Hamiltonicity of graphs on surfaces in terms of toughness and scattering number – A survey

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Abstract

This paper aims to survey Hamiltonicity of graphs on surfaces, including stronger (e.g. Hamiltonian-connectedness) and weaker (e.g. containing Hamiltonian paths and spanning trees with certain conditions) properties. Toughness and scattering number conditions are necessary conditions for graphs to have such properties. Since every k-connected graph on a surface F^2 satisfies some toughness and scattering number condition, we can expect that "every k-connected graph on a surface F^2 satisfies the property $\mathcal{P}$". We explain which triple $(k, F^2, \mathcal{P})$ makes the statement true in the viewpoint of the toughness and scattering number of graphs,

Keywords: Hamiltonian cycles, Spanning trees, Toughness, Scattering number, Plane graphs
MSC 2010: 05C05, 05C10, 05C38, 05C45,

1 Introduction

This paper aims to survey Hamiltonicity of graphs on surfaces, including stronger (e.g. Hamiltonian-connectedness) and weaker (e.g. containing Hamiltonian paths and spanning trees with certain conditions) properties. In the viewpoint of the toughness and scattering number of graphs, we discuss which triple $(k, F^2, \mathcal{P})$ makes the statement "every k-connected graph on a surface F^2 satisfies the property $\mathcal{P}$" be true.

2 Preliminaries

2.1 Hamiltonicity

A cycle or a path C in a graph G is said to be *Hamiltonian* if C visits all vertices in G. A graph with a Hamiltonian cycle is said to be *Hamiltonian*. Because of its relation to several problems such as the Travelling Salesman Problem, Hamiltonicity of graphs is one of the important topics in graph theory. In particular, Hamiltonicity of planar graphs or graphs on surfaces is related to the Four Color Theorem, and hence this has attracted many researchers. For example, Tait [95] claimed to show in 1880 that "if every 3-connected planar cubic graph is Hamiltonian, then every planar graph can be colored by four colors". Note that it was later shown that the assumption of Tait's theorem does not hold (see [103] for example), and hence this approach does not reach to a solution to the Four Color Problem. However, since then, Hamiltonicity of graphs on surfaces has attracted wide attention.

For example, Whitney [105] proved in 1931 that every 4-connected plane triangulation is Hamiltonian, and Tutte [101] improved this result to 4-connected planar graphs;

Theorem 1 (Tutte [101]) *Every 4-connected planar graph is Hamiltonian.*

Many results have been shown, and we will introduce those in this survey.

A graph G is said to be *Hamiltonian-connected* if for every pair of distinct vertices x and y in G, there is a Hamiltonian path in G between x and y. Note that any Hamiltonian-connected graph is Hamiltonian and any Hamiltonian graph contains a Hamiltonian path. We denote properties related to Hamiltonicity by $\mathcal{P}_{\mathrm{H}}, \mathcal{P}_{\mathrm{HP}}, \mathcal{P}_{\mathrm{HC}}$ as follows;

$$\begin{aligned} \mathcal{P}_{\mathrm{H}} &: \text{ Containing a Hamiltonian cycle,} \\ \mathcal{P}_{\mathrm{HP}} &: \text{ Containing a Hamiltonian path,} \\ \mathcal{P}_{\mathrm{HC}} &: \text{ Being Hamiltonian-connected.} \end{aligned}$$

2.2 The condition $(*)$, toughness and scattering number

We denote by $c(H)$ the number of components of a graph H. For a graph G and real numbers a and b, consider the following condition;

$$(*) \qquad c(G-S) \le a|S| + b \text{ for any } S \subseteq V(G) \text{ with } c(G-S) \ge 2.$$

The *toughness* of a graph G is the minimum of a satisfying $(*)$ with $b = 0$. Chvátal [26] (see also [27]) in 1973 introduced the concept of toughness and conjectured that there exists a constant t_0 such that every graph satisfying $(*)$ with $(a, b) = (\frac{1}{t_0}, 0)$ is Hamiltonian. This conjecture is still open. Chvátal first conjectured $t_0 > 3/2$ suffices, but some counterexamples were found (e.g. see [38]). Bauer, Broersma and Veldman [8] showed that there exist non-Hamiltonian graphs satisfying $(*)$ with $(a, b) = \left(\frac{4}{9}, \frac{11}{9}\right)$. Thus, $t_0 \ge 9/4$ even if Chvátal's conjecture holds. See [7, 15] for more information on toughness.

On the other hand, the *scattering number* of a graph G is defined as the minimum of b satisfying $(*)$ with $a = 1$. This concept was defined by Jung [56], as the "additive dual" for the concept of toughness.

As pointed out in [26, 56], it is easy to see the following necessary condition for Hamiltonicity.

Proposition 2 *Let G be a graph.*

- *If G contains a Hamiltonian path, then G satisfies $(*)$ with $(a, b) = (1, 1)$.*
- *If G is Hamiltonian, then G satisfies $(*)$ with $(a, b) = (1, 0)$.*
- *If G is Hamiltonian-connected, then G satisfies $(*)$ with $(a, b) = (1, -1)$.*

In this survey, we focus on several properties which have the condition $(*)$ as a necessary condition, such as those in Proposition 2. For a property $\mathcal{P}$, we define the real numbers $a_{\mathcal{P}}$ and $b_{\mathcal{P}}$ as follows;

$$(a_{\mathcal{P}}, b_{\mathcal{P}}) = \inf\Big\{(a, b) : \text{Every graph with the property } \mathcal{P} \text{ satisfies } (*) \text{ with } (a, b)\Big\},$$

where the infimum is taken in the lexicographical order. Note that some properties $\mathcal{P}$ do not have $a_{\mathcal{P}}$ and $b_{\mathcal{P}}$, but we do not consider such properties $\mathcal{P}$ in this survey.

For example, Proposition 2 implies $(a_{\mathcal{P}_\mathrm{H}}, b_{\mathcal{P}_\mathrm{H}}) \leq (1, 0)$, where $\leq$ represents the lexicographical order. On the other hand, consider the complete bipartite graph $K_{m,m}$ with $m \geq 2$, which is Hamiltonian. We see that for any $a < 1$ and any real number b, if we take a large integer m satisfying $(1-a)m > b$, then $c(K_{m,m} - S) = m > am + b = a|S| + b$, where S is the set of vertices contained in one partite set. Furthermore, for any $b < 0$, $c(K_{m,m} - S) = m > m + b = |S| + b$ for the same vertex set S. Thus, for any $a < 1$ and any b, and for $a = 1$ and any $b < 0$, there exists a Hamiltonian graph that does not satisfy $(*)$ with (a, b). These show that $(a_{\mathcal{P}_\mathrm{H}}, b_{\mathcal{P}_\mathrm{H}}) = (1, 0)$. Similarly, we see $(a_{\mathcal{P}_\mathrm{HP}}, b_{\mathcal{P}_\mathrm{HP}}) = (1, 1)$ and $(a_{\mathcal{P}_\mathrm{HC}}, b_{\mathcal{P}_\mathrm{HC}}) = (1, -1)$, by $K_{m,m+1}$ and the graph obtained from the complete bipartite graph $K_{m,m-1}$ by adding all possible edges in the larger bipartite set for a sufficiently large integer m, respectively.

