List orientation number of graphs

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Abstract

The *orientation number* of a graph G, denoted by $o(G)$, is the minimum integer k such that there exits an orientation D with $d^+_D(v) \le k$ for every $v \in V(G)$, where $d^+_D(v)$ is the out-degree of v in the directed graph G with respect to the orientation O. In this paper, we define its list-analog as the *list-orientation number* $o_\ell(G)$, and conjectured that $o_\ell(G) \le o(G) + 2$ for every graph G. We give partial solutions to this conjecture for some fundamental classes of graphs, such as bipartite graphs, graphs with $o(G) \le 2$, and planar graphs. In addition, we consider the same problem for signed graphs.

Key words. orientation; list-orientation; planar graph; Combinatorial Nullstellensatz; the Alon-Tarsi number

1 Introduction

In this paper, we deal with only finite and simple graphs. Let G be a graph. We denote by $d_G(v)$ the degree of a vertex v in G. The maximum degree of G is denoted by $\Delta(G)$. If $d_G(v) = k$ for every $v \in V(G)$, then G is said to be *k-regular*. An *orientation* of G is an assignment of a direction to each edge of G. For an orientation O of G, we denote by $d^+_O(v)$ (respectively, $d^-_O(v)$) the out-degree (respectively, the in-degree) of v in the directed graph G with respect to the orientation O. The maximum out-degree of O is denoted by Δ^+_O. The *orientation number*, denoted by $o(G)$, is the minimum integer k such that G admits an orientation O with $d^+_O(v) \le k$ for every $v \in V(G)$. Hakimi obtained the following criteria for the orientation number in terms of the maximum average degree, where the *maximum average degree* of a graph G denoted by $\mathrm{Mad}(G)$, is defined as

$$\mathrm{Mad}(G) = \max\left\{\frac{2|E(H)|}{|V(H)|} : H \text{ is a non-empty subgraph of } G\right\}.$$

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Theorem 1.1 ([3]) *Let G be a graph. Then $o(G) \le k$ if and only if* $\mathrm{Mad}(G) \le 2k$

The following corollary is obtained directly from Theorem 1.1.

Corollary 1.2 *Let G be a graph. Then $o(G) \le \left\lceil \frac{\Delta(G)}{2} \right\rceil$, where this upper bound is tight.*

As we extend an ordinally coloring to a list-coloring, we consider a list-orientation in this paper. It was introduced by Akbrai et al. [1] as a complement manner, so-called an *F-avoiding orientation*. The details of an F-avoiding orientation are mentioned in Section 2.

Let G be a graph. We call a map $L : V(G) \to 2^{\mathbb{N}}$ a *list-assignment* of G. A list-assignment L is said to be *valid* if L satisfies $L(v) \subseteq \{0, 1, \ldots, d_G(v)\}$ for every $v \in V(G)$. In this paper, we only consider valid list-assignments. If G admits an orientation D such that $d^+_D(v) \in L(v)$ for every vertex $v \in V(G)$, then G is said to be *L-orientable* and D is an *L-orientation* of G. (See Figure 1)

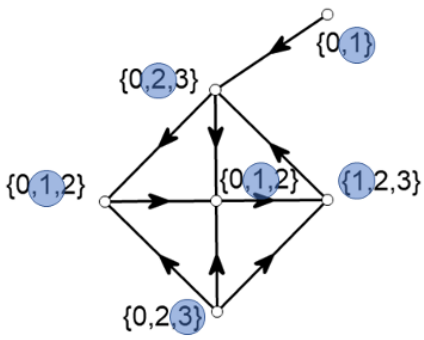

Figure 1: A graph G, where the set close to each vertex represents its list by the valid list-assignment L. The blue circles show an L-orientation of G.

A graph G is said to be *k-list-orientable* if G is L-orientable for every valid list-assignment L such that $|L(v)| \ge \min\{d_G(v) + 1, k\}$ for every $v \in V(G)$. The *list-orientation number*, denoted by $o_\ell(G)$, is the minimum integer k such that G is k-list-orientable. For the list-orientation number, it is easy to see the following.

Proposition 1.3 *Let G be a graph. Then $o(G) + 1 \le o_\ell(G)$.*

On the other hand, the following conjecture was posed, in terms of F-avoiding orientations in [1], and in the same paper, a weaker statement was proven as follows.

Conjecture 1.4 ([1]) *Let G be a graph. Then G is L-orientable for every valid list-assignment L with $|L(v)| \geq \dfrac{d_G(v)+4}{2}$ for every $v \in V(G)$.*

Theorem 1.5 ([1]) *Let G be a graph. Then G is L-orientable for every valid list-assignment L with $|L(v)| \geq \dfrac{3d_G(v)}{4} + 1$ for every $v \in V(G)$.*

Conjecture 1.4 implies that $o_\ell(G) \leq \left\lceil \dfrac{\Delta(G)}{2} \right\rceil + 2$ for any graph G, and this upper bound is best possible if it is true. Theorem 1.5 implies that $o_\ell(G) \leq \left\lceil \dfrac{3\Delta(G)}{4} \right\rceil + 1$ for every $v \in V(G)$.

It seems difficult to solve Conjecture 1.4 in general, so we consider other upper bound on the list-orientation number $o_\ell(G)$, in terms of the orientation number $o(G)$ with Proposition 1.3 in mind. The following is our main conjecture.

Conjecture 1.6 *Let G be a graph. Then $o(G) + 1 \leq o_\ell(G) \leq o(G) + 2$.*

Conjecture 1.6 also implies $o_\ell(G) \leq \left\lceil \dfrac{\Delta(G)}{2} \right\rceil + 2$ if it is true. The aim of this paper is to give partial solutions to Conjecture 1.6 for some fundamental classes of graphs, as follows.

