2-Factors of cubic bipartite graphs

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Abstract

For a positive integer $k \geq 3$, a $\{C_n | n \geq k\}$-factor in a graph G is a spanning subgraph of G in which each component is a cycle of length at least k. Kano, Lee and Suzuki proved that every 2-connected bipartite cubic graph contains a $\{C_n | n \geq 6\}$-factor. In this paper, we prove that every bipartite cubic graph G contains a $\{C_n | n \geq 8\}$-factor, provided that G satisfies some conditions related to 6-cycles.

Keywords: Cubic graphs, 2-factors without short cycles,
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1 Introduction

In this paper, we consider only finite and simple graphs, except for otherwise stated.

A 2-*factor* of a graph G is a spanning subgraph of G in which all vertices have degree exactly 2. Note that a 2-factor consists of vertex-disjoint cycles such that every vertex in G is contained in some of them. It is easy to see that any two cycles in a cubic graph either share an edge or are vertex-disjoint. Thus, in the remaining of this paper, we simply use the term "disjoint" for two cycles in cubic graphs regardless the meanings of vertex-disjoint and edge-disjoint.

Petersen [8] proved that every 2-connected cubic graph contains a 2-factor. For a set $\mathcal{F}$ of connected graphs, an $\mathcal{F}$-factor of a graph G is a spanning subgraph in which every component is a member of $\mathcal{F}$. Note that a 2-factor is nothing but a $\{C_n : n \geq 3\}$-factor, where we denote by C_n an n-cycle, that is, a cycle of length exactly n. Thus, Petersen's theorem can be re-stated as follows.

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Theorem 1. *[Petersen [8]] Every 2-connected cubic graph has a $\{C_n | n \geq 3\}$-factor.*

Because of the relation to several applications, such as Travelling Salesman Problem (see [1]), many researchers have been interested in the length of each cycle in a 2-factor, that is, a $\{C_n | n \geq k\}$-factor for a large integer k. There are several known theorems.

Theorem 2. *Let G be a 2-connected cubic graph. Then the following holds.*

(1) *G has a $\{C_n | n \geq 4\}$-factor. (Kawarabayashi, Matsuda, Oda and Ota [6])*

(2) *If G is 3-connected, then G has a $\{C_n | n \geq 5\}$-factor. (Jackson and Yoshimoto [4])*

(3) *If G is a bipartite graph, then G has a $\{C_n | n \geq 6\}$-factor. (Kano, Lee and Suzuki [5])*

Note that any connected bipartite cubic graph is 2-connected, see [5, Lemma 8]. So in Theorem 2(3), 2-connectedness in the assumption is not necessary.

In this paper, we focus on bipartite cubic graphs and attempt to find a $\{C_n | n \geq 8\}$-factor. As in Figure 1, which is same as [5, Figure 1(c)], there are 2-connected bipartite cubic graphs in which every 2-factor contains a 6-cycle. Thus, it is impossible to guarantee for every 2-connected bipartite cubic graph to have a $\{C_n | n \geq 8\}$-factor, but we show that with additional conditions.

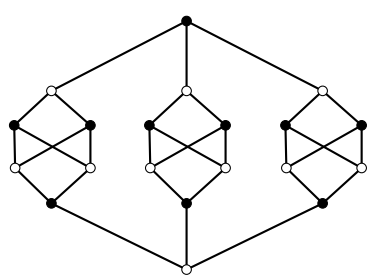

Figure 1: A bipartite cubic graph in which every 2-factor contains a 6-cycle.

An edge e in a graph G is called a *chord* of a cycle C in G if e is not an edge in C, but both end vertices of e are contained in C. We are now ready to state our main theorem.

Theorem 3. *Let G be a cubic bipartite graph satisfying the following conditions.*

(G1) Each 6-cycle of G has at most one chord, and

(G2) for each 6-cycle D, if D shares exactly one edge with another 6-cycle, then D does not share an edge with any other 6-cycles.

Then G has a $\{C_n|n \geq 8\}$-factor.

Note that considering the example in Figure 1, it seems that the condition (G1) is essential while we need condition (G2) only for technical reason for our proofs. So, we expect the following conjecture:

Conjecture 1 *Let G be a cubic bipartite graph such that each 6-cycle of G has at most one chord. Then G has a $\{C_n|n \geq 8\}$-factor.*

Tutte [9] suggested the statement that every 3-connected cubic bipartite graph contains a Hamiltonian cycle. There are several known counterexamples to this statement, but the following weaker version might hold.

Conjecture 2 *Let G be a 3-connected cubic bipartite graph of order at least 8. Then G has a $\{C_n|n \geq 8\}$-factor.*

If there exists a 6-cycle D of a cubic graph G with at least two chords, then either G is $K_{3,3}$ or the two vertices in D that are not incident with the chords of D form a cut set of order 2. This shows that if Conjecture 1 is true, then Conjecture 2 is also true. We hope that Theorem 3 would be a step toward to solve those Conjectures.

Note that in several literatures, the reduction of short cycles is a powerful method to show the existence of a $\{C_n|n \geq k\}$-factor for some k. For example, the reduction of 3-cycles and 4-cycles is a key idea to prove Theorem 2(1)–(3) in [6], and in [4, 5], respectively. In fact, this paper follows the latter reduction to reduce 4-cycles, and in addition, we also reduce 6-cycles, depending on the neighbouring situations. (In order to restrict the neighbouring situations of 6-cycles, we need the condition (G2).) Note that reduction to other structures can be found in [2, 3], but their reductions are more restricted. As long as the authors know, this paper is the first to attempt to reduce 6-cycles to find a 2-factor in which each cycle has length longer than 6.

2 Preliminary

In this section, we define some terminology. Let G be a cubic graph. For two 6-cycles D and D' in G sharing at least one edge, their union $\widetilde{D}$ is defined as $V(\widetilde{D}) = V(D) \cup V(D')$ and $E(\widetilde{D}) = E(D) \cup E(D')$. We call $\widetilde{D}$ *a 6-cycle pair of Type-i*, where i is the number of edges shared by D and D'.

For an edge subset T in G, we say that a subgraph H in G is *T-intersecting* if H contains at least one edge in T. If H is not T-intersecting, we say that H is *T-disjoint*. Furthermore, a $\{C_n|n \geq 6\}$-factor F in G is *T-good* if each 6-cycle in F either is T-intersecting or shares exactly one edge with some 4-cycle. We show the following lemma.

Lemma 1. *Let G be a cubic bipartite graph, and let $T = \emptyset$. If G contains a T-good $\{C_n | n \geq 6\}$-factor in G, then G contains a $\{C_n | n \geq 8\}$-factor.*

Proof. Let F be a T-good $\{C_n | n \geq 6\}$-factor in G. We choose such F so that the number of components in F is as few as possible. Suppose that F contains a 6-cycle $D = abcdefa$. Since F is T-good and $T = \emptyset$, D shares exactly one edge with a 4-cycle. By symmetry, we may assume that the edge ab is shared by a 4-cycle, say $D' = abb'a'$. Since F is a 2-factor in the cubic graph G, the edge $a'b'$ is contained in another cycle in F. Then, by switching the two edges ab and $a'b'$ in F with the two edges aa' and bb', we obtain a T-good $\{C_n | n \geq 6\}$-factor with fewer number of components, contradicting the choice of F. Thus, F contains no 6-cycle, and hence F is a $\{C_n | n \geq 8\}$-factor in G. $\square$

The following lemma is almost trivial, but useful. We will use it implicitly several times in our proof.

Lemma 2. *Every minimal edge cut of a 2-factor contains even number of edges.*

We briefly explain an outline of our proof. From a cubic bipartite graph G satisfying the conditions (G1) and (G2), we recursively reduce 6-cycle pairs of Type-1 (Section 3.1), those of Type-3 (Section 3.2), 6-cycles (Section 3.3) and then 4-cycles (Section 3.4), if they satisfy certain conditions. To explain those reduction, we let $G_0 = G$ and $T_0 = \emptyset$. For $i \geq 1$, let G_i be the graph obtained from G_{i-1} by the ith reduction, and T_i be the set of edges in G_i created by the reductions we have done before obtaining G_i from G_0. For each $i \geq 1$, we also show the following in each section:

- The new graph G_i is a simple cubic bipartite graph.
- If the new graph G_i contains a T_i-good $\{C_n | n \geq 6\}$-factor, then G_{i-1} contains a T_{i-1}-good $\{C_n | n \geq 6\}$-factor.

By Lemma 1, those claims enable us to focus on the simple cubic bipartite graph G_s obtained finally by the all reductions. By using Theorem 2(3), G_s contains a $\{C_n | n \geq 6\}$-factor F_s. In the last section (Section 3.5), we show that G_s contains a T_s-good $\{C_n | n \geq 6\}$-factor, which completes the proof.

