Topological Graph Theory from Japan

Seiya Negami*

Abstract

This is a survey of studies on topological graph theory developed by Japanese people in the recent two decades and presents a big bibliography including almost all papers written by Japanese topological graph theorists.

Introduction

A *graph* is just a combinatorial structre over a finite set expressing a binary relation, so graph theory is usually regarded as one of branches in combinatorics or discrete mathematics. However, people always draw a picture consisting of "points" and "lines" on paper to present a graph. Although such a picture is not essential to discuss an abstract structure of a graph, it stimulates the intuition of geometors and lead them to attemption of establishing a kind of geometry dealing with pictures or shapes of graphs. Topological graph theory is one of the results of such attemption.

One of main subjects in topological graph theory is "embeddings of graphs on surfaces". Any nonplanar graph cannot be drawn on the plane or the sphere so that any two edges do not cross each other, so we need another stage to put graphs without edge crossings. The torus, the projective plane, the Klein bottle and so on are such stages and topological graph theorists are interested in what happens with those graphs placed on them.

"Map Color Theorem" [123] is a typical example in topological graph theory, and it gave us a formula on the minimum number of colors that we need to paint all maps on a given closed surface so that two regions get different colors if they contact each other along their boundary lines. It is well-known that the proof of "Map Color Theorem" is equivalent to determining the minimum genus of closed surfaces in which the complete graph K_n can be embedded.

However, "Map Color Theorem" may be said to be classical today and many new types of studies have been already developed. Our purpose in this paper is to show those studies made in *Japan* in particular. The author and people around him have developed and presented most of them in the annual workshop on topological graph theory held at Yokohama National University since 1989, which is the first year of "Heisei" era in Japan.

The studies made in Japan can be categorized into the following six topics.

*Department of Mathematics, Faculty of Education and Human Sciences, Yokohoma National University, 79-2 Tokiwadai, Hodogaya-Ku, Yokohama 240-8501, Japan. E-mail: negami@edhs.ynu.ac.jp

- Re-embedding structures of graphs on surfaces
- Diagonal flips of triangulations on surfaces
- Coverings and planarity of graphs
- Ramsey theorems for spatial graphs
- Polynomial invariants of graphs
- Miscellaneous

Each section below will present an outline for each topic. Although such studies are expanding beyond the boundary of Japan, we shall show only the results given by Japanese topological graph theorists. The bibliography includes almost all of their papers.

1. Re-embedding structures of graphs on surfaces

First, let us look at Figure 1. It is clear that two pictures in the figure exhibit the same graph, say G. Thus, G can be said to have two different embeddings on the plane. Turning a part of G around two vertices deforms one into the other. This is a typical type of “re-embedding structures”.

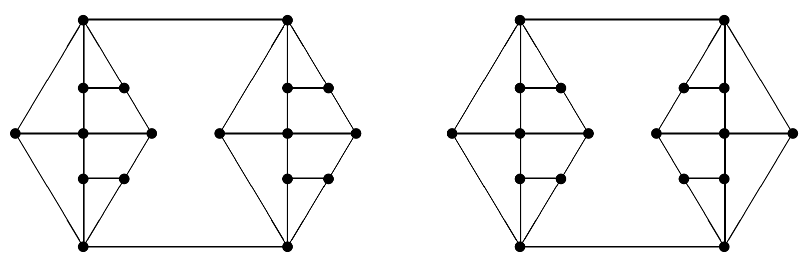

Figure 1: Turning around two vertices

In fact, one of classical theorems, proved by Whitney [138] in 1933, can be translated into that all embeddings of a connected graph on the plane can be obtained from one by repeating such deformations. This implies that any 3-connected planar graph has a unique embedding on the sphere, which is equivalent to the uniqueness of combinatorial duals of 3-connected planar graphs, proved in [137].

The property of G being 3-connected does not guarantee the uniqueness of embeddings of G on other surfaces in general. However, highly connected graphs might have a unique embedding on a closed surface. This is a motivation of Negami's studies [89, 90, 91, 92, 93, 94, 95, 100] on the uniqueness and faithfulness of embeddings. He formulated these notions as follows.

Let $f_i : G \to F^2$ be embeddings of a graph G into a closed surface F^2, which are regarded as injective continuous maps in the topological sense. Two embeddings f_1 and f_2 are said to be *equivalent* to each other if there exist a homeomorphism $h : F^2 \to F^2$

and an automorphism $\sigma : G \to G$ with $hf_1 = f_2\sigma$. A graph G is said to be *uniquely embeddable* in F^2 if any two embeddings are equivalent to each other. On the other hand, an embedding $f : G \to F^2$ is said to be *faithful* if there is a homeomorphism $h : F^2 \to F^2$ with $hf = f\sigma$ for any automorphism $\sigma : G \to G$. A graph G is said to be *faithfully embeddable* in F^2 if G has a faithful embedding in F^2.

In this terminology, Whitney's result [137] can be restated in such a way that every 3-connected planar graph is uniquely and faithfully embeddable on the sphere. For the torus, Negami has established the following theorem in [89] with the above formulation. See [89] for the notation $T(p,q,r)$, which expresses 6-regular triangulations on the torus and is basically the same as in [1]. See the next section for triangulations of surfaces.

THEOREM 1. (Negami [89]) *Every 6-connected toroidal graph is uniquely embeddable in the torus. It is faithfully embeddable in the torus if it is isomorphic to none of $T(7,2,1)$, $T(8,2,1)$ and $T(3,0,3)$.*

Furthermore, Negami has proved the following theorems for the projective plane and the Klein bottle. The notation $Kh(p,k)$ stands for one of two types of 6-regular triangulations on the Klein bottle, which have been classified in [91, 92].

THEOREM 2. (Negami [90]) *Every 5-connected projective-planar triangulation, not isomorphic to K_6, is uniquely and faithfully embeddable in the projective plane.*

THEOREM 3. (Negami [93]) *Every 5-connected projective-planar graph which contains a subdivision of K_6 as its subgraph is uniquely embeddable in the projective plane. It is faithfully embeddable in the projective plane unless it is isomorphic to K_6.*

THEOREM 4. (Negami [91]) *Every 6-connected Klein-bottlal graph is uniquely embeddable in the Klein bottle. It is faithfully embeddable in the Klein bottle unless it is isomorphic to $Kh(3,3)$.*

These theorems suggest the following problems; find the minimum number n such that every n-connected graphs embeddable in F^2 is uniquely (or faithfully) embeddable in a given closed surface F^2, with finitely many exceptions. Summarizing Negami's results in this topic, we have the following table, where each number enclosed by brackets is the number of those exceptions and "?" stands for a finite number, but not specified.

	Uniqueness	Faithfulness
The sphere	3 [0]	3 [0]
The projective plane	6 [0]	6 [0]
The torus	6 [0]	6 [3]
The Klein bottle	6 [0]	6 [1]
Others	7 [?]	7 [?]

Table 1: Embeddings and connectivity

This table presents a complete answer in a sense, but it contains something vain. For example, there is no 6-connected graph embeddable in the projective plane and there are only a finite number of 7-connected graphs embeddable in a closed surface in general. Thus, the connectivity is uncongenial to arguments on embeddings of graphs in surfaces. It is the "incompressibility" of embeddings, defined by Negami [100], that has appeared to act in its place and it can be called the roots of "representativity", which has been studied with "graph minor" recently. (See [125] for the representativity.)

Let G be a graph embedded on a closed surface F^2. A simple closed curve ℓ on F^2 is called an *n-compressing curve* for G if ℓ meets G in precisely n vertices and if ℓ does not bounds a 2-cell on F^2 which contains no vertex of G. If there is no m-compressing curve ($m < n$) for G, then G is said to be *n-incompressible*. An abstract graph is *n-incompressibly embeddable* in F^2 if it has an n-incompressible embedding on F^2.

For example, any triangulation on a closed surface is 3-incomporessible and so is K_6 on the projective plane. This implies that the following theorem includes Theorems 2 and 3 as its corollaries.

Theorem 5. (Negami [100]) *Every* 5*-connected* 3*-incompressibly embeddable, projective-planar graph is uniquely embeddable in the projective plane. Furthermore, it is faithfully embeddable in the projective plane unless it is isomorphic to* K_6*.*

Furthermore, Theorem 5 has been proved as a corollary of the following more general theorem. This theorem classifies re-embedding structures of projective-planar graphs completely, but is so complicated that we cannot explain all shortly here. Let G be a graph embedded on F^2. A *re-embedding* $f : G \to F^2$ is another embedding of G in F^2 which does not extend to an auto-homeomorphism over F^2. See [100] for the details.