Theorem 1.7 *Let G be a bipartite graph. Then $o_\ell(G) = o(G) + 1$.*

Theorem 1.8 *Let G be a graph with $o(G) \leq 2$. Then $o_\ell(G) \leq o(G) + 2 = 4$. where this upper bound is tight.*

Theorem 1.9 *Let G be a planar graph. Then $o_\ell(G) \leq o(G) + 2$, where this upper bound is tight.*

As in these theorems, the upper bound $o(G) + 2$ is best possible, if Conjecture 1.6 is true, even for graphs G with $o(G) \leq 2$, and planar graphs.

One of the important differences between Conjectures 1.4 and 1.6 is the size of a list of each vertex. By Theorem 1.1, we intuitively think that Conjecture 1.6 is stronger than Conjecture 1.4 when the maximum average degree of G is small, but some vertices have high degree. For example, let us focus on the *wheel* W_n of order n. For Conjecture 1.4, the center vertex is required to have a large list compared with n, while $o_\ell(W_n) = 4$, which does not depend on n, as expected by Conjecture 1.6.

This paper is organized as follows. In Section 2, we prepare some terminologies and then prove Theorem 1.7. In Sections 3 and 4, we show Theorems 1.8 and Theorem 1.9, respectively. In Section 5, we extend some results of list-orientations to signed graphs.

2 Preliminaries and a proof of Theorem 1.7

In this section, we introduce some terminologies, and then prove Theorem 1.7. Let $\mathbb{N}$ be a set of natural numbers including zero. At first, we introduce the Combinatorial Nullstellensatz and its applications. Alon and Tarsi proved the following important theorem.

Theorem 2.1 (Combinatorial Nullstellensatz, [2]) *Let $\mathbb{F}$ be an arbitrary field and let $f = f(x_1, x_2, \dots, x_n)$ be a polynomial in $\mathbb{F}[x_1, x_2, \dots, x_n]$. Suppose that the degree $\deg(f)$ of f is $\sum_{i=1}^n t_i$ where each t_i is a nonnegative integer, and suppose that the coefficient of $\prod_{i=1}^n x_i^{t_i}$ of f is nonzero. Then if $S_1, S_2, \dots, S_n$ are subsets of $\mathbb{F}$ with $|S_i| \geq t_i + 1$, then there are $s_1 \in S_1$,$s_2 \in S_2$,...,$s_n \in S_n$ so that $f(s_1, s_2, \dots, s_n) \neq 0$.*

Let G be a graph and let '$<$' be an arbitrary fixed ordering of the vertices of G. The *graph polynomial* of G is defined as

$$P_G(\boldsymbol{x}) = \prod_{u \sim v, u<v} (x_u - x_v),$$

where $u \sim v$ means that vertices u and v are adjacent in G, and $\boldsymbol{x} = (x_v)_{v \in V(G)}$ is a vector of $|V(G)|$ variables indexed by the vertices of G. We note that a graph polynomial $P_G(\boldsymbol{x})$ is a homogeneous polynomial and $\deg(P_G)$ is equal to $|E(G)|$. Jensen and Toft [4] defined the *Alon-Tarsi number* of a graph as follows.

Definition 2.2 The *Alon-Tarsi number* of a graph G, denoted by $AT(G)$, is the minimum integer k for which there exists a monomial $c\prod_{v \in V(G)} x_v^{t_v}$ in the expansion of $P_G(\boldsymbol{x})$ such that $c \neq 0$ and $t_v < k$ for all $v \in V(G)$.

Let H be a subgraph of a graph G with orientation D, and let D_H be the restriction of D to H. The subgraph H is called *Eulerian* if $d^+_{D_H}(v) = d^-_{D_H}(v)$ for ever vertex $v \in V(G)$. The subgraph H is called *even* if $|E(H)| \equiv 0 \pmod 2$ (respectively *odd* if $|E(H)| \equiv 1 \pmod 2$). Let $EE(D)$ (respectively $OE(D)$) denote the set of spanning even (respectively odd) Eulerian subgraphs of G with respect to D. We say that an orientation D is an *AT-orientation* if $|EE(D)| - |OE(D)| \neq 0$. Alon and Tarsi gave following characterization in terms of orientations.

Theorem 2.3 ([2]) *Let G be a graph. Then $AT(G)$ is equal to the minimum integer k such that there exists an AT-orientation in G with maximum out-degree $k-1$.*

Next, we focus on the application of AT-orientation to a list-orientation, through an F-avoiding orientation. Let G be a graph and let F be a valid list-assignment of G. If G has an orientation D such that $d_D^+(v) \notin F(v)$ for every vertex $v \in V(G)$, then D is said to be an F*-avoiding* orientation of G. Note that an orientation of a graph G is an L-orientation if and only if it is F-avoiding, where L and F are valid list-assignments of G satisfying $L(v) = \{0, 1, \dots, d_G(v)\} - F(v)$ for each vertex v.

An algebraic method was used to obtain an F-avoiding orientations in [1].

Lemma 2.4 (Theorem 9, [1]) *Let G be a graph and let F be a valid list-assignment of G. If there exists an AT-orientation D satisfying $|F(v)| \leq d_D^+(v)$ for every vertex $v \in V(G)$, then G admits an F-avoiding orientation.*

Lemma 2.4 can be translated in terms of list-orientations as follows.

Lemma 2.5 *Let G be a graph and let D be an AT-orientation of G. Then G is L-orientable for each valid list-assignment L with $|L(v)| \geq d_D^+(v) + 1$ for every vertex $v \in V(G)$.*

Proof. Let $F(v) = \{0, 1, \dots, d_G(v)\} - L(v)$. Then we have

$$|F(v)| \leq (d_G(v) + 1) - \left(d_D^+(v) + 1\right) = d_G(v) - d_D^+(v) = d_D^-(v)$$

for every vertex $v \in V(G)$.

