

Least conflict choosability*

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Abstract

Given a multigraph, suppose that each vertex is given a local assignment of k colours to its incident edges. We are interested in whether there is a choice of one local colour per vertex such that no edge has both of its local colours chosen. The least k for which this is always possible given any set of local assignments we call the *conflict choosability* of the graph. This parameter is closely related to separation choosability and adaptable choosability. We show that conflict choosability of simple graphs embeddable on a surface of Euler genus g is $O(g^{1/4} \log g)$ as $g \rightarrow \infty$. This is sharp up to the logarithmic factor.

Keywords: graph colouring, list colouring, conflict choosability, DP colouring, adaptable choosability, separation choosability, graphs on surfaces.

1 Introduction

Dvořák and Postle [4] and Fraigniaud, Heinrich and Kosowski [7] independently defined the *conflict k -colouring* problem as follows. Given a (simple) graph $G = (V, E)$, each edge $uv \in E$ is assigned a list $K(u, v)$ of ordered pairs—called *conflicts*—of colours from $[k] = \{1, \dots, k\}$. The question is

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whether G admits a colouring $c : V \rightarrow [k]$ of the vertices so that no edge is in a conflict, i.e. there is no edge $uv \in E$ and conflict $(c_u, c_v) \in K(u, v)$ such that $c(u) = c_u$ and $c(v) = c_v$. The authors in [4] and [7] also imposed further natural restrictions based on contrasting goals and perspectives, but here instead we only prescribe the maximum number μ of conflicts per edge.

In fact, this is equivalent to the “least conflict” version of the problem, with one conflict per edge, provided we pass to a *multigraph* of maximum edge multiplicity μ . Let us be more precise. Let $G = (V, E)$ be a multigraph. For any positive integer k , a *local k -partition* of G is a collection $\{L_v\}_{v \in V}$ of maps of the form $L_v : E(v) \rightarrow [k]$, where $E(v)$ denotes the set of edges incident to v . So each L_v is a partition of $E(v)$ into k parts, and for each $e \in E(v)$ the colour $L_v(e)$ can be thought of as the *local colour*¹ of v associated to e . Given such a collection $\{L_v\}$, we say G is *conflict $\{L_v\}$ -colourable* if there is some colouring $c : V \rightarrow [k]$ of the vertices so that no edge $e = uv \in E$ has $L_u(e) = c(u)$ and $L_v(e) = c(v)$. Observe we may assume each L_v is onto, or else it is safe to colour v with any colour not in the image. The (*least*) *conflict choosability* $\text{ch}_{\leftrightarrow}(G)$ of G is the smallest k such that G is conflict $\{L_v\}$ -colourable for any local k -partition $\{L_v\}_{v \in V}$.

As we discuss in Section 2, conflict choosability considerably strengthens upon two notable list colouring parameters, separation choosability (cf. [13]) and adaptable choosability (cf. [11]), and so its study could potentially yield new insights into these two parameters.

Before continuing, we give two easy but instructive examples. First, for a square integer μ , consider two vertices with μ edges between them. Take the local $\sqrt{\mu}$ -partition which lists all μ possible pairwise conflicts between the two vertices. So this is a μ -edge planar multigraph with maximum degree and multiplicity both μ that has conflict choosability strictly greater than $\sqrt{\mu}$. Second, for a positive integer μ , consider a star with centre u and leaves v_1, \dots, v_μ , where each edge has multiplicity μ . Take the local μ -partition where the edges between u and v_i include all μ possible conflicts having i as the local colour for u . Since u cannot be coloured, this is a μ^2 -edge planar multigraph with maximum degree μ^2 and maximum multiplicity μ that has conflict choosability strictly greater than μ .

Besides introducing conflict choosability and setting down some of its basic behaviour, our main task in this paper is to treat it in a classic setting for chromatic graph theory. We prove the following.

Theorem 1. *For some constant $C_1 > 0$, if G is a multigraph of maximum multiplicity $\mu \geq 1$ that is embeddable on a surface of Euler genus g , then $\text{ch}_{\leftrightarrow}(G) \leq \max\{C_1 \sqrt{\mu}(g+1)^{1/4} \log(\mu^2(g+2)), 8\mu\}$.*

The 8μ term cannot be improved by more than a constant factor due to the

¹By relabelling, we alternatively may define the L_v as maps from $E(v)$ to \mathbb{N} each image set of which contains at most k elements, so not necessarily the same image for every v .

second example above, and we will see below that the other term is sharp up to at most a polylogarithmic factor.