Theorem 6. (Negami [100], "Re-embedding Theorem") *A re-embedding of a nonplanar* 3*-connected graph embedded in the projective plane is either a throwing-in or -out of some bridge for a cycle, or one of the re-embeddings of types I to IV.*

Based on these results for re-embeddings of projective-planar graphs, Kitakubo [46, 48] has classified those structures that generate all embeddings of 5-connected projective-planar graphs and determined the precise number of their inequivalent embeddings, as follows.

Theorem 7. (Kitakubo [48]) *Every* 5*-connected projective-planar graph admits exactly* 1*,* 2*,* 3*,* 4*,* 6*,* 9 *or* 12 *mutually inequivalent embeddings into the projective plane.*

In this theorem, two embeddings $f_i : G \to F^2$ are *equivalent* to each other if there is a homeomorphism $h : F^2 \to F^2$ with $hf_1 = f_2$. This equivalence is slightly different from Negami's. Call the latter *weak equivalence* here. The only 5-connected projective-planar graph attaining "12" is K_6. We can omit "12" from this theorem if we count the embeddings up to weak equivalence, since K_6 is uniquely embeddable in the projective plane in Negami's sense.

Kitakubo's thesis includes a proof of Theorem 7 but his proof is so long that it has been never published in any journal. Recently, Kitakubo and Negami [49] have given another brief proof of Theorem 7.

Another re-embedding theorem can be found in [76], where Nakamoto, Negami and Tanuma have developed a theory about re-embedding structures of triangulations on closed surfaces, called "panel structures". They have classified the panel structures of triangulations on the projective plane and determined the number of their inequivalent embeddings, whose values are the same as in Theorem 7.

Finally, it should be pointed out that enumeration of inequivalent embeddings of graphs in the projective plane [96] is linked to the "1-2-∞ Conjecture" presented in Section 3.

2. Diagonal flips of triangulations on surfaces

A *triangulation* G on a surface is a simple graph embedded on the surface so that each face is triangular and any two faces share at most one edge. A *diagonal flip* is a local deformation of a triangulation G which replaces a diagonal edge ac with the other bd in a quadrilateral region obtained from two triangular faces abc and acd, as shown in Figure 2. We however forbid flipping a diagonal ac if it breaks the simpleness of a graph, that is, if b and d are already adjacent in G. Two triangulations are said to be *equivalent under diagonal flips* if they can be transformed into each other by a finite sequence of diagonal flips.

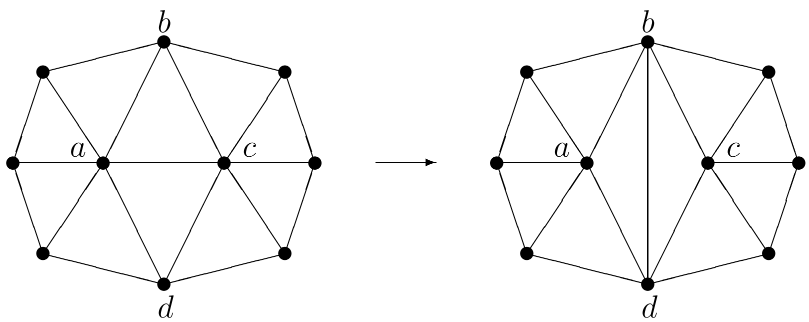

Figure 2: A diagonal flip in a triangulation

Classically, Wagner [132] proved in 1936 that any two triangulations on the sphere with the same number of vertices are equivalent under diagonal flips. Dewdney [24], Negami and Watanabe [102] also have shown the same results for the torus, the projective plane and the Klein bottle. Some of their arguments work in some general situation, but an essential part of their proofs strongly depends on those individual surfaces. So, it is hardly possible to generalize or extend their proofs for other general surfaces. Actually, the same does not hold in general.

However, Negami [109] has found some "breakthroughs" and opened a way to a general theory on diagonal flips of triangulations with the following theorem:

Theorem 8. (Negami [109]) *Given a closed surface F^2, there exists a natural number N such that two triangulations G_1 and G_2 on F^2 are equivalent to each other under diagonal flips, up to homeomorphism, if $|V(G_1)| = |V(G_2)| \geq N$.*

This theorem is a starting point of many studies on triangulations of surfaces, by Komuro, Nakamoto, Negami, Ota and Watanabe; strengthening or extending the statement, estimating the value of N and the number of diagonal flips that we need to transform one to the other, and so on [17, 21, 22, 23, 50, 51, 108, 117, 118, 119, 121, 133]. Those studies have branched toward many other studies on triangulations of surfaces. A study on "irreducible triangulations" is one of them.

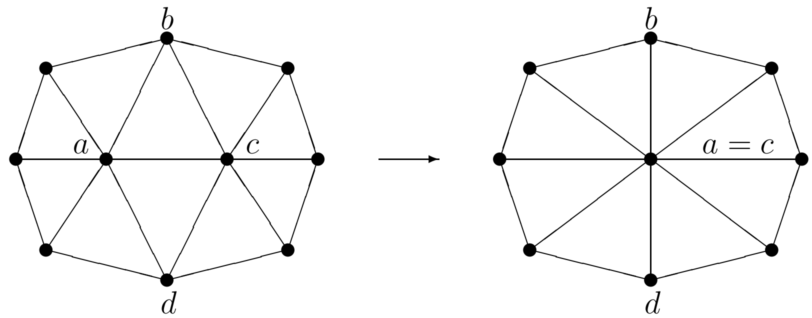

Figure 3: Contracting an edge in a triangulation

A *contraction* of an edge ac in a triangulation on a closed surface is to shrink ac into a point as shown in Figure 3. We do not carry out it if it breaks the simpleness of a graph as well as a diagonal flip and call an edge *contractible* if its contraction is applicable. A triangulation on a closed surface is said to be *irreducible* if it has no contractible edge. It is clear that any triangulation on a closed surface can be obtained from an irreducible one by the inverse process of contractions, called *vertex splittings*. (Nakamoto and Negami [83] have shown a generating theorem for triangulations of this style with some degree condition.) Thus, investigating irreducible triangulations about a given property will be a starting point to prove a theorem on triangulations with the property.

For example, the only irreducible triangulation of the sphere is K_4, which forms the tetrahedron, while there are precisely two irreducible triangulations of the projective plane, which are isomorphic to K_6 and $K_4 + \overline{K}_3$, as shown in [14]. Furthermore, Lawrencenko [52] has identified all irreducible triangulations of the torus, which are 21 in number. For the Klein bottle, Lawrencenko and Negami [54] have classified those, as follows:

Theorem 9. (Lawrencenko and Negami [54]) *There are precisely* 25 *irreducible triangulations of the Klein bottle, up to homeomorphism;* 21 *handle types* Kh1 *to* Kh21, *and* 4 *crosscap types* Kc1 *to* Kc4.

Usig this theorem, Brunet, Nakamoto and Negami [18] have proved that every 5-connected triangulation of the Klein bottle has a hamilton cycle although a vertex splitting does not preserve the hamiltonicity in general. Also, Lawrencenko and Negami [55] have determined the structure of those graphs that triangulate both the torus and the Klein bottle.

It can be proved that the number of irreducible triangulations of a closed surface is finite, as a consequence of "Wagner's conjecture" [124]. There are many papers which show an elementary proof of this fact or an upper bound for the order of irreducible triangulations. The best result among those is the following theorem at present, where $\chi(F^2)$ stands for the Euler characteristic of F^2:

Theorem 10. (Nakamoto and Ota [67]) *Any irreducible triangulations of a closed surface F^2 with $\chi(F^2) < 2$ has at most $171(2 - \chi(F^2)) - 72$ vertices.*

The finiteness of irreducible triangulations on a closed surface in number plays an essential role in Negami's proof of Theorem 8 and an upper bound for their order is necessary to estimate that for the minimum value of N. On the other hand, Lawrencenko and Negami's results [53] on inequivalent embeddings of complete graphs contributes to estimation of a lower bound for N. Furthermore, the *crossing number* of graph embedding pairs [120], the *looseness* of triangulations [57, 130, 131] and the notion of *d-covered triangulations* [82] have been devised in the stream of studies on diagonal flips. See a survey [118] for detailed connection of these topics with the theory on diagonal flips of triangulations.

