Kempe equivalence classes of cubic graphs embedded on the projective plane

Kenta Ozeki

Faculty of Environment and Information Sciences,
Yokohama National University, Japan.
e-mail: ozeki-kenta-xr@ynu.ac.jp

Abstract

Let G be a cubic graph with 3-edge-colorings and let φ be a 3-edge-coloring of G. If we take an alternating cycle (i.e. a 2-edge-colored cycle) C and swap the colors of all edges in C, then we obtain another 3-edge-coloring of G. This operation is called a *Kempe switch at* C and two 3-edge-colorings of G are *Kempe equivalent* if one is obtained from the other by a sequence of Kempe switches. Mohar asked in 2007 which cubic bipartite graphs have only one Kempe equivalence class. In this paper, we partially answer this question using several topological arguments. We prove that a cubic bipartite graph G embedded on the projective plane admits only one Kempe equivalence class if and only if the dual G^* is not 4-vertex-colorable. As a by-product, we also show that the list-edge-coloring conjecture holds for cubic graphs G embedded on the projective plane such that the dual G^* is not 4-vertex-colorable.

Keywords: 4-Color Theorem, 3-edge-coloring, Kempe equivalence,
MSC 2010: 05C10, 05C15

1 Introduction

Let G be a cubic graph with 3-edge-colorings and let φ be a 3-edge-coloring of G. For an *alternating* cycle (i.e. a 2-edge-colored cycle) C with respect to φ, a *Kempe switch at* C is an operation to obtain another 3-edge-coloring by swapping the colors of all edges in C. See Figure 1 for an example. Two 3-edge-colorings of G are said to be *Kempe equivalent* if one is obtained from the other by a sequence of Kempe switches. This is indeed an equivalence relation on the set of all 3-edge-colorings of G. Note that if we admit Kempe switch at all alternating cycles of two particular colors, then we have the 3-edge-coloring obtained by the permutation of those colors.

This notion comes from Kempe's wrong "proof" [17] of the 4-Color Theorem, where an error was pointed out by Heawood. However, his idea is so strong that it can prove the 5-Color Theorem, and it was used in the proofs [3, 31] of the 4-Color Theorem. (To be exact, the "dual" form was used to show the reducibility of configurations for the proof of the 4-Color Theorem. See Section 3.1 for the duality of 3-edge-colorings.)

Note that the statement of the 4-Color Theorem can be rephrased that "no planar snark exists", where a *snark* is a non-3-edge-colorable cubic graph. (Note that the definition of a snark is sometimes required more conditions on the connectivity and girth.) The class of

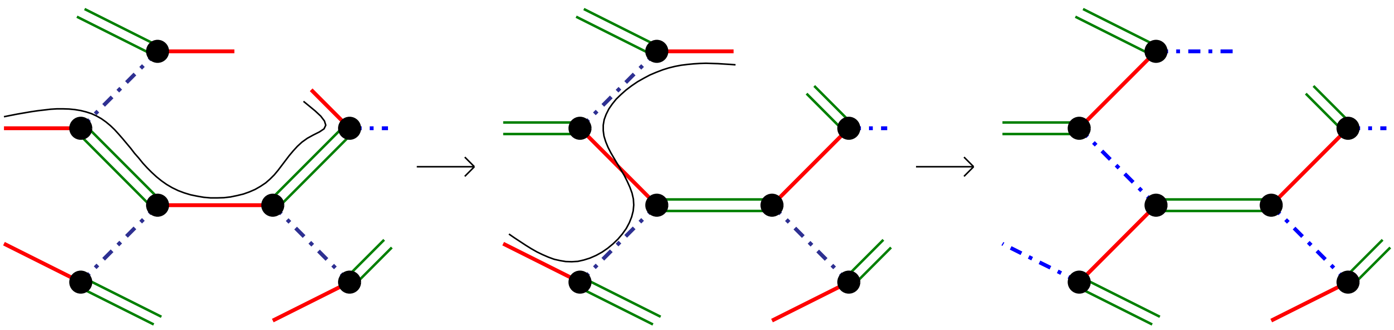

Figure 1: An example of a Kempe switch. In each step, we perform the Kempe switch at the cycle indicated by the thin curve.

snarks is known to be important, since this class forms a possible counterexample to several well-known conjectures, such as the cycle double cover conjecture, the nowhere zero 5-flow conjectures, and the 4-connected line graph Hamiltonian conjecture, and so on. So, we are interested in some properties on snarks. To do that, Nedela and Škoviera [29] used Kempe switches in subcubic graphs. This method appears in several papers, see [14, 22, 33].

Related to those works, it is important to study Kempe equivalence classes of 3-edge-colorings, and in particular, we would like to check whether a given 3-edge-colorable cubic graph has only one Kempe equivalence class. Mohar [25] asked the following problem:

Problem 1 *Characterize all cubic bipartite graphs that have only one Kempe equivalence class.*

Note that there does not seem to exist a positive reason to restrict ourselves to bipartite graphs. However, we point out that the problem for general graphs is too difficult: One big issue is, before focusing on Kempe equivalence classes, we have to first consider the 3-edge-colorability of cubic graphs. This is a well-known NP-complete problem [12]. Oppositely, any cubic bipartite graphs admit a 3-edge-coloring, and hence we can avoid such an issue.

As a partial solution to Problem 1, we have the following theorem. As pointed out in [25], Fisk ([9, Theorem 6] and [11, Theorem 1]) originally expressed the statement in the dual version.

Theorem 1 (Fisk [9]) *Any 3-edge-colorings in a cubic bipartite plane graph are Kempe equivalent.*

In this paper, we will consider the case of the projective plane, and give several results on Kempe equivalence. The following, which is an analogy of Theorem 1, can be obtained from our results.

Theorem 2 *Let G be a cubic bipartite graph embedded on the projective plane. The any 3-edge-colorings of G are Kempe equivalent if and only if the dual triangulation G^* is not 4-vertex-colorable.*

Since there are infinitely many cubic bipartite graphs G embedded on the projective plane such that G^* is not 4-vertex-colorable, this gives a new family for the answer to Problem 1.

As Mohar [25] explained, the number of Kempe equivalence classes has some applications in statistical physics (see [26, 40] for example) and Markov chain [39]. Furthermore, there are several related works on the Kempe equivalence classes: belcastro and Haas [4] construct some families of cubic graphs with particular number of Kempe equivalence classes, and McDonald, Mohar and Scheide [23] considered Kempe equivalence classes for 4-edge-colorings in cubic graphs. The vertex-coloring version has been also considered in several papers [6, 8, 38].

We organize this paper as follows: In the next section, we give some notation. In Section 3, we define a *type* of a 3-edge-coloring, which can distinguish Kempe equivalence classes under certain conditions, and in Section 4, we introduce a *signature* of a 3-edge-coloring and show its relation to types. As a by-product, this relation shows that the list-edge-coloring conjecture holds for cubic graphs G embedded on the projective plane such that the dual G^* is not 4-vertex-colorable. In Sections 5 and 6, we prove Theorem 2 showing several properties on types of 3-edge-colorings. In the last section, we give several related open problems.

2 Preliminary

2.1 Terminology on topology

A *surface* F^2 is a connected compact 2-dimensional manifold without boundary. A closed curve γ on F^2 is *essential* if γ does not bound a 2-cell region on F^2. Otherwise, γ is *contractible*. A closed curve γ on F^2 is 1-*sided* if a tubular neighborhood of γ is a Möbius strip (i.e., along γ left and right interchange); otherwise it is 2-*sided*. Two closed curves are *homotopic* if one can be continuously deformed into the other on the surface. Note that when the surface F^2 is the projective plane, it is known that a closed curve is essential if and only if it is 1-sided. In fact, the projective plane has the unique essential cycle, up to homotopy. The following well-known fact for the projective plane is used several times in this paper:

Fact 3 *For two closed curves on the projective plane, they are both essential (i.e.* 1*-sided) if and only if they intersect transversally an odd number of times.*

For a set Γ of pairwise non-intersecting closed curves on a surface F^2, we denote by $F^2-\Gamma$ the topological space obtained from F^2 by deleting all closed curves in Γ. The topological space $P^2-\Gamma$ is said to be 2-*region-colorable*, if the regions of $P^2-\Gamma$ can be colored by 2-colors so that two regions sharing a closed curve in Γ have different colors. We also need the following proposition

Proposition 4 *Let Γ be a set of pairwise non-intersecting closed curves on the projective plane P^2. Then exactly one of the following holds:*

(I) All closed curves in Γ are contractible. Furthermore, $P^2-\Gamma$ is 2*-region-colorable.*

(II) Γ contains exactly one essential closed curve. Furthermore, $P^2-\Gamma$ is not 2*-region-colorable.*

Proof. If there are two essential curves in Γ, then Fact 3 implies that they intersect transversally an odd number of times, which contradicts that they are non-intersecting. Therefore, there is at most one essential cycle in Γ. If there are no essential cycles in Γ, then the first half of (I) holds. Furthermore, since all closed curves in Γ are contractible, it is easy to find the desired coloring of the regions of $P^2-\Gamma$. On the other hand, suppose that there is an exactly one essential closed curve γ_0 in Γ. Since all essential closed curves on the projective plane are 1-sided, γ_0 is 1-sided. Thus, if we color the regions of $P^2-\Gamma$ by 2-colors, then both side of γ_0 must be contained in the same face, and in fact colored by the same color. Therefore, $P^2 - \Gamma$ is not 2-region-colorable. $\square$

2.2 Terminology on graphs embedded on surfaces

In this paper, a graph *embedded on a surface* F^2 always means its 2-cell embedding on F^2 without edge crossings (i.e., an embedding on F^2 such that each face is homeomorphic to an open disc).

A *triangulation* of a surface F^2 is a graph embedded on F^2 with each face triangular. A *facial walk* in a graph embedded on a surface is the boundary walk of some face. If it is a cycle, then it is called a *facial cycle* in particular. Note that a cycle of a graph embedded on a surface F^2 can be regarded as a closed curve on F^2. Thus, we sometimes say that a cycle is essential/contractible in the sense of closed curves.