Allow us to reiterate the $\mu = 1$ case, which may be interpreted as an analogue of Heawood's classic formula for the chromatic number [9].

Corollary 2. *There is a constant $C > 0$ such that $\text{ch}_{\leftrightarrow}(G) \leq C(g + 1)^{1/4} \log(g + 2)$ for every simple graph G that is embeddable on a surface of Euler genus g .*

The $\mu = \Theta(\sqrt{g})$ case of Theorem 1 is of special interest, hinting at the following possible version of Heawood's.

Conjecture 3. *For any $0 < \varepsilon \leq 1$, there exists $C > 0$ such that the following holds. For any simple graph G that is embeddable on a surface of Euler genus g , if every edge is assigned at most $(1 - \varepsilon)k$ conflicts from $[k]^2$, then G is conflict k -colourable, provided $k \geq C\sqrt{g}$.*

Theorem 1 follows from the following perhaps more general result.

Theorem 4. *For some constant $C_2 > 0$, if G is a multigraph with $m \geq 3$ edges and maximum multiplicity $\mu \geq 1$, then $\text{ch}_{\leftrightarrow}(G) \leq C_2(\mu m)^{1/4} \log(\mu m)$.*

We prove Theorems 1 and 4 in Section 4. The proof of Theorem 4 is partly probabilistic in nature. It relies on a stronger version (see Lemma 13 below) of the following simple bound.

Proposition 5. *If G is a multigraph of maximum degree $\Delta \geq 1$, then $\text{ch}_{\leftrightarrow}(G) \leq \lceil \sqrt{e(2\Delta - 1)} \rceil$.*

For completeness, we prove Proposition 5 in Section 3 by a standard application of the Lovász Local Lemma. This has the following strong yet still partial converse, also shown in Section 3.

Proposition 6. *If G is a multigraph of average degree $d \geq 3$, then $\text{ch}_{\leftrightarrow}(G) \geq \lfloor \sqrt{d/\log d} \rfloor$.*

The last two assertions alone highlight a clear distinction between conflict choosability and, say, ordinary choosability, for which the behaviour of the complete graphs K_{d+1} is linear in d , while that of the complete bipartite graphs $K_{d,d}$ is logarithmic in d [5].

Notice that Proposition 6 helps to provide a broad certificate of sharpness of Theorems 1 and 4 up to polylogarithmic factors. This is akin to the two-vertex example exhibited earlier. In particular, consider the complete multigraph on n vertices of uniform edge multiplicity μ . It is a $\mu(n - 1)$ -regular graph, so with $\mu \binom{n}{2}$ edges, that has $\Theta(n^2)$ Euler genus. By Proposition 6, the conflict choosability is $\Omega(\sqrt{\mu n / \log(\mu n)})$, and this is not far from the $O(\sqrt{\mu n \log(\mu n)})$ upper bound implied in both Theorems 1 and 4.

It may be challenging to eliminate the logarithmic factors in Theorems 1 and 4. Since we do not know the correct asymptotics in these results, we have made no effort to optimise the values of C_1 and C_2 . On the other hand, we managed to avoid the logarithmic factors for separation and adaptable choosability (see Theorems 11 and 12 below). The simpler argument uses Proposition 5 directly (rather than needing Lemma 13), and we present it in Section 4 as a warm up to proving our main result.

One might wonder if degeneracy could be an alternative way to prove Theorem 1, at least in the $\mu = 1$ case. That was essentially Heawood's original approach to bounding the chromatic number. As we will see in Section 2, density considerations have some use (see Lemma 7 below); however, a construction of Kostochka and Zhu [11] for the adaptable chromatic number shows that there are graphs of degeneracy d which have conflict choosability greater than d . There might yet be some constant $C' > 0$ such that the conflict choosability of any d -degenerate graph on n vertices is at most $C'\sqrt{d}\log n$ (which would imply the $\mu = 1$ case of Theorem 1), but we have not been able to prove this thus far. Theorem 4 implies an upper bound of $C_2(nd)^{1/4}\log(nd)$ in this situation.

For small g , it would be interesting to precisely determine the optimal upper bound on $\text{ch}_{\leftrightarrow}(G)$ over all simple graphs G embeddable on a surface of Euler genus g . As we will indicate in Section 2 it is easy to verify that the extremal conflict choosability is 4 for planar graphs.