3 Proof of Theorem 3

Proof. Let G be a cubic bipartite graph satisfying the conditions (G1) and (G2). As explained in the above outline of the proof, we recursively reduce some configurations in G one by one. Let $G_0 = G$ and $T_0 = \emptyset$.

3.1 Reduction for 6-cycle pairs of Type-1 with particular properties

Let $i \geq 1$, and suppose that the graph G_{i-1} contains a T_{i-1}-disjoint 6-cycle pair $\widetilde{D_i}$ of Type-1 that is disjoint from any 4-cycles in G_{i-1}. Then we reduce $\widetilde{D_i}$ and update T_{i-1} as follows. Let $\widetilde{D_i}$ consist of the 10-cycle $abcdefghkla$ with a chord bg. Suppose that two vertices in $\{a, c, d, e, f, h, k, l\}$ are adjacent in G_{i-1} but not in $\widetilde{D_i}$. Since $\widetilde{D_i}$ is disjoint from any 4-cycles and G_{i-1} is bipartite, either $dk \in E(G_{i-1})$ or $el \in E(G_{i-1})$. Note that one of the edges dk and el do not exist, since otherwise $delkd$ is a 4-cycle in G_{i-1}, contradicting the condition of $\widetilde{D_i}$. This arguments distinguish the T_{i-1}-disjoint 6-cycle pair $\widetilde{D_i}$ of Type-1 into two types, namely, Type-1(i): neither dk nor el exist in G_{i-1} and Type-1(ii): exactly one of dk and el exist in G_{i-1}. We reduce $\widetilde{D_i}$ of each type as follows.

Type-1(i): Neither dk nor el exist in G_{i-1}.

In this type, for $x \in \{a, c, d, e, f, h, k, l\}$, let x' be the vertex in G_{i-1} such that xx' is an edge in G_{i-1} but not in $\widetilde{D_i}$, see the left of Figure 2. Note that some vertices of $a', c', \ldots, l'$ might possibly coincide. Then remove $V(\widetilde{D_i})$ and add four new edges $c'd', e'f', h'k'$ and $l'a'$, see the right of Figure 2. Let G_i be the obtained graph, and let T_i be the set of edges in G_i that are not in G. In other words, T_i is obtained from T_{i-1} by deleting all edges xx' for $x \in \{a, c, d, e, f, h, k, l\}$ (if contained in T_{i-1}) and adding the edges $c'd', e'f', h'k'$ and $l'a'$. As we will see in Figure 2, G_i is still bipartite (and hence it does not have any loops), and since the degree of any vertices in G_i is same as that in G_{i-1}, G_i is still cubic. Since $\widetilde{D_i}$ is disjoint from any 4-cycles in G_{i-1}, G_i is still a simple graph.

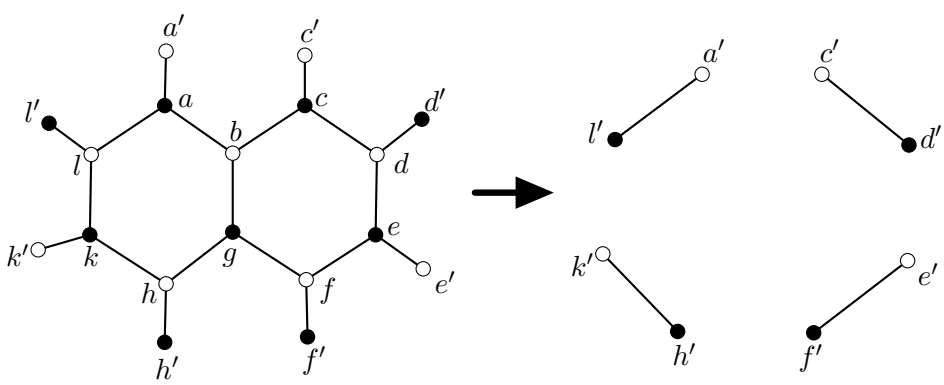

Figure 2: Reduction of the 6-cycle pair $\widetilde{D_i}$ of Type-1(i).

Type-1(ii): Exactly one of dk and el exist in G_{i-1}.

In this type, we may assume $dk \in E(G_{i-1})$ by symmetry. Note that $abcdkla$ is a 6-cycle sharing an edge with the 6-cycle $abghkla$ in $\widetilde{D_i}$. Since no edge in $\widetilde{D_i}$ is contained in T_{i-1}, it follows from the condition (G2) that $abcdkla$ is not a 6-cycle in G. This implies that $dk \in T_{i-1}$.

For $x \in \{a, c, e, f, h, l\}$, let x' be the vertex in G_{i-1} such that xx' is an edge in G_{i-1} but not in $\widetilde{D_i}$, see the left of Figure 3. Then remove $V(\widetilde{D_i})$ and add six new vertices u_a, u_c, u_e, u_f, u_h and u_l together with 12 new edges $a'u_a$, $c'u_c$, $e'u_e$, $f'u_f$, $h'u_h$, $l'u_l$, u_au_l, u_lu_c, u_cu_h, u_hu_e, u_eu_f and u_fu_a, see the right of Figure 3. Let G_i be the obtained graph, and let T_i be the set of edges in G_i that are not in G. In other words, T_i is obtained from T_{i-1} by deleting dk and all edges xx' for $x \in \{a, c, e, f, h, l\}$ (if contained in T_{i-1}) and adding the 12 new edges. As we will see in Figure 3, G_i is still bipartite and cubic. It is clear from the construction that G_i is still a simple graph.

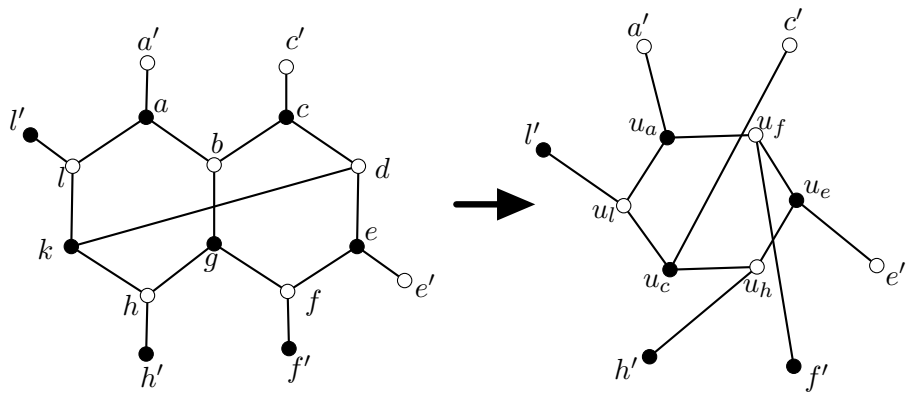

Figure 3: Reduction of the 6-cycle pair $\widetilde{D_i}$ of Type-1(ii).

Let G_p be the graph obtained by the above reduction as long as we can. So, any T_p-disjoint 6-cycle pair $\widetilde{D}$ of Type-1 in G_p shares an edge with a 4-cycle. Note that G_p is a simple, cubic and bipartite graph. In order to show our main theorem, we first prove the following claim.

Claim 1. *Let $1 \leq i \leq p$. If G_i contains a T_i-good $\{C_n | n \geq 6\}$-factor, then G_{i-1} contains a T_{i-1}-good $\{C_n | n \geq 6\}$-factor.*

To show this claim, suppose that G_i has a T_i-good $\{C_n | n \geq 6\}$-factor F_i. Depending on types of $\widetilde{D_i}$, we modify each edge in $T_i - T_{i-1}$ contained in F_i into some suitable paths in G_{i-1} as follows. This modification converts F_i in G_i into a T_{i-1}-good $\{C_n | n \geq 6\}$-factor F_{i-1} in G_{i-1}. By the condition of $\widetilde{D_i}$, no 4-cycle in G_{i-1} shares an edge with $\widetilde{D_i}$. If there is a 6-cycle in G_{i-1} sharing an edge with $\widetilde{D_i}$, except for $abghkla$ and $bcdefgb$, then it is T_{i-1}-intersecting, since otherwise, the 6-cycle $abghkla$ or $bcdefgb$ contradicts the condition (G2). Therefore, no 4-cycle nor T_{i-1}-disjoint 6-cycle in G, except for $abghkla$ and $bcdefgb$, shares an edge with $\widetilde{D_i}$. In the following, this property ensures that the obtained $\{C_n | n \geq 6\}$-factor F_{i-1} in G_{i-1} is in fact T_{i-1}-good.

Type-1(i): Recall that G_i is obtained from G_{i-1} by reducing $\widetilde{D_i}$ as in Figure 2. In this

case, we divide the proof into five cases, depending on how many edges in $\{c'd', e'f', h'k', l'a'\}$ are used in F_i.

Case 1(i)-1: No edges in $\{c'd', e'f', h'k', l'a'\}$ are used in F_i.

In this case, we add the 10-cycle $abcdefghkla$ as a component of F_i.