The study on diagonal flips as above motivated Nakamoto's work on quadrangulations of closed surfaces [2, 3, 66, 68, 69, 71, 72, 70, 74, 75, 78]. Some of them go in parallel with arguments for triangulations, but there are several special features of quadrangulations. For example, an algebraic invariant called a *cycle parity* controls the equivalence over quadrangulations, as follows. See his own survey [86] for the details.

Theorem 11. (Nakamoto [72]) *For any closed surface F^2, there is a natural number M such that two quadrangulations G_1 and G_2 on F^2 with $|V(G_1)| = |V(G_2)| \geq M$ can be transformed into each other, up to homeomorphism, by a sequence of diagonal slides and diagonal rotations if their cycle parities are congruent.*

Very recently, Nakamoto, Negami, Ota, Širáň and Suzuki [84, 122] have investigated those triangulations on closed surfaces that quadrangulate other closed surfaces, establishing a combinatorial method to analyze them. Their papers will lead us to further studies on triangulations.

3. Coverings and planarity of graphs

A graph $\tilde{G}$ is called a *cover* or a *covering* of a simple graph G if there is a surjection $p : V(\tilde{G}) \to V(G)$, called a *projection*, which maps the neighbors of each vertex $v \in V(\tilde{G})$ bijectively to those of $p(v)$. In particular, if there is a subgroup A of the automorphism group $\mathrm{Aut}(\tilde{G})$ such that $p(u) = p(v)$ whenever $\tau(u) = v$ for some automorphism $\tau \in A$, then $\tilde{G}$ is called a *regular covering* of G. A regular covering of a graph is a notion equivalent to what is called a "voltage graph". The latter has been used to construct embeddings of complete graphs in the solution of "Map Color Theorem" [123] and its theory can be found in [33].

These are just combinatorial interpretations of the corresponding notions for "covering spaces" in topology. For example, the sphere covers the projective plane doubly. Consider the unit sphere S^2 in the 3-space. Then the projective plane P^2 can be obtained from S^2 by identifying each pair of antipodal points $\boldsymbol{x}$ and $-\boldsymbol{x}$ to a point $[\boldsymbol{x}]$. The natural map $q : S^2 \to P^2$ defined by $q(\boldsymbol{x}) = [\boldsymbol{x}]$ is a covering projection. Furthermore, suppose that a graph G is embedded on P^2. Then, the pull-back $\tilde{G} = q^{-1}(G)$ is a planar graph embedded on S^2 and can be regarded as a double covering of G.

Combining this fact and the uniqueness and faithfulness of 3-connected planar graphs, Negami proved that there is a bijection between the equivalence classes of a projective-planar graph and the isomorphism classes of its planar double coverings. This implies the following characterization of a projective-planar graph:

Theorem 12. (Negami [96]) *A connected graph is embeddable in the projective plane if and only if it has a planar double covering.*

Furthermore, Negami [99] proved in 1986 that if a connected graph has a finite planar regular covering, then it is embeddable in the projective plane, and proposed the following conjecture, called "the 1-2-∞ Conjecture" or "Negami's Planar Cover Conjecture":

Conjecture 1. (Negami [99], "the 1-2-∞ Conjecture") *If a connected graph has a finite planar covering, then it is embeddable in the projective plane.*

Define $\mathrm{sph}(G)$ to be the minimum number n that G has an n-fold planar covering. If this conjecture is true, then $\mathrm{sph}(G)$ will take only three values; 1, 2 and ∞. This is the reason why the above is called "the 1-2-∞ Conjecture".

It is easy to see that if a graph has a finite planar covering, then so does each of its minors. Thus, it suffices to prove that none of minor-minimal connected graphs among those not embeddable in the projective plane has a finite planar coverings, in order to solve the conjecture affirmatively. Such minor-minimal graphs has been already determined in [5] and [31]; they are 32 in number. Furthermore, they can be classified into 11 families, up to $Y\Delta$-transformations, and it has been proved that 10 of them do not include any counterexample to the conjecture, in a series of papers [10, 30, 39, 40, 44, 99, 101]. The remaining family consists of $K_{1,2,2,2}$ and others that can be transformed into it by $Y\Delta$-transformations, and we have:

Theorem 13. (Archdeacon, Fellows, Hliněný and Negami) *If $K_{1,2,2,2}$ has no finite planar covering, then Conjecture 1 is true.*

Recently, Hliněný and Thomas [42, 43] have analyzed the last family and concluded that there are at most 16 possible counterexamples to the conjecture, up to obvious constructions called "1-, 2- and 3-extensions".

There are several other papers related to planar coverings. For example, Archdeacon and Richter [6] has proved that if a nonplanar graph has an n-fold planar covering, then n is an even number. Negami [105, 106] has given a characterization for those graphs that can be embedded on the projective plane with only even faces, focusing on the planarity of their bipartite coverings.

There are some analogues of Conjecture 1, as follows. A graph $\tilde{G}$ is called a *branched covering* of G with a projection $p : V(\tilde{G}) \to V(G)$ if the surjection p maps the neighbors of $v \in V(\tilde{G})$ onto those of $p(v)$. A *regular branched covering* is defined as in the previous. Kitakubo [47] has proved that if a connected graph has a finite planar regular branched covering, then it is embeddable in the projective plane, and proposed the following conjecture of the same style as Conjecture 1:

Conjecture 2. (Kitakubo [47]) *If a connected graph has a finite planar branched covering, then it is embeddable in the projective plane*

Fellows [30] calls a branched covering an *emulator* and has conjectured that if a graph has a finite planar emulator, then it has a finite planar covering. Also, Hliněný [41] has presented some conjectures extending Conjecture 1 for other nonorientable closed surfaces, but there is not enough evidence for them at present.

A *k-factor* of a graph G is a spanning subgraph in G each of whose vertices has degree k. The following theorem is not so related to the topics in this section, but it has the same flavor as "the 1-2-∞ Conjecture".

Theorem 14. (Negami [98]) *Let G be a connected graph and k a positive integer. If G admits a finite covering that has a k-factor, then there is a* 1- *or* 2*-fold covering of G that has a k-factor.*

To prove this theorem, Negami has defined the *canonical bipartite covering* $B(G)$ of a graph G, as follows. Let V_1 and V_2 be two copies of $V(G)$ and let v_i denote a copy of a vertex $v \in V(G)$ belonging to V_i. Join v_i to u_{3-i} with an edge for $i = 1, 2$ whenever v is adjacent to u in G. It is clear that $B(G)$ is a bipartite graph with partite sets V_1 and V_2, and the natural map $b : V_1 \cup V_2 \to V(G)$ becomes a covering projection. If G itself is bipartite, then $B(G)$ consists of two disjoint copies of G. It is an essential phenomenon that any bipartite covering $p : \tilde{G} \to G$ factors through $B(G)$, that is, there is a covering projection $p' : \tilde{G} \to B(G)$ with $p = bp'$ unless G is bipartite.

One might expect that there would be a kind of the "canonical planar double covering" of any graph G so that any finite planar covering factors through it, from which Conjecture 1 follows. However, this is not true. For, a nonplanar graph has nonisomorphic planar double coverings as many as inequivalent embeddings in the projective plane, as proved in [96]. Nevertheless, the idea of canonical bipartite coverings led to Negami's work [105, 106] on projective-planar embeddings with even duals.

Finally, we should refer to a series of studies on enumeration of coverings of graphs by Mizuno and Sato [58, 59, 60, 126]. They have established many formulas to enumerate equivalence classes of regular coverings with additional conditions. See Sato's survey [127] for the details.

4. Ramsey theorems for spatial graphs

An embedding $f : G \to \boldsymbol{R}^3$ of a graph G into the 3-space $\boldsymbol{R}^3$ is called a *spatial embedding* of G and its image $f(G)$ is called a *spatial graph*. Conway and Gordon [20] have shown the following interesting theorem on knots and links contained in spatial complete graphs.

Theorem 15. (Conway and Gordon [20])

(i) *Any spatial embedding of* K_6 *contains a nonsplittable link.*

(ii) *Any spatial embedding of* K_7 *contains a nontrivial knot.*

This theorem can be regarded as a kind of "Ramsey theorems" in combinatorial theory, which present those phenomena that any system of sufficiently large size always contains a certain structure. (See [32] for Ramsey theory.)

More precisely, they showed that any spatial K_6 contains a 2-component link with odd linking number while K_7 contains a knot of Arf invariant 1. Can we establish theorems in the same style for prescribed link types or knot types? For example, does a spatial embedding of K_n with n sufficiently large always contain the Hopf link or the trefoil knot?