In addition, the property of containing a spanning tree with bounded maximum degree is also related to the condition $(*)$. For $k \geq 2$, a *k-tree* is a tree with maximum degree at most k. A Hamiltonian path is nothing but a spanning 2-tree, and hence a spanning k-tree is related to Hamiltonicity. We denote by $\mathcal{P}_{k\text{-tree}}$ the property of containing a spanning k-tree. For this property, it is known that $(a_{\mathcal{P}_{k\text{-tree}}}, b_{\mathcal{P}_{k\text{-tree}}}) = (k-1, 1)$, see [79, Proposition 5] for a proof. We refer the readers to a survey [79] on spanning trees. Note that Win [106] showed that every graph satisfying $(*)$ with $(a, b) = (k-2, 2)$ contains a spanning k-tree, and a short proof was found in [37].

As an extension of a spanning k-tree, we define the following concept. For $k \geq 2$ and a tree T, the *total excess of T from k*, denoted by $\mathrm{te}(T, k)$, is defined as

$$\mathrm{te}(T, k) = \sum_{v \in V(T)} \max\left\{ \deg_T(v) - k, 0 \right\},$$

where $\deg_T(v)$ is the degree of a vertex v in T. Note that a tree T satisfies $\mathrm{te}(T, k) = 0$ if and only if T is a k-tree. Furthermore, a tree T satisfies $\mathrm{te}(T, 2) \leq t$ if and only if T has at most $(t+2)$ leaves. We denote by $\mathcal{P}_{(k,t)\text{-tree}}$ the property of containing a spanning tree of which the total excess from k is at most t. Similarly to the case of $\mathcal{P}_{k\text{-tree}}$, it can be shown that $(a_{\mathcal{P}_{(k,t)\text{-tree}}}, b_{\mathcal{P}_{(k,t)\text{-tree}}}) = (k-1, t+1)$.

A *spanning closed k-walk*, which is a spanning closed walk visiting every vertex at most k times, is related to a spanning k-tree. By the definition, a spanning closed 1-walk is nothing but a Hamiltonian cycle. Note that if a graph contains a spanning k-tree, then it contains a spanning closed k-walk, and if a graph contains a spanning closed k-walk, then it contains a spanning $(k+1)$-tree (see [54]). Let $\mathcal{P}_{k\text{-walk}}$ be the property of containing a spanning closed k-walk. Then we see $(a_{\mathcal{P}_{k\text{-walk}}}, b_{\mathcal{P}_{k\text{-walk}}}) = (k, 0)$. Recall that $(a_{\mathcal{P}_{k\text{-tree}}}, b_{\mathcal{P}_{k\text{-tree}}}) = (k-1, 1)$. Thus, we expect that the property of containing a spanning closed k-walk is close to the property of containing a spanning $(k+1)$-tree, while the former is stronger than the latter. This expectation will appear in Section 5.2 (see also [54]).

2.3 Graphs on Surfaces

We refer to Mohar and Thomassen [66] for the detail of surfaces and graphs on them.

A *surface* F^2 is a connected compact 2-dimensional manifold without boundary. By the classification of surfaces, F^2 is either an orientable surface of genus $t \geq 0$, denoted by S_t, or a nonorientable surface of genus $k \geq 1$, denoted by N_k. The *Euler genus* of a surface F^2, denoted by $g(F^2)$, is defined as

$$g(F^2) = \begin{cases} 2t & \text{if } F^2 \text{ is the orientable surface } S_t, \\ k & \text{if } F^2 \text{ is the nonorientable surface } N_k. \end{cases}$$

Note that S_0 is the sphere, N_1 is the projective plane, S_1 is the torus, and N_2 is the Klein bottle, where their Euler genera are $0, 1, 2$ and 2, respectively.

A graph G *on a surface* F^2 means an embedding of G in F^2, that is, a drawing of G on F^2 in which the edges do not cross each other. We can regard plane graphs also as graphs on the sphere through the inverse of the stereographic projection. Euler's formula states that any graph G on F^2 satisfies that

$$|V(G)| - |E(G)| + |F(G)| \geq 2 - g(F^2).$$

(Note that if G is 2-cell embedding in F^2, that is, every face is homeomorphic to a disk, then the equality holds.) The following is am important relation between the condition $(*)$ and graphs on surfaces.

Proposition 3 (Schmeichel and Bloom [91]) *Let $k \geq 3$ and let G be a k-connected graph on a surface F^2. Then G satisfies $(*)$ with $(a, b) = \left(\dfrac{2}{k-2}, \dfrac{2g(F^2)-4}{k-2}\right)$.*

Proof. We follows the proof by Goddard, Plummer and Swart [47].

Let $S \subseteq V(G)$, and let c be the number of components in $G - S$ (that is, $c = c(G - S)$). We construct the graph H by contracting each component in $G - S$ to one vertex, deleting all obtained multiple edges and loops, and deleting all edges connecting two vertices in S. Then H is a bipartite graph still on the surface F^2. Hence, Euler's formula directly implies $|E(H)| \leq 2(|S| + c) + 2g(F^2) - 4$. Since G is k-connected, each component in $G - S$ has at least k neighbors in S, and hence $|E(H)| \geq kc$. These two inequalities imply that $c(G - S) = c \leq \frac{2}{k-2}|S| + \frac{2g(F^2)-4}{k-2}$, as desired. □

Consider a triangulation H of F^2, and take the *face subdivision* of H, that is, add a new vertex to each face and connected it to all the vertices on the facial cycle. Let G be the obtained graph, which is a triangulation of F^2. By Euler's formula, we can calculate $c(G - V(H)) = |F(H)| = 2|V(H)| + 2g(F^2) - 4$. This means that G satisfies the bound obtained in Proposition 3 with equality when $S = V(H)$ for $k = 3$. Similarly, if we begin with quadrangulations with minimum degree at least 2, as H, then we can construct triangulations G of F^2 satisfying the bound obtained in Proposition 3 with equality when $S = V(H)$ for $k = 4$. To construct 5-connected ones, we may be more careful to choose graphs H, but at least some pentangulations with minimum degree at least 3 work. These show the following, that is, Proposition 3 is in some sense best possible.