Case 1(i)-2: Exactly one edge in $\{c'd', e'f', h'k', l'a'\}$ is used in F_i.

By symmetry, we may assume that the edge $c'd'$ is used in F_i. Then we replace the edge $c'd'$ with the path $c'cbalkhgfedd'$, see Figure 4.

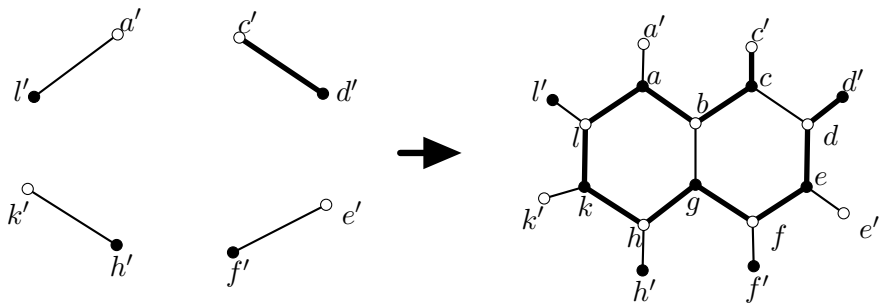

Figure 4: Conversion of the 6-cycle pair $\widetilde{D_i}$ of Type-1(i) in Case 1(i)-2.

Case 1(i)-3: Exactly two edges in $\{c'd', e'f', h'k', l'a'\}$ are used in F_i.

In this subcase, we have three possibilities up to symmetry.

(1) $\{c'd', e'f', h'k', a'l'\} \cap E(F_i) = \{c'd', e'f'\}$.
In this case, we replace the edges $c'd'$ and $e'f'$ with the paths $c'cbalkhgff'$ and $d'dee'$, respectively, see Figure 5.

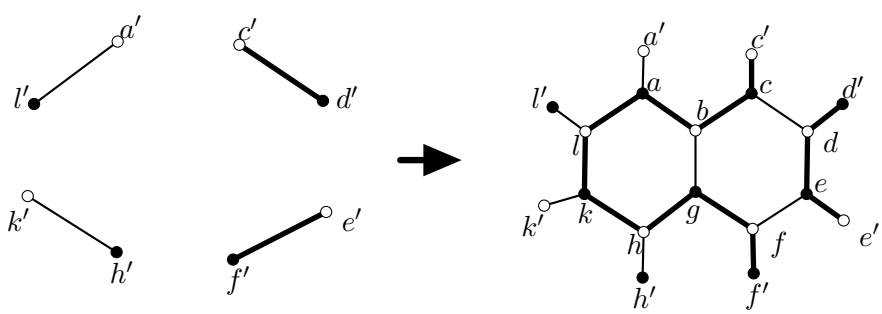

Figure 5: Conversion of the 6-cycle pair $\widetilde{D_i}$ of Type-1(i) in Case 1(i)-3 (1).

(2) $\{c'd', e'f', h'k', a'l'\} \cap E(F_i) = \{c'd', h'k'\}$.
In this case, we replace the edges $c'd'$ and $h'k'$ with the paths $c'cbalkk'$ and $d'defghh'$,

respectively, see Figure 6.

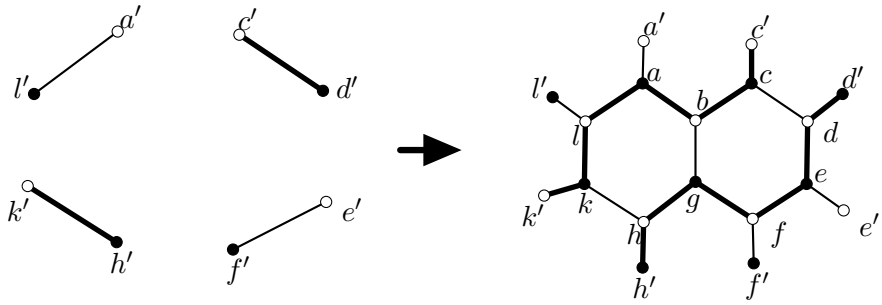

Figure 6: Conversion of the 6-cycle pair $\widetilde{D_i}$ of Type-1(i) in Case 1(i)-3 (2).

(3) $\{c'd', e'f', h'k', a'l'\} \cap E(F_i) = \{c'd', a'l'\}$.
In this case, we replace the edges $c'd'$ and $a'l'$ with the paths $a'abcc'$, and $d'defghkll'$, respectively, see Figure 7.

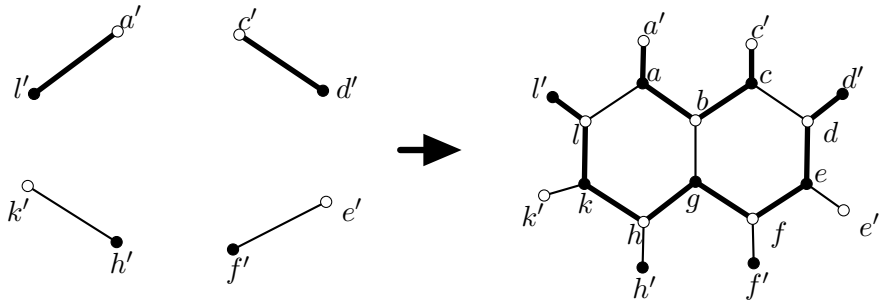

Figure 7: Conversion of the 6-cycle pair $\widetilde{D_i}$ of Type-1(i) in Case 1(i)-3 (3).

Case 1(i)-4: Exactly three edges in $\{c'd', e'f', h'k', l'a'\}$ are used in F_i.

By symmetry, we may assume that $\{c'd', e'f', h'k', a'l'\} \cap E(F_i) = \{c'd', e'f', a'l'\}$. Then we replace the edges $c'd', e'f'$ and $a'l'$ with the paths $a'abcc'$, $d'dee'$ and $f'fghkll'$, respectively, see Figure 8.

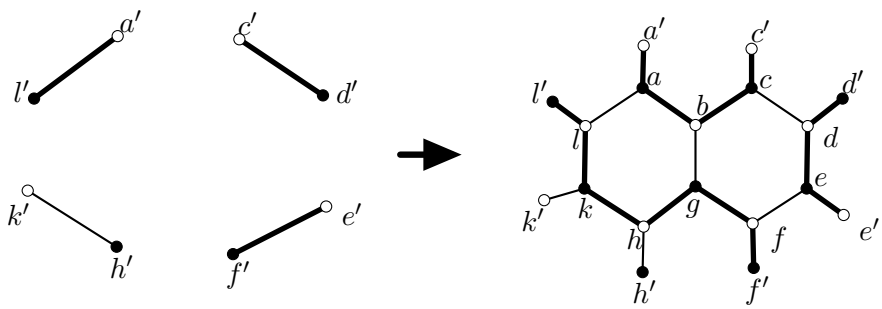

Figure 8: Conversion of the 6-cycle pair $\widetilde{D_i}$ of Type-1(i) in Case 1(i)-4.

Case 1(i)-5: All edges in $\{c'd', e'f', h'k', l'a'\}$ are used in F_i.

In this case, we replace the edges $c'd', e'f', h'k'$ and $a'l'$ with the paths $a'abcc'$, $d'dee'$, $f'fghh'$ and $k'kll'$, respectively, see Figure 9.

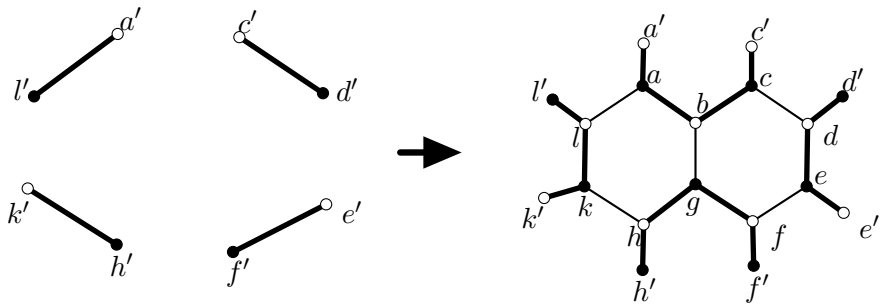

Figure 9: Conversion of the 6-cycle pair $\widetilde{D_i}$ of Type-1(i) in Case 1-5.

Type-1(ii): Recall that G_i is obtained from G_{i-1} by reducing $\widetilde{D_i}$ as in Figure 3. In this case, we divide the proof into four cases, depending on how many edges in $\{a'u_a,\ c'u_c,\ e'u_e,\ f'u_f,\ h'u_h,\ l'u_l\}$ are used in F_i. By Lemma 2, its number must be even, so $0, 2, 4$ or 6. Note that the vertices u_a and u_f, the vertices u_c and u_h, and the vertices u_e and u_l are symmetric, respectively.

Case 1(ii)-1: No edges in $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\}$ are used in F_i.