We focus on the orientation D^c obtained from D by reversing the direction of all edges. If C is a directed cycle in D, then the cycle C^c, which is obtained by reversing the direction of all edges in C is also directed. Therefore, $|EE(D^c)| = |EE(D)|$, $|OE(D^c)| = |OE(D)|$ and hence D^c is also an AT-orientation of G. Moreover, D^c satisfies $d_{D^c}^+(v) = d_D^-(v)$. By applying D^c to Lemma 2.4, G admits an F-avoiding orientation O, i.e. $d_O^+(v) \notin F(v)$. This implies $d_O^+(v) \in L(v)$ for every vertex $v \in V(G)$ and we complete the proof. □

From Lemma 2.5, we obtain the following corollary.

Corollary 2.6 *Let G be a graph. Then $o_\ell(G) \leq AT(G)$.*

Proof. Let $AT(G) = k$. It suffices to prove that G is L-orientable for every valid list-assignment L with $|L(v)| = k$ for every vertex $v \in V(G)$. Since $AT(G) = k$, it follows from Theorem 2.3 that G admits an AT-orientation D such that $d_D^+(v) \leq k-1$ for every $v \in V(G)$. By applying D to Lemma 2.5, we see that G is L-orientable and hence $o_\ell(G) \leq k = AT(G)$. □

Now, we are ready to prove Theorem 1.7, which states that $o_\ell(G) = o(G) + 1$ holds for any bipartite graph G.

Proof of Theorem 1.7. Let D be an orientation of G with maximum out-degree $o(G)$. Since G is bipartite, there are no odd cycles in G. Thus we have $|EE(D)| \geq 1$, $|OE(D)| = 0$ and hence $|EE(D)| - |OE(D)| \neq 0$. Therefore, we see

$$o_\ell(G) \leq AT(G) \leq o(G) + 1$$

by Corollary 2.6. □

3 Graphs with small orientation number

First, we observe the upper bound of list-orientation number.

Lemma 3.1 *Let G be a graph and let v be a vertex with $d_G(v) \leq k-1$. If $o_\ell(G-v) \leq k$, then $o_\ell(G) \leq k$.*

Proof. Let L be a valid list-assignment of G. Since $d_G(v) \leq k-1$, we have $L(v) = \{0, 1, \ldots, d_G(v)\}$. We show that G admits an L-orientation.

Let $u \in N_G(v)$. If $d_G(u) \leq k-1$, then we see $L(u) = \{0, 1, \ldots, d_G(u)\}$. In this case, we let $L'(u) = \{0, 1, \ldots, d_G(u)-1\}$. On the other hand, suppose $d_G(u) \geq k$. If $d_G(u) \notin L(u)$, then we let $L'(u) = L(u)$. Otherwise, $d_G(u) \in L(u)$ and there exists an integer $a_u \in \{0, 1, \ldots, d_G(u)-1\}$ such that $a_u \notin L(u)$ and $a_u + 1 \in L(u)$. In this case, we let $L'(u) = \big(L(u) - \{d_G(u)\}\big) \cup \{a_u\}$. For a vertex $w \in V(G) - N_G(v) - \{v\}$, we assign $L'(w) = L(w)$. Note that L' is a valid list-assignment of $G - v$. Since $o_\ell(G-v) \leq k$, $G - v$ admits an L'-orientable D'.

For each vertex u in $N_G(v)$, if $d_{D'}^+(u) \in L(u)$, then we assign to the edge uv the direction from v to u so that the out-degree of u is not changed. Otherwise, since $d_{D'}^+(u) \in L'(u)$, we have $d_{D'}^+(u) = a_u$. Then we assign to the edge uv the direction from u to v, so that the out-degree of u is $a_u + 1 \in L(u)$.

It is easy to see that the obtained orientation is an L-orientation of G. $\Box$

By Lemma 3.1, if we would like to prove $o_\ell(G) \leq k$, then we may assume $\delta(G) \geq k$. This is an important property for the proofs of the main theorems.

A graph G is *k-degenerate* if every non-empty subgraph of G has a vertex of degree at most k. Lemma 3.1 directly shows the following corollary.

Corollary 3.2 *If G is k-degenerate, then $o_\ell(G) \leq k+1$*

From now on, we focus on the graphs with small orientation number. A *unicyclic* graph is a connected graph containing exactly one cycle.

Lemma 3.3 *Let G be a graph with $o(G) = 1$. Then $o_\ell(G) \leq 3$, where this upper bound is tight.*

Proof. We may assume that G is connected. Since $o(G) = 1$, it follows from Theorem 1.1 that $|E(G)| \leq |V(G)|$, and hence G is isomorphic to a tree or a unicycle graph. In either case, G is 2-degenerate and $o_\ell(G) \leq 3$ by Corollary 3.2. Moreover, if G is an odd cycle, then $o_\ell(G) = 3$ by [1, Lemma 1]. $\Box$

Next, we focus on the case with $o(G) = 2$. First we prepare the following lemma, which is well-known in graph theory, but in case, we prove it.

Lemma 3.4 *Let G be a connected graph and let $S \subseteq V(G)$ such that $|S|$ is even. Then G contains a spanning subgraph H such that for each vertex v, $d_H(v)$ is odd if and only if $v \in S$.*

Proof. Let $S = \{v_{1,1}, v_{1,2}, v_{2,1}, \ldots, v_{t,1}, v_{t,2}\}$ and consider a path P_i in G connecting $v_{i,1}$ and $v_{i,2}$ for $1 \leq i \leq t$. Let

$$w_i(e) = \begin{cases} 1 & \text{if } e \in E(P_i), \\ 0 & \text{otherwise.} \end{cases}$$

and let $w(e) = \sum_{i=1}^{t} w_i(e)$. Moreover, let H be a spanning subgraph of G with $E(H) = \{\ e \in E(G) : w(e) \equiv 1 \pmod 2\}$.