The answer to this question is negative in general. Embed K_n in an arbitrary way and make a local knot on each edge. Then, any cycle in this spatial K_n must involve many local knots and cannot be a prime knot. However, Negami [104] gave a positive answer, restricting spatial embeddings of K_n, as follows. A spatial embedding of a graph G is said to be *rectilinear* if each edge of G is straight line segment in the 3-space:

Theorem 16. (Negami [104]) *Given a spatial graph* H*, there exists a natural number* N *such that any rectilinear spatial embedding of* K_N *contains a subgraph which is ambient isotopic to a subdivision of* H*.*

Similarly, Miyauchi [61] has proved the following theorem on spatial embeddings of complete bipartite graphs:

Theorem 17. (Miyauchi [61]) *Given a spatial graph* H*, there exists a pair of natural numbers* (M, N) *such that any rectilinear spatial embedding of* $K_{M,N}$ *contains a subgraph which is ambient isotopic to a subdivision of* H*.*

Negami's original proof consists of many steps and is slightly complicated. It may be said that Miyauchi purified his arguments and extracted a more essential phenomenon, focusing on complete bipartite graphs $K_{M,N}$ instead of K_N. Actually, this led to similar theorems, relaxing the restrictions on spatial embeddings, as follows.

A *projective-rectilinear* spatial embedding of G admits an ambient-isotopic image which projects to a *rectilinear* projection, that is, each of whose edges projects to a straight line segment on the plane. A *good* spatial embedding of G admits an ambient-isotopic image which projects to a *good drawing* on the plane. That is, it admits a projection on the plane such that:

(i) The points presenting vertices are all distinct.

(ii) Each edge is a simple arc.

(iii) Any adjacent pair of edges intersect only in their ends.

(iv) Any nonadjacent pair of edges cross each other in at most one point.

(v) No three edges meet together in one point.

This is the same one as a familiar notion in topological graph theory, related to "the crossing numbers" of graphs.

THEOREM 18. (Negami [115]) *Given a spatial graph H, there exists a pair of natural numbers (M,N) such that any projective-rectilinear spatial embedding of $K_{M,N}$ contains a subgraph which is ambient isotopic to a subdivision of H.*

THEOREM 19. (Negami [116]) *Given a spatial graph H, there exists a pair of natural numbers (M,N) such that any good spatial embedding of $K_{M,N}$ contains a subgraph which is ambient isotopic to a subdivision of H.*

Since it is clear that the following implications hold, each of Theorems 16 to 18 is a corollary of its successor.

$$\text{“rectilinear”} \implies \text{“projective-rectilinear”} \implies \text{“good”}$$

It is easy to construct an example of a spatial graph which is projective-rectilinear but is not isotopic to any rectilinear embedding. Negami and Tsukamoto [114] have proved that the converse of the second arrow does not hold, showing the following theorem:

THEOREM 20. (Negmai and Tsukamoto [114]) *Every graph admits a good spatial embedding which is not projective-rectilinear if it contains an even cycle of length at least* 6*.*

Back to Theorem 16, let $R(H)$ be the minimum number N that makes the theorem valid and call it the *Ramsey number* of a spatial graph H. We can give only a theoretical upper bound for $R(H)$, but it is hardly possible to determine its precise value as well as a usual Ramsey number in combinatorial theory. We know only $R(\text{“Hopf link”}) = 6$ and $R(\text{“trefoil knot”}) = 7$. See [104] for the details.

There are many other studies on spatial graphs by Japanese mathematicians, but they are knot theorists rather than topological graph theorists. So we shall omit introducing their work here and the bibliography also does not include their papers.

5. Polynomial Invariants of graphs

Negami defined in [97] a polynomial $f(G) = f(G;t,x,y)$ associated with a graph G, called the *Negami polynomial*, by the following recursive formula:

(i) $f(\overline{K}_n) = t^n$

(ii) $f(G) = xf(G/e) + yf(G-e)$ for any edge $e \in E(G)$.

Although this definition is so simple, this polynomial gives us many information; the number of vertices, edges, components, spanning trees, cliques, self-loops, the edge connectivity, the chromatic number, being eulerian or not, and so on. For example, the Negami polynomial of Petersen graph includes $2000tx^9y^6$, as one term. This means that Petersen graph has exactly 2000 spanning trees! (See [97] for the details.)

Suppose that a graph G decomposes into two subgraphs with only two vertices in common. Split G at these two vertices and identify them in different pairs. This is one of 2-*isomorphic deformations*, defined in [138], which preserve the cycle matroids of graphs. (See [136] for matroid theory.) As is shown in [97], this deformation does not change the Negami polynomial $f(G)$. In other words, two graphs have the same polynomial $f(G)$ if they are 2-isomorphic and have the same number of components. For example, all trees T with the same number of vertices, say n, are 2-isomorphic and have the same polynomial $f(T) = t(x + txy)^{n-1}$.

The *Tutte polynomial* $T(G)$ also is another famous 2-isomorphic invariant of graphs. These two polynomials can be translated into each other by suitable substitution of variables, so they can be said to be nearly equivalent. However, the Negami polynomial has a good structure described with "splitting formula" while the Tutte polynomial does not.

Negami and Kawagoe [111] have discussed the Negami polynomial $f(G)$ from a point of view of "state models" and defined another polynomial $\tilde{f}(G; t, x, z, y)$, called the *extended Negami polynomial*, with the following expansion:

$$\tilde{f}(G; t, x, z, y) = \sum_{A \subset V(G)} f(G - A; t - 1, z, y)(x + y)^{e(A)} y^{|[A, \overline{A}]|}$$

where $e(A)$ stands for the number of edges both of whose ends belong to A and $[A, \overline{A}]$ is the cutset separating A and its complement $\overline{A} = V(G) - A$. Note that $f(\emptyset) = 1$ for the empty graph $\emptyset = G - V(G)$ in the above.

In fact, the extended Negami polynomial $\tilde{f}(G)$ can be reduced to the original polynomial by unifying two variables x and z.

$$f(G; t, x, y) = \tilde{f}(G; t, x, x, y)$$

For example, the complete graph K_3 has the following two polynomials:

$$\begin{aligned} f(K_3; t, x, y) &= tx^3 &+&\ 3tx^2y + 3t^2xy^2 + t^3y^3 \\ \tilde{f}(K_3; t, x, z, y) &= tz^3 &+&\ 3tz^2y + 3t^2zy^2 + t^3y^3 \\ & &+&\ 3txy^2 - 3tzy^2 + x^3 - z^3 + 3x^2y - 3z^2y \end{aligned}$$

The extended Negami polynomial $\tilde{f}(G)$ has so many terms in general that it is not so easy to get many information on G from it, so well as $f(G)$. However, Negami and Ota [113] have given a method to recognize the independence number $\alpha(G)$ and the degree sequence of G from $\tilde{f}(G)$ and shown that all trees up to 10 vertices can be distinguished from one another by their extended Negami polynomials.

The Negami polynomial $f(G)$ itself is a purely combinatorial object, rather than one in topological graph theory. To connect this with a theory on embeddings of graphs on surfaces, Negami has defined two other polynomials $f^*(G)$ and $h^*(G)$. The first one $f^*(G)$ was introduced in a combinatorial way together with $f(G)$ in [97] and is called the *dual polynomial*. Although the dual polynomial can be defined for any graph, if G is a graph embedded on the sphere, then it holds that

$$f^*(G) = f(G^*)$$

for the dual G^* of G on the sphere. However, this does not hold for other surfaces, while $h^*(G)$ satisfies this condition. That is,

$$h^*(G) = f(G^*)$$

where G^* stands for the dual of G taken on the closed surface where G is embedded.

To define $h^*(G)$ recursively, Negami [112] has defined several transformations which deform not only graphs but also the surfaces including them. This polynomial $h^*(G)$ is not 2-isomorphisc invariant, that is, it deos not preserve a cycle matroid of G in general. What $h^*(G)$ preserves is the matroid consisting of all cycles in G which bound faces of G on the surface. See [112] for the details.

The reader might know the connection of these polynomials of graphs with polynomial invariants of knots and links on which knot theorists have been studying recently. For example, description on the Negami polynomial can be found in [64], where Murasugi has developed a theory on invariants of graphs which can be applied to knot theory.

Also, we should refer here to Yamada's work [139, 140] on polynomial invariants for spatial graphs. The Yamada polynomial is not an invariant for combinatorial structures. It depends on spatial embeddings of a graph and need some restrictive or additional structures on vertices. Furthermore, there are many studies on polynomial invariants for spatial graphs, but they should belong to recent knot theory.