For a graph G embedded on a surface, the *dual* of G is denoted by G^*. For $S \subseteq E(G)$, we denote by S^* the set of dual edges e^* in G^* taken over all edges e in S. In this paper, we sometimes regard a cycle also as the set of all edges in the cycle. For example, if D is a cycle in a graph embedded on a surface, then it sometimes means $E(D)$ and we write D^* for $\big(E(D)\big)^*$.

2.3 Terminology on 3-edge-colorings of a cubic graph

In this paper, we always use a, b and c for the colors of 3-edge-colorings. In all figures, the red thick line edges, the blue dashed-dotted line edges, and the green double line edges represent those colored by a, b and c, respectively. Note that each color x induces a perfect matching of the cubic graph, which is always denoted by M_x. Thus, any 3-edge-coloring of a cubic graph G can be regarded as a partition of $E(G)$ into three perfect matchings M_a, M_b and M_c.

A 2-*factor* of a graph G is a spanning subgraph in which every vertex has degree exactly 2. A 2-factor consists of pairwise vertex-disjoint cycles in G. Note that T is a 2-factor in a cubic graph G if and only if $E(G) - E(T)$ is a perfect matching. In particular, for a 3-edge-coloring of a cubic graph G, any two colors form a 2-factor, that is, $G - M_x$ is a 2-factor of G for any $x \in \{a, b, c\}$, in which all cycles are alternating by the two colors in $\{a, b, c\} - \{x\}$.

Let T be a 2-factor of a graph G embedded on the projective plane. In this case, all cycles in a 2-factor can be regarded as closed curves on the projective plane. Thus, by Proposition 4, either (I) all cycles in T are contractible, or (II) T contains exactly one essential cycle. In the former case (I), T is said to be *contractible*, while in the latter case (II), T is *essential*. We see that Fact 3 holds with closed curves replaced by 2-factors.

3 "Type" of a 3-edge-coloring of cubic graphs

In this section, we give a method to distinguish Kempe equivalence class of 3-edge-colorings. This plays the central role in this paper. To do that, we need some definitions.

3.1 The duality of 3-edge-colorings

Let G be a cubic plane graph, and let $\varphi : E(G) \to \{a, b, c\}$ be a 3-edge-coloring of G. As explained by Tait [35], a 3-edge-coloring φ of G *produces* a mapping $f_\varphi : V(G^*) \to \{1, 2, 3, 4\}$

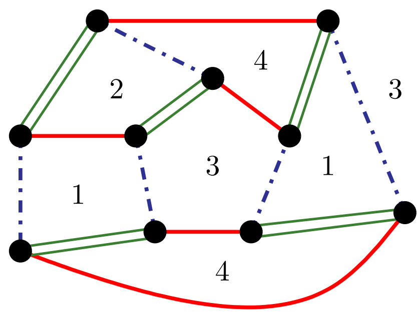

Figure 2: A 3-edge-coloring of a cubic plane graph G and a 4-vertex-coloring of G^* (a 4-face-coloring of G). One produces the other, and vice versa. Recall that the red thick line edges, the blue dashed-dotted line edges, and the green double line edges represent those colored by a, b and c, respectively.

of the dual triangulation G^* of G. To see this, first, fix a vertex of G^* and color it by 1, and then we extend the colors in the following rule: Let h and h' be two adjacent vertices of G^* such that h is already colored but h' has not been colored yet, and let e^* be the edge connecting them in G^*. Then we color h' so that the pair of colors of h and h' is either $\{1,2\}$ or $\{3,4\}$ if $\varphi(e) = a$, either $\{1,3\}$ or $\{2,4\}$ if $\varphi(e) = b$, and either $\{1,4\}$ or $\{2,3\}$ if $\varphi(e) = c$. See Figure 2 for an example. It was shown by Tait [35] that f_φ is indeed a 4-vertex-coloring of G^*. Conversely, when we are given a 4-vertex-coloring f in G^*, we can define the mapping $\varphi_f : E(G) \to \{a, b, c\}$ in G by the opposite rules, and the obtained φ_f is a 3-edge-coloring of G.

Similarly to the planar case above, we may consider the duality between a 3-edge-coloring of a cubic graph G on a non-spherical surface F^2 and a 4-vertex-coloring in the dual G^*. In fact, any 4-vertex-colorings of the dual G^* always produce 3-edge-colorings, but the converse does not hold. See the 3-edge-coloring φ_1 in the left of Figure 3 [1]. We explain its detail, defining the *type* of 3-edge-colorings in Section 3.3

3.2 "Type" of a perfect matching

Let G be a cubic graph, and let X be a subset of the edge set of G. Then for any perfect matching M of G, define

$$\sigma_M(X) \equiv |X| - |X \cap M| \pmod 2$$

with $\sigma_M(X) \in \{0, 1\}$. See Figure 3 for an example. In particular, when G is embedded on a surface F^2 and $X = D$, where D is the dual of an essential cycle D^* in G^*, the type σ_M represents the topological properties of the perfect matching M, as follows.

Theorem 5 (See [19, 21, 28]) *Let G be a cubic graph embedded on a surface F^2, and let M be a perfect matching of G. Then the following hold:*

(I) $\sigma_M(D_1) = \sigma_M(D_2)$ *for any two homotopic cycles D_1^* and D_2^* in G^*.*

[1]The projective plane is obtained by identifying the antipodal points of the outer dotted circle. Suppose that we put the color 1 into the central hexagon. By the coloring rule, the top and the bottom faces must have color 4 and 3, respectively, as in Figure 3. However, those faces coincide, a contradiction. On the other hand, the 3-edge-coloring φ_2 in the right produces a 4-vertex-coloring in the dual G^*.

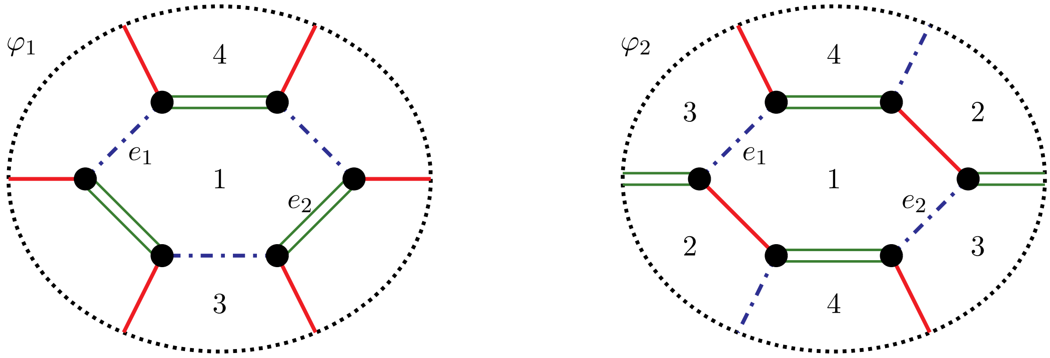

Figure 3: Two 3-edge-colorings of $K_{3,3}$ embedded on the projective plane. The projective plane is obtained by identifying the antipodal points of the outer dotted circle. Let $D = \{e_1, e_2\}$, and let M_a, M_b, M_c be the perfect matchings consisting all edges of the color a, b, c in φ_1, respectively. We have $\sigma_{M_a}(D) = 2 - 0 \equiv 0 \pmod 2$, $\sigma_{M_b}(D) = 2 - 1 \equiv 1 \pmod 2$, and $\sigma_{M_c}(D) = 2 - 1 \equiv 1 \pmod 2$, and hence $\rho_{\varphi_1}(D) = (0, 1, 1)$. Similarly, we see $\rho_{\varphi_2}(D) = (0, 0, 0)$. Thus, the left one is an odd 3-edge-coloring, while the right one is an even 3-edge-coloring. They are not Kempe equivalent by Theorem 8.

(II) *In particular, $\sigma_M(D) = 0$ if D^* is a contractible cycle in G^*.*

(III) *The following three are equivalent:*

- *For any essential cycle D^* in G^*, $\sigma_M(D) = 0$.*
- *$G - M$ is a 2-factor such that if we regard $G - M$ as a set of pairwise non-intersecting closed curves on F^2, then $F^2 - (G - M)$ is 2-region-colorable.*
- *$G^* - M^*$ is a bipartite quadrangulation on F^2.*

3.3 "Type" of a 3-edge-coloring

Using the type of perfect matchings, we define the *type* of 3-edge-colorings. Let φ be a 3-edge-coloring of G and let X be a subset of the edge set of G. Recall that φ can be regarded as a partition of $E(G)$ into three perfect matchings M_a, M_b and M_c. Then, we define the *type with respect to* X, denoted by $\rho_\varphi(X)$, of φ as the triple of $\sigma_{M_x}(X)$ for $x \in \{a, b, c\}$. Namely,

$$\rho_\varphi(X) = \big(\sigma_{M_a}(X), \sigma_{M_b}(X), \sigma_{M_c}(X)\big) \in \{0, 1\}^3.$$

See Figure 3 for an example. Then the following holds.

Proposition 6 *For any cubic graph G with a 3-edge-coloring φ, and any subset X of $E(G)$, we have*

$$\rho_\varphi(X) = (0, 0, 0),\ (1, 1, 0),\ (1, 0, 1),\ \textit{or}\ (0, 1, 1).$$

Proof. Since $\{M_a, M_b, M_c\}$ is a partition of $E(G)$, we see that $|X \cap M_a| + |X \cap M_b| + |X \cap M_c| = |X|$. Therefore,

$$\sigma_{M_a}(X) + \sigma_{M_b}(X) + \sigma_{M_c}(X) \equiv 3|X| - |X| \equiv 0 \pmod 2.$$

This directly implies Proposition 6. $\square$

By Proposition 6, we see that $\rho_\varphi(X)$ can attain only four types. In addition, if $\rho_\varphi(X) \neq (0,0,0)$, say $\rho_\varphi(X) = (1,1,0)$, then we obtain 3-edge-colorings φ', φ'' with $\rho_{\varphi'}(X) = (1,0,1)$ and $\rho_{\varphi''}(X) = (0,1,1)$ by a permutation of colors. This means that the three types $(1,1,0), (1,0,1), (0,1,1)$ cannot distinguish by Kempe equivalence classes. On the other hand, we will show in the next subsection that the types $(0,0,0)$ and others can distinguish for a certain edge set X.