1.1 Probabilistic preliminaries

We make use of the following basic probabilistic tools. We refer the reader to the monograph of Molloy and Reed [14] for further details.

The Chernoff Bound. For any $0 \leq t \leq np$,

$$\mathbb{P}(|\text{Bin}(n, p) - np| > t) < 2 \exp(-t^2/(3np)).$$

The Lovász Local Lemma. Consider a set \mathcal{E} of (bad) events such that for each $A \in \mathcal{E}$

- (i) $\mathbb{P}(A) \leq p < 1$, and
- (ii) A is mutually independent of a set of all but at most d of the other events.

If $ep(d+1) \leq 1$, then with positive probability none of the events in \mathcal{E} occur.

The General Local Lemma. Consider a set $\mathcal{E} = \{A_1, \dots, A_n\}$ of (bad) events such that each A_i is mutually independent of $\mathcal{E} - (\mathcal{D}_i \cup A_i)$, for some $\mathcal{D}_i \subseteq \mathcal{E}$. If we have reals $x_1, \dots, x_n \in [0, 1)$ such that for each i

$$\mathbb{P}(A_i) \leq x_i \prod_{A_j \in \mathcal{D}_i} (1 - x_j),$$

then the probability that none of the events in \mathcal{E} occur is at least $\prod_i(1-x_i) > 0$.

2 Definitions

In this section, we give some more definitions, one of conflict choosability, one of adaptable choosability, and one of separation choosability. We also show how these three parameters are related, and give a few comments related to planar graphs.

First we give an alternative definition of conflict choosability, which may be insightful. Let $G = (V, E)$ be a multigraph. Given a local k -partition $\{L_v\}$ of G , we say G is *conflict $\{L_v\}$ -orientable* if there is some orientation of all edges of G such that for every vertex $v \in V$, the set of local colours of v associated to the (oriented) edges leaving v does not contain all of $[k]$. Then the conflict choosability $\text{ch}_{\leftrightarrow}(G)$ of G is the least k such that G is conflict $\{L_v\}$ -orientable for any local k -partition $\{L_v\}_{v \in V}$.

Proof of equivalence. Let $G = (V, E)$ and fix a local k -partition $\{L_v\}$ of G . It suffices to show that G is conflict $\{L_v\}$ -orientable if and only if it is conflict $\{L_v\}$ -colourable. If it has a conflict $\{L_v\}$ -orientation, then for every $v \in V$ choose a colour from $[k]$ that is absent from the local colours of v associated to the edges leaving v to produce a conflict $\{L_v\}$ -colouring. If it has a conflict $\{L_v\}$ -colouring c , then orient towards v all incident edges e such that $L_v(e) = c(v)$ to produce a conflict $\{L_v\}$ -orientation. \square

From this equivalence, the following proposition becomes plain.

Proposition 7. *If there is an orientation of G such that every vertex has maximum outdegree less than k , then $\text{ch}_{\leftrightarrow}(G) \leq k$.*

This implies $\text{ch}_{\leftrightarrow}(G) \leq 1 + \max_{S \subseteq V} [|E(G[S])|/|S|]$, cf. e.g. [1, Lem. 3.1].

Corollary 8. *If G is a planar graph, then $\text{ch}_{\leftrightarrow}(G) \leq 4$. If G is a triangle-free planar graph, then $\text{ch}_{\leftrightarrow}(G) \leq 3$. If G is a simple graph embeddable on a surface of Euler genus $g > 0$, then $\text{ch}_{\leftrightarrow}(G) \leq H_g/2 + 1$, where H_g is Heawood's formula for Euler genus g .*

Recall that every k -degenerate graph has an orientation of maximum out-degree at most k . So Proposition 7 cannot be improved in general, since there are k -degenerate graphs with adaptable chromatic number greater than k [11] (and, as we will shortly see, the same then is true of conflict choosability).

Next we discuss how conflict choosability is connected to two colouring parameters, both of which are weaker versions of *list colouring*, as introduced independently by Erdős, Rubin and Taylor [5] and by Vizing [15].

