph in B^* if $|S_B| = 1$, where Q_B is a subpath of C_B from z_B to y_B with $x_B \in V(Q_B)$ and $\{z_B\} = S_B$ (by Theorem (II-a)), or such that $T_B \cup S_B$ is a $C[x_B, y_B]$-Tutte subgraph in B^* if $|S_B| = 2$ (by Theorem (II-b)). Note that when $|S_B| = 1$, then $E\big(C[x_B, y_B]\big) \subset E\big(C_B[z_B, y_B]\big)$ and $x_B \in V\big(C_B[z_B, y_B]\big)$. In particular, in either case, $T_B \cup S_B$ is a $C[x_B, y_B]$-Tutte subgraph in B^*.

Let
$$T = \Big(C[x, u_1] - \bigcup_{B \in \widetilde{\mathscr{B}_2}} C[x_B, y_B]\Big) \cup \bigcup_{B \in \widetilde{\mathscr{B}_2}} T_B \cup \{u_1 u_2\} \cup T_H.$$
See Figure 2. By Claim 2, T is a path in G from x to y through e. We will show that T is a C-Tutte path in G.

Let D be a non-trivial T-bridge of G. Note that either (i) D is a non-trivial $\big(T_B \cup S_B\big)$-bridge of B^* for some $B \in \widetilde{\mathscr{B}_2}$, or (ii) D is a non-trivial $\big(C[x, u_1] \cup T_H\big)$-bridge of G having at most one attachment on $C[x, u_1]$.

Suppose first that D satisfies (i). Since $T_B \cup S_B$ is a $C[x_B, y_B]$-Tutte subgraph in B^*, D has at most three attachments, and at most two attachments if D contains an edge in $C[x_B, y_B]$. Hence if $E(C) \cap E(B^*) \subseteq E(C[x_B, y_B])$, then we are done. Thus, suppose that $\big(E(C) \cap E(B^*)\big) - E(C[x_B, y_B]) \neq \emptyset$. Then B^* contains an edge of $C[u_2, x] - \{x\}$, and hence by Claim 3, $|S_B| \leq 1$. This implies that D has at most two attachments on $T_B \cup S_B$ if $S_B = \emptyset$; otherwise, that is, if $|S_B| = 1$, then D contains an edge in Q_B (see the

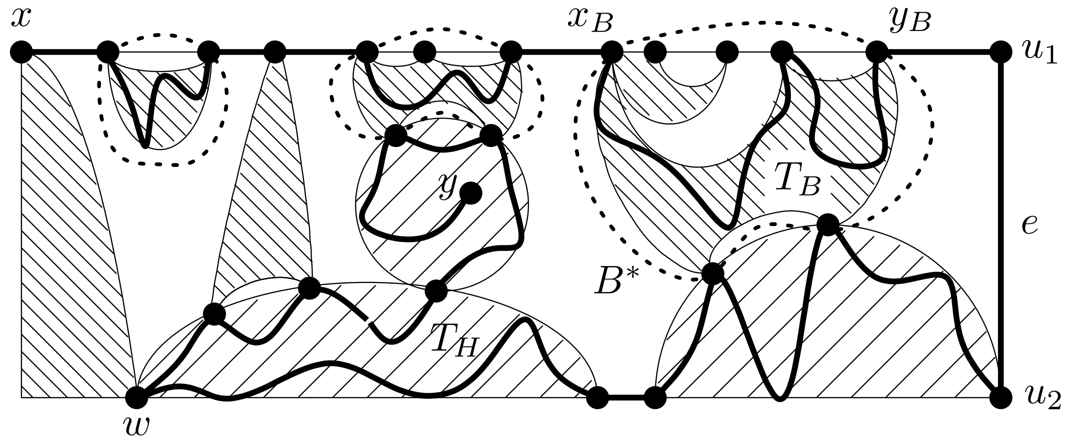

Figure 2: The C-Tutte path T in G.

definition of Q_B), and hence D also has at most two attachments on $T_B \cup S_B$. Note that $E(C) \cap E(B^*) \subset E(C_B)$, and furthermore $E(C) \cap E(B^*) = E(Q_B)$ if $|S_B| = 1$. Then in either case, we obtain that D has at most three attachments on T and at most two attachments on T if D contains an edge of C.

Suppose next that D satisfies (ii). Since $|S_D| \leq 2$, D has at most three attachments on $C[x, u_1] \cup T_H$ such that at most one of them is on $C[x, u_1]$. Furthermore, if D contains an edge of C, then by the planarity of G, D contains an edge of $C[u_2, x] - \{x\}$. Then D has at most two attachments by the same arguments as in the previous paragraph.

These imply that T is a C-Tutte path in G from x to y through e, and we complete the proof of statement (i). $\square$

Proof of statement (ii). Let G be a plane graph and let C be the outer walk of G. Let $x, y \in V(C)$ with $x \neq y$, and $S \subset V(C) - \{x, y\}$ with $|S| \leq 2$.

(a) Suppose first that $|S| = 1$ and C has a subpath Q_1 from z to y with $x \in V(Q_1)$, where $S = \{z\}$. Let G^* be the graph obtained from G by adding an edge connecting z and y so that Q_1 and the edge zy form the outer walk of G^*, say C^*. Since we assumed that Theorem 2 (I) holds for all graphs G' with $|G'| \leq |G|$, G^* has a C^*-Tutte path T^* from z to x through the edge zy. Let T be the path obtained from T^* by deleting z. Note that T is a path in G from x to y with $V(T) \cap S = \emptyset$. Since $E(Q_1) = E(C^*) - \{zy\}$, $T \cup S$ is a Q_1-Tutte subgraph in G. This completes the proof of (a).

(b) Suppose that $|S| = 2$ and C has a subpath Q_2 from x to y with $V(Q_2) \cap S = \emptyset$. Let $\{z_1, z_2\} = S$, and by symmetry, we may assume that x, z_1, z_2, y appear in C in this clockwise order. Let G^* be the graph obtained from G by deleting z_2 and adding an edge connecting z_1 and x so that Q_2, the edge z_1x and all neighbors of z_2 appear in the outer walk of G^*, say C^*. Since we assumed that Theorem 2 (I) holds for all graphs G' with $|G'| \leq |G|$, G^* has a C^*-Tutte path T^* from z_1 to y through the edge z_1x. Let T be the path obtained from T^* by deleting z_1. Note that $V(T) \cap S = \emptyset$. Let B be a $(T \cup S)$-bridge of G. If z_2 is not an attachment of B, then B has at most three attachments and at most two attachments if B contains an edge of C^*. Notice that $E(Q_2) \subset E(C^*)$. On the other hand, if z_2 is an attachment of B, then $B - z_2$ is a T^*-bridge of G^* containing an edge of C^* because all neighbors of z_2 appear in C^*. Thus, $B - z_2$ has at most two attachments on $T \cup \{z_1\}$, and at most three attachments on $T \cup S$ one of which is z_2. In particular, by the planarity of G and the fact that Q_2 is a subpath of the outer walk of G, B contains no edge of Q_2. These imply that $T \cup S$ is a Q_2-Tutte subgraph in G, and this completes the proof of (b). $\square$

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