Proposition 4 *For $k = 3, 4$ or 5 and any surface F^2, there exist infinitely many k-connected triangulations G on F^2 with $S \subseteq V(G)$ such that $c(G - S) \geq 2$ and*

$$c(G - S) = \frac{2}{k-2}\,|S| + \frac{2g(F^2) - 4}{k-2}.$$

To show the same conclusion for the case $k = 6$, we need to begin with some hexangulations of a surface F^2 with minimum degree at least 3. (Otherwise, after the face subdivision, the vertices in H will have degree at most 4, and hence it is not 5-connected.) Such hexangulations exist for any surfaces F^2 with $g(F^2) \geq 2$, but do not exist for the sphere and the projective plane. Similarly, for the case $k = 7$, we need to begin with a heptangulation with minimum degree at least 4. Note that for each surface F^2, there are only finitely many heptangulations. In fact, there are only finitely many 7-connected graphs on F^2. Thus, we do not consider the case $k \geq 7$ in this survey.

On the other hand, it should be worthwhile to notice that for $k = 3, 4$ or 5 and any surface F^2, there exist infinitely many k-connected triangulations G on F^2 that satisfy the bound obtained in Proposition 3 with strict inequality for any $S \subset V(G)$. For example, consider the triangulation obtained from the face subdivision of a 4-connected triangulation by removing some added vertices. We can check that such triangulations satisfy the above condition. We leave the detail to the readers.

A closed curve γ on a surface is said to be *contractible* if γ bounds a 2-cell region on F^2; Otherwise, γ is *essential.* The *representativity* $\text{rep}(G)$ of a graph G on a surface F^2 is the minimum number of intersecting points of G and γ, where γ ranges over all essential closed curves on F^2. (If F^2 is the sphere, then we define $\text{rep}(G) = \infty$.) Graphs on surfaces with sufficiently large representativity are sometimes informally called *locally planar* graphs, and known to have properties similar to plane graphs. Actually, by the same way as Proposition 3, we obtain the next proposition.

Proposition 5 *Let $k \geq 3$, $a_0 > \frac{2}{k-2}$ and b_0 be a real number. Then for any surface F^2, there exists a constant r such that every k-connected graph G on F^2 with $\text{rep}(G) \geq r$ satisfies $(*)$ with $(a, b) = (a_0, b_0)$.*

Note that for each k, Archdeacon, Hartsfield and Little [1] constructed a k-connected triangulation G_k embedded in some orientable surface F^2 with $S \subseteq V(G_k)$ such that $\text{rep}(G_k) \geq k$ and $c(G_k - S) \geq k|S|$. Note that the genus of F^2 depends on K, and this shows that the constant r in Proposition 5 must depend on the surface F^2.

2.4 Purpose of the study

This survey investigates for $k \geq 3$, a surface F^2, and several properties $\mathcal{P}$, such as $\mathcal{P}_{\text{H}}$, $\mathcal{P}_{\text{HP}}$, $\mathcal{P}_{\text{HC}}$, whether the statement that

every k-connected graph on F^2 satisfies $\mathcal{P}$

holds or not. In fact, this statement does not hold for some triples $(k, F^2, \mathcal{P})$, since as in Proposition 2, some properties $\mathcal{P}$ requires $(*)$ with $(a, b) = (a_{\mathcal{P}}, b_{\mathcal{P}})$ as a necessary condition, but Proposition 4 shows that for some surface F^2, there are k-connected graphs on F^2 that do not satisfy $(*)$ with $a_{\mathcal{P}}$ and $b_{\mathcal{P}}$.

However, we expect that the statement above holds for $(k, F^2, \mathcal{P})$ if all k-connected graphs on F^2 satisfy $(*)$ with $(a, b) = (a_{\mathcal{P}}, b_{\mathcal{P}})$. We introduce several results and conjectures, which supports this expectation. In this sense, the relation between Hamiltonicity and the connectivity of graphs on surfaces seems to be based on the condition $(*)$. See also the survey [32].

For example, consider the sphere S^2, which satisfies $g(S^2) = 0$. Proposition 4 implies that there are infinitely many 4-connected graphs on the sphere S^2 satisfying the equality in $(*)$ with $(a, b) = (1, -2)$. Since $b_{\mathcal{P}_{\text{H}}} = 0 \geq -2 = g(S^2) - 2$, all 4-connected graphs on the sphere S^2 satisfy $(*)$ with $(a, b) = (a_{\mathcal{P}_{\text{H}}}, b_{\mathcal{P}_{\text{H}}})$, which is a necessary condition to be $\mathcal{P}_{\text{H}}$. Thus, one can expect that all 4-connected graphs on the sphere S^2 are Hamiltonian. In fact, Tutte showed that this expectation indeed holds, see Theorem 1.

This situation can be seen for several triples $(k, F^2, \mathcal{P})$, and we discuss which triple makes the statement "every k-connected graph on F^2 satisfy $(*)$ with $(a, b) = (a_{\mathcal{P}}, b_{\mathcal{P}})$ satisfies $\mathcal{P}$" true.

2.5 Remark on non-Hamiltonian planar graphs

As in the previous subsection, our aim is to show some properties related to Hamiltonicity in k-connected graphs on a surface in terms of the condition $(*)$. One may wonder that instead of the k-connected condition, we can use directly the condition $(*)$ and guarantee Hamiltonicity. However, this does not seem to have a good behavior.

It is well-known that every graph satisfying $(*)$ with $(a, b) = (t, 0)$ is $\lceil 2/t \rceil$-connected (see [26]). Thus, Theorem 1 directly implies that every planar graph satisfying $(*)$ with $(a, b) = (t, 0)$ for $t < \frac{2}{3}$ is Hamiltonian. Nishizeki [71] constructed non-Hamiltonian plane triangulations satisfying $(*)$ with $(a, b) = (1, 0)$. This was later extended to non-Hamiltonian planar graphs satisfying $(*)$ with $(a, b) = \left(\frac{2}{3}, 0\right)$ by Harant [51] and also to non-Hamiltonian plane triangulations satisfying $(*)$ with $(a, b) = \left(\frac{2}{3}, \frac{2}{3}\right)$ by Owens [74]. (It is an open problem whether there exists a non-Hamiltonian plane triangulation satisfying $(*)$ with $(a, b) = \left(\frac{2}{3}, 0\right)$.) Thus, in this sense, Theorem 1 in terms on 4-connected planar graphs is more essential than the statement in terms of the condition $(*)$.

3 4-connectedness and properties $\mathcal{P}$ with $a_{\mathcal{P}} = 1$

In this section, we focus on 4-connected graphs on surfaces. By Proposition 3,

every 4-connected graph on a surface F^2 satisfies $(*)$ with $(a, b) = (1, g(F^2) - 2)$,

and by Proposition 4, there are infinitely many 4-connected graphs on F^2 that satisfy the equality in $(*)$ with $(a, b) = (1, g(F^2) - 2)$. Therefore, for properties $\mathcal{P}$ with $a_{\mathcal{P}} = 1$, such as $\mathcal{P}_{\mathrm{H}}$, $\mathcal{P}_{\mathrm{HP}}$ and $\mathcal{P}_{\mathrm{HC}}$, there are some surfaces F^2 such that every 4-connected graph on F^2 satisfies $(*)$ with $(a, b) = (a_{\mathcal{P}}, b_{\mathcal{P}})$. On the other hand, depending on the value of $b_{\mathcal{P}}$, Proposition 4 implies that there are some surfaces F^2 such that some 4-connected graphs on F^2 do not satisfy $(*)$ with $(a, b) = (a_{\mathcal{P}}, b_{\mathcal{P}})$.