Now, let us focus on each vertex. If $v \in S$, then v is exactly one end-point of the paths $P_1, \ldots, P_t$, possibly some paths pass through v, and hence $\sum_{e \in E(v)} w(e) \equiv 1 \pmod 2$, where $E(v)$ is the edge set which is incident to v. Thus we see $d_H(v) \equiv 1 \pmod 2$. On the other hand, if $v \notin S$, then each path P_i goes through or do not touch at v and hence we see $\sum_{e \in E(v)} w(e) \equiv 0 \pmod 2$. This implies $d_H(v) \equiv 0 \pmod 2$. $\Box$

Proof of the first half of Theorem 1.8. The case $o(G) = 1$ follows from Lemma 3.3, and hence we may assume that $o(G) = 2$.

We may assume that G is connected and $\delta(G) \geq 4$ by Lemma 3.1. If there exists a vertex $v \in V(G)$ with $d_G(v) > 4$, then we see $|E(G)| > 2|V(G)|$ and this implies $o(G) > 2$, a contradiction. Therefore, G must be a 4-regular graph.

Let L be a valid list-assignment such that $L(v) \subseteq \{0, 1, 2, 3, 4\}$ with $|L(v)| \geq 4$, and let

$$V_{\text{even}} = \{v \in V(G) : \{0, 2, 4\} \subseteq L(v)\}, \text{ and } V_{\text{odd}} = V(G) - V_{\text{even}}.$$

Note that for every $v \in V_{\text{odd}}$, we have $\{1, 3\} \subseteq L(v)$.

Since G is 4-regular and connected, there is an orientation D_0 of G such that $d^+_{D_0}(v) = 2$ for each $v \in V(G)$. Suppose first that $|V_{\text{odd}}|$ is even. By Lemma 3.4, G contains a spanning subgraph H_1 such that for each vertex v, $d_{H_1}(v)$ is odd if and only if $v \in V_{\text{odd}}$. Then, we obtain the orientation D_1 from D_0 by reversing the direction of each edge in H_1. We see that for each vertex v, its out-degree in D_1 is odd if and only if $d_{H_1}(v)$ is odd. Thus, D_1 satisfies that the out-degree of a vertex in V_{even} is $0, 2$ or 4, and the out-degree of a vertex in V_{odd} is 1 or 3. By the definition of V_{even} and V_{odd}, this is an L-orientation of G.

Thus, we may assume that $|V_{\text{odd}}|$ is odd. We next claim that $V_{\text{even}} = \emptyset$. Suppose otherwise that there exists a vertex w in V_{even}. By applying Lemma 3.4 to $V_{\text{odd}} \cup \{w\}$, G contains a spanning subgraph H_2 such that for each vertex v, $d_{H_2}(v)$ is odd if and only if $v \in V_{\text{odd}} \cup \{w\}$. By the same way as above, we obtain an orientation D_2 from D_0 such that the out-degree of a vertex in $V_{\text{even}} - \{w\}$ is $0, 2$ or 4, and the out-degree of a vertex in $V_{\text{odd}} \cup \{w\}$ is 1 or 3. Thus, the out-degree of each vertex meets its list, possibly except for w. Note that the out-degree of w is 1 or 3. If the out-degree of w is contained in $L(w)$, then we are done. Otherwise, the orientation D_2^c, which is obtained from D_2 by reversing the direction of every edge, is an L-orientation of G. Therefore, we may further assume that $V_{\text{even}} = \emptyset$ as claimed.

Note that $V_{\text{odd}} = V(G)$. Since G is connected, there exists a vertex w such that $G' = G - w$ is connected. Since G' is connected and only four vertices have degree three, there exists an orientation D' of G' such that two vertices of $N_G(w)$, say v_1 and v_2, have out-degree one, and all other vertices have out-degree two. Since G' is connected, by applying Lemma 3.4 with $V_{\text{odd}} - \{w, v_1, v_2\}$, G contains a spanning subgraph H_3 such that for each vertex v, $d_{H_3}(v)$ is odd if and only if $v \in V_{\text{odd}} - \{w, v_1, v_2\}$. Let D_3 be the orientation obtained from D' by reversing the direction of each edge in H_3. We see that D_3 satisfies that the out-degree of all vertices in G' are odd. Thus, by giving the direction to the edges incident with w from w to a vertex in $N_G(w)$, we obtain an orientation of G such that the out-degree of w is four and the out-degree of all other vertices in G are odd. If $4 \in L(w)$, then this is an L-orientation of G. Otherwise, by reversing the direction of every edge, we obtain an L-orientation of G. □

The second half of Theorem 1.8 will be shown in the next section.

4 Planar graphs and graphs embedded on closed surfaces

4.1 Planar graphs

In this subsection, we focus on planar graphs. Before proving Theorem 1.9, we consider the upper bound of the list-orientation number. For a planar graph G, it is easy to see $o(G) \leq 3$ by Theorem 1.1 and Euler's formula.

Zhu proved the following.

Theorem 4.1 ([6]) *Let G be a planar graph. Then $AT(G) \leq 5$.*

Together with Corollary 2.6, we have the following corollary.

Corollary 4.2 *Let G be a planar graph. Then $o_\ell(G) \leq 5$.*

Proof of the first half of Theorem 1.9. Let G be a planar graph. By Theorem 1.8, the theorem holds when $o(G) \leq 2$. Since $o_\ell(G) \leq 5$ by Corollary 4.2, the case $o(G) = 3$ also holds □

Next, let us focus on the second half of Theorem 1.9, tightness of the upper bound. It is natural to ask whether there exists a graph G with $o_\ell(G) = o(G) + 2$ especially for planar case. We answer this question in affirmative for both cases of $o(G) = 2$ and 3, in Theorems 4.3 and 4.4, respectively. Note that Theorem 4.3 shows the second half of Theorem 1.8 as well.