6. Miscellaneous

In this section, we shall introduce other topics on topological graph theory from Japan. Although they have never formed a big stream alone so well as the previous, some of them are related to popular topics being studied all over the world and others will give us a new seed for further studies. See each paper listed up in each topic for the details.

• Kodama's classic work

The first Japanese mathematician who can be found in papers on topological graph theory as their authors is Kodama [19, 15, 16, 37]. His work in 1960's however belongs to the "classic", and it is hardly possible to find its connection with recent studies on topological graph theory from Japan. There was not a significant work by Japanese people in 1970's. Thus, we can say that the activity of topological graph theory in Japan began with Negami's pioneer work in 1980's.

• Negami's splitter theorem

Negami's first theorem [88] was published in 1982, as a complement of what is called "Tutte's splitter theorem". A *vertex splitting* in the following is to replace a vertex v with two adjacent vertices v' and v'' and to join each neighbor of v to exactly one of v' and v'' with an edge. Note that this is not the same one as for triangulations in Section 2.

Theorem 21. (Negami [88], "Negami's splitter theorem") *Let G and H be 3-connected graphs and suppose that H is not isomorphic to any wheel. If H is a minor of G, then G can be obtained from H by a sequence of vertex splittings and edge additions.*

This theorem is purely combinatorial, but has been established in Negami's master thesis [87] for analyzing the uniqueness and faithfulness of embeddings in fact. Not only those but also many other properties studied in topological graph theory are hereditary from a graph to another graph including it as a subgraph or a minor. Thus, the above theorem is useful for analyzing those properties.

• Crossing number and thickness

The *crossing number* $\mathrm{cr}(G)$ of a graph G is one of most popular invariants in topological graph theory and is defined as the minimum number of crossings in drawings of G on the plane. It is so difficult to determine the precise value of $\mathrm{cr}(G)$ in general, but Asano [11] has given a formula for the crossing number of $K_{1,3,n}$ and $K_{2,3,n}$ with $n \leq 6$.

The *thickness* $\theta(G)$ of a graph G is another popular invariant in topological graph theory and is defined as the minimum number of planar graphs whose union forms G. Asano has shown in [12] the following inequality giving an upper bound for $\theta(G)$ and proved in [13] that any graph of genus 2 has thickness 2 or 3. Let $\gamma(G)$ denote the *genus* of a graph G, that is, the minimum number n such that G can be embedded on the orientable closed surface of genus n.

THEOREM 22. (Asano [12]) $\theta(G) \leq \gamma(G)+1$ *if a graph* G *has no triangle or is toroidal.*

• Book embeddings

A *book* is another topological object, instead of a closed surface, which we can use to exclude crossing points in a drawing of a graph G on the plane, and it consists of several half planes with common boundary, called its *spine*. A *book embedding* of G is usually defined as a drawing of G on a book with some pages so that all vertices lie along its spine and each edge is placed on one page without crossing. The *page number* $\mathrm{pg}(G)$ is the minimum number of pages that we need to embed G in a book.

Book embeddings have been studied along two contexts. One is related to spatial embeddings of graphs. For example, Endo and Otsuki [25] have discussed them in this context and Endo [26] has proved in particular that a toroidal graph can be embedded in a book of 7 pages. The other context is related to a kind of "computer layout problem" and there are Enomoto, Miyauchi and Ota's work [27, 28]. Allowing edges to cross the spine of a book, they have shown that there is an embedding of a graph G in a book with 3 pages such that each edge crosses its spine at most $O(\log n)$ times, where n stands for the number of vertices of G. Furthermore, Enomoto, Nakamigawa and Ota [29] have shown an upper bound for the paper number of $K_{n,m}$, which improved a known result given in [63].

• Linear arboricity

Also, there are several studies on interaction between combinatorial invariants and embeddings of graphs on surfaces. The *linear arboricity* $\mathrm{la}(G)$ of a graph G is the minimum number of disjoint sets into which one can decompose $V(G)$ so that each of the sets induces a disjoint union of paths. The maximum of $\mathrm{la}(G)$ taken over all graphs G embedded on a closed surface F^2 is called the *linear arboricity* of F^2 and is denoted by $\mathrm{la}(F^2)$. Hara, Ohyama and Yamashita [36] have determined the values of $\mathrm{la}(F^2)$ for the following surfaces and given an upper bound for others. (Note that la("sphere") = 3.)

$$\text{la}(\text{"projective plane"}) = 3, \quad \text{la}(\text{"torus"}) = 4, \quad \text{la}(\text{"Klein bottle"}) = 3 \text{ or } 4$$

• Achromatic number

A map $c : V(G) \to \{1, 2, 3, \ldots\}$ is called an *n-coloring* of G if $c(u) \neq c(v)$ whenever u and v are adjacent in G and if $|c(V(G))| \leq n$. A usual way is to consider the minimum number n such that G has an n-coloring, which is called the *chromatic number* $\chi(G)$ of G. To consider the possible maximum number of colors, we need suitable restriction on colorings. A *complete n-coloring* $c : V(G) \to \{1, 2, 3, \ldots\}$ is an n-coloring such that any pair of distinct colors appears at the ends of an edge. The *achromatic number* $\psi(G)$ of a graph G is the maximum number n such that G has a complete n-coloring.

There are many studies on the chromatic number of graphs on surfaces as like Map Color Theorem while there are few studies on the achromatic number of those. Hara [34] has shown that a triangulation on a closed surface has achromatic number 3 if and only if it is isomorphic to $K_{n,n,n}$ for some $n \geq 1$ while Hara and Nakamoto [35] have given an upper bound for the achromatic number of maximal outerplanar graphs.

• Bipartite graphs on surfaces

As more topological ones, there are studies on special features of embeddings of graphs on closed surfaces. We shall introduce them shortly here in order.

When we make a picture of a bipartite graph G with partite sets X and Y, we usually draw two circles to separate black vertices in X and white vertices in Y. Can we draw such circles for a fixed embedding of a bipartite graph on a closed surface? Negami [107] has given a positive answer to this question. This is not however a deep result.

• Self-dual graphs

A graph G embedded on a closed surface F^2 is said to be *self-dual* if there is a homeomorphism $h : F^2 \to F^2$ such that $h(G) = G^*$ for its dual G^* on F^2. Archdeacon [7, 8, 9] is a central figure for a study on self-dual graphs. Especially, Negami [8] has classified self-dual graphs on the projective plane, as a joint work with him. Their classification is based on that of periodic maps which exchange the partite sets of bipartite quadrangulations on the projective plane, The latter is closely related to the group actions on closed surfaces and there is a connection with Negami's arguments [99] for planar regular coverings of graphs, at least in his brain.

• Full-symmetric embeddings

Let G be a graph embedded on a closed surface F^2. To describe a group action on F^2 preserving G, a triple $\{v, e, A\}$ of a vertex $v \in V(G)$, an edge $e \in E(G)$ and a face A of G is often used and is called a *flag* associated with e. These elements v, e and A must be mutually incident, that is, e lies along the boundary of A and v is one of the ends of e. Such a flag $\{v, e, A\}$ corresponds to the triangular region with v, the middle point of e and the central point of A as its vertices. Thus, there are precisely $4|E(G)|$ flags of G.

Any homeomorphism $h : F^2 \to F^2$ sends flags to flags and can be controled by specifying the image of a fixed flag. This implies that the group $\text{Sym}(G)$ consisting of all automorphisms of G which extend to auto-homeomorphisms over F^2, called the *symmetry group* of G, has order at most $4|E(G)|$. We say that G is *full-symmetric* if

$|\mathrm{Sym}(G)| = 4|E(G)|$. Nakamoto and Negami [81] have identified the complete list of full-symmetric graphs on the projective plane and those on the torus, as follows:

- K_6, Petersen graph, K_4 with three quadrilateral faces and its dual, a single essential cycle and a wedge of loops dual to the cycle on the projective plane
- 6-regular triangulations $T(p, 0, p)$, $T(3r, r, r)$, 4-regular quadrangulations $Q(p, 0, p)$, $Q(2r, r, r)$ and 3-regular hexagonal maps $T^*(p, 0, p)$, $T^*(3r, r, r)$ on the torus.

Furthermore, they have shown that there is no full-symmetric graph on the Klein bottle. Their work should be however joined to a well-developed study on "Cayley maps".

• Strong embeddings

An embedding of a graph on a closed surface is said to be *strong* or is called a *closed* 2-*cell embedding* if each face is bounded by a cycle, that is, if each vertex on the boundary of a face appears there no more than once. This is equivalent to a "2-representative embedding of a 2-connected graph".