Similarly to the type σ_M of a perfect matching M, the type ρ_φ represents the topological property of the 3-edge-coloring of φ. Indeed, the following holds (see [10, Theorem 2]).

Theorem 7 *Let G be a cubic graph embedded on a surface, and let φ be a 3-edge-coloring of G. Then φ produces a 4-vertex-coloring of the dual G^* if and only if $\rho_\varphi(D) = (0,0,0)$ for any essential cycle D^* in G^*.*

3.4 Types and Kempe equivalence classes

Now we consider cubic graphs G embedded on the projective plane. Recall that the projective plane has the unique essential cycle, up to homotopy. Therefore, by Theorems 5 (I) and 7, the types of 3-edge-colorings φ of G only attain one of the following two:

- $\rho_\varphi(D) = (0,0,0)$ for any essential cycles D^* in G^*. Then φ is said to be of *even* type. In this case, φ produces a 4-vertex-coloring of the dual G^*.
- $\rho_\varphi(D) \neq (0,0,0)$ for any essential cycles D^* in G^*. Then φ is said to be of *odd* type. In this case, φ does not produce a 4-vertex-coloring of the dual G^*.

Indeed, these two types can distinguish Kempe equivalence classes. See Figure 3 for an example. By Theorem 8, the two 3-edge-colorings φ_1 and φ_2 in Figure 3 are not Kempe equivalent.

Theorem 8 *Let G be a cubic graph embedded on the projective plane. Then two 3-edge-colorings of G are not Kempe equivalent if their types are different.*

Proof of Theorem 8. Let G be a cubic graph embedded on the projective plane and let φ be an odd 3-edge-coloring of G. Then for any essential cycle D^* in G^*, we have $\rho_\varphi(D) \neq (0,0,0)$. By symmetry, we may assume that $\rho_\varphi(D) = (1,1,0)$. Let C be an alternating cycle with respect to φ, and let φ' be the 3-edge-coloring of G obtained by Kempe switch at C. It suffices to prove that $\rho_{\varphi'}(D) \neq (0,0,0)$.

If C consists of edges of color a and c, then we still have $(\varphi')^{-1}(b) = \varphi^{-1}(b) = M_b$, so it follows from Proposition 6 that $\rho_{\varphi'}(D) = (1,1,0)$ or $(0,1,1)$. The same also occurs if C consists of edges of color b and c. Therefore, we may assume that C consists of edges of color a and b.

Suppose that C is contractible on the projective plane. By Fact 3, C and D^* intersect transversally an even number of times. This implies that $|C \cap D|$ is even. Since $|C \cap D| = |C \cap D \cap M_a| + |C \cap D \cap M_b|$, we obtain $|C \cap D \cap M_a| \equiv |C \cap D \cap M_b| \pmod 2$. Therefore, for the matching M'_a obtained from M_a by the Kempe switch at C, we have

$$\begin{aligned}
\sigma_{M'_a}(D) &\equiv |D| - |D \cap M'_a| \\
&= |D| - \Big(|D \cap M_a| - |C \cap D \cap M_a| + |C \cap D \cap M_b|\Big) \\
&\equiv |D| - |D \cap M_a| \quad \equiv \sigma_{M_a}(D) \qquad \pmod 2.
\end{aligned}$$

The same also holds for M_b, and we obtain $\rho_{\varphi'}(D) = (1, 1, 0)$.

Thus, we may assume that C is essential on the projective plane. By Fact 3, C and D^* intersect transversally an odd number of times, and hence $|C \cap D| \equiv 1 \pmod 2$. Since $\sigma_{M_c}(D) = 0$,

$$\begin{aligned} |D \cap (M_a \cup M_b) - C| &= |D \cap (M_a \cup M_b)| - |C \cap D| \\ &= |D| - |D \cap M_c| - |C \cap D| \\ &\equiv \sigma_{M_c}(D) - 1 = 1 \pmod 2. \end{aligned}$$

Since $M_a \cup M_b$ forms a set of vertex-disjoint cycles, this implies that there exists a cycle C' in $M_a \cup M_b$ such that $C' \neq C$ and $|C' \cap D|$ is odd. Since D^* is essential, it follows from Fact 3 that C' is also essential. This implies that C and C' are two vertex-disjoint essential cycles, contradicting Fact 3. □

3.5 Questions about the types

In Theorem 8, we give a method to distinguish two 3-edge-colorings belonging to different Kempe equivalence classes for cubic graphs embedded on the projective plane. This naturally gives us the following questions.

- Do any cubic graphs embedded on the projective plane contain an odd 3-edge-coloring and an even 3-edge-coloring?
- For any cubic graphs embedded on the projective plane, are any 3-edge-colorings always Kempe equivalent if their types are the same?

With Problem 1 in mind, we give answers to those questions for bipartite graphs in Sections 5 and 6, respectively.

4 Signature of 3-edge-colorings

Here we introduce the signature of 3-edge-colorings, which will be used for our proofs. The idea was used in several papers, e.g. [13, 32] for counting the number of 3-edge-colorings, list-edge-colorings [1, 7] and Pfaffian labelings [30, 36]. We explain about list-edge-colorings, together with a corollary of our results in Section 4.3.

4.1 Definition of the signature

For two 3-edge-colorings φ_1 and φ_2 of a cubic graph G, we define the *signature* between them as follows. First, we focus on a vertex v in G, and let e_1, e_2, e_3 be the three edges incident with v in G. Then some element, say π_v, in the symmetric group of order three acts on the colors by φ_1 and φ_2. Namely, $\pi_v(\varphi_1(e_i)) = \varphi_2(e_i)$ for any $1 \leq i \leq 3$. See Figure 4 for an example. Recall that the *signature* of π_v is defined as $\mathrm{sign}(\pi_v) = +1$ (resp. $= -1$) if π_v is an even (resp. odd) permutation. Then let

$$\mathrm{sign}(\varphi_1, \varphi_2) = \prod_{v \in V(G)} \mathrm{sign}(\pi_v),$$

which is called the *signature* between φ_1 and φ_2.

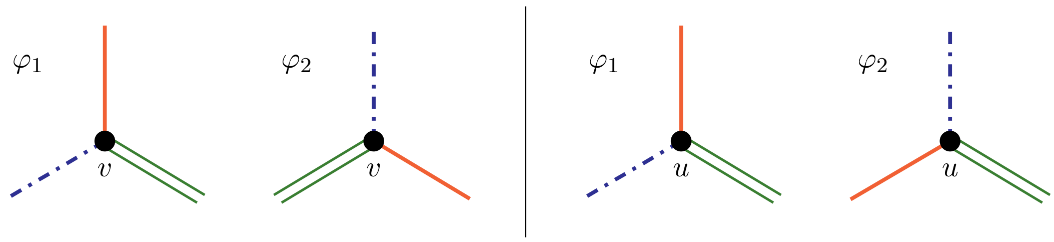

Figure 4: Two 3-edge-colorings φ_1 and φ_2. For the vertex v in the left, we have $\pi_v(a) = b$, $\pi_v(b) = c$ and $\pi_v(c) = a$, which is an even permutation (so $\mathrm{sign}(\pi_v) = +1$), while for the vertex u in the right, we have $\pi_u(a) = b$, $\pi_u(b) = a$ and $\pi_u(c) = c$, which is an odd permutation (so $\mathrm{sign}(\pi_u) = -1$).

The following proposition suggests that this can distinguish the Kempe equivalence. This appeared in [7, p. 346] without proofs. Since the proof is not difficult nor long, we here prove it for self-containedness.

Proposition 9 *Let G be a cubic graph, and let φ_1 and φ_2 be two 3-edge-colorings of G. If φ_1 and φ_2 are Kempe equivalent, then $\mathrm{sign}(\varphi_1, \varphi_2) = +1$.*

Proof. Let φ_1' be the 3-edge-coloring of G obtained from φ_1 by a Kempe switch at an alternating cycle C. It is enough to show that $\mathrm{sign}(\varphi_1, \varphi_1') = +1$. For each vertex $v \in V(G) - V(C)$, π_v is the identity permutation, and hence $\mathrm{sign}(\pi_v) = +1$. Let $u \in V(C)$. Since we swap the colors of the edges in C and exactly two edges in C are incident with u, we have $\mathrm{sign}(\pi_u) = -1$. (See the right side of Figure 4.) Then it follows from the fact $|C|$ is even that

$$\mathrm{sign}(\varphi_1, \varphi_1') = \prod_{v \in V(G)} \mathrm{sign}(\pi_v) = (-1)^{|C|} = +1. \qquad \Box$$

It was shown that any 3-edge-colorings of every cubic plane graph have the same signature. See [13, Proposition 1] and [15, Theorem 3.2]. [16, Parity Lemma] also gives it with another context. From this and Proposition 9, one may expect the converse also holds, that is, "two 3-edge-colorings in a cubic plane graph belong to the same Kempe equivalence class" if the signature between them is +1. This is generally false, but true for "bipartite" cubic plane graphs (see Theorem 1).

4.2 Relation between type and signature of 3-edge-colorings

As mentioned in the previous section, for two 3-edge-colorings, having the same types is a necessary condition to be Kempe equivalent. In this section, we explain its relation to the signature. In fact, we prove the following, which together with Proposition 9 gives another proof of Theorem 8.