In this case, we add the 10-cycle $abcdefghkla$ as a component of F_i.

Case 1(ii)-2: Exactly two edges in $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\}$ are used in F_i.

In this subcase, we have four possibilities up to symmetry. We will show each possibility one by one.

(1) $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{a'u_a, f'u_f\}$.
In this case, we replace the path $a'u_au_lu_cu_hu_eu_ff'$ with the path $a'alkhgbcdeff'$, see

Figure 10.

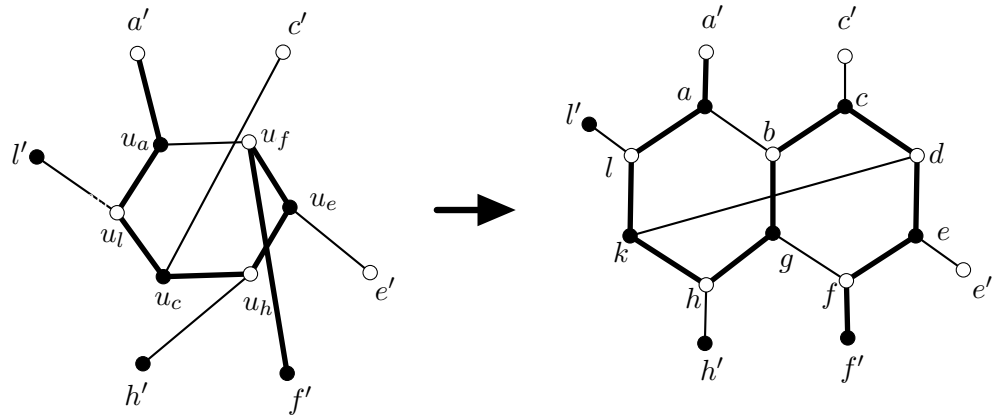

Figure 10: Conversion of the 6-cycle pair $\widetilde{D_i}$ of Type-1(ii) in Case 1(ii)-2 (1).

(2) $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{f'u_f, e'u_e\}$.
In this case, we replace the path $f'u_f u_a u_l u_c u_h u_e e'$ with the path $f'fghklabcdee'$, see Figure 11.

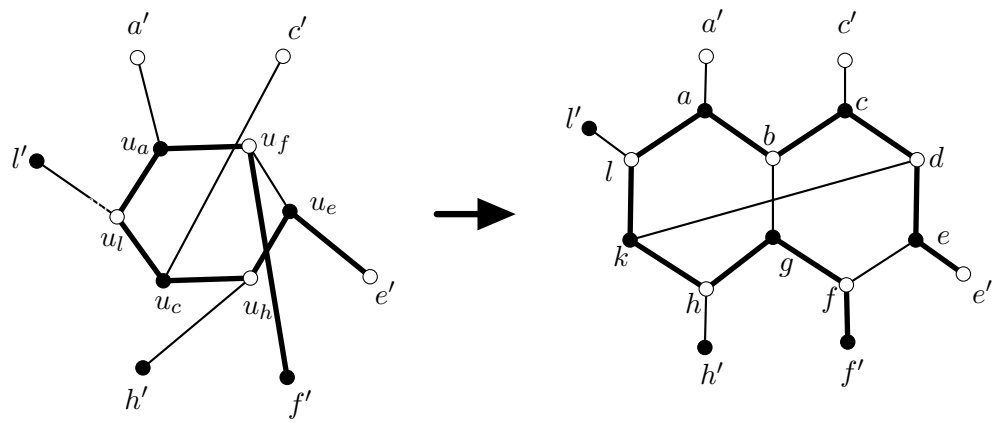

Figure 11: Conversion of the 6-cycle pair $\widetilde{D_i}$ of Type-1(ii) in Case 1(ii)-2 (2).

(3) $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{e'u_e, h'u_h\}$.
In this case, we replace the path $e'u_e u_f u_a u_l u_c u_h h'$ with the path $e'efghh'$ and then add the 6-cycle $abcdkla$, see Figure 12. Note that the 6-cycle $abcdkla$ contains the edge dk that belongs to T_{i-1}, and hence it is T_{i-1}-intersecting.

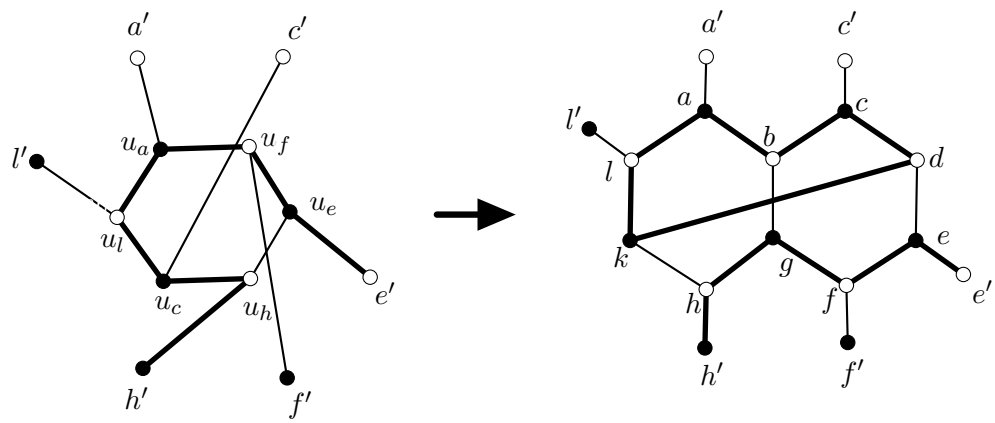

Figure 12: Conversion of the 6-cycle pair $\widetilde{D_i}$ of Type-1(ii) in Case 1(ii)-2 (3).

(4) $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{h'u_h, c'u_c\}$.
In this case, we replace the path $h'u_hu_eu_fu_au_lu_cc'$ with the path $h'hklabgfedcc'$, see Figure 13.

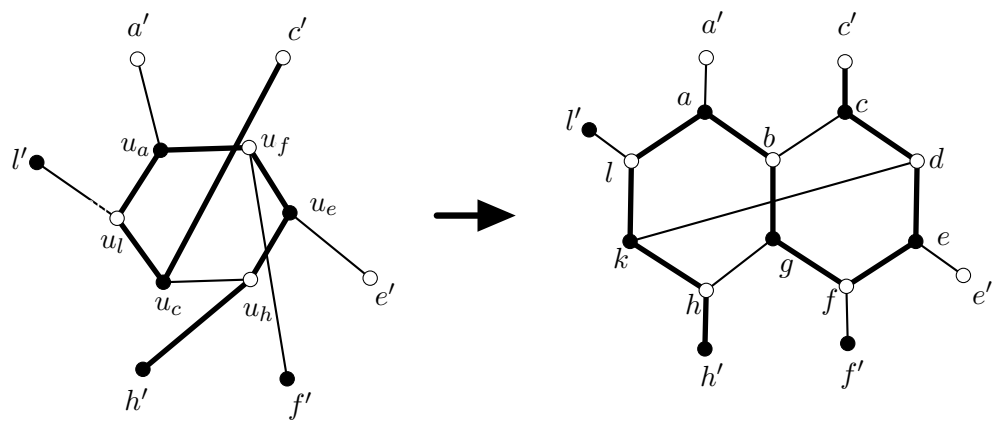

Figure 13: Conversion of the 6-cycle pair $\widetilde{D_i}$ of Type-1(ii) in Case 1(ii)-2 (4).

Note that the case $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{c'u_c, l'u_l\}$ is symmetric to (3), and the case $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{l'u_l, a'u_a\}$ is symmetric to (2), respectively. This completes the proof for Case 1(ii)-2.

Case 1(ii)-3: Exactly four edges in $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\}$ are used in F_i.

In this subcase, we have six possibilities up to symmetry. We will show each possibility one by one.

(1) $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{l'u_l, a'u_a, f'u_f, e'u_e\}$.
In this case, we replace the paths $l'u_lu_cu_hu_ee'$ and $a'u_au_ff'$ with the paths $l'lkhgff'$ and

$e'edcbaa'$, see Figure 14.

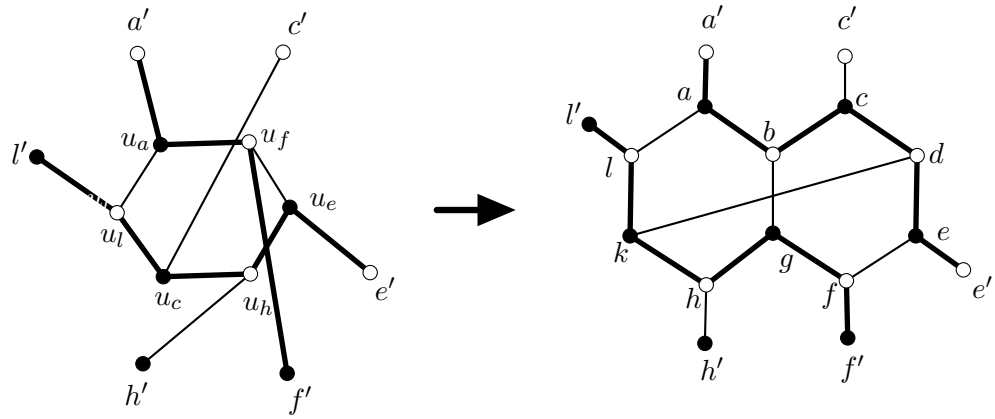

Figure 14: Conversion of the 6-cycle pair $\widetilde{D_i}$ of Type-1(ii) in Case 1(ii)-3 (1).