For example, consider the torus T^2, which satisfies $g(T^2) = 2$. Proposition 4 implies that there are infinitely many 4-connected graphs on the torus T^2 satisfying the equality in $(*)$ with $(a, b) = (1, 0)$. Since $b_{\mathcal{P}_{\mathrm{HC}}} = -1 < 0 = g(T^2) - 2$, those are not Hamiltonian-connected. Because of this reason, we will deal with only the triple $(4, F^2, \mathcal{P})$ that satisfy $a_{\mathcal{P}} = 1$ and $b_{\mathcal{P}} \geq g(F^2) - 2$.

3.1 Hamiltonian paths, Hamiltonian cycles and Hamiltonian-connectedness

We first focus on the properties $\mathcal{P}_{\mathrm{H}}$, $\mathcal{P}_{\mathrm{HP}}$ and $\mathcal{P}_{\mathrm{HC}}$. We summarize the known results and conjectures in Table 1. As explained above, depending on the Euler genus $g(F^2)$, some surfaces F^2 may admit 4-connected graphs on F^2 that do not satisfy $(*)$ with $(a, b) = (a_{\mathcal{P}}, b_{\mathcal{P}})$. When such graphs exist, we represent "×" for the corresponding cell in Table 1, which means that there exist infinitely many counterexamples to the corresponding statement. The cell with "◯" means that the corresponding statement holds, which was shown in the reference in the cell. (e.g. Thomassen [99] proved that every 4-connected planar graph is Hamiltonian-connected.) The cell with "?" means that it is still open whether the corresponding statement holds, which has been conjectured in designated literature.

As explianed in Section 2.1, Whitney [105] proved that every 4-connected plane triangulation is Hamiltonian before Tutte's theorem for 4-connected planar graphs (Theorem 1). Note that later, Tutte [102] himself gave a simpler proof.

[1] This statement was known as Grünbaum's conjecture [50] before Thomas and Yu [96].

[2] This statement was known as Dean's conjecture [29] before Kawarabayashi and Ozeki [61].

Table 1: Hamiltonicity of 4-connected graphs on a surface.

	$\mathcal{P}_{\mathrm{HP}}$ ($b_{\mathcal{P}_{\mathrm{HP}}} = 1$)	$\mathcal{P}_{\mathrm{H}}$ ($b_{\mathcal{P}_{\mathrm{H}}} = 0$)	$\mathcal{P}_{\mathrm{HC}}$ ($b_{\mathcal{P}_{\mathrm{HC}}} = -1$)
Sphere (plane) $g(F^2) = 0$	○	○ Tutte [101]	○ Thomassen [99]
Projective plane $g(F^2) = 1$	○	○ Thomas & Yu [96][1]	○ Kawarabayashi & Ozeki [61][2]
Torus $g(F^2) = 2$	○ Thomas, Yu & Zang [98]	? Grünbaum [50] Nash-Williams [70]	×
Klein bottle $g(F^2) = 2$	?	?	×
N_3 $g(F^2) = 3$	?	×	×
Other surfaces $g(F^2) \geq 4$	×	×	×

Chiba and Nishizeki [23] pointed out an omission in the proof of Thomassen's result [99] stating that every 4-connected planar graph is Hamiltonian-connected, and fixed it. See also [72, Chapter 10]. Other proofs for the theorem can be found in [64, 75]. Based on Thomassen's proof, Chiba and Nishizeki [24] presented an algorithm to find a Hamiltonian cycle in 4-connected planar graphs, which was an improvement of the algorithms in [2, 49]. Recently, further algorithms can be found in [10, 92, 93].

As in Table 1, there are several results on Hamiltonicity of 4-connected graphs on surfaces, but still some open problems remain. The following is the most important conjecture in this field.

Conjecture 6 (Grünbaum [50], Nash-Williams [70]) *Every 4-connected graph on the torus is Hamiltonian.*

The difficulty of Conjecture 6 seems to come from the proof method for 4-connected graphs on the sphere and the projective plane to be $\mathcal{P}_{\mathrm{H}}$. Because of inductive arguments, all such results have been shown by proving a stronger property $\mathcal{P}'$ than $\mathcal{P}_{\mathrm{H}}$ (e.g. the property of containing a Hamiltonian cycle passing through a prescribed edge), which satisfies $(a_{\mathcal{P}'}, b_{\mathcal{P}'}) = (1, -1)$. However, this method does not work for the toroidal case, since there are infinitely many 4-connected graphs on the torus that do not satisfy $(*)$ with $(a, b) = (1, -1)$ as in Proposition 4. Thus, we cannot expect to use the same idea. This seems to be a difficulty in proving Conjecture 6.

Note that Proposition 3 implies that for a 4-connected graph G on the torus, G does not satisfy $(*)$ with $(a, b) = (1, -1)$ if and only if the equality in $(*)$ with $(a, b) = (1, 0)$ holds for some $S \subseteq V(G)$. Considering this situation, Fujisawa, Nakamoto and Ozeki proved the following theorem.

Theorem 7 (Fujisawa, Nakamoto and Ozeki [41, 68]) *Let G be a 4-connected graph on the torus. If the equality in $(*)$ with $(a, b) = (1, 0)$ holds for some $S \subseteq V(G)$, then G is Hamiltonian.*

Thus, to solve Conjecture 6, it suffices to focus on 4-connected graphs on the torus that satisfy $(*)$ with $(a, b) = (1, -1)$, and we may be able to use the proof method that has been succeeded for the case of the sphere and the projective plane. Conjecture 6 is still open, but the author expects that it will be solved by this argument.

3.2 Planar graphs and properties $\mathcal{P}$ with $(a_{\mathcal{P}}, b_{\mathcal{P}}) = (1, -2)$

If we restrict ourselves to 4-connected planar graphs G, then Proposition 3 implies that G satisfies $(*)$ with $(a, b) = (1, -2)$. Since $(a_{\mathcal{P}_{\mathrm{HC}}}, b_{\mathcal{P}_{\mathrm{HC}}}) = (1, -1)$, we expect that every 4-connected planar graph has properties stronger than being Hamiltonian-connected such that they require $(*)$ with $(a, b) = (1, -2)$ as a necessary condition. In fact, the following results are obtained;

- Every 4-connected planar graph G with at least 5 vertices satisfies that for any $S \subseteq V(G)$ with $|S| \leq 2$, $G - S$ is Hamiltonian[3]. (This statement was conjectured by Plummer [82], and shown by Thomas and Yu [96].)
- For any two vertices x, y and an edge e with $e \neq xy$ in a 4-connected planar graph G, there exists a Hamiltonian path in G connecting x and y through e. (This statement was shown by Sanders [88].)
- For any vertex x in a 4-connected planar graph G, $G - x$ is Hamiltonian-connected. (This statement can be shown using a result by Sanders [88].)
- For vertices u_1, u_2, v_1, v_2, possibly $u_i = v_j$ for some $i, j \in \{1, 2\}$, $G + \{u_1u_2, v_1v_2\}$ contains a Hamiltonian cycle through both u_1u_2 and v_1v_2. (This statement was shown by Ozeki and Vràna [78].)