Theorem 10 *Let G be a cubic graph embedded on the projective plane, and let φ_1, φ_2 be two 3-edge-colorings of G. Then φ_1 and φ_2 have the same type if and only if $\mathrm{sign}(\varphi_1, \varphi_2) = +1$.*

Before starting the proof, we prepare a definition, so-called *singular* and *non-singular* edges. Note that this was defined by Fisk [9] in the dual version. Let G be a cubic graph embedded on a surface, and let φ be a 3-edge-coloring of G. An edge e is said to be *singular* (with respect to φ) if for a face h containing e, the two edges incident with h and adjacent

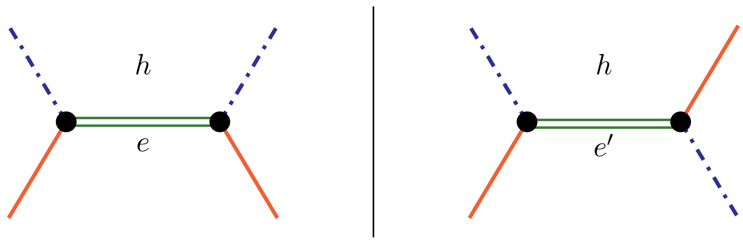

Figure 5: A singular edge e in the left side, and a non-singular edge e' in the right side. In the left, the 2-factors $T_a = G - M_a$ and $T_b = G - M_b$ intersect at e non-transversally, while in the right, they intersect at e non-transversally.

to e have the same color by φ: Otherwise, e is said to be *non-singular*. See Figure 5 for an example. The following is easily obtained from the definition.

Fact 11 *For a cycle C in a cubic graph embedded on a surface, any two of the following conditions imply the other (see the cycle $v_2v_3v_5v_4$ in Figure 6, for example);*

- *C is a facial cycle.*
- *C is an alternating cycle.*
- *All edges in C is singular.*

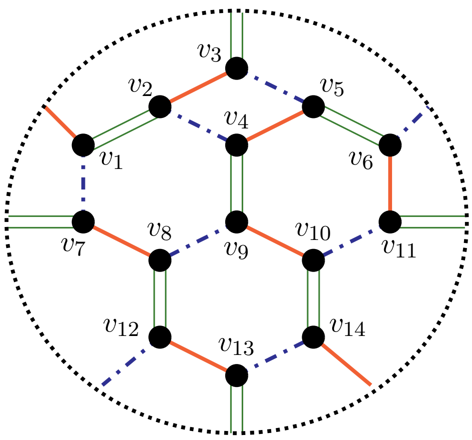

Figure 6: An essential cycle D^* in G^* represented by the outer dotted circle, and an odd 3-edge-coloring φ with $\rho_\varphi(D) = (1,1,0)$. Note that $\mathrm{sign}_D(\pi_{v_i}) = +1$ for $i = 2,5,8,9,10,12,13,14$, and $\mathrm{sign}_D(\pi_{v_i}) = -1$ for $i = 1,3,4,6,7,11$. These imply that $\mathrm{sign}_D(\varphi) = (+1)^8(-1)^6 = +1$.

Let G be a cubic graph embedded on the projective plane. For a 3-edge-coloring φ, we denote by $NS(\varphi)$ the set of non-singular edges with respect to φ, and furthermore, for each color x, denote by $NS_x(\varphi)$ the set of non-singular edges of color x.

Let D^* be an essential cycle in G^*. Similarly to the signatures of two 3-edge-colorings, we define the *signature* of a 3-edge-coloring φ with respect to D, denoted by $\mathrm{sign}_D(\varphi)$, as follows: Deleting all edges in D from G, we obtain a spanning subgraph G' of G that is contained in a disk of the projective plane. On the disk, we can give a consistent clockwise rotation to each vertex of G. So, for an edge uv, u and v have the same consistent clockwise rotation if and only if $uv \notin D$. For a vertex v, if the colors a, b and c appear in the edges incident with v along the consistent clockwise rotation on the disk containing G', then let $\mathrm{sign}_D(\pi_v) = +1$: Otherwise let $\mathrm{sign}_D(\pi_v) = -1$. Then let

$$\mathrm{sign}_D(\varphi) = \prod_{v \in V(G)} \mathrm{sign}_D(\pi_v).$$

See Figure 6 for an example.

We now prove the following Lemma:

Lemma 12 *Let G be a cubic graph embedded on the projective plane, let φ be a 3-edge-coloring of G, and let D^* be an essential cycle in G^*. Then*

$$\mathit{sign}_D(\varphi) = \begin{cases} (-1)^{\frac{1}{2}|G|+|D|+1} & \text{if } \varphi \text{ is an odd type,} \\ (-1)^{\frac{1}{2}|G|+|D|} & \text{if } \varphi \text{ is an even type,} \end{cases} \pmod 2.$$

Since $\mathrm{sign}(\varphi_1, \varphi_2) = \mathrm{sign}_D(\varphi_1) \cdot \mathrm{sign}_D(\varphi_2)$, Theorem 10 directly follows from Lemma 12. Thus, it suffices to prove this lemma.

Proof of Lemma 12. Let φ be a 3-edge-coloring of G. We first prove the case when φ is an odd type.

By Proposition 6 and symmetry, we may assume that $\rho_\varphi(D) = (1, 1, 0)$. Denote $T_a = E(G) - M_a$ (resp. $T_b = E(G) - M_b$), which is a 2-factor of G such that each cycle is alternating by the colors b and c (resp. by the colors a and c) in φ. Since

$$|D \cap T_a| \;=\; |D| - |D \cap M_a| \;\equiv\; \sigma_{M_a}(D) \;=\; 1 \pmod 2,$$

T_a transversally intersects with D^* an odd number of times. Then Fact 3 implies that T_a is essential. By symmetry, T_b is also essential. Therefore, again by Fact 3, T_a and T_b intersect transversally an odd number of times. Note that T_a and T_b only share edges in M_c. It is easy to see that T_a and T_b intersect transversally at $e \in M_c$ if and only if e is non-singular. (See Figure 5.) Therefore, we have $|NS_c(\varphi)| \equiv 1 \pmod 2$.

Now we focus on each edge uv in M_c. Suppose first that $uv \notin D$. Then by the definition, u and v have the same consistent rotations. This directly implies that $\mathrm{sign}_D(\pi_u) \neq \mathrm{sign}_D(\pi_v)$ if and only if the edge uv is singular. (See Figure 5.) On the other hand, if $uv \in D$, then the same argument implies that $\mathrm{sign}_D(\pi_u) \neq \mathrm{sign}_D(\pi_v)$ if and only if the edge uv is non-singular. Therefore, for an edge $uv \in M_c$, $\mathrm{sign}_D(\pi_u) \cdot \mathrm{sign}_D(\pi_v) = -1$ if and only if one of the following hold:

- $uv \notin D$ and $uv \notin NS_c(\varphi)$.
- $uv \in D$ and $uv \in NS_c(\varphi)$.

Since M_c is a perfect matching of G, we obtain the following:

$$\begin{aligned} \mathrm{sign}_D(\varphi) &= \prod_{uv \in M_c} \Big(\mathrm{sign}_D(\pi_u) \cdot \mathrm{sign}_D(\pi_v)\Big) \\ &= (-1)^{|M_c - (D \cup NS_c(\varphi))|} \cdot (-1)^{|D \cap NS_c(\varphi)|}. \end{aligned}$$

Since $\sigma_{M_c}(D) = |D| - |M_c \cap D| = 0$, $|NS_c(\varphi)| \equiv 1 \pmod 2$ and $|M_c| = \frac{1}{2}|G|$, we have the following, which completes the proof for an odd 3-edge-coloring:

$$\begin{aligned}
&|M_c - (D \cup NS_c(\varphi))| + |D \cap NS_c(\varphi)| \\
&= \Big(|M_c| - |M_c \cap D| - |NS_c(\varphi) - D|\Big) + \Big(|NS_c(\varphi)| - |NS_c(\varphi) - D|\Big) \\
&\equiv |M_c| - \big(|D| - \sigma_{M_c}(D)\big) + |NS_c(\varphi)| \\
&\equiv \frac{1}{2}|G| + |D| + 1 \pmod 2.
\end{aligned}$$

Example: We explain the above argument using an example shown in Figure 6. Recall that an essential cycle D^* in G^* is represented by the outer dotted circle. The 3-edge-coloring φ is odd satisfying $\rho_\varphi(D) = (1, 1, 0)$. The 2-factor $T_a = E(G) - M_a$ consists of the cycle $v_1v_2v_4v_9v_8v_{12}v_6v_5v_3v_{13}v_{14}v_{10}v_{11}v_7$, which is essential. Thus, T_a is an essential 2-factor. Similarly, the 2-factor $T_b = E(G) - M_b$ is essential. Within edges in M_c, the edges $v_1v_2, v_4v_9, v_5v_6, v_7v_{11}$ are singular, while the edges $v_3v_{13}, v_8v_{12}, v_{10}v_{14}$ are non-singular. Thus, $NS_c(\varphi) = \{v_3v_{13}, v_8v_{12}, v_{10}v_{14}\}$. We have $M_c \cap D = \{v_3v_{13}, v_7v_{11}\}$. Note that an edge $v_iv_j \in M_c$ satisfies $\mathrm{sign}_D(\pi_u) \cdot \mathrm{sign}_D(\pi_v) = -1$ if and only if $v_iv_j = v_1v_2, v_4v_9, v_5v_6$ or v_3v_{13}. Therefore,

$$\mathrm{sign}_D(\varphi) = (-1)^4 = (-1)^{14/2+4+1} = (-1)^{\frac{1}{2}|G|+|D|+1}.$$

The case when φ is an even 3-edge-coloring can be shown similarly. In this case, we see that both T_a and T_b are contractible. This implies that they intersect transversally an even number of times, and hence $|NS_c(\varphi)| \equiv 0 \pmod 2$. On the other hand, we have

$$\begin{aligned}
|M_c - (D \cup NS_c(\varphi))| + |D \cap NS_c(\varphi)| &\equiv |M_c| - \big(|D| - \sigma_{M_c}(D)\big) + |NS_c(\varphi)| \\
&\equiv \frac{1}{2}|G| + |D| \pmod 2.
\end{aligned}$$

By the same calculation as in the previous case, this shows the case when φ is an even 3-edge-coloring. □

Remark: As mentioned in Section 4.1, any 3-edge-colorings of every cubic plane graph have the same signature. Note that Lemma 12 corresponds to the projective planar version of this fact. Actually, the proof of Lemma 12 is based on the idea of the proofs in [13, Proposition 1] and [15, Theorem 3.2].