(2) $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{a'u_a, f'u_f, e'u_e, h'u_h\}$.
In this case, we replace the paths $a'u_au_lu_cu_hh'$ and $f'u_fu_ee'$ with the paths $a'alkhh'$ and $f'fgbcdee'$, see Figure 15.

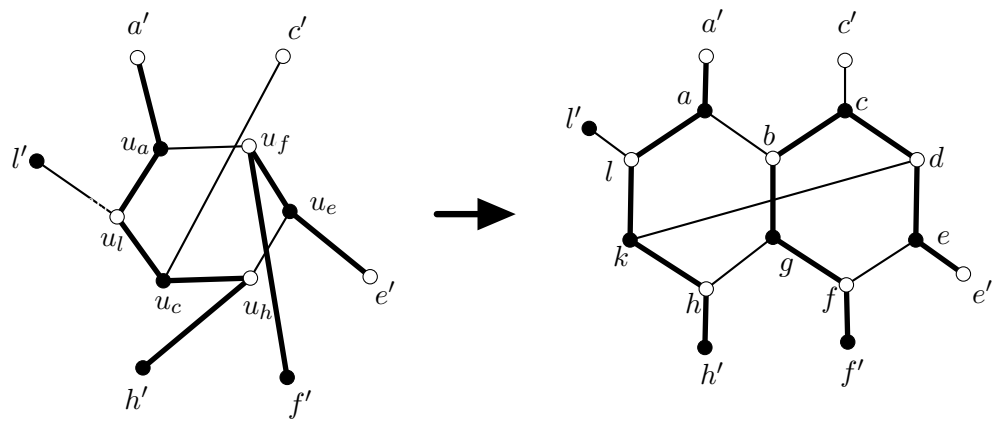

Figure 15: Conversion of the 6-cycle pair $\widetilde{D_i}$ of Type-1(ii) in Case 1(ii)-3 (2).

(3) $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{f'u_f, e'u_e, h'u_h, c'u_c\}$.
In this case, we replace the paths $f'u_fu_au_lu_cc'$ and $e'u_eu_hh'$ with the paths $f'fgbalkhh'$ and $c'cdee'$, see Figure 16.

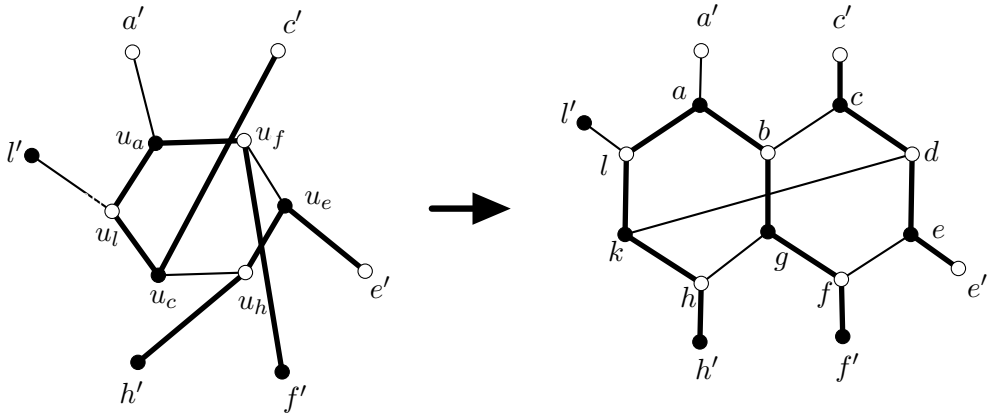

Figure 16: Conversion of the 6-cycle pair $\widetilde{D_i}$ of Type-1(ii) in Case 1(ii)-3 (3).

(4) $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{e'u_e, h'u_h, c'u_c, l'u_l\}$.
In this case, we replace the paths $e'u_eu_fu_au_ll'$ and $h'u_hu_cc'$ with the paths $e'efgball'$ and $h'hkdcc'$, see Figure 17.

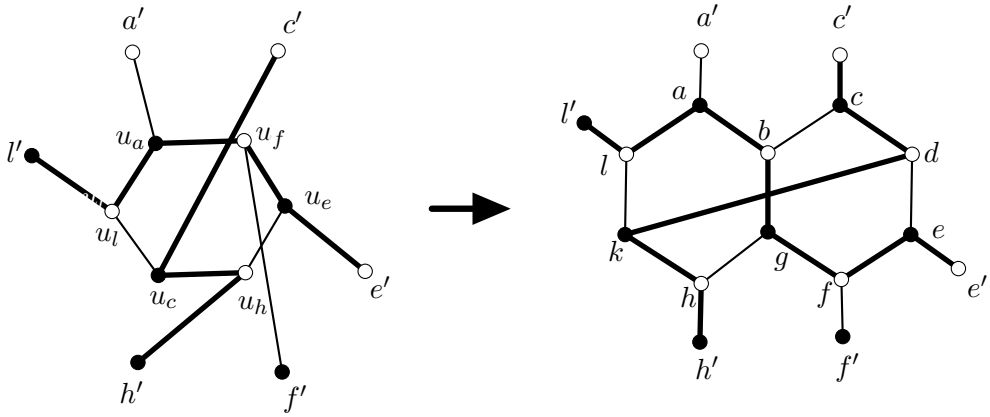

Figure 17: Conversion of the 6-cycle pair $\widetilde{D_i}$ of Type-1(ii) in Case 1(ii)-3 (4).

Note that the case $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{h'u_h, c'u_c, l'u_l, a'u_a\}$ is symmetric to (3), and the case $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{c'u_c, l'u_l, a'u_a, f'u_f\}$ is symmetric to (2), respectively.

(5) $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{f'u_f, e'u_e, c'u_c, l'u_l\}$.
In this case, we replace the paths $f'u_fu_au_ll'$ and $c'u_cu_hu_ee'$ with the paths $f'fghkdee'$ and $l'labcc'$, see Figure 18.

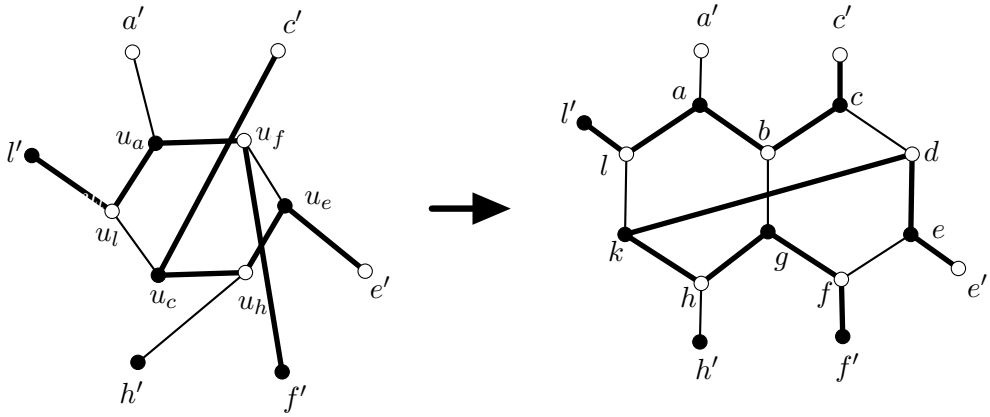

Figure 18: Conversion of the 6-cycle pair $\widetilde{D_i}$ of Type-1(ii) in Case 1(ii)-3 (5).

(6) $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{a'u_a, f'u_f, h'u_h, c'u_c\}$.
In this case, we replace the paths $a'u_au_lu_cc'$ and $h'u_hu_eu_ff'$ with the paths $a'alkdeff'$ and $c'cbghh'$, see Figure 19.