As one of famous conjectures in topological graph theory, we have Jaeger's *strong embedding conjecture*, which states that every 2-connected simple graph has a strong embedding in some closed surface. This implies what is called "Double cycle cover conjecture" and is still open. Yamashita [141] has contributed to the conjecture, showing some conjectures equivalent to it.

On the other hand, Nakamoto [77] has established a theorem on strong embeddings with prescribed properties in the same style as Theorem 8 on diagonal flips of triangulations, introducing two kinds of transformations, called *edge slides* and *P_2-B_2 exchanges.* This is motivated by his own work on quadrangulations. For, a bipartite quadrangulation on a closed surface works as the "radial graph" of a graph embedded there.

• Kernels on the torus

Let G be a graph embedded on a closed surface F^2 and let $f_G(\ell)$ denote the minimum value of $|G \cap \ell'|$ taken over all closed curves ℓ' homotopic to a given closed curve ℓ on F^2. This function f_G is called the *width* of G. Then, G is called a *kernel* if $f_G(\ell) > f_H(\ell)$ with some ℓ for any proper minor H of G. Schrijver [128] has already proved that any two kernels on an orientable closed surface with the same width function can be transformed into each other by a sequence of Y-Δ exchanges and taking duals. Although the two types of transformations are needed in general, Nakamoto [79] has shown that only Y-Δ exchanges suffice for the torus.

• Map Color Theorem, again

Very recently, "Map Color Theorem" has been studied again with the "representativity" of graphs on surfaces. For example, Hutchinson, Richter and Seymour [45] have shown that an *even triangulation* (or one whose vertices have even degree) on any orientable closed surface has chromatic number at most 4 if it has sufficiently large representativity, and that there are even triangulations on the projective plane of chromatic number 5. Such examples can be constructed from non-bipartite projective-planar quadrangulations by adding a vertex of degree 4 to each face. This suggests a connection to Nakamoto's work on quadrangulations introduced in Section 2.

Actually, Archdeacon, Hutchinson, Nakamoto, Negami and Ota [4] discussed it and proved the corresponding result for nonorientable closed surfaces, during Slovenia conference in 1999. Furthermore, they have shown a characterization for quadrangulations on the torus and the Klein bottle of given chromatic number. Extending their arguments with cycle parities, Nakamoto, Negami and Ota [85] have proved that any quadrangulation on a nonorientable closed surface F^2 of sufficiently large representativity is 4-colorable and that it is 4-chromatic if and only if it contains an odd cycle which cuts open F^2 into an orientable surface.

In a different context, Watanabe [134] has established a generating theorem for even triangulations on the projective plane, using two local deformations called 4-*contraction* and *removing octahedra*. There are 20 types of those triangulations irreducible to these deformations and some of them consist of an infinite series. Using this result, he has shown a constructive characterization for 5-chromatic even triangulations on the projective plane in [135].

• **Others**

There are many other participants of the annual workshop on topological graph theory at Yokohama National University mentioned in introduction. Their works also have geometric flavor, but may not stay within topological graph theory. For example, Taniyama [129] and Nagasaka [65] has studied on a differential geometric notion called the *total curvature* of a graph embedded in the Euclidean space. Higuchi and Shirai's work [38] from an analytic point of view includes a study on a kind of "curvatures of graphs", too. Also, Tsuchiya [110] has been working for *posets*, or partially ordered sets, and has shown a theory on "manifold posets" with Negami. Motivated by Watanabe's work on discrete geometry, Nakamoto, Ota and Tanuma [73, 80] have studied on orientations in planar graphs.

Furthermore, Maehara has established geometry of *frameworks*, which are graphs consisting of rigid edges and free joints at vertices. This is another stream running in the workshop. See his own survey [56] for the details. He also has studied on distance graphs on the plane, graphs representing spatial configurations of many unit spheres or balls and so on.

7. Conclusion

Exactly 10 years ago, the author wrote a survey [103] on topological graph theory in Japan, which includes four topics from Negami's work in those days. It will be interesting to compare its contents with this survey, say the number of researchers and the diversity of their studies in this field. Actually, the 10th anniversary of the annual workshop on topological graph theory at Yokohama National University involved 30 speakers and 70 participants.

It should be however pointed out that this survey does not support ultranationalism for Japan and uniformity over the earth of infinite entropy. The group of topological graph theorists in Japan should and will play its own role as one of the nodes that form a big network connecting all topological graph theorists over the world.

References

[1] A. Altshuler, Construction and enumeration of regular maps on the torus, *Discrete Math.* **4** (1973), 201–217.

[2] K. Ando, H. Komuro and A. Nakamoto, Plane quadrangulations with minimum degree three, to appear in *Discrete Math.*

[3] K. Ando and A. Nakamoto, On quadrangulations of closed surfaces covered by vertices of degree 3, to appear in *Ars Combinatoria*.

[4] D. Archdeacon, J. Hutchinson, A. Nakamoto, S. Negami and K. Ota, Chromatic numbers of quadrangulations on closed surfaces, submitted.

[5] D. Archdeacon, A Kuratowski theorem for the projective plane, *J. Graph Theory* **5** (1981), 243–246.

[6] D. Archdeacon and R.B. Richter, On the parity of planar covers, *J. Graph Theory* **14** (1990), 199–204.

[7] D. Archdeacon and R.B. Richter, The construction and classification of self-dual spherical polyhedra, *J. Combin. Theory, Ser. B* **54** (1992), 37–63.

[8] D. Archdeacon and S. Negami, The construction of self-dual projective polyhedra, *J. Combinatorial Theory, Ser. B*. **59** (1993), 122–131.

[9] D. Archdeacon, Self-dual embeddings of complete multipartite graphs, *J. Graph Theory*, **18** (1994), 735-749.

[10] D. Archdeacon, Two graphs without planar covers, manuscript, 1998.

[11] K. Asano, The crossing number of $K_{1,3,n}$ and $K_{2,3,n}$, *J. Graph Theory* **10** (1986), 1–8.

[12] K. Asano, On the genus and thickness of graphs, *J. Comb. Theory, Ser. B* **43** (1987), 287–292.

[13] K. Asano, On the thickness of graphs with genus 2, *Ars Comb.* **38** (1994), 87–95.

[14] D. Barnette, Generating the triangulations of the projective plane, *J. Combin. Theory, Ser. B* **33** (1982), 222-230.

[15] J. Battle, F. Harary, Y. Kodama and J.W.T. Youngs, Additivity of the genus of a graph, *Bull. Amer. Math. Soc.* **68** (1962), 565–568.

[16] J. Battle, F. Harary and Y. Kodama, Every planar graph with nine points has a nonplanar complement, *Bull. Amer. Math. Soc.* **68** (1962), 569–571.

[17] R. Brunet, A. Nakamoto and S. Negami, Diagonal transformations of triangulations on closed surfaces preserving specified properties, *J. Combin. Theory, Ser. B* **68** (1996), 295–309.

[18] R. Brunet, A. Nakamoto and S. Negami, Every 5-connected triangulation of the Klein bottle is hamiltonian, *Yokohama Math. J.* **47**, special issue (1999), 239–244.

[19] D.E. Cohen, F. Harary, Y. Kodama, On the embedding of complete graphs in orientable surfaces, *Mathematika, London* **10** (1963), 79–83.

[20] J.H. Conway and C.McA. Gordon, Knots and links in spatial graphs, *J. Graph Theory* **7** (1983), 445-453.

[21] C. Cortés and A. Nakamoto, Diagonal flips in outer-torus triangulations, *Discrete Math.* **216** (2000), 71–83.

[22] C. Cortés and A. Nakamoto, Diagonal flips in outer-Klein-bottle triangulations, to appear in *Discrete Math.*

[23] C. Cortés, C. Glima, A. Márquez and A. Nakamoto, Diagonal flips in outer-triangulations of surfaces, submitted.

[24] A.K. Dewdney, Wagner's theorem for the torus graphs, *Discrete Math.*, **4** (1973), 139–149.

[25] T. Endo and T. Otsuki, Notes on spatial representations of graphs, *Hokkaido Math. J.* **23** (1994), 383–398.

[26] T. Endo, The page number of toroidal graphs is at most seven, *Discrete Math.* **175** (1997), 87–96.

[27] H. Enomoto and M.S. Miyauchi, Embedding graphs nto a three page book with $O(m \log n)$ crossings of edges over the spine, *SIAM J. Discrete Math.* **12** (1999), 337–341.

[28] H. Enomoto, M.S. Miyauchi and K. Ota, Lower bounds for the number of edge-crossings over the spine in a topological book embedding of a graph, *Discrete Appl. Math.* **92** (1999), 149–155.