4.3 Application to list-edge-colorings

We show a corollary of Lemma 12 on list-edge-coloring conjecture. For a positive integer k, a *edge-list* $L : E(G) \to 2^{[k]}$ of a graph G is a mapping that assigns a set of colors to each edge. An *L-edge-coloring* of G is a k-edge-coloring f such that $f(e) \in L(e)$ for any edge $e \in E(G)$. If G admits L-edge-coloring for any edge-list L with $|L(e)| \geq k$ for each edge $e \in E(G)$, then G is said to be *k-list-edge-colorable.* Since a k-edge-coloring is an L-edge-coloring with $L(u) = [k]$ for any edge $e \in E(G)$, any k-list-edge-colorable graph is k-edge-colorable. The converse is an open problem, which is known as list-edge-coloring conjecture:

Conjecture 13 (List-edge-coloring conjecture, see [5]) *Any k-edge-colorable graph is k-list-edge-colorable.*

To attack Conjecture 13, Alon [1] used a method using the signature of 3-edge-colorings.

Lemma 14 (Alon [1]) *For any 3-edge-colorable cubic graph G, if all pairs of two 3-edge-colorings φ_1 and φ_2 in G satisfy sign$(\varphi_1, \varphi_2) = +1$, then G is 3-list-edge-colorable.*

Thus, Theorem 10 and Lemma 14, together with Theorem 7, imply the next corollary. This give a new classes of graphs for which Conjecture 13 hold.

Corollary 15 *Let G be a 3-edge-colorable cubic graph embedded on the projective plane. If the dual G^* is not 4-vertex-colorable, then G is 3-list-edge-colorable.*

Note that the signature can be similarly defined for k-edge-colorings in k-regular graphs with $k \geq 4$. To be exact, Alon [1] showed that Lemma 14 also holds even for k-edge-colorable k-regular graph. Ellingham and Goddyn [7] proved that for every k-regular plane graph, all k-edge-colorings have the same signature. This together with Lemma 14 implies that every k-edge-colorable k-regular plane graph is k-list-edge-colorable.

5 The existence of a 3-edge-coloring of prescribed type

In this section, we show the following theorem, which proves the existence of an odd or even 3-edge-coloring.

Theorem 16 *Let G be a cubic bipartite graph embedded on the projective plane. Then both of the following hold:*

(I) *There exists an odd 3-edge-coloring.*

(II) *There exists an even 3-edge-coloring if and only if G^* is 4-vertex-colorable.*

Note that Theorem 16 (II) directly follows from Theorem 7. Thus, we only prove Theorem 16 (I), using the following result, which was obtained in several papers, see [19, 21, 28]. Note that the following theorem was in [19, 28] stated in the context of spanning bipartite quadrangulations as in Theorem 5 (III).

Theorem 17 ([19, 21, 28]) *Let G be a cubic bipartite graph embedded on the projective plane, and let D^* be an essential cycle in G^*. Then G admits a perfect matching M with $\sigma_M(D) = 1$.*

Proof of Theorem 16 (I). It follows from Theorem 17 that G admits a perfect matching M_a such that $\sigma_{M_a}(D) = 1$. Since $G - M_a$ is a 2-regular bipartite graph, $E(G) - M_a$ can be naturally colored by two colors b and c and we can extend it to the 3-edge-coloring of G, say φ. Since $\sigma_{M_a}(D) = 1$, we see that $\rho_\varphi(D) = (1, 1, 0)$ or $(1, 0, 1)$, that is φ is an odd 3-edge-coloring. $\square$

We also give another proof of Theorem 16 (I), using a *generating theorem* shown in [18, Theorem 10] (see also [34] for a related work). To do that, we define some terminology. An *even map* is a graph embedded on a surface such that every facial walk has even length. Note that any bipartite graph embedded on a surface is an even map, while the converse does not generally hold. The two cubic even maps T_1^* and T_2^* and the family $\{(T_3^k)^* : k \geq 1\}$ of cubic even maps are depicted in Figure 7. A 2-*bridging* for a cubic even map on a surface is the

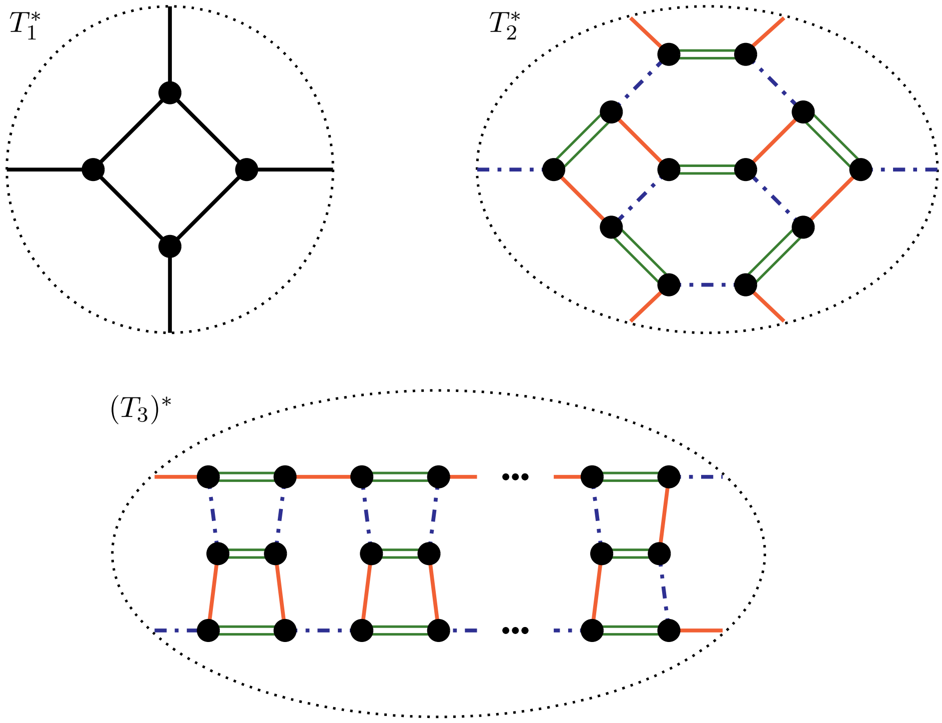

Figure 7: The two cubic even maps T_1^* and T_2^* on the projective plane and the family $\{(T_3^k)^* : k \geq 1\}$ of cubic even maps, where k represents the number of hexagons in $(T_3^k)^*$. Note that each of T_2^* and $(T_3^k)^*$ has a 3-edge-coloring.

operation for two edges contained in a same facial cycle to subdivide them twice and add two new edges as in the left of Figure 8. A *cube addition* is the operation for a vertex to replace it with seven vertices and nine edge as in the right of Figure 8. A graph G on a non-spherical surface F^2 is *k-representative* if all essential closed curves on F^2 intersect with G at least k times.

Theorem 18 (Kobayashi, Nakamoto and Yamaguchi [18]) *Every 3-edge-connected 2-representative cubic even map on the projective plane can be obtained from T_1^*, T_2^* or $(T_3^k)^*$ for some $k \geq 1$ by a sequence of 2-bridgings and cube additions.*

For the even maps $T_1^*, T_2^*, (T_3^k)^*$ and the two operations, we observe the following.

Fact 19 (i) *The even map T_1^* is not bipartite, while both T_2^* and $(T_3^k)^*$ for $k \geq 1$ are bipartite.*

(ii) *Both T_2^* and $(T_3^k)^*$ for $k \geq 1$ admit an odd 3-edge-coloring. (See Figure 7.)*

(iii) *Let G be a 3-edge-connected 2-representative cubic even map on the projective plane and let G' be the graph obtained from G by a 2-bridging or a cube addition. Then G is bipartite if and only if so is G'.*

(iv) *Let G and G' be as in (iii). If G admits an odd 3-edge-coloring, then so does G'.*

For Fact 19 (iv), we show an example how to obtain an odd 3-edge-coloring in G' in Figure 8. For a 2-bridging, two middle edges in the original graph might have the same color, but even in such a case, we can easily find an odd 3-edge-coloring in the new map.

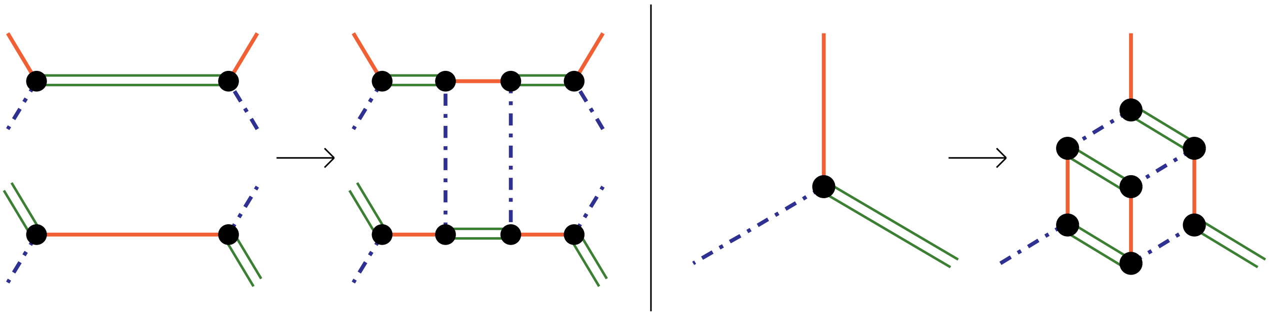

Figure 8: A 2-bridging (left) and a cube addition (right), together with extensions of a 3-edge-coloring.

Using Theorem 18 and Fact 19, we have another proof of Theorem 16 (I).

Another proof of Theorem 16 (I). Suppose first that G is 2-representative and 3-edge-connected. By Theorem 18, G is obtained from T_1^*, T_2^* or $(T_3^k)^*$ for some $k \geq 1$ by a sequence of 2-bridgings and cube additions. Since G is bipartite, it follows from Fact 19 (i) and (iii) that G is obtained from T_2^* or $(T_3^k)^*$ for some $k \geq 1$ by a sequence of 2-bridgings and cube additions. Then by Fact 19 (ii) and (iv), G admits an odd 3-edge-coloring, and we are done.