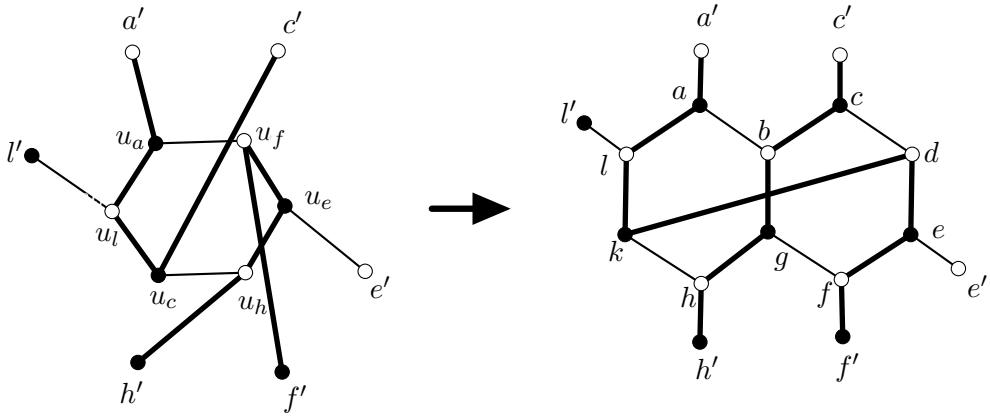

Figure 19: Conversion of the 6-cycle pair $\widetilde{D_i}$ of Type-1(ii) in Case 1(ii)-3 (6).

Note that the case $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\} \cap E(F_i) = \{e'u_e, h'u_h, l'u_l, a'u_a\}$ is symmetric to (5). This completes the proof for Case 1(ii)-3.

Case 1(ii)-4: All edges in $\{a'u_a, c'u_c, e'u_e, f'u_f, h'u_h, l'u_l\}$ are used in F_i.

In this case, we replace the paths $a'u_au_ff'$, $e'u_eu_hh'$ and $c'u_cu_ll'$ (or the paths $l'u_lu_aa'$, $f'u_fu_ee'$ and $h'u_hu_cc'$) with the paths $l'lkhh'$, $a'abgff'$ and $c'cdee'$, see Figure 20.

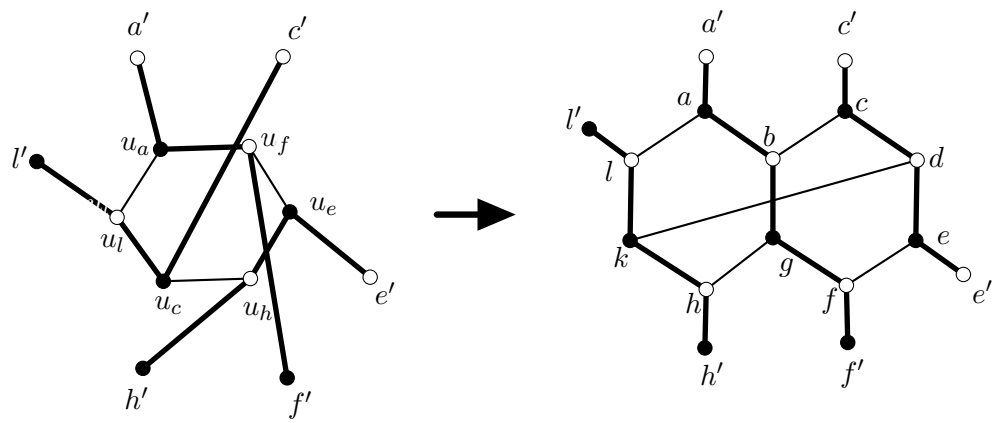

Figure 20: Conversion of the 6-cycle pair $\widetilde{D_i}$ of Type-1(ii) in Case 1(ii)-4.

Recall that no 4-cycle nor T_{i-1}-disjoint 6-cycle in G, except for $abghkla$ and $bcdefgb$, shares an edge with $\widetilde{D_i}$. Thus, in either case, the obtained $\{C_n|n \geq 6\}$-factor F_{i-1} in G_{i-1} is T_{i-1}-good. This completes the proof of Claim 1.

By Lemma 1, it suffices to find a T_0-good $\{C_n|n \geq 6\}$-factor in G_0. Thus, by recursively applying Claim 1, it suffices to show that G_p has a T_p-good $\{C_n|n \geq 6\}$-factor.

3.2 Reduction for 6-cycle pairs of Type-3 with particular properties

Let $i \geq p+1$, and suppose that the graph G_{i-1} contains a T_{i-1}-disjoint 6-cycle pair $\widetilde{D_i}$ of Type-3 that is disjoint from any 4-cycles in G_{i-1}. Then we reduce $\widetilde{D_i}$ in G_{i-1} and update T_{i-1} as follows. Since $\widetilde{D_i}$ is disjoint from any 4-cycles, the three edges shared by two 6-cycles in $\widetilde{D_i}$ must be consecutive and we name each vertex in $\widetilde{D_i}$ as in the left of Figure 21. For $x \in \{b, c, e, f, g, h\}$, let x' be the vertex such that xx' is an edge in G_{i-1} but not in $\widetilde{D_i}$. If $x' = y$ or $x' = y'$ for some vertices $x, y \in \{b, c, e, f, g, h\}$, then we can find a 4-cycle sharing an edge with $\widetilde{D_i}$, contradicting the condition of $\widetilde{D_i}$. Thus, all vertices in the left of Figure 21 are pairwise distinct. This implies that the following reduction does not create multiple edges: Remove $V(\widetilde{D_i})$ and add two new vertices u and v together with six new edges ub', ug', uf', vc', ve' and vh', see the right of Figure 21. Let G_i be the obtained graph, and let T_i be the set of edges in G_i that are not in G. In other words, T_i is obtained from T_{i-1} by deleting all edges xx' for $x \in \{b, c, e, f, g, h\}$ (if contained in T_{i-1}) and adding the six new edges. As in Figure 21, the obtained graph is still cubic and bipartite. As explained in the above, G_i is still a simple graph.

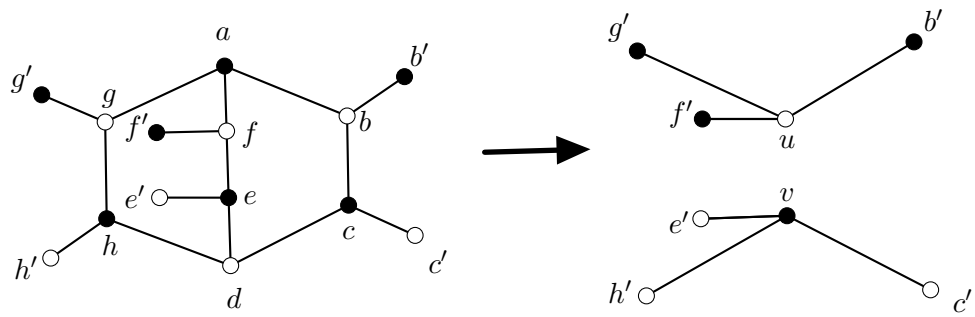

Figure 21: Reduction of the 6-cycle pair $\widetilde{D_i}$ of Type-3.

Let G_q be the graph obtained by the above reduction as long as we can. So, any T_q-disjoint 6-cycle pair $\widetilde{D}$ of Type-3 in G_q shares an edge with a 4-cycle. Note that G_q is a simple, cubic and bipartite graph. We next show the following claim.

Claim 2. *Let $p+1 \leq i \leq q$. If G_i contains a T_i-good $\{C_n | n \geq 6\}$-factor, then G_{i-1} contains a T_{i-1}-good $\{C_n | n \geq 6\}$-factor.*

To show this claim, suppose that G_i has a T_i-good $\{C_n | n \geq 6\}$-factor F_i. Similarly to the proof of Claim 1, we will convert F_i in G_i into a T_{i-1}-good $\{C_n | n \geq 6\}$-factor F_{i-1} in G_{i-1}.

Recall that G_i is obtained from G_{i-1} by reducing $\widetilde{D_i}$ as in Figure 21. By the condition of $\widetilde{D_i}$, no 4-cycle in G_{i-1} shares an edge with $\widetilde{D_i}$. Since F_i is a 2-factor of G_i, F_i contains exactly two edges incident with u and exactly two edges incident with v. Note that the pairs $\{bb', cc'\}$, $\{ee', ff'\}$ and $\{hh', gg'\}$ are pairwise symmetric, and hence we may assume that the edges ub' and vc' are both used in F_i. Then, depending on how other edges are used in F_i, we have two subcases, namely, Case 2-1: $\{ub', ug', vc', vh'\} \subseteq E(F_i)$ and Case 2-2: $\{ub', ug', vc', ve'\} \subseteq E(F_i)$.

Case 2-1: The paths $b'ug'$ and $c'vh'$ are used in F_i.

In this case we replace these paths with the paths $b'bcc'$ and $g'gafedhh'$, respectively, see Figure 22. Suppose that the new cycle using the path $b'bcc'$, say D, is a T_{i-1}-disjoint 6-cycle. Then the 6-cycle $abcdefa$ shares exactly one edge with D and three edges with $afedgha$. Since those three 6-cycles are all T_{i-1}-disjoint, they are also contained in G, which contradicts the condition (G2). Thus the new cycle using the path $b'bcc'$ is either of length at least 8 or T_{i-1}-intersecting. On the other hand, the new cycle using the path $g'gafedhh'$ contains the vertices g', g, a, f, e, d, h, h', and hence its length is at least 8.