If we write those properties as $\mathcal{P}_{2\mathrm{H}}$, $\mathcal{P}_{\mathrm{eHC}}$, $\mathcal{P}_{1\mathrm{HC}}$ and $\mathcal{P}_{2\mathrm{eHC'}}$, respectively, then we obtain $(a_{\mathcal{P}_{2\mathrm{H}}}, b_{\mathcal{P}_{2\mathrm{H}}}) = (a_{\mathcal{P}_{\mathrm{eHC}}}, b_{\mathcal{P}_{\mathrm{eHC}}}) = (a_{\mathcal{P}_{1\mathrm{HC}}}, b_{\mathcal{P}_{1\mathrm{HC}}}) = (a_{\mathcal{P}_{2\mathrm{eHC'}}}, b_{\mathcal{P}_{2\mathrm{eHC'}}}) = (1, -2)$ by the same way as Proposition 2, and the graph obtained from the complete bipartite graph $K_{m,m+2}$ by adding all possible edges in the larger bipartite set for a sufficiently large integer m.

Note that any graph with the property $\mathcal{P}_{2\mathrm{eHC'}}$ satisfies the properties $\mathcal{P}_{\mathrm{eHC}}$ and $\mathcal{P}_{1\mathrm{HC}}$ (the case when $u_1 = x$, $u_2 = y$ and $v_1v_2 = e$ corresponds to $\mathcal{P}_{\mathrm{eHC}}$, and the case when $u_2 = v_1 = x$ corresponds to $\mathcal{P}_{1\mathrm{HC}}$). However, it is unknown whether any graph with the property $\mathcal{P}_{2\mathrm{eHC'}}$ satisfies the property $\mathcal{P}_{2\mathrm{H}}$. At least there exist infinitely many graphs that satisfy the property $\mathcal{P}_{2\mathrm{H}}$, but do not satisfy the property $\mathcal{P}_{2\mathrm{eHC'}}$.

A *figure-eight* graph consists of two edge-disjoint cycles sharing exactly one vertex. The shared vertex is called its *center*. Rosenfeld [86] proved that every 4-connected planar graph contains a figure-eight graph as a spanning subgraph with prescribed vertex as a center. This property $\mathcal{P}_8$ is also related to the condition $(*)$, which satisfies $(a_{\mathcal{P}_8}, b_{\mathcal{P}_8}) = (1, 0)$. Thus, we may expect properties stronger than $\mathcal{P}_8$ for 4-connected planar graphs.

3.3 Surfaces F^2 with $g(F^2) \geq 4$ and properties $\mathcal{P}$ with $b_{\mathcal{P}} > 0$

As in Table 1, for any surface F^2 with $g(F^2) \geq 4$, there exist 4-connected graphs on F^2 without Hamiltonian path. This is shown by Proposition 4 and the fact $b_{\mathcal{P}_{\mathrm{HP}}} = 1 < g(F^2) - 2$. However, we would like to find some properties that may be weaker than the property $\mathcal{P}_{\mathrm{HP}}$ but are related to Hamiltonicity. In this subsection, we focus on $\mathcal{P}_{(2,t)\text{-tree}}$, which is equivalent with the property of containing a spanning tree with at most $(t + 2)$ leaves. Recall that $(a_{\mathcal{P}_{(2,t)\text{-tree}}}, b_{\mathcal{P}_{(2,t)\text{-tree}}}) = (1, t + 1)$. Since every 4-connected graph G on F^2 satisfies $(*)$ with

[3] This statement is related to Malkevitch's conjecture [65] stating that every 4-connected planar graph G contains a cycle of length l for any $3 \leq l \leq |V(G)|$, possibly except for $l = 4$. See also [21].

$(a,b) = \big(1, g(F^2)-2\big)$ (by Proposition 3), we expect the following conjecture, which was posed by the author.

Conjecture 8 (Ozeki [76]) *For every 4-connected graph G on a surface F^2 with $g(F^2) \geq 3$, there exists a spanning tree with at most $(g(F^2)-1)$ leaves.*

Recently, using a similar idea to the proof of Theorem 7, Nakamoto and Ozeki [69] showed a partial solution for 4-connected graphs on a surface F^2 with $g(F^2) \geq 3$ such that the equality in $(*)$ with $(a,b) = \big(1, g(F^2)-2\big)$ holds for some $S \subseteq V(G)$.

Considering Proposition 5, every 4-connected locally planar graph on a surface satisfies $(*)$ with $a = 2$. Ellingham, Gao [33] and Yu [104] focused on this fact, and proved that every 4-connected locally planar graph on a surface contains a spanning 3-tree and a spanning closed 2-walk. Mohar (see [66, P. 181]) posed the problem whether every 4-connected locally planar graph on a surface contains a spanning tree with few leaves, which corresponds to Conjecture 8.

Problem 9 (Mohar, see [66]) *For any surface F^2 with $g(F^2) \geq 3$, does there exist a constant $r = r(F^2)$ such that every 4-connected graph G on F^2 with rep$(G) \geq r$ contains a spanning 3-tree with $O(g(F^2))$ leaves?*

Böhme, Mohar and Thomassen [12] showed a weaker statement than Problem 9 stating that for any surface F^2 with $g(F^2) \geq 3$ and any $\varepsilon > 0$, there exists a constant $r = r(F^2, \varepsilon)$ such that every 4-connected graph G on F^2 with rep$(G) \geq r$ contains a spanning 3-tree with $\varepsilon|V(G)|$ leaves.

4 5-connectedness and properties $\mathcal{P}$ with $a_{\mathcal{P}} \geq \frac{2}{3}$

In this section, we focus on 5-connected graphs on surfaces. By Proposition 3,

any 5-connected graph on F^2 satisfies $(*)$ with $(a,b) = \left(\frac{2}{3}, \frac{2g(F^2)-4}{3}\right)$.

Regarding Hamiltonicity of 4-connected graphs on surfaces, there are some open problems represented by "?" in Table 1 and false statements represented by "×". However, assuming the stronger assumption 5-connectedness, we may obtain a result that guarantees Hamiltonicity. For example, Thomas and Yu [97] proved that Conjecture 6 holds if we replace "4-connected" in the assumption with "5-connected". See Table 2 for more results.

Note that before those results in Table 2, Barnette [5] and Brunet and Richter [18] proved that every 5-connected triangulation of the torus contains a Hamiltonian path, and a Hamiltonian cycle, respectively. Brunet, Nakamoto and Negami [17] proved the same conclusion holds for 5-connected triangulation of the Klein bottle.