Suppose next that G is not 2-representative. Then there exist an edge e and an essential cycle D^* in G^* that consists only e^* (which is the dual edge of e and is a loop in G^*). Since G is bipartite and cubic, G admits a 3-edge-coloring φ. If we let $\varphi(e) = a$ by symmetry, then $\sigma_{M_a}(D) = |D| - |D \cap M_a| = 1 - 1 = 0$ and $\sigma_{M_b}(D) = \sigma_{M_c}(D) = 1$. Therefore, we have $\rho_\varphi(D) = (0, 1, 1)$, and hence φ is an odd 3-edge-coloring of G.

Suppose finally that G is not 3-edge-connected. Since G is bipartite and cubic, G cannot have a bridge. Therefore, there exists a 2-edge-cut $\{u_1u_2, v_1v_2\}$ in G, where $G - \{u_1u_2, v_1v_2\}$ has two components, say G_1 and G_2, with $u_1, v_1 \in V(G_1)$ and $u_2, v_2 \in V(G_2)$. Then for $i \in \{1, 2\}$, let G_i' be the graph obtained from G_i by adding the edge u_iv_i. Since G is bipartite and embedded on the projective plane, it is easy to see that for some $i \in \{1, 2\}$, say $i = 1$, G_1' is a cubic bipartite graph embedded on the projective plane and G_2' is a cubic bipartite graph contained in a disk on the projective plane. Note that any 3-edge-coloring of G is obtained from a 3-edge-coloring of G_1' and that of G_2' with u_iv_i having the same color. Furthermore, it is easy to see that G admits an odd 3-edge-coloring if and only if G_1' admits an odd 3-edge-coloring. Since G_1' is smaller than G, we can show by inductive argument that G_1' (and hence G) admits an odd 3-edge-coloring. (The base case was done by the arguments in the first paragraph.) $\square$

6 Kempe equivalence class for odd 3-edge-colorings

Considering the second problem in Section 3.5, we first give the answer for odd 3-edge-colorings. Indeed, there exists only one Kempe equivalence class for odd 3-edge-colorings.

Theorem 20 *Let G be a cubic bipartite graph embedded on the projective plane. Then any two odd 3-edge-colorings in G are Kempe equivalent.*

Note that this corresponds to Theorem 1, which is the planar case. We prove Theorem

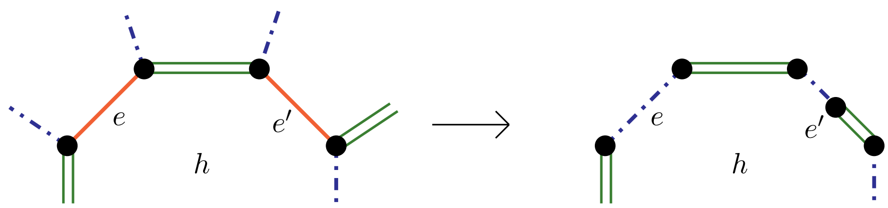

Figure 9: The face h in the proof of Lemma 21, where e is singular, while e' is non-singular.

20 with several steps as in the following subsections, and the first part is in fact based on the proof of Theorem 1 in [9, 11].

6.1 Canonical 3-edge-colorings

Before starting the proof, we prepare the following important lemma on non-singular edges.

Lemma 21 (Fisk [9]) *Let G be a cubic graph embedded on a surface and let φ be a 3-edge-coloring of G. Then for any face h and any color x,*

$$|E(h) \cap NS_x(\varphi)| \equiv |E(h)| \pmod 2,$$

where $E(h)$ is the set of all edges in G contained in the facial cycle of h.

A proof of Lemma 21 can be found in [9, Lemma 5] and [11, Lemma 2], but we here give a proof for self-containedness.

Proof. Let h be a face and let x be a color. We may assume that $x = a$. Now we modify the facial cycle of h together with its colors. See Figure 9 for an example for the following argument.

Let e be an edge in $E(h)$ colored by a. If e is singular, then the two edges in $E(h)$ adjacent to e have the same color, say c. In this case, we recolor e by the color b. On the other hand, if e is non-singular, then the two edges in $E(h)$ and adjacent to e have different colors. In this case, subdivide e and recolor the two obtained edges by b and c so that edges with the same color are not consecutive. After recoloring all edges in $E(h)$ colored by a, we finally obtain an alternating cycle colored by b and c, and hence its length is even. On the other hand, since the length of the new cycle is $|E(h)| + |E(h) \cap NS_a(\varphi)|$, we have the desired equality. $\square$

Let G be a cubic even map on a surface F^2, and let φ be a 3-edge-coloring. Then $|E(h)|$ is even for each face h, and hence Lemma 21 implies that for each color x, $\big(NS_x(\varphi)\big)^* \subseteq E(G^*)$ forms an Eulerian subgraph of G^*. Thus, $\big(NS_x(\varphi)\big)^*$ can be divided into edge-disjoint cycles in G^*. Suppose that some cycle C^* in $\big(NS_x(\varphi)\big)^*$, say $x = a$, is contractible. Then C^* divides the surface F^2 into the *interior*, which is homeomorphic to a disk, and the *exterior*. Note that all edges in $M_b \cup M_c$ do not intersect with C^*, and hence the edges in $M_b \cup M_c$ that are contained in the interior of C^* form a set of vertex-disjoint contractible cycles. Then let φ' be the 3-edge-coloring obtained from φ by the Kempe switches at all such cycles. Let $C = (C^*)^* \subseteq NS_a(\varphi)$. It is easy to see the following:

- Since each edge in C is incident with exactly one vertex in the interior of C^* and the colors b and c are all swapped in the interior of C^*, any edge in C will be singular.

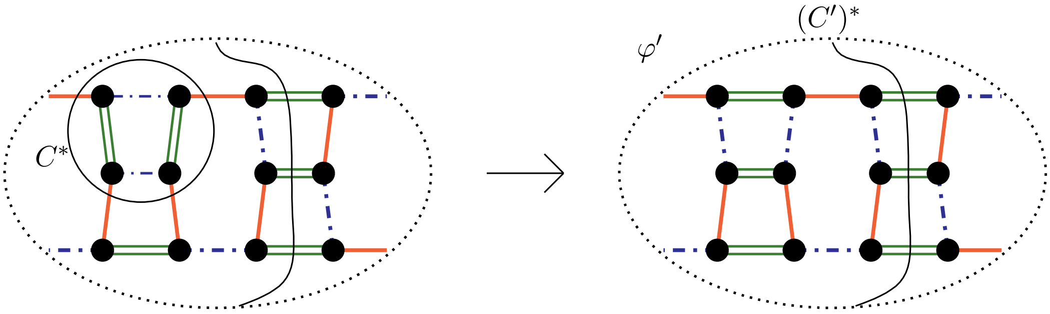

Figure 10: Two 3-edge-colorings of a cubic even map G on the projective plane. The thin lines represent the cycles in $\big(NS(\varphi)\big)^*$. The left has two such cycles: C^* is one of them, which is of color a and contractible, while the other is of color c and essential. If we perform Kempe switches in the cycles of $M_b \cup M_c$ inside C^*, then we obtain the 3-edge-coloring in the right with fewer non-singular edges. In particular, the 3-edge-coloring in the right is canonical.

- For any other edges of color a, Kempe switches happened at both end vertices or at neither. Therefore, such edges do not change the property (singular or non-singular).
- Take an essential cycle D^* in G^*. Since C^* is contractible, it follows from Fact 3 that C^* and D^* intersects an even number of times. Then it is easy to see that $\rho_\varphi(D) = (0,0,0)$ if and only if $\rho_{\varphi'}(D) = (0,0,0)$.

Therefore, we obtain the new 3-edge-coloring φ' with fewer non-singular edges such that for any cycle D^* in G^*, $\rho_\varphi(D) = (0,0,0)$ if and only if $\rho_{\varphi'}(D) = (0,0,0)$. See Figure 10. This operation can be performed as long as there exists a contractible cycle in $\big(NS_x(\varphi)\big)^*$ for some color x. This leads us to a *canonical* 3-edge-coloring, which is a 3-edge-coloring such that for each color x, either $\big(NS_x(\varphi)\big)^*$ consists of only essential cycles or the empty set. (See [9, p.331].) Actually, we obtain the following lemma.

Lemma 22 *Let G be a cubic even map on a surface. For any 3-edge-coloring φ of G, there exists a canonical 3-edge-coloring φ' such that φ' is Kempe equivalent with φ and for any cycle D^* in G^*, $\rho_\varphi(D) = (0,0,0)$ if and only if $\rho_{\varphi'}(D) = (0,0,0)$.*

Suppose that the surface is the sphere. In this case, no essential cycles exist, and hence a canonical 3-edge-coloring φ must satisfy that there are no non-singular edges. It is easy to see that for the 4-vertex-coloring f_φ of G^* produced from φ, we only use three colors. (In fact, since all edges are singular, it follows from Fact 11 that every facial cycle in G is alternating by φ. The three colors in f_φ correspond to the (a,b)-faces, the (a,c)-faces, and the (b,c)-faces, where an (x,y)-face is a face whose facial cycle is alternating by the colors x and y.) Since such a 3-vertex-coloring of G^* is unique except for the permutation of colors, Lemma 22 directly implies Theorem 1.

For the case of the projective plane, Lemma 22 states that any 3-edge-coloring φ of a cubic even map G on the projective plane is Kempe equivalent with some canonical 3-edge-coloring φ' with the same type as φ.

6.2 Topological properties of non-singular edges

Now we consider a canonical 3-edge-coloring φ of a cubic bipartite graph embedded on the projective plane and the topological property on $\big(NS_x(\varphi)\big)^*$. We prove the following.

Lemma 23 *Let G be a cubic bipartite graph embedded on the projective plane, let D^* be an essential cycle in G^*, and let φ be a canonical 3-edge-coloring of G. Then for $x \in \{a, b, c\}$, $\big(NS_x(\varphi)\big)^*$ consists of only one essential cycle if and only if $\sigma_{M_x}(D) = 0$*

Proof. In this proof, we denote by P^2 the projective plane.