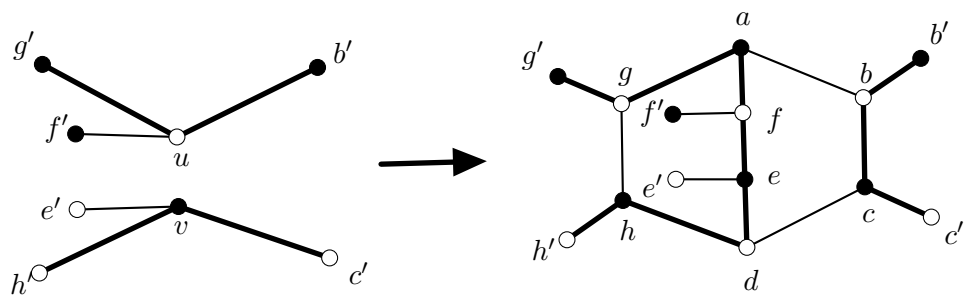

Figure 22: Conversion of the 6-cycle pair $\widetilde{D_i}$ of Type-3 in Case 2-1.

Case 2-2: The paths $b'ug'$ and $c've'$ are used in F_i.

In this case, we replace these paths with $b'bafee'$ and $c'cdhgg'$, respectively, see Figure 23. If the new cycle using the path $b'bafee'$, say D, is a T_{i-1}-disjoint 6-cycle, then by the same argument as in Case 2-1, the 6-cycle $abcdhga$ shares exactly one edge with D and three edges with $abcdefa$, which contradicts the condition (G2). Thus, the new cycle using the path $b'bafee'$ either has length at least 8 or is T_{i-1}-intersecting. Similarly, the new cycle using the path $c'cdhgg'$ either has length at least 8 or is T_{i-1}-intersecting.

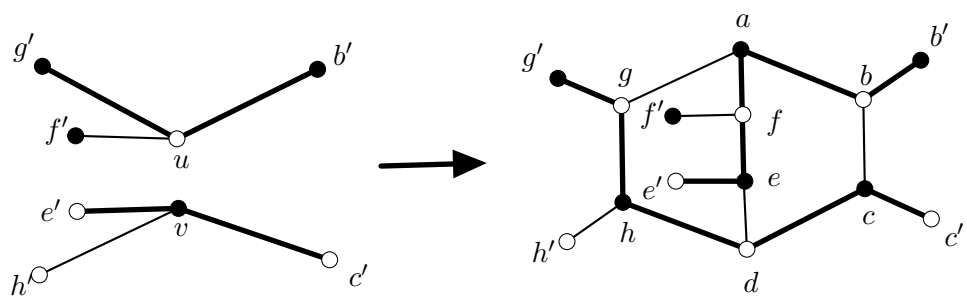

Figure 23: Conversion of the 6-cycle pair $\widetilde{D_i}$ of Type-3 in Case 2-2.

This completes the proof of Claim 2.

As in the end of Section 3.1, it suffices to show that G_p has a T_p-good $\{C_n | n \geq 6\}$-factor. Thus, by recursively applying Claim 2, it also suffices to show that G_q has a T_q-good $\{C_n | n \geq 6\}$-factor.

3.3 Reduction for 6-cycles with particular properties

We next reduce 6-cycles with particular properties. Let $i \geq q+1$, and suppose that G_{i-1} contains a T_{i-1}-disjoint 6-cycle D_i that is disjoint from any 4-cycles in G_{i-1}. Then we reduce D_i from G_{i-1} and update T_{i-1} as follows. Let $D_i = abcdefa$, and for

$x \in \{a, b, c, d, e, f\}$, let x' be the vertex such that xx' is an edge in G_{i-1} but not in D_i, see the left of Figure 24. If $x' = y$ or $x' = y'$ for some vertices $x, y \in \{a, b, c, d, e, f\}$, then we can find a 4-cycle sharing an edge with D_i, contradicting the condition of D_i. Thus, all vertices in the left of Figure 24 are pairwise distinct. This implies that the following reduction does not create multiple edges: Remove $V(D_i)$ and add two new vertices u and v together with six new edges ua', uc', ue', vb', vd' and vf', see the right of Figure 24. Let G_i be the obtained graph, and let T_i be the set of edges in G_i that are not in G. In other words, T_i is obtained from T_{i-1} by deleting all edges xx' for $x \in \{a, b, c, d, e, f\}$ (if contained in T_{i-1}) and adding the six new edges. As in Figure 24, the obtained graph is still cubic and bipartite. As explained in the above, G_i is still a simple graph.

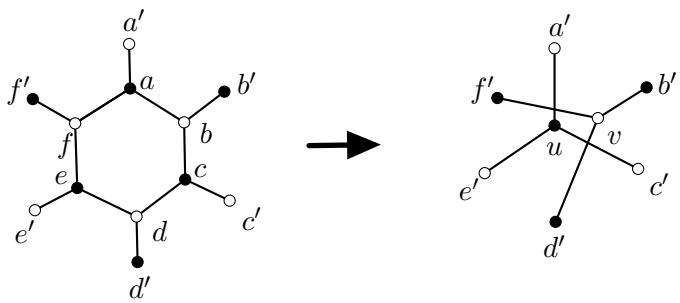

Figure 24: Reduction of the 6-cycle D_i.

Let G_r be the graph obtained by the above reduction as long as we can. So, any T_r-disjoint 6-cycle D in G_r shares an edge with a 4-cycle. Note that G_r is a simple, cubic and bipartite graph. We next show the following claim.

Claim 3. *Let $q + 1 \leq i \leq r$. If G_i contains a T_i-good $\{C_n | n \geq 6\}$-factor, then G_{i-1} contains a T_{i-1}-good $\{C_n | n \geq 6\}$-factor.*

Similarly to the proofs of Claims 1 and 2, suppose that G_i has a T_i-good $\{C_n | n \geq 6\}$-factor F_i, and we will convert F_i into a T_i-good $\{C_n | n \geq 6\}$-factor F_{i-1} in G_{i-1}.

Recall that G_i is obtained from G_{i-1} by reducing D_i as in Figure 24. By the condition of D_i, there is no 4-cycle sharing an edge with D_i in G_{i-1}. In this case, by symmetry, we have two subcases, namely, Case 3-1: $\{ua', ue', vb', vf'\} \subseteq E(F_i)$ and Case 3-2: $\{ua', ue', vb', vd'\} \subseteq E(F_i)$.

Case 3-1: The paths $a'ue'$ and $b'vf'$ are used in F_i.

In this case, we can replace these paths with the paths $a'aff'$ and $b'bcdee'$, respectively, see Figure 25.

Let D be the new cycle in F_{i-1} using the path $a'aff'$, and suppose that D is a T_{i-1}-disjoint 6-cycle. Since both D_i and D are T_{i-1}-disjoint, D_i and D form a 6-cycle pair of

Type-1 in G, and also in G_p. By the condition of G_p, the 6-cycle pair of Type-1 must share an edge with some 4-cycle in G_p. Since D_i is disjoint from any 4-cycles, D shares exactly one edge with the 4-cycle. This shows that D is either of length at least 8 or a T_{i-1}-intersecting or shares exactly one edge with a 4-cycle.

Let D' be the new cycle in F_{i-1} using the path $b'bcdee'$ and suppose that D' is a T_{i-1}-disjoint 6-cycle. Since both D_i and D' are T_{i-1}-disjoint, D_i and D' form a 6-cycle pair of Type-3 in G, and hence in G_q. By the condition of G_q, the 6-cycle pair of Type-3 must share an edge with some 4-cycle in G_q. Thus, by the same argument as above, D' is either of length at least 8 or a T_{i-1}-intersecting or shares exactly one edge with a 4-cycle.

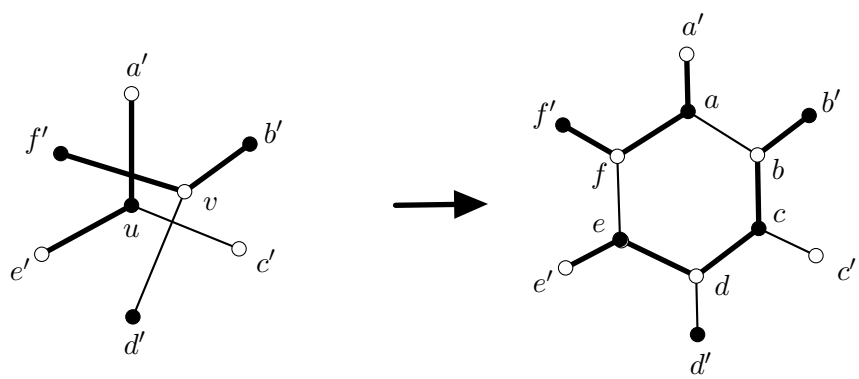

Figure 25: Conversion of the 6-cycle D_i in Case 3-1.