We would like to guarantee Hamiltonicity of 5-connected graphs on a surface F^2 with $g(F^2) \geq 4$. As explained in the previous section, this is impossible with the assumption of "4-connectedness" by Proposition 4, but Proposition 5 implies that every 5-connected locally planar graph on a surface satisfies $(*)$ with $(a,b) = (1,0)$, which is a necessary condition for $\mathcal{P}_{\mathrm{H}}$. Base on this fact, we expect that the following conjecture holds. (See [66, P. 182] and [104].)

Conjecture 10 *For any surface F^2 with $g(F^2) \geq 3$, there exists a constant $r = r(F^2)$ such that every 5-connected graph G on F^2 with rep$(G) \geq r$ is Hamiltonian.*

Table 2: Hamiltonicity of 5-connected graphs on a surface.

	$\mathcal{P}_{\mathrm{HP}}$	$\mathcal{P}_{\mathrm{H}}$	$\mathcal{P}_{\mathrm{HC}}$
Sphere (plane) $g(F^2) = 0$	◯	◯	◯
Projective plane $g(F^2) = 1$	◯	◯	◯
Torus $g(F^2) = 2$	◯	◯ Thomas & Yu [97]	◯ Kawarabayashi & Ozeki [62]
Klein bottle $g(F^2) = 2$	?	?	?
Others (Locally planar) $g(F^2) \geq 3$	?	? Thomassen [100]	?

Conjecture 10 is still open. Yu [104] showed that Conjecture 10 is true if we restrict ourselves to triangulations. This was conjectured by Thomassen [100]. Kawarabayashi [58] showed that Yu's result holds even if we replace the conclusion by "Hamiltonian-connected".

On the other hand, Proposition 3 guarantees that every 5-connected graph on a surface satisfies $(*)$ with $(a, b) = (\frac{2}{3}, t)$ for some t. This seems to suggest that such graphs satisfy some properties $\mathcal{P}$ with $1 > a_{\mathcal{P}} \geq \frac{2}{3}$. As far as the author knows, there are no results on such properties, and hence we leave this as an open problem.

Problem 11 *Which properties $\mathcal{P}$ satisfy $1 > a_{\mathcal{P}} \geq \frac{2}{3}$? Does every 5-connected planar graph (or 5-connected graph on a surface) satisfy the property $\mathcal{P}$?*

5 3-connectedness and properties $\mathcal{P}$ with $a_{\mathcal{P}} = 2$

As explained in the previous sections, 4-connected graphs on surfaces have some properties related to Hamiltonicity. In this section, we consider the similarity for 3-connected graphs. By Proposition 3,

any 3-connected graph on F^2 satisfies $(*)$ with $(a, b) = (2, 2g(F^2) - 4)$,

and by Proposition 4, there are infinitely many 3-connected graphs on F^2 that satisfy the equality in $(*)$ with $(a, b) = (2, 2g(F^2) - 4)$. Therefore, it is natural to show properties $\mathcal{P}$ with $a_{\mathcal{P}} \leq 2$ in 3-connected graphs on surfaces.

5.1 Spanning trees

We first focus on the property $\mathcal{P}_{(3,t)\text{-tree}}$, which satisfies $a_{\mathcal{P}_{(3,t)\text{-tree}}} = 2$, together with some works on $\mathcal{P}_{k\text{-tree}}$ and $\mathcal{P}_{(2,t)\text{-tree}}$. We summarize those results in Table 3. The cell with "◯" means that the corresponding statement holds with best possible value on k or t, while "△" means that the corresponding statement holds but the sharpness is still unknown.

Barnette [6] proved that every 3-connected planar graph contains a spanning 3-tree. Similarly to the extension from $\mathcal{P}_{\mathrm{H}}$ to $\mathcal{P}_{\mathrm{HC}}$, this result was extended to show the property that for any vertex x, there exists a spanning 3-tree in which x is a leaf [4, 67]. If we denote this property by $\mathcal{P}_{3\text{-tree}^+}$, then we see that $(a_{\mathcal{P}_{3\text{-tree}^+}}, b_{\mathcal{P}_{3\text{-tree}^+}}) = (2, -1)$. Since every

Table 3: Spanning trees and closed walks in 3-connected graphs on a surface.

	$\mathcal{P}_{k\text{-tree}}$	$\mathcal{P}_{(2,t)\text{-tree}}$	$\mathcal{P}_{k\text{-walk}}$
Sphere (plane) $g(F^2) = 0$	○ Barnette [6] $(k = 3)$	△ N. O. O. [67] $\left(t = \frac{\lvert G\rvert-7}{3}\right)$ 11	○ Gao & Richter [43] $(k = 2)$
Projective plane $g(F^2) = 1$	○ Barnette [4] $(k = 3)$	○ N. O. O. [67] $\left(t = \frac{\lvert G\rvert-7}{3}\right)$	○ Gao & Richter [43] $(k = 2)$
Torus/K-bottle $g(F^2) = 2$	○ Barnette [4] $(k = 3)$	○ N. O. O. [67] $\left(t = \frac{\lvert G\rvert-3}{3}\right)$	○ B. E. G. M. R. [16] $(k = 2)$
Others (locally planar) $g(F^2) \geq 3$	○ Yu [104] $(k = 4)$	△ $\left(t = \frac{\lvert G\rvert+O(1)}{3}\right)$	○ Yu [104] $(k = 3)$
Others $g(F^2) \geq 3$	○ Sanders & Zhao [90] Ota & Ozeki [73] $\left(k = \lceil\frac{2g(F^2)+4}{3}\rceil\right)$	?	○ Sanders & Zhao [90] Hasanvand [52] $\left(k = \lceil\frac{2g(F^2)+2}{3}\rceil\right)$

3-connected planar graph satisfies $(*)$ with $(a, b) = (2, -4)$ by Proposition 3, we may expect that a further stronger statement holds.

Proposition 5 implies that there exist infinitely many 3-connected triangulations on a surface F^2 that satisfy the equality in $(*)$ with $(a, b) = (2, 2g(F^2) - 4)$ for some $S \subseteq V(G)$. Since $(a_{\mathcal{P}_{3\text{-tree}}}, b_{\mathcal{P}_{3\text{-tree}}}) = (2, 1)$, we cannot expect that every 3-connected graph on a surface F^2 with $g(F^2) \geq 3$ contains a spanning 3-tree. Thus, we consider the existence of a spanning tree with higher (but bounded) maximum degree in such graphs. Yu [104] proved that for any surface F^2 with $g(F^2) \geq 3$, there exists a constant $r = r(F^2)$ such that every 3-connected graph G on F^2 with $\mathrm{rep}(G) \geq r$ contains a spanning 4-tree. Note that before Yu [104], Thomassen [100] proved the same statement for triangulations of orientable surfaces. Sanders and Zhao [90], and Ota and Ozeki [73] proved that every 3-connected graph G on F^2 with $g(F^2) \geq 3$ contains a spanning $\lceil \frac{2g(F^2)+4}{3} \rceil$-tree. (Sanders and Zhao [90] proved the case $g(F^2) \geq 38$, and later Ota and Ozeki [73] extended this to the case $g(F^2) \geq 3$.) Note that the upper bound "$\lceil \frac{2g(F^2)+4}{3} \rceil$" of the maximum degree in the obtained spanning tree is best possible, which is obtained also by the condition $(*)$. Before [73], Ellingham [32] obtained a weaker conclusion.