[29] H. Enomoto, T. Nakamigawa and K. Ota, On the pagenumber of complete bipartite graphs, *J. Comb. Theory, Ser. B* **71** (1997), 111-120.

[30] M. Fellows, Planar emulators and planar covers, manuscript, 1989.

[31] H.H. Glover, J.P. Huneke and C.S. Wang 103 graphs that are irreducible for the projective plane, *J. Comb. Theory, Ser. B* **27** (1979), 332–370.

[32] R.L. Graham, B.L. Rothschild and J.H. Spencer, *"Ramsey Theory"*, Wiley, 1980.

[33] J.L. Gross and T.W. Tucker, *"Topological Graph Theory"*, John Wiley & Sons, 1987.

[34] S. Hara Trianqurations of closed surfaces with achromatic number 3, *Yokohama Math. J.* **47**, special issue (1999), 225–229.

[35] S. Hara and A. Nakamoto, Achromatic numbers of maximal outerplanar graphs, submitted.

[36] M. Hara, Y. Ohyama and S. Yamashita, On the linear vertex-arboticity of a surface, *J. Comb. Math. Comb. Comput.* **18** (1995), 3–10.

[37] F. Harary and Y. Kodama, On the genus of an n-connected graph, *Fundam. Math.* **54** (1964), 7–13.

[38] Y. Higuchi and T. Shirai, A remark on the spectrum of magnetic Laplacian on a graph, **47**, special issue (1999), 129–141.

[39] P. Hliněný $K_{4,4} - e$ has no finite planar cover, *J. Graph Theory* **27** (1998), 51–60.

[40] P. Hliněný, Planar covers of Graphs: Negami's conjecture, Ph.D. Dissertation, Georgia Institute of Technology, 1999

[41] P. Hliněný, A note on possible extensions of Negami's conjecture, *J. Graph Theory* **32** (1999), 234–240.

[42] P. Hliněný, Another two graphs with no planar covers, manuscript, 1999.

[43] P. Hliněný and R. Thomas, On possible counterexamples to Negami's planar cover conjecture, manuscript, 1999.

[44] J.P. Huneke, A conjecture in topological graph theory, "Graph Structure Theory", edited by N. Robertson and P. Seymour, *Contemporary Math.* **147** (1993), 363–379.

[45] J.P. Hutchinson, R.B. Richter and P. Seymour, Coloring Eulerian triangulations, preprint.

[46] S. Kitakubo, Bounding the number of embeddings of 5-connected projective-planar graphs, *J. Graph Theory* **15** (1991), 199–205.

[47] S. Kitakubo, Planar branched covering of graphs, *Yokohama Math. J.* **38** (1991), 113–120.

[48] S. Kitakubo, "Embeddings of graphs into the projective plane", Doctor thesis, Tokyo Institute of Technology, 1992.

[49] S. Kitakubo and S. Negami, Re-embedding structures of 5-connected projective-planar graphs, submitted.

[50] H. Komuro, The diagonal flips of triangulations on the sphere, *Yokohama Math. J.*, **44** (1997), 115–122.

[51] H. Komuro, A. Nakamoto and S. Negami, Diagonal flips in triangulations on closed surfaces with minimum degree at least 4, *J. Combin. Theory, Ser. B* **76** (1999), 68–92.

[52] S. Lawrencenko, The irreducible triangulations of the torus, *Ukrain. Geom. Sb.*, **30** (1987), 52–62. [In Russian; MR 89c:57002; English translation: J. Soviet Math. **51**, No.5 (1990), 2537–2543.]

[53] S. Lawrencenko, S. Negami and A.T. White, Three nonisomorphic triangulations of an orientable surface with the same complete graph, *Discrete Math.* **135** (1994), 367–369.

[54] S. Lawrencenko and S. Negami, Irreducible triangulations of the Klein bottle, *J. Combin. Theory, Ser. B* **70** (1997), 265–291.

[55] S. Lawrencenko and S. Negami, Constructing the graphs that triangulate both the torus and the Klein bottle, *J. Combin. Theory, Ser. B* **77** (1999), 211–218.

[56] H. Maehara, Geometry of frameworks *Yokohama Math. J.* **47**, special issue (1999), 41–65.

[57] T. Midorikawa and S. Negami, Loosely tightness of triangulations of closed surfaces, *Sci. Rep. Yokohama Nat. Univ., Sec. I* **43** (1996), 25–41.

[58] H. Mizuno and I. Sato, Characteristic polynomials of some graph coverings, *Discrete Math.* **142** (1995), 295–298.

[59] H. Mizuno and I. Sato, Characteristic polynomials of some covers of symmetric digraphs, *Ars Combinatoria* **45** (1997), 3–12.

[60] H. Mizuno and I. Sato, Isomorphisms of cyclic abelian covers of symmetric digraphs, to appear in *Ars Combinatoria.*

[61] M.S. Miyauchi, Topological Ramsey theorem for complete bipartite graphs, *J. Combin. Theory, Ser. B* **62** (1994), 164–179.

[62] M.S. Miyauchi, Topological Ramsey theorem for complete bipartite graphs with linked circles, Y. Alavi (ed.) et al., Graph theory, combinatorics, algorithm and applications Vol.2, Proceedings of the seventh quadrennial international conference on the theory and applications of graphs, Kalamazoo, MI, USA, June 1–5, 1992. New York, NY: Wiley, (1995), 1043–1052.

[63] D.J. Muder, M.L. Weaver and D.B. West, Pagenumber of complete bipartite graphs, *J. Graph Theory* **12** (1988), 469–489.

[64] K. Murasugi, On invariants of graphs with applications to knot theory, *Trans. Amer. Math. Soc.* **314** (1989), 1–49.

[65] H. Nagasaka, Vertex dimension of complete graphs on even vertices, *Differ. Geom. Appl.* **8** (1998), 163–170.

[66] A. Nakamoto, Diagonal transformations in quadrangulations of surfaces, *Master thesis*, Yokohama National University, 1993.

[67] A. Nakamoto and K. Ota, Note on irreducible triangulations of surfaces, *J. Graph Theory* **20** (1995), 227–233.

[68] A. Nakamoto, Irreducible quadrangulations of the Klein bottle, *Yokohama Math. J.* **43** (1995), 125–139.

[69] A. Nakamoto, Irreducible quadrangulations of the torus, *J. Combin. Theory, Ser. B* **67**(1996), 183–201.

[70] A. Nakamoto, Triangulations and quadrangulations of surfaces, *Doctor thesis*, Keio University, 1996.

[71] A. Nakamoto, Diagonal transformations in quadrangulations of surfaces, *J. Graph Theory* **21** (1996), 289–299.

[72] A. Nakamoto, Diagonal transformations and cycle parities of quadrangulations on surfaces, *J. Combin. Theory, Ser. B* **67** (1996), 202–211.

[73] A. Nakamoto, K. Ota and T. Tanuma, Three-cycle reversions in oriented planar triangulations, *Yokohama Math. J.* **44** (1997), 123–139.

[74] A. Nakamoto and K. Ota, Diagonal transformations in quadrangulations and Dehn twists preserving cycle parities, *J. Combin. Theory, Ser. B* **69** (1997), 125–141.

[75] A. Nakamoto and K. Ota, Diagonal transformations of graphs and Dehn twists of surfaces, *J. Combin. Theory, Ser. B* **70** (1997), 292–300.

[76] A. Nakamoto, S. Negami and T. Tanuma, Re-embedding structures of triangulations on closed surfaces, *Sci. Rep. Yokohama Nat. Univ., Sec. I* **44** (1997), 41–55.

[77] A. Nakamoto, Transformations in closed 2-cell embeddings on surfaces preserving specified properties, *Discrete Math.* **197/198** (1999), 607–615.

[78] A. Nakamoto, Generating quadrangulations of surfaces with minimum degree at least 3, *J. Graph Theory* **30** (1999), 223–234.

[79] A. Nakamoto, The $Y\Delta$-equivalence of kernels on the torus without taking duals, *Yokohama Math. J.* **47**, special issue (1999), 183–190.

[80] A. Nakamoto and M. Watanabe, Cycle reversals in orientated plane quadrangulations and orthogonal plane partitions, *J. Geometry*, in press.