Case 3-2: The paths $a'ue'$ and $b'vd'$ are used in F_i.

In this case, we can replace these paths with the paths $a'afee'$ and $b'bcdd'$, respectively, see Figure 26. Since F_i is a $\{C_n | n \geq 6\}$-factor of G_i, the cycle using the path $a'ue'$ in F_i has length at least 6. Since the length of the new cycle using the path $a'afee'$ is increased by two, its length is at least 8. In the same manner, we can show that the new cycle using the path $b'bcdd'$ also has length at least 8, as desired.

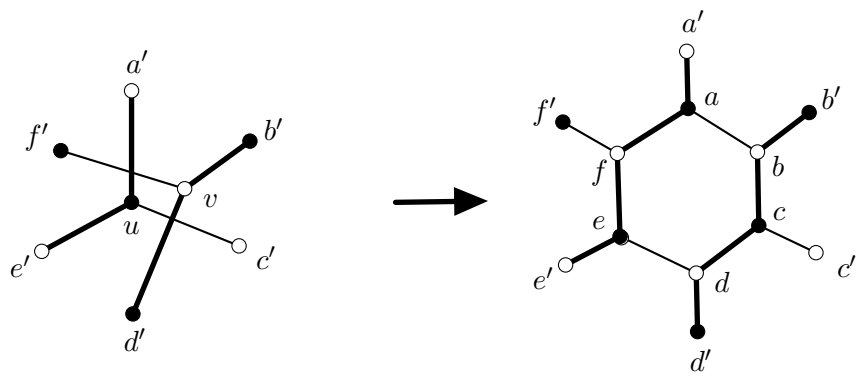

Figure 26: Conversion of the 6-cycle D_i in Case 3-2.

This completes the proof of Claim 3.

Thus, again by recursively applying Claim 3, it also suffices to show that G_r has a T_r-good $\{C_n | n \geq 6\}$-factor.

3.4 Reduction for 4-cycles with particular properties

Finally, we reduce 4-cycles with particular properties. Let $i \geq r+1$, and suppose that G_{i-1} contains a T_{i-1}-disjoint 6-cycle D with exactly one chord. Let $D = abcdefa$ with a chord $ad \in E(G_{i-1})$. Then $abcda$ is a 4-cycle, and let $D_i = abcda$. We reduce D_i from G_{i-1} and update T_{i-1} as follows. For $x \in \{b, c\}$, let x' be the vertex such that xx' is an edge in G_{i-1} but not in D_i. Since the 6-cycle D has exactly one chord in G_{i-1}, $b' \neq e$ and $c' \neq f$. Then, we remove the four vertices a, b, c, d and add two new vertices u and v together with five new edges $fu, c'u, b'v, ev$ and uv, see Figure 27. Let G_i be the obtained graph, and let T_i be the set of edges in G_i that are not in G. In other words, T_i is obtained from T_{i-1} by deleting the edges ad, bb', cc' (if contained in T_{i-1}) and adding the five new edges. As in Figure 27, the obtained graph is still cubic and bipartite. As explained in the above, G_i is still a simple graph.

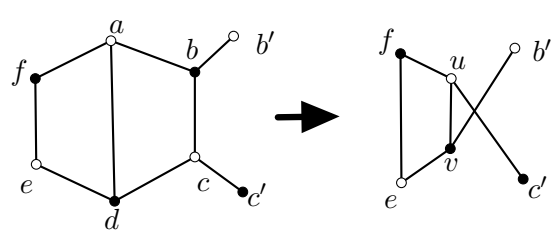

Figure 27: Reduction of the 4-cycle D_i.

Let G_s be the graph obtained by the above reduction as long as we can. Note that

G_s is a simple, cubic and bipartite graph. Similarly to the other reductions, we show the following claim.

Claim 4. *Let $r+1 \leq i \leq s$. If G_i contains a T_i-good $\{C_n | n \geq 6\}$-factor, then G_{i-1} contains a T_{i-1}-good $\{C_n | n \geq 6\}$-factor.*

Similarly to the proofs of Claims 1–3, suppose that G_i has a T_i-good $\{C_n | n \geq 6\}$-factor F_i, and we will convert F_i into a T_{i-1}-good $\{C_n | n \geq 6\}$-factor F_{i-1} in G_{i-1}. Recall that G_i is obtained from G_{i-1} by reducing D_i as in Figure 27. In this case, we have two subcases, depending on whether $uv \in E(F_i)$ or not.

Case 4-1: The edge uv is used in F_i.
Suppose that F_i passes through uv with the path $b'vuc'$. Then we replace the path $b'vuc'$ in F_i with the path $b'badcc'$, see Figure 28. Note that the length of the cycle using uv in F_i is increased by two, and hence the new cycle has length at least 8, as desired. In the case when F_i passes through uv with the other ways, we can similarly convert the cycle using uv in F_i to a cycle of length at least 8.

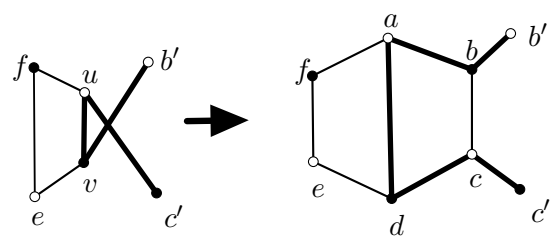

Figure 28: Conversion of the 4-cycle D_i in Case 4-1.

Case 4-2: The edge uv is not used in F_i.
In this case, F_i contains the two paths fuc' and $b've$. Then we first replace them with the paths $fabb'$ and $c'cde$, respectively, see Figure 29. If the new cycle using the path $fabb'$ does not contain the path $c'cde$, then we further change the edges ab and cd with ad and bc. In either case, the new cycle using the edge fa contains all the vertices a, b, c, d, e, f, b' and c', and hence the new cycle has length at least 8, as desired.

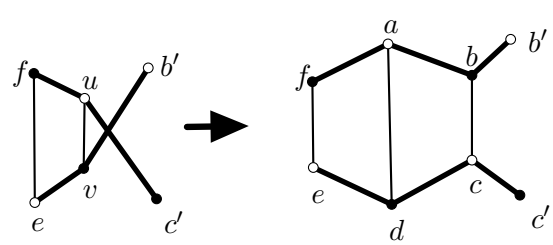

Figure 29: Conversion of the 4-cycle D_i in Case 4-2.

This completes the proof of Claim 4.

Thus, again by recursively applying Claim 4, it also suffices to show that G_s has a T_s-good $\{C_n|n \geq 6\}$-factor.

3.5 Finding a T_s-good $\{C_n|n \geq 6\}$-factor in G_s

Then we proceed to the final step of the proof. Since G_s is a simple cubic bipartite graph, we can apply Theorem 2(3) to G_s and obtain a $\{C_n|n \geq 6\}$-factor F_s in G_s. We take such a $\{C_n|n \geq 6\}$-factor F_s so that the number of T_s-disjoint 6-cycles is as small as possible.

Now we prove that every 6-cycle in F_s either is T_s-intersecting or shares exactly one edge with some 4-cycle. Suppose that there is a T_s-disjoint 6-cycle Let $D = abcdefa$ in F_s. This means that D is also a 6-cycle in G. Thus, if D shares no edges with any 4-cycles in G_s, then we must reduce D in G_r (or G_p or G_q if D forms a 6-cycle pair of Type-1 or -3 that is disjoint from any 4-cycles, respectively), a contradiction. Therefore, D shares an edge with a 4-cycle in G_s. If D shares exactly one edge with a 4-cycle, then there is nothing to prove. Thus, we may assume that such a 4-cycle does not exist.

Suppose that D shares exactly three edges with a 4-cycle D'. Since G_s is a cubic graph, those three edges must be consecutive on D. So, we may assume by symmetry that ab, bc and cd are the three edges, which means that ad is a chord of D. By the condition of G_s, D has another chord, say be by symmetry. By the condition (G1), at least one of ad and be is not an edge in G, that is, it is contained in T_s. Thus, we can replace the 6-cycle D in F_s with the T_s-intersecting 6-cycle $adcbefa$, which contradicts the choice of F_s. This implies that D shares exactly three edges with no 4-cycles. Since D shares an edge with a 4-cycle in G_s, there is a 4-cycle D' that shares exactly two edges with D. If sush two edges are not consecutive in D, then D has two chords and there is a 4-cycle sharing three edges with D, a contradiction. Thus, D shares two consecutive edges, say ab and bc by symmetry, with a 4-cycle D'. Let $D' = abca'a$. Since G_s is a cubic graph and the two neighbors a and c of a' are both contained in D, a' cannot be contained in any cycles in the 2-factor F_s, a contradiction. This shows that F_s is a T_s-good $\{C_n|n \geq 6\}$-factor in G_s. This completes the proof of Theorem 3.

□

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