Furthermore, similarly to the extension mentioned in Section 3.3, those results on a spanning 3-tree were extended to a spanning tree with bounded total excess from 3 for the case of surfaces F^2 with $g(F^2) \geq 3$. Recall that $(a_{\mathcal{P}_{(3,t)\text{-tree}}}, b_{\mathcal{P}_{(3,t)\text{-tree}}}) = (2, t + 1)$. Since every 3-connected graph on F^2 satisfies $(*)$ with $(a, b) = (2, 2g(F^2) - 4)$, we expect that such graphs contain a spanning tree T with $\mathrm{te}(T, 3) \leq 2g(F^2) - 5$. This expectation was indeed true; Kawarabayashi, Nakamoto and Ota [60] proved that every 3-connected locally planar graph G on a surface F^2 contains a spanning 4-tree T with $\mathrm{te}(T, 3) \leq 2g(F^2) - 5$, and Ozeki [76] proved that every 3-connected graph G on F^2 contains a spanning $\lceil \frac{2g(F^2)+4}{3} \rceil$-tree T with $\mathrm{te}(T, 3) \leq 2g(F^2) - 5$.

Nakamoto, Oda and Ota [67] considered the existence of a spanning 3-tree with few number of leaves. Their results are displayed in the column of $\mathcal{P}_{(2,t)\text{-tree}}$ in Table 3. The bound on t obtained in [67] is almost best possible, but in the planar case, there is a gap. They showed that every 3-connected planar graph contains a spanning 3-tree T with $\mathrm{te}(T, 2) \leq \max\left\{\frac{|G|-7}{3}, 0\right\}$ (that is, the number of leaves is at most $\max\left\{\frac{|G|-1}{3}, 2\right\}$), while they conjectured that the best possible value would be $\max\left\{\frac{|G|-11}{3}, 0\right\}$ (that is, the number of leaves could be reduced to at most $\max\left\{\frac{|G|-5}{3}, 2\right\}$).

5.2 Spanning closed walks

Recall that we expect that the property of containing a spanning closed k-walk is closed to the property of containing a spanning $(k+1)$-tree, while the former is stronger than the latter. With this expectation in mind, almost all results on $\mathcal{P}_{k\text{-tree}}$ in Table 3 were extended to the existence of a spanning closed k-walk. Gao and Richter [43] proved that every 3-connected graph on the sphere or the projective plane contains a spanning closed 2-walk. Cui [28] gave later a short proof to this result. This was extended to 3-connected graphs on the torus and the Klein bottle by Brunet, Ellingham, Gao, Metzlar and Richter [16]. Some results on 3-connected graphs on surfaces were obtained as in Table 3.

As explained in the previous section, Kawarabayashi, Nakamoto and Ota [60] proved that every 3-connected locally planar graph G on a surface F^2 contains a spanning 4-tree T with $\text{te}(T,3) \leq 2g(F^2) - 5$. They proved the similar result also for a spanning closed 3-walk.

By the above situation, it is natural to think an extension of the result of the property $\mathcal{P}_{(2,t)\text{-tree}}$ to spanning closed 2-walks, Nakamoto, Oda and Ota [67] implicitly posed the following problem.

Problem 12 *Does there exist a constant s_0 such that every 3-connected planar graph G contains a spanning closed 2-walk by which the number of vertices visited twice is at most $\frac{1}{3}|G| + s_0$?*

This has not been solved yet. Gao, Richter and Yu [44] (see also [45]) showed that every 3-connected planar graph G contains a spanning closed 2-walk by which every vertex visited twice is contained in a 3-cut of G. However, since there exist infinitely many 3-connected planar graphs such that almost all vertices are contained in a 3-cut, this result does not give a non-trivial bound on the number of vertices visited twice in a spanning closed 2-walk.

Note that for a spanning closed 2-walk W, the number of vertices visited twice by W is at most $\frac{1}{3}|G| + s_0$ if and only if the length of W is at most $\frac{4}{3}|G| + s_0$. With this relation in mind, Kawarabayashi and Ozeki [63] proved that every 3-connected planar graph G contains a spanning closed walk of length at most $\frac{4|G|-1}{3}$, but some vertices may be visited three or more times.

5.3 Prism-Hamiltonicity

Another related property, so-called *prism-Hamiltonian*, has been studied. The *prism over a graph G* is defined as the Cartesian product of G and K_2. Thus, it consists of two copies of G and a matching joining the corresponding vertices. A graph G is said to be *prism-Hamiltonian* if the prism over G is Hamiltonian. It is easy to see that any graph with a Hamiltonian path is prism-Hamiltonian, and any prism-Hamiltonian graph contains a spanning closed 2-walk, see [57]. Furthermore, if we denote by $\mathcal{P}_{\text{prism}}$ the property of being prism-Hamiltonian, then we see that $(a_{\mathcal{P}_{\text{prism}}}, b_{\mathcal{P}_{\text{prism}}}) = (a_{\mathcal{P}_{\text{2-walk}}}, b_{\mathcal{P}_{\text{2-walk}}}) = (2, 0)$.

Motivated the theorem that every 3-connected planar graph contains a spanning closed 2-walk (see Section 5.2), Rosenfeld and Barnette [85] (see also [57]) conjectured that every 3-connected planar graph is prism-Hamiltonian, and they proved that any 3-connected cubic planar graph is prism-Hamiltonian[4]. Biebighauser and Ellingham [9] proved that this

[4]Their proof depends on the Four Color Theorem, which was still unsolved at that time. Without using

conjecture holds for triangulations on the sphere, and also the same conclusion holds for triangulations on the projective plane, the torus, and the Klein bottle. Recently, Spacapan [94] constructed a counterexample to the Rosenfeld-Barnette conjecture.

6 Related results

6.1 Graphs without complete bipartite minors

It is well known that for a 3-connected graph G of order at least 6, G is planar if and only if G does not contain $K_{3,3}$ as a minor (see [66, Lemma 2.5.5], together with Kuratowski's characterization of planar graphs). Considering this characterization, we here focus on 3-connected $K_{3,t}$-minor-free graphs. Note that the lower bound of the genus of the complete bipartite graphs, which is directly obtained by Euler's formula, shows that for any positive integer g, there is an integer t such that any graph on a surface F^2 with $g(F^2) = g$ does not have a $K_{3,t}$-minor[5]. On the other hand, as pointed out in [22], for any g, Böhme, Maharry and Mohar [11] constructed 3-connected $K_{3,7}$-minor-free graphs G that cannot be embedded in an orientable surface F^2 with $g(F^2) = g$. These two results show that studying 3-connected $K_{3,t}$-minor graphs is in fact different from studying 3-connected graphs on surfaces with large Euler genus.