We first show the "only if" part for $x = a$. Suppose that $\big(NS_a(\varphi)\big)^*$ consists of only one essential cycle. Then deleting the cycle in $\big(NS_a(\varphi)\big)^*$ from the projective plane results in a disk. Since all cycles in $M_b \cup M_c = G - M_a$ are contained in this disk, they are contractible. This implies that $P^2 - (M_b \cup M_c)$ is region-2-colorable (see Proposition 4 (I)). Then it follows from Theorem 5 (III) that $\sigma_{M_a}(D) = 0$. The cases for $x = b$ and $x = c$ are symmetric, and hence the "only if" part holds.

Next we show the "if" part. Suppose that $\sigma_{M_a}(D) = 0$, but $\big(NS_a(\varphi)\big)^*$ does not consist of an essential cycle. Since φ is canonical, we have $\big(NS_a(\varphi)\big)^* = \emptyset$, so all edges of color a are singular. Since $\sigma_{M_a}(D) = 0$, it follows from Theorem 5 that $P^2 - (G - M_a)$ is 2-region-colorable, where P^2 denotes the projective plane. By Proposition 4, this implies that all cycles in $M_b \cup M_c$ are contractible.

We color each vertex in one partite set of G by black, and each vertex in the other partite set by white. Let C be an essential cycle in G, and let $u_1v_1, u_2v_2, \ldots, u_rv_r$ be the sequence of the edges in $M_a \cap E(C)$ such that they appear in C in that order and v_i and u_{i+1} are contained in the same cycle in $M_b \cup M_c$ for $1 \le i \le r$, where $u_{r+1} = u_1$. We take such an essential cycle C so that r is as small as possible. By this choice, each cycle in $M_b \cup M_c$ are incident with either exactly two edges in C or no edges in C.

For each vertex w in $\{u_i, v_i : 1 \le i \le r\}$, we give a local rotation to w in the following manner.

- If w is a black vertex, then the colors a, b and c appear in this order in the edges incident with w along the rotation.
- If w is a white vertex, then the colors a, b and c appear in the opposite order (the order of a, c, b) in the edges incident with w along the rotation.

See Figure 11 for an example.

Let $1 \le i \le r$. Since u_i and v_i belong to different partite sets of G and the edge u_iv_i is singular, both u_i and v_i have the same rotation. On the other hand, note that v_i and u_{i+1} are contained in the same cycle in $M_b \cup M_c$. Since all cycles in $M_b \cup M_c$ are contractible, both edges v_iu_i and $v_{i+1}u_{i+1}$ leaves from the cycle in $M_b \cup M_c$. Thus, both u_i and v_i have the same rotation, regardless the partite set to which they belong. These give the consistent rotations along the sequence $u_1, v_1, \ldots, u_r, v_r, u_1$. However, this is impossible, since the sequence corresponds to an essential cycle on the projective plane, which is 1-sided. This completes the proof of "if" part. $\square$

6.3 The color factor in the case of the projective plane

Before proceeding the proof, we introduce an important notation, called a *color factor*. Let H be an even map on the projective plane. Put a new vertex in each face of H and join it to

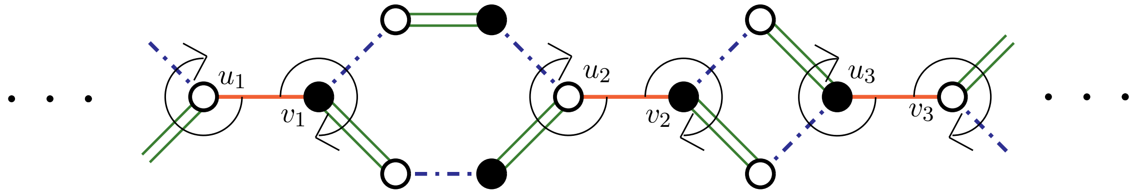

Figure 11: Rotation on the vertices $u_1, v_1, u_2, \ldots$ in the proof of Lemma 23. White and black vertices represent bipartite sets of G. We see that all vertices $u_1, v_1, u_2, \ldots, u_r, v_r$ have the consistent rotation.

all the vertices on the corresponding facial walk. Then the resulting map K is an Eulerian triangulation, which is the *face subdivision* of H and denoted by $K = \mathrm{FS}(H)$. The vertex set $U = V(K) - V(H)$ is the *color factor* of K. It is easy to see that U is a color factor of K if and only if it satisfies all of the following conditions:

(CF1) $U \neq \emptyset$ and independent.

(CF2) For any two triangular facial cycles uxy and vxy in K sharing an edge xy, $u \in U$ if and only if $v \in U$.

We use the following theorem.

Theorem 24 (Mohar [24]) *Every Eulerian triangulation K of the projective plane has a color factor. Moreover, if K is not 3-vertex-colorable, the color factor is uniquely determined.*

Let G be a cubic bipartite graph embedded on the projective plane, let D^* be an essential cycle in G^*, and let φ be a canonical odd 3-edge-coloring in G. Since each facial cycle in G has even length, each vertex in G^* has even degree, that is, G^* is an Eulerian triangulation.

We claim that G^* is not 3-vertex-colorable. In fact, suppose that G^* is 3-vertex-colorable, that is, the faces of G can be colored by three colors, say A, B, C, such that every two adjacent faces have different colors. We color each vertex in one partite set of G by black, and each vertex in the other partite set by white. Then for each vertex v, we give a rotation around v so that

- the colors A, B, C appear in the faces incident with v in this order along the rotation if v is a black vertex,
- and the colors A, B, C appear in the faces incident with v in the opposite order along the rotation if v is a white vertex.

This choice implies that all vertices admit a consistent rotation, which is a contradiction since G is a graph embedded on the projective plane. Therefore, the claim holds.

By Proposition 6 and symmetry, we may assume that $\rho_\varphi(D) = (1, 1, 0)$. Therefore, it follows from Lemma 23 that $NS_a(\varphi) = NS_b(\varphi) = \emptyset$ and $\left(NS_c(\varphi)\right)^*$ consists of only one essential cycle, say C^*. Since $NS_a(\varphi) = NS_b(\varphi) = \emptyset$, all edges colored by a or b are singular. Thus, by Fact 11, any cycle in $M_a \cup M_b$ is facial. Let U^* be the set of vertices in G^* that correspond to each cycle in $M_a \cup M_b$. Then it is easy to see that U^* satisfies both conditions (CF1) and (CF2). Therefore, it follows from Theorem 24 that U^* is the unique color factor of G^*. In particular, since $C^* \subseteq \left(NS_c(\varphi)\right)^*$, C^* does not pass through any face bounded by

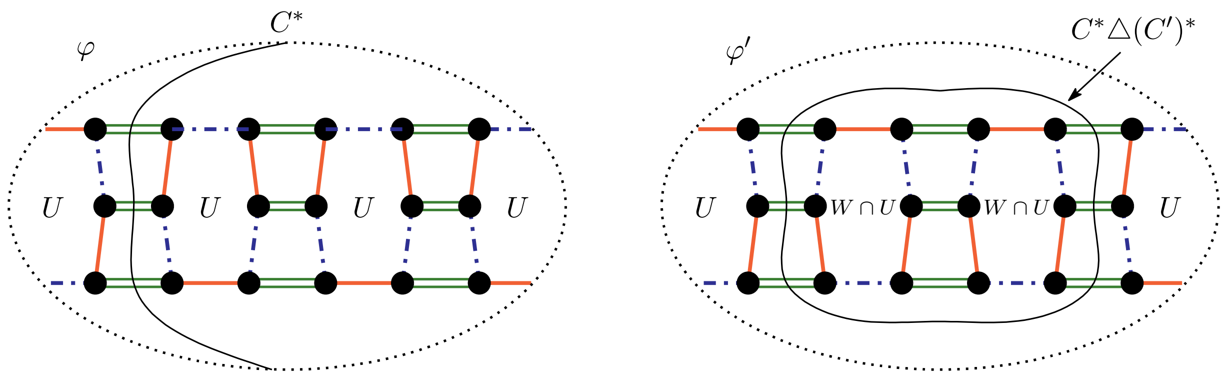

Figure 12: Two canonical odd 3-edge-colorings φ and φ' in a cubic bipartite graph G embedded on the projective plane. The cycles in $M_a \cup M_b$ correspond to the color factor U^* in the dual G^*. Together with the unique cycle $(C')^*$ in $\big(NS_c(\varphi')\big)^*$ as in the right of Figure 10, we take the curve $C^*\triangle(C')^*$ in the right. The set of all faces inside $C^*\triangle(C')^*$ is W. From φ', by Kempe switch at the cycles corresponding to $W \cap U$, we obtain the 3-edge-coloring φ.

a cycle in $M_a \cup M_b$, and hence $V(C^*) \cap U^* = \emptyset$. See the left of Figure 12 for an example. This shows the next lemma.

Lemma 25 *Let G be a cubic bipartite graph embedded on the projective plane, let D^* be an essential cycle in G^*, let φ be a canonical odd 3-edge-coloring in G with $\rho_\varphi(D) = (1,1,0)$, and let U^* be the color factor of G^*. Then both of the following hold:*

(i) *U^* corresponds to the set of the cycles in $M_a \cup M_b$.*

(ii) *$V(C^*) \cap U^* = \emptyset$, where C^* is the unique essential cycle in $\big(NS_c(\varphi)\big)^*$.*

6.4 Proof of Theorem 20

Then we are ready to prove Theorem 20.

Proof of Theorem 20. Let G be a cubic bipartite graph embedded on the projective plane, let D^* be an essential cycle in G^*, and let φ, φ' be two odd 3-edge-colorings in G. By Lemma 22, it suffices to prove the case where both φ, φ' are canonical odd 3-edge-colorings. By swapping the colors (which can be expressed as Kempe switches) if necessary, we may assume that $\rho_\varphi(D) = \rho_{\varphi'}(D) = (1,1,0)$. It suffices to prove that φ and φ' are Kempe equivalent.

Let C^* and $(C')^*$ be the unique essential cycles in $\big(NS_c(\varphi)\big)^*$ and the one in $\big(NS_c(\varphi')\big)^*$, respectively. Since both C^* and $(C')^*$ are essential cycles on the projective plane, $C^*\triangle(C')^*$ bounds some disks on the projective plane. Let W^* be the vertices of G^* that appear in the disks, excluding its boundary. (So, W^* does not contain any vertices in $C^* \cup (C')^*$). See the right side of Figure 12.