Theorem 4.3 *There exists a triangle-free planar graph G with $o(G) = 2$ and $o_\ell(G) = 4$.*

Proof of Theorem 4.3. Let G' be a quadrangulation obtained by embedding pairwise vertex-disjoint five quadrangles $Q_1, \ldots, Q_5$ with $V(Q_i) = \{a_i, b_i, c_i, d_i\}$ for $1 \leq i \leq 5$ on the plane so that the infinite face is incident to all vertices of Q_i's and adding edges arbitrary in the infinite face so that all faces will be quadrangular. Then we have the following claim.

Claim 1 *For any orientation D' of G' with $\Delta^+_{D'} \leq 2$, there exists an integer $i \in \{1,2,3,4,5\}$ such that $d^+_{D'}(a_i) = d^+_{D'}(b_i) = d^+_{D'}(c_i) = d^+_{D'}(d_i) = 2$.*

Proof. For $j \in \{0,1,2\}$, let n_j denote the number of vertices with out-degree j in D'. By counting the number of vertices and edges, we have

$$\sum_{j=0}^{2} n_j = 20 \quad \text{and} \quad \sum_{j=0}^{2} j n_j = 36.$$

Thus, we have $2n_0 + n_1 = 4$ and hence there exist at most four vertices with out-degree less than 2. This concludes the existence of an integer i with desired condition. ■

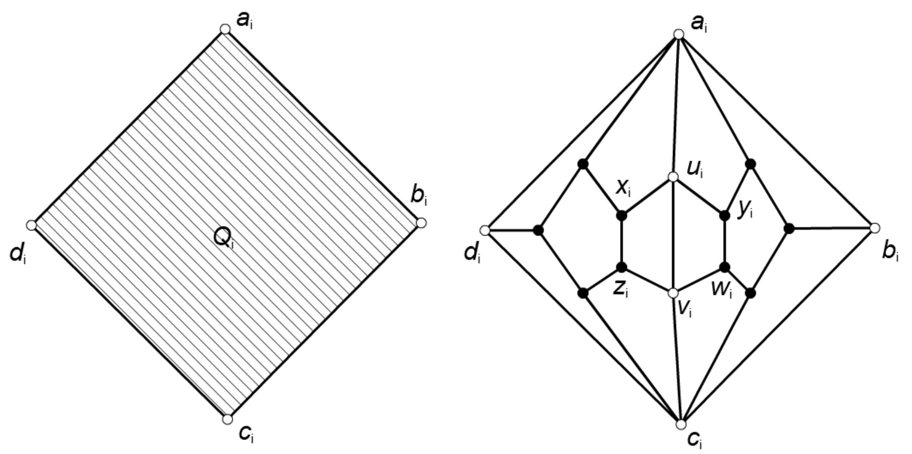

Figure 2: The construction from G' to G. The shaded area in the left shows the face of Q_i and we insert the graph in the right to Q_i.

Next, we construct the graph G by inserting the graph depicted in the right of Figure 2 to Q_i for $1 \leq i \leq 5$. We assign the valid list-assignment L such that $L(x) = \{0,1,3\}$ to the black vertices in Figure 2 and $L(u) = \{0,1,2\}$ for other vertices u. Then we have the following claim.

Claim 2 *G is not L-orientable.*

Proof. Suppose that G has an L-orientation D. First, we focus on the spanning subgraph G'. By Claim 1, for the orientation D' that is the restriction of D to G', there exists an integer $i \in \{1,2,3,4,5\}$ such that

$d^+_{D'}(a_i) = d^+_{D'}(b_i) = d^+_{D'}(c_i) = d^+_{D'}(d_i) = 2$. By symmetry, we may assume that $i = 1$. Therefore, the vertices a_1, b_1, c_1, d_1 cannot have edges directed from Q_1 to its inside.

Next, we focus on the vertices u_1 and v_1 as in the right of Figure 2. By the symmetry, we may assume that the edge u_1v_1 has the direction from u_1 to v_1. Since $L(u_1) = \{0, 1, 2\}$, the edges x_1u_1 and y_1u_1 have the direction to u_1. Moreover, since $L(v_1) = \{0, 1, 2\}$, at least one of z_1 or w_1 has the direction to v_1. Without loss of generality, we may assume that z_1 has the direction to v_1. Now we focus on the facial cycle C of length 5 containing the edge x_1z_1. Since all edges betwee C and a_1, b_1, c_1, u_1, v_1 are directed out-going from the vertices of C, C is oriented so that each vertex has out-degree either 0 or 2, which is impossible since C is an odd cycle. Thus, we conclude that G is not L-orientable. ■

Therefore, we have $o_\ell(G) \geq 4$. On the other hand, since G is triangle-free and planar, it follows from Theorem 1.1 and Euler's formula that $o(G) = 2$. □

Theorem 4.4 *There exists a planar graph G with $o(G) = 3$ and $o_\ell(G) = 5$.*

Proof of Theorem 4.4. Let G' be a plane triangulation obtained by embedding pairwise vertex-disjoint seven triangles $T_1, T_2, \ldots, T_7$ with $V(T_i) = \{a_i, b_i, c_i\}$ for each $i \in \{1, 2, \ldots, 7\}$ on the plane so that the infinite face is incident to all vertices of T_i's and adding edges arbitrary in the infinite face so that all faces will be triangular. For the graph G', we see the following claim.