[81] A. Nakamoto and S. Negami, Full-symmetric embeddings of graphs on closed surfaces, to appear in *Memoirs of Osaka Kyoiku Univ., Ser. III, Natural Science and Applied Science.*

[82] A. Nakamoto and S. Negami, Triangulations on closed surfaces covered by vertices of given degree, to appear in *Graph and Combin.*

[83] A. Nakamoto and S. Negami, Generating triangulations on closed surfaces with minimum degree 4, submitted, 1999.

[84] A. Nakamoto, S. Negami, K. Ota and J. Širáň, Planar triangulations which quadrangulate other surfaces, submitted, 1999.

[85] A. Nakamoto, S. Negami and K. Ota, Chromatic numbers and cycle parities of quadrangulations on nonorientable closed surfaces, preprint, 2000.

[86] A. Nakamoto, Quadrangulations on closed surfaces, preprint.

[87] S. Negami, A characterization of 3-connected graphs containg a given graph and its applications. *Master thesis,* Tokyo Institute of Technology, 1981.

[88] S. Negami, A characterization of 3-connected graphs containing a given graph, *J. Combin. Theory, Ser. B* **32** (1982), 69–74.

[89] S. Negami, Uniqueness and faithfulness of embedding of toroidal graphs, *Discrete Math.* **44** (1983), 161–180.

[90] S. Negami, Uniquely and faithfully embeddable projective-planar triangulations, *J. Combin. Theory, Ser. B* **36** (1984), 189–193.

[91] S. Negami, Classification of 6-regular Klein-bottlal graphs, *Res. Rep. Inf. Sci. T.I.T.* **A-96** (1984).

[92] S. Negami, Uniqueness and faithfulness of embedding of graphs into surfaces, *Doctor thesis,* Tokyo Institute of Technology, 1985.

[93] S. Negami, Unique and faithful embeddings of projective-planar graphs, *J. Graph Theory* **9** (1985), 235–243.

[94] S. Negami, Construction of graphs which are not uniquely and not faithfully embeddable in surfaces, *Yokohama J. Math.* **33** (1985), 67–91.

[95] S. Negami, Heredity of uniqueness and faithfulness of embedding of graphs into surfaces, *Res. Rep. Inf. Sci. T.I.T.* **A-103** (1986).

[96] S. Negami, Enumeration of projective-planar embeddings of graphs, *Discrete Math.* **62** (1986), 299–306.

[97] S. Negami, Polynomial invariants of graphs, *Trans. Amer. Math. Soc.* **299** (1987), 601–622.

[98] S. Negami, The virtual k-factorability of graphs, *J. Graph Theory* **11** (1987), 359–365.

[99] S. Negami, The spherical genus and virtually planar graphs, *Discrete Math.* **70** (1988), 159–168.

[100] S. Negami, Re-embedding of projective-planar graphs, *J. Combinatorial Theory, Ser. B* **44** (1988), 276–299.

[101] S. Negami, Graphs which have no finite planar covering, *Bull. Inst. Math. Academia Sinica* **16** (1988), 377–384.

[102] S. Negami and S. Watanabe, Diagonal transformations of triangulations on surfaces, *Tsukuba J. Math.* **14** (1990), 155–166.

[103] S. Negami, Topological graph theory, Four topics, *Proc. TGRC-KOSEF* **1** (1990), 85–103.

[104] S. Negami, Ramsey theorems for knots, links and spatial graphs, *Trans. Amer. Math. Soc.* **324** (1991), 527–541.

[105] S. Negami, Projective-planar graphs with even duals, *J. Graph Theory* **16** (1992), 287–295.

[106] S. Negami, Projective-planar graphs with even duals II, "Graph Structure Theory", edited by N. Robertson and P. Seymour, *Contemporary Math.* **147** (1993), 363–379.

[107] S. Negami, Topological bipartite graphs, "*Proceedings of the JSPS Workshop on Graph Theory and Combinatorics*", edited by J. Akiyama and M.P. Ruiz, (1993), 61–64.

[108] S. Negami and A. Nakamoto, Diagonal transformations of graphs on closed surfaces, *Sci. Rep.Yokohama Nat. Univ., Sec. I* **40** (1993), 71–97.

[109] S. Negami, Diagonal flips in triangulations on surfaces, *Discrete Math.* **135** (1994), 225–232.

[110] S. Negami and M. Tsuchiya, Manifold posets, *Sci. Rep.Yokohama Nat. Univ., Sec. I* **41** (1994), 23–32.

[111] S. Negami and K. Kawagoe, Polynomial invariants of graphs with state models, *Discrete Applied Math.* **56** (1994), 323-331.

[112] S. Negami, Polynomial invarinats of embeddings of graphs on surfaces, *Sci. Rep. Yokohama Nat. Univ., Sec. I* **42** (1995), 1–10.

[113] S. Negami and K. Ota, Polynomial invariants of graphs II, *Graph and Combinatorics* **12** (1996), 189–198.

[114] S. Negami and T. Tsukamoto, Note on spatial graphs with good drawings, *Proceedings of Knots96*, edited by S. Suzuki, (1997), 91–98.

[115] S. Negami, Ramsey-type theorem for spatial graphs, *Graph and Combin.* **14** (1997), 75–80.

[116] S. Negami, Ramsey-type theorems for spatial graphs and good drawings, *J. Combin. Theory, Ser. B* **72** (1998), 53–62.

[117] S. Negami, Diagonal flips in triangulations of closed surfaces, estimating upper bounds, *Yokohama Math. J.* **45** (1998), 113–124.

[118] S. Negami, Diagonal flips of triangulations on surfaces, a survey, *Yokohama Math. J.* **47**, special issue (1999), 1–40.

[119] S. Negami, Note on frozen triangulations on closed surfaces, *Yokohama Math. J.* **47**, special issue (1999), 191–202.

[120] S. Negami, Crossing numbers of graph embedding pairs on closed surfaces, to appear in *J. Graph Theory*.

[121] S. Negami, Diagonal flips in pseudo-triangulations of closed surfaces, to appear in *J. Combin. Theory, Ser. B*.

[122] S. Negami and Y. Suzuki, 5-Connected planar triangulations quadrangulating other surfaces, preprint, 1999.

[123] G. Ringel, "Map Color Theorem", Springer Verlag, 1974.

[124] N. Robertson and P. Seymour, Graphs Minors XVI, "Wagner's conjecture", preprint.

[125] N. Robertson and R. Vitray, Representativity of surface embedding, *Paths, Flows, and VLSI-Layout*, ("Algorithm and Combinatorics" Vol. 9, B. Korte, L. Lovász, H.J. Prömel and A. Schrijver, eds), Springer-Verlag, Berlin, (1990), 293–328.

[126] I. Sato, Isomorphisms of some graph coverings, *Discrete Math.* **128** (1994), 317–326.

[127] I. Sato, Graph covering and its generalization, *Yokohama Math. J.* **47**, special issue (1999), 67–88.

[128] A. Schrijver, On the uniqueness of kernels, *J. Combin. Theory, Ser. B* **55** (1992), 9–17.

[129] K. Taniyama, Total curvature of graphs in Euclidean spaces, *Differ. Geom. Appl.* **8** (1998), 135–155.

[130] T. Tanuma, One-loosely tight triangulations of closed surfaces, *Yokohama Math. J.* **47**, special issue (1999), 203–211.

[131] T. Tanuma, Looseness of triangulations of the projective plane, preprint.

[132] K. Wagner, Bemekungen zum Vierfarbenproblem, *J. der Deut. Math.*, Ver. **46**, Abt. 1, (1936), 26–32.

[133] T. Watanabe and S. Negami, Diagonal flips in pseudo-triangulations of closed surfaces without loops, *Yokohama Math. J.* **47**, special issue (1999), 213–223.

[134] T. Watanabe, Generating even triangulations on the projective plane, preprint, 1999.

[135] T. Watanabe, Chromatic number of even triangulations on the projective plane, preprint, 2000.

[136] D.J.A. Welsh, *"Matroid theory"*, Academic Press, London, New York, San Franciscom 1976.

[137] H. Whitney, Congruent graphs and the connectivity of graphs, *Amer. J. Math.* **54** (1932), 150–168.

[138] H. Whitney, 2-isomorphic graphs, *Amer. J. Math.* **55** (1933), 245–254.

[139] S. Yamada, An invariant of spatial graphs, *J. Graph Theory* **13** (1989), 537–551.

[140] S. Yamada, A topological invariant of spatial regular graphs, *Knots 90, Proc. Int. Conf. Knot Theory Rel. Topics, Osaka/Japan 1990*, (1992), 447–454.

[141] S. Yamashita, The strong embedding theorem for 3-representative chain graphs, *Yokohama Math. J.* **41** (1994), 175–186.