Chen, Egawa, Kawarabayashi, Mohar and Ota [20] showed that every 3-connected $K_{3,t}$-minor-free graph satisfies $(*)$ with $(a, b) = (t-1, -2t+2)$, and for each odd integer t with $t \geq 3$, there exist infinitely many 3-connected $K_{3,t}$-minor-free graphs that satisfy the equality in $(*)$ with $(a, b) = (t-1, -2t+2)$. Recall that $(a_{\mathcal{P}_{k\text{-tree}}}, b_{\mathcal{P}_{k\text{-tree}}}) = (k-1, 1)$. Ota and Ozeki [73] showed the following theorem as expected; for $t \geq 3$, every 3-connected $K_{3,t}$-minor-free graph contains a spanning $(t-1)$-tree when t is even, and a spanning t-tree when t is odd.

Related to this result, it was shown in [36, 35] that every 3-connected $K_{2,4}$-minor-free graph and every 3-connected $K_{2,5}$-minor-free planar graph are Hamiltonian, respectively. Using some properties of 3-connected graphs with no $K_{2,t}$-minor, which was shown in [25][6], we can prove that for any integer $t \geq 3$, there exists a constant $c(t)$ such that every 3-connected $K_{2,t}$-minor-free graph contains a spanning tree with at most $c(t)$ leaves. We leave the proof to the readers.

6.2 Hamiltonicity of 4-connected graphs with small crossing number

The *crossing number* of a graph G, denoted by $\mathrm{cr}(G)$, is the minimum number of edge crossings over all possible drawings of G on the plane. Note that a graph G is planar if and only if $\mathrm{cr}(G) = 0$. By the same way to obtain Proposition 3, we can show that every 4-connected graph G satisfies $(*)$ with $(a, b) = \left(1, \frac{\mathrm{cr}(G)-4}{2}\right)$, which is best possible. With this in mind, recently Ozeki and Zamfirescu [80] conjectured the following;

Conjecture 13 (Ozeki and Zamfirescu [80]) *Every 4-connected graph G with $\mathrm{cr}(G) \leq 5$ is Hamiltonian.*

Since any graph G with $\mathrm{cr}(G) \leq 1$ has an embedding in the projective plane, Conjecture 13 holds by [96] if we replace the condition $\mathrm{cr}(G) \leq 5$ with $\mathrm{cr}(G) \leq 1$. Ozeki and Zamfirescu

the Four Color Theorem, the same result was proven by Goodey and Rosenfeld [48] with additional conditions and then finally, by Fleischner [40]. The most general result on this direction is the following due to Paulraja [81]; any 3-connected cubic graph is prism-Hamiltonian.

[5] This is related to the map color theorem, see [83, 84].

[6] Ding [31] gave a complete structure of $K_{2,t}$-minor-free graphs.

[80] extended this to the case of $\mathrm{cr}(G) \leq 2$. Since any graph G with $\mathrm{cr}(G) \leq 2$ can be embedded in the Klein bottle, this is a special case of the open problem stating that every 4-connected graph on the Klein bottle is Hamiltonian, see Table 1. Conjecture 13 is generally open.

6.3 Hamiltonicity of 3-connected planar graphs

More results on Hamiltonian cycles in planar graphs have been obtained. By Proposition 4, there are some 3-connected planar graphs that do not satisfy (∗) with $(a, b) = (1, 0)$, and hence are not Hamiltonian. However, assuming some conditions which guarantees the necessary condition, some results are obtained.

As a corollary of Thomassen's theorem [99], we obtain that every plane triangulation with at most one 3-cut is Hamiltonian (which is also shown by Chen [19]). It was shown that the condition on "at most one 3-cut" can be replaced with "at most two 3-cuts" (Helden [53]) and also "at most three 3-cuts" (Jackson and Yu [55]). The same conclusion holds for 3-connected planar graphs with at most three 3-cuts (Brinkmann and Zamfirescu [14]). There are 3-connected plane triangulations with six 3-cuts that do not satisfy (∗) with $(a, b) = (a, 0)$, and hence are not Hamiltonian (see [13]). Thus, it is natural to conjecture that every 3-connected planar graph with at most five 3-cuts is Hamiltonian, and this is still open. The similar work for the properties of containing a Hamiltonian path or of being Hamiltonian-connected can be found in a survey [77].

Dillencourt [30] and Sanders [87] gave another type of sufficient condition for planar graphs to be Hamiltonian.

6.4 Spanning 2-connected subgraph with bounded degree

A spanning 2-connected subgraph with maximum degree at most k in a graph G is called a *k-covering* (or sometimes a *k-trestle*) of G. While the property of containing a k-covering does not seem to be directly related to the condition (∗), this is an extension of a spanning k-tree, and there are some works on this topic. First, Barnette [3] proved that every 3-connected planar graph contains a 15-covering. This result was later improved by Gao [42] to a 6-covering in a 3-connected graph on a surface F^2 with $g(F^2) \leq 2$, which is best possible. Kawarabayashi, Nakamoto and Ota [60, 59] and Sanders and Zhao [89] considered the existence of a k-covering in 3-connected graphs on surfaces with higher genera.

It is also known that with additional conditions, we can reduce the bound on maximum degree in a k-covering. Ellingham and Kawarabayashi [34] showed the existence of a 3-covering in 4-connected locally planar graphs. Enomoto, Iida and Ota [39] proved that every 3-connected planar graph with minimum degree at least 4 contains a connected subgraph in which every vertex has degree 2 or 3.

Note that Gao and Wormald [46] proved that every plane triangulation contains a spanning closed trail (i.e. a spanning closed walk that does not use an edge twice or more) by which every vertex is visited at most four times. Similarly to a k-covering, this does not seem to be directly related to the condition (∗), but this is an extension of a spanning closed walk.

7 Conclusion

In this survey, we introduce a relation between Hamiltonicity and connectivity of graphs on surfaces in terms of the condition (∗), which is related to a toughness and scattering number. In fact, we pose the following informal meta-conjecture.

Meta-conjecture *For any graph property $\mathcal{P}$ related to Hamiltonicity, if all k-connected graphs on a surface F^2 satisfy the condition $(*)$ with $(a, b) = (a_{\mathcal{P}}, b_{\mathcal{P}})$, then every k-connected graph on F^2 satisfies the property $\mathcal{P}$.*

As explained in Sections 3–5, this meta-conjecture holds for several properties $\mathcal{P}$, and in Section 6 we show some related results and conjectures that support the meta-conjecture. In this sense, the relation between Hamiltonicity and the connectivity of graphs on surfaces seems to be based on the condition $(*)$.

Acknowledgments

I would thank the referees for carefully reading the paper and helpful comments. This work was supported by JSPS KAKENHI Grant Number JP18K03391.

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