Claim 3 *For any orientation D' of G' with $\Delta^+_{D'} \leq 3$, there exists an integer $i \in \{1, 2, \ldots, 7\}$ such that $d^+_{D'}(a_i) = d^+_{D'}(b_i) = d^+_{D'}(c_i) = 3$.*

Proof. Let n_j denote the number of vertices with out-degree j in D'. By counting the number of vertices and edges, we have

$$\sum_{j=0}^{3} n_j = 21 \quad \text{and} \quad \sum_{j=0}^{3} jn_j = 21 \times 3 - 6 = 57.$$

Thus, we have $3n_0 + 2n_1 + n_2 = 6$ and hence there exist at most six vertices with out-degree less than 3. Therefore, there exists an integer i with $1 \leq i \leq 7$ such that $d^+_{D'}(a_i) = d^+_{D'}(b_i) = d^+_{D'}(c_i) = 3$. ■

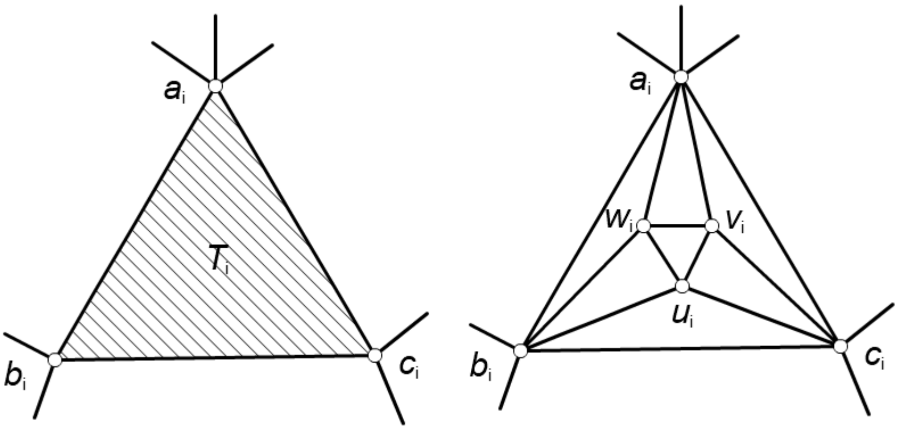

Figure 3: The construction from G' to G. The shaded area in the left shows the face of T_i and we insert the octahedrons into T_i.

Next, we construct the graph G by inserting the octahedron into each T_i as in Figure 3. We assign the valid list-assignment L so that $L(a_i) = L(b_i) = L(c_i) = \{0, 1, 2, 3\}$ for $1 \leq i \leq 3$ and $L(v) = \{0, 1, 2, 4\}$ for other vertices v. Then we have the following.

Claim 4 *G is not L-orientable.*

Proof. Suppose that G has an L-orientation D. Let us focus on the spanning subgraph G'. By Claim 3, for the orientation D' that is the restriction of D to G', there exists an integer $i \in \{1, 2, \ldots, 7\}$ such that $d^+_{D'}(a_i) = d^+_{D'}(b_i) = d^+_{D'}(c_i) = 3$. By symmetry, we may assume that $i = 1$. Thus, all edges between the vertices in $\{a_1, b_1, c_1\}$ and the vertices in $\{u_1, v_1, w_1\}$ are directed out-going from u_1, v_1 or w_1. Thus, the cycle consisting of u_1, v_1, w_1 must be oriented so that each vertex has out-degree 0 or 2, but this is not possible, since the cycle has odd length. ■

Thus, there exists a valid list-assignment L with $|L(v)| = 4$ such that G is not L-orientable. Moreover, it is not hard to see that G is planar and $o(G) = 3$. Therefore, we complete the proof. □

4.2 Graphs on non-spherical closed surface

In addition to planar graphs, we consider graphs on non-spherical closed surfaces. For a planar graph G, it follows from Theorem 4.1 that $AT(G) \leq 5$, which was the main part to obtain $o_\ell(G) \leq 5$. However, the same bound for the Alon-Tarsi number $AT(G)$ does not hold for graphs G on non-spherical closed surfaces, such as the projective plane, the torus or the Klein bottle. Considering Euler's formula, we pose the following conjecture, which is the restriction of Conjecture 1.6 to graphs on closed surfaces with non-negative Euler characteristic.

Conjecture 4.5 *Let G be a graph on the projective plane, the torus or the Klein bottle. Then $o_\ell(G) \leq 5$.*

If this conjecture is true, then the bound is tight.

An *even-embedding* of a graph on a closed surface is one with each face having a facial walk of even length. If G is embedded on the sphere, then G must be bipartite and we see that Conjecture 1.6 is true for G by Theorem 1.7. However, non-bipartite graphs possibly admit an even-embedding on non-spherical closed surface. Thus, it seems natural to think whether or not Conjecture 1.6 holds for such graphs. We answer this question in affirmative if G is embedded on the torus or the Klein bottle, and also show that the bound is best possible.

Theorem 4.6 *Let G be an even-embedding of the torus or the Klein bottle. Then $o_\ell(G) \leq o(G) + 2$, where this upper bound is tight.*

Proof of Theorem 4.6. Since G is an even-embedding on the torus or the Klein bottle, then we see $o(G) = 2$ by Euler's formula. Thus by Theorem 1.8, we see $o_\ell(G) \leq 4 = o(G) + 2$.

For the tightness, first we consider the toroidal case. Let us focus on C_n^2, which is obtained from the cycle C_n of length n by joining vertices of distance exactly two in C_n. As in the left of Figure 4, C_n^2 has an even-embedding on the torus for $n \geq 5$. Moreover, consider the case $n \notin 3\mathbb{Z}$, which implies $\alpha(G) < \dfrac{n}{3}$. For the valid list-assignment L with $L(v) = \{0, 1, 4\}$ for each vertex in C_n^2, it follows from [1, Lemma 10] that there is no L-orientation.