Let U^* be the color factor of G^*. By Lemma 25 (ii), $V(C^*) \cap U^* = \emptyset$ and $V\big((C')^*\big) \cap U^* = \emptyset$. This implies that $W^* \cap U^* \neq \emptyset$. By Lemma 25 (i), any vertex in $W^* \cap U^*$ corresponds to a cycle in $M_a \cup M_b$. Then performing Kempe switches to φ' at all cycles corresponding to $W^* \cap U^*$, we obtain the following.

- For each edge in $C \triangle C'$, Kempe switches are performed at exactly one end vertex.

- For any other edges, Kempe switches are performed at both or neither end vertices.

Let φ'' be the obtained 3-edge-coloring. Then, φ'' satisfies that C^* is the unique essential cycle in $NS_c(\varphi'')$, and hence φ'' is indeed equivalent with φ. This completes the proof of Theorem 20. $\square$

6.5 Proof of Theorem 2

Proof of Theorem 2. Let G be a cubic bipartite graph embedded on the projective plane.

We first prove the "only if" part. Suppose that G^* is 4-vertex-colorable. Then it follows from Theorem 16 (i) and (ii) that G admits both an odd 3-edge-coloring and an even 3-edge-coloring. By Theorem 8, they are not Kempe equivalent, and hence there are at least two Kempe equivalence classes. This completes the proof of the "only if" part.

We next prove the "if" part. Suppose that G is not 4-vertex-colorable. Then it follows from Theorem 16 (i) and (ii) that G admits an odd 3-edge-coloring, but no even 3-edge-colorings. By Theorem 20, all odd 3-edge-colorings are Kempe equivalent. This completes the proof of the "if" part. $\square$

7 Open problems

7.1 Discussion for even 3-edge-colorings

We have explained several properties on "odd" 3-edge-colorings in a cubic bipartite graph embedded on the projective plane. In this section, we explain that the similar (but not same) properties hold with odd 3-edge-colorings replaced by even 3-edge-colorings. Recall that Lemmas 22 and 23 hold also for even 3-edge-colorings. Then by the similar way to the proof of Lemma 25, we can give the following lemma, which shows the relation to the color factor of G^*. We leave the detail for the readers.

Lemma 26 *Let G be a cubic bipartite graph embedded on the projective plane, let φ be a canonical even 3-edge-coloring in G, and let U^* be the color factor of G^*. By Lemma 23, for each color x in $\{a, b, c\}$, $\big(NS_x(\varphi)\big)^*$ forms an essential cycle, say C_x^*. Then the following holds:*

(U1) *For each $x \in \{a, b, c\}$, C_x^* alternates a vertex in U^* and not in U^*.*

(U2) *There exists a unique vertex v_0^* in U^* such that the three essential cycles C_a^*, C_b^* and C_c^* pairwise transversally intersect at v_0^*.*

In contrast with odd 3-edge-colorings, two even 3-edge-colorings in a cubic bipartite graph embedded on the projective plane are not necessarily Kempe equivalent. For example, see the two even 3-edge-colorings in Figure 13. Note that the graph in Figure 13 contains exactly three Kempe equivalence classes, two of which consist of even 3-edge-colorings and the other consists of odd ones. We leave the problem to find the upper bound on the number of Kempe equivalence classes as an open problem.

Problem 2 *Do all cubic bipartite graphs embedded on the projective plane admit at most three Kempe equivalence classes?*

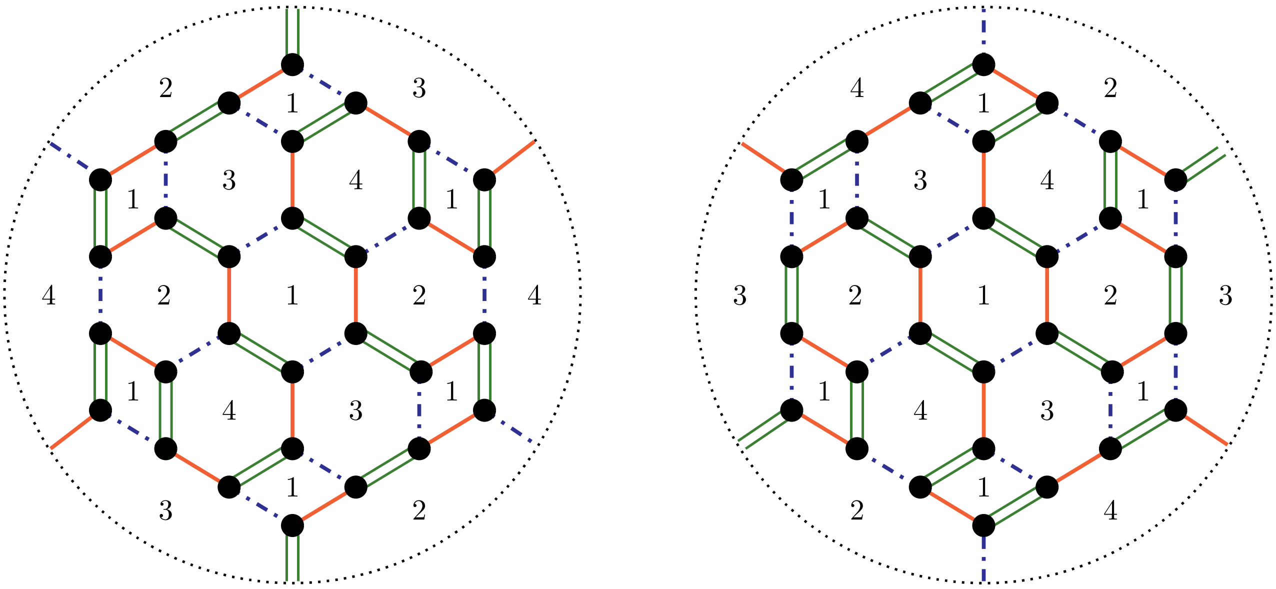

Figure 13: Two even 3-edge-colorings in a cubic bipartite graph G on the projective plane such that they are not Kempe equivalent. (Since in each 3-edge-coloring, the edges of any two colors form a Hamiltonian cycle, any Kempe switch gives a 3-edge-coloring obtained by the permutation of colors.) The numbers on faces represent the produced 4-vertex coloring of G^*, and "1" faces form a color factor of G^*. The central hexagon corresponds to the vertex v_0^* in (U2) in Lemma 26.

7.2 The case of surfaces other than the sphere nor the projective plane

In this paper, we have mainly focused on cubic graphs embedded on the projective plane. Then it is natural to consider other surfaces. However, there do not seem to exist nice properties on the Kempe equivalence classes even for cubic bipartite graphs embedded on the torus.

Consider the two 3-edge-colorings in Figure 14. Since the Kempe switch at the cycle depicted by the thin black line in the left gives the right, they are Kempe equivalent. Note that the left one does not produce a 4-vertex-coloring in the dual, while the right one does. Therefore, Theorem 8 does not hold for a cubic graph embedded on the torus, even if it is bipartite.

Furthermore, as the counterpart of Problem 2, it is natural to ask how many Kempe

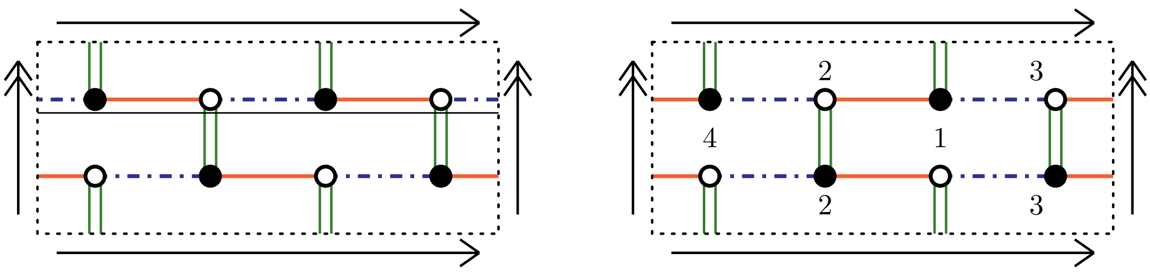

Figure 14: Two 3-edge-colorings of a cubic bipartite graphs embedded on the torus. We obtain the torus by identifying the top and the bottom, and the left and the right, respectively.

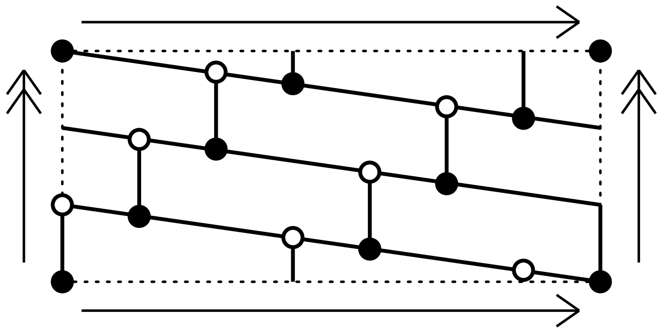

Figure 15: A cubic bipartite graph embedded on the torus.

equivalence classes a cubic bipartite graph embedded on the torus can have. For a possible answer to this question, we give the graph in Figure 15. It is the dual of the complete graph K_7, and is sometimes called the *Heawood graph.* It has exactly forty-eight 3-edge-colorings all of which have the property that the edges of any two colors form a Hamiltonian cycle. Because of the property above, the number of 3-edge-colorings contained in each Kempe equivalence class is exactly $3! = 6$, and hence the graph in Figure 15 has exactly eight Kempe equivalence classes. With this example in mind, we pose the next problem.

Problem 3 *Do all cubic bipartite graphs embedded on the torus admit at most eight Kempe equivalence classes?*

Note that belcastro and Haas [4] proved that for any integer t, there exists a cubic bipartite graph such that the number of Kempe equivalence classes is at least t. In their construction, they use t copies of $K_{3,3}$ and its genus is exactly t. Considering this, Problems 2 and 3 suggest the following more general problem.

Problem 4 *For any surface F^2, does there exist a positive integer $t(F^2)$ such that all cubic bipartite graphs on F^2 admit at most $t(F^2)$ Kempe equivalence classes?*

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