Next, we consider the case of the Klein bottle. The graph in the right of Figure 4 satisfies $o(G) = 2$ and $o_\ell(G) = 4$. If we replace a disk at which an edge-crossing occurs with a cross-cap, we obtain an even-embedding of G on the Klein bottle. □

5 Orientations of signed graphs

In this section, we consider a list-orientation of a signed graph. In particular, we extend Corollary 2.6 to a signed graph, and then prove the singed analog of Corollary 4.2.

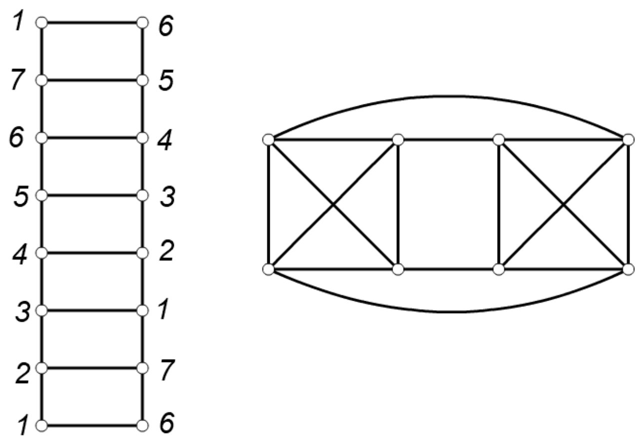

Figure 4: The left shows an embedding of C_7^2 on the torus, where by identifying the top and bottom, and also the left and right so that the vertices with same numbers coincide, we obtain the quadrangulation on the torus. The right represents a quadrangulation with $o(G) = 2$ and $o_\ell(G) = 4$ on the Klein bottle, where we obtain an even-embedding of G on the Klein bottle by replacing a disk at which an edge-crossing occurs with a cross-cap.

Let G be a graph and $\sigma : E(G) \to \{\pm 1\}$ be a mapping. An edge e with $\sigma(e) = +1$ is said to be *positive*, while one with $\sigma(e) = -1$ is *negative*. We call a pair (G, σ) a *singed graph*. For an orientation of signed graphs, each edge has a direction from one endpoint to the other in a positive edge and both endpoints in-directed or out-directed in a negative edge. We note that orientations of unsigned graph can be regarded as a special case where $\sigma(e) = +1$ for every $e \in E(G)$. The *orientation number* $o(G, \sigma)$, the *list-orientation number* $o_\ell(G, \sigma)$ and the Alon-Tarsi number $AT(G, \sigma)$ of the singed graph are defined similarly, see [5].

Let (G, σ) be a signed graph and D be an orientation of the graph G. Moreover, let $\sigma EE(D)$ (respectively, $\sigma OE(D)$) denote the set of spanning Eulerian subgraph of (G, σ) with even (respectively, odd) number of positive edges. We say that the orientation D is a σAT-orientation if $|\sigma OE(D)| - |\sigma EE(D)| \neq 0$. Similarly to the unsigned case, the Alon-Tarsi number of signed graphs is also characterized by the orientation.

Theorem 5.1 ([5]) *Let (G, σ) be a signed graph. Then $AT(G, \sigma)$ is equal to the minimum integer k such that there exists a σAT-orientation in G with maximum out-degree $k - 1$.*

The following is the main theorem in this section.

Theorem 5.2 *Let (G,σ) be a signed graph. Then $o_\ell(G,\sigma) \leq AT(G,\sigma)$.*

The next lemma is a key to prove Theorem 5.2, which can be proven similarly to the argument in [1].

Lemma 5.3 *Let (G,σ) be a signed graph. If there exists a σAT-orientation D of G, then (G,σ) is F-avoiding for every valid list-assignment F with $|F(v)| \leq d_D^+(v)$ for every $v \in V(G)$.*

Proof. It suffices to prove this lemma when $|F(v)| = d_D^+(v)$ for every $v \in V(G)$. Let $c_D : E(G) \times V(G) \to \{-1,0,1\}$ as follows.

$$c_D(e,v) = \begin{cases} -1 & \text{if } e \in E_D^-(v), \\ 0 & \text{if } e \notin E_G(v), \\ 1 & \text{if } e \in E_D^+(v), \end{cases}$$

where $E_G(v), E_D^+(v)$ and $E_D^-(v)$ denote the set of all edges, the set of all out-going edges and the set of incoming edges incident with v with respect to D, respectively.

Let $E(G) = \{e_1,\dots,e_m\}$. Then we assign a variable x_i for each edge e_i and define $P \in \mathbb{Z}[x_1,\dots,x_m]$ as follows.

$$P(x_1,\dots,x_m) = \prod_{v\in V(G)} \prod_{f\in F(v)} \left(\sum_{i=1}^{m} c_D(e_i,v)\cdot x_i + d_G(v) - 2f \right).$$

Then we have the following.

Claim 5 *All of the following hold.*

(i) $\deg(P) \leq m$.

(ii) If $P(t_1,\dots,t_m) \neq 0$ for some $(t_1,\dots,t_m) \in \{-1,1\}^m$, then G is F-avoiding.

Proof. $\deg(P) = \sum_{v\in V(G)} |F(v)| \leq \sum_{v\in V(G)} d_D^+(v) = |E(G)| = m$ by the assumption. Moreover, if there exists $(t_1,\dots,t_m) \in \{-1,1\}^m$ such that $P(t_1,..,t_m) \neq 0$, then we consider the orientation O obtained from D by reversing the direction of all edges e_i with $t_i = -1$. Note that when $\sigma(e_i) - 1$, then the direction of e_i changed so that if both endpoints of e_i are in-directed, then both will be out-directed, and vice versa. If $c_D(e_i,v)t_i = 1$ at a vertex v, then the edge e_i has out-going direction at v regardless of the signature of e_i. Similarly, if $c_D(e_i,v)t_i = -1$, then the edge e_i has in-coming direction at v. Therefore, we see that $\sum_{i=1}^{m} c_D(e_i, v) \cdot t_i = d_O^+(v) - d_O^-(v)$ for every $v \in V(G)$. Since $d_O^+(v) + d_O^-(v) = d_G(v)$, we have

$$0 \neq \sum_{i=1}^{m} c_D(e_i, v) \cdot t_i + d_G(v) - 2f = 2(d_O^+(v) - f)$$

for every $v \in V(G)$. Thus we have $d_O^+(v) \neq f$ for every $v \in V(G)$ and $f \in F(v)$, and hence orientation O is F-avoiding of G. ■

By Claim 5 and Theorem 2.1, it suffices to show that the coefficient of the monomial $x_1 \dots x_m$ in the expansion of P is non-zero. First, we expand P without summations and fix the monomial $h = x_{i_1} \dots x_{i_m}$, where $\{i_1, \dots, i_m\}$ is a permutation of $\{1, \dots, m\}$. If we choose the variable x_{i_j} in the j-th parenthesis of P at v, then we orient the edge e_{i_j} from v to the other endpoint. Since the degree of x_i in the monomial h is exactly equal to 1, this operation is well-defined and we obtain an orientation D' of G. We call such an orientation D' *suitable*. It is not hard to see that D' is suitable if and only if $d_{D'}^+(v) = |F(v)|$ for every vertex $v \in V(G)$.

On the other hand, if we fix a suitable orientation D'' and v_j are incident out-going edge $e_{j_1}, \dots, e_{j_{|F(v)|}}$, then we obtain the monomial $x_{i_1} \dots x_{i_m}$ by choosing the variables $x_{j_1}, \dots x_{j_{|F(v_j)|}}$ at v_j from the product

$$\prod_{f \in F(v_j)} \left(\sum_{i=1}^{n} c_D(e_i, v_j) \cdot x_i + d_G(v_j) - 2f \right).$$

Since the number of choices of variables $x_{j_1}, \dots x_{j_{|F(v_j)|}}$ at v_j is equal to $|F(v_j)|!$, the sum of the monomial which is generated from D'', denoted by $r(D'')$, is

$$r(D'') = \prod_{v \in V(G)} \left(|F(v)|! \prod_{e \in E_{D'}^+(v)} c_D(e, v) \right) = C \prod_{v \in V(G)} \prod_{e \in E_{D'}^+(v)} c_D(e, v),$$

where $C = \prod_{v \in V(G)} |F(v)|! \neq 0$ is a constant. Therefore, the coefficient of the monomial $x_{i_1} \dots x_{i_m}$ is equal to $\sum_{D'} r(D')$, where Σ is taken over all suitable orientation of G.

Next, we focus on the signature of $r(D')$. For a suitable orientation D', we consider the orientation $D - D'$ of the spanning subgraph of G with the same directon as in D, which is obtained by removing all edges having a common direction in D and D'. Since $d_{D'}^+(v) = |F(v)| = d_D^+(v)$ by the suitability of

D' and assumption of D, we see that $D - D'$ is also Eulerian. Moreover, for every edge $e = uv \in E(G)$, we have

$$c_D(e, v) = \begin{cases} -c_D(e, u) & \text{if } \sigma(e) = +1, \\ c_D(e, u) & \text{if } \sigma(e) = -1. \end{cases}$$

Thus, if $D - D'$ has an even (respectively, odd) number of positive edges, then $r(D') = r(D)$ (respectively, $r(D') = -r(D)$). On the other hand, for every spanning Eulerian subgraph D' of D, we see that $D' \cup (D - D')$ is a suitable orientation of G. Thus, we have that $D - D' \in \sigma EE(D)$ if and only of $r(D) = r(D')$ for a suitable orientation D'. Therefore, we obtain

$$\left| \sum_{D'} r(D') \right| = |\sigma EE(D) - \sigma OE(D)| \neq 0$$

and we complete the proof. $\square$

Similarly to the proof of Lemma 2.5, we have the followings.

Proposition 5.4 *Let (G, σ) be a singed graph and D be a σAT-orientation of the graph G. Then (G, σ) is L-orientable for each valid list-assignment L with $|L(v)| \geq d_D^+(v) + 1$ for every $v \in V(G)$.*

Let us focus on the planar graph again. The following is shown in [5].

Theorem 5.5 ([5]) *$AT(G, \sigma) \leq 5$ for every planar signed graph (G, σ).*

Therefore, we have the following Corollary.

Corollary 5.6 *$o_\ell(G, \sigma) \leq 5$ for every planar signed graph (G, σ).*

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References

[1] S. Akbari, M. Dalirruuyfard, K. Ehsani, K. Ozeki and R. Sherkati, Orientations of graphs avoiding given lists on out-degrees, *J. Graph Theory*, **93** (2020) 483–502.

[2] N. Alon and M. Tarsi, Colorings and orientations of graphs, *Combinatorica* **12** (1992) 125–134.

[3] S.L. Hakimi, On the degrees of the vertices of a directed graph, *J. Franklin Inst*, **4** (1965) 290–308.

[4] T. Jensen and B. Toft, Graph Coloring Problems, *Wiley, New York*, 1995.

[5] W. Wang, J. Qian and T. Abe, Alon-Tarsi Number and Modulo Alon-Tarsi Number of Signed Graphs, *Graphs Combin.* **35** (2019) 1051–1064.

[6] X. Zhu, The Alon-Tarsi number of planar graphs, *J. Combin. Theory Ser. B* **134** (2019) 354–358.