For completeness, we recall the classic definition. Let $G = (V, E)$ be a (multi)graph. For a positive integer k , a mapping $L : V \rightarrow \binom{\mathbb{Z}^+}{k}$ is called a k -list-assignment of G , and a colouring c of V is called an L -colouring if $c(v) \in L(v)$ for any $v \in V$. We say G is k -choosable if there is a proper L -colouring of G for any k -list-assignment L . The *choosability* $\text{ch}(G)$ of G is the least k such that G is k -choosable.

2.1 Adaptable choosability

The following list colouring parameter was proposed by Kostochka and Zhu [11]. Let $G = (V, E)$ be a multigraph. Given a labelling $\ell : E \rightarrow \mathbb{Z}^+$ of the edges, a (not-necessarily-proper) vertex colouring $c : V \rightarrow \mathbb{Z}^+$ is *adapted* to ℓ if for every edge $e = uv \in E$ not all of $c(u)$, $c(v)$ and $\ell(e)$ are the same value. We say that G is *adaptably k -choosable* if for any k -list-assignment L and any labelling ℓ of the edges of G , there is an L -colouring of G that is adapted to ℓ . The *adaptable choosability* $\text{ch}_a(G)$ of G is the least k such that G is adaptably k -choosable. Every proper colouring is adapted to any labelling ℓ , so $\text{ch}(G) \geq \text{ch}_a(G)$ always.

We observe adaptable choosability is at most conflict choosability.

Observation 9. *For any multigraph G , $\text{ch}_{\leftrightarrow}(G) \geq \text{ch}_a(G)$.*

Proof. Fix $G = (V, E)$ and let $k = \text{ch}_{\leftrightarrow}(G)$. Let L be a k -list-assignment and let ℓ be a labelling of the edges of G . For each $v \in V$, locally colour each edge e incident to v with colour a if $a \in L(v)$ and $\ell(e) = a$. This yields a local k -partition $\{L_v\}$ (as mentioned in the introduction, it is not important that the image of each map L_v is equal to $[k]$, the image of each L_v can be different sets of k elements for each vertex v). By the choice of k there must be a conflict $\{L_v\}$ -colouring. It follows from our definition of $\{L_v\}$ that this corresponds to an L -colouring that is adapted to ℓ . \square

We remark that adaptable choosability is in turn a strengthening of the adaptable chromatic number (for which the list assignment always takes all lists equal) and Hell and Zhu [10] have exhibited planar graphs with adaptable chromatic number at least 4. So conflict choosability is also exactly 4 for such graphs.

2.2 Separation choosability

The following list colouring parameter was proposed by Kratochvíl, Tuza and Voigt [13]. Let $G = (V, E)$ be a graph. We say a k -list-assignment L has *maximum separation* if $|L(u) \cap L(v)| \leq 1$ for every edge uv of G . We say G is *separation k -choosable* if there is a proper L -colouring of G for any k -list-assignment L that has maximum separation. The *separation choosability* $\text{ch}_{\text{sep}}(G)$ of G is the least k such that G is separation k -choosable. Since

the choosability $\text{ch}(G)$ of G omits any separation requirement on the lists, $\text{ch}(G) \geq \text{ch}_{\text{sep}}(G)$ always.

Let us see that separation choosability is at most adaptable choosability. This observation was made earlier [6], but we include it here for cohesion.

Observation 10. *For any simple graph G , $\text{ch}_a(G) \geq \text{ch}_{\text{sep}}(G)$.*

Proof. Fix $G = (V, E)$ and let $k = \text{ch}_a(G)$. Let L be a k -list-assignment of maximum separation. Let ℓ be a labelling defined for each $uv \in E$ by taking $\ell(uv)$ as the unique element of $L(u) \cap L(v)$ if it is nonempty, and arbitrary otherwise. By the choice of k , there is guaranteed to be an L -colouring c that is adapted to ℓ . Due to the maximum separation property of L and the definition of ℓ , the colouring c must be proper. \square

Conflict choosability is a direct strengthening of separation choosability, in the same way that “DP-colouring” is a strengthening of choosability [4].

Alternative proof that $\text{ch}_{\text{sep}}(G) \leq \text{ch}_{\leftrightarrow}(G)$ for any simple graph G . Fix $G = (V, E)$ and let $k = \text{ch}_{\leftrightarrow}(G)$. Let L be a k -list-assignment of maximum separation. Let $\{L_v\}$ be a local k -partition of G defined as follows. For each edge $e = uv \in E$, if i is the unique colour in $L(u) \cap L(v)$, then let $L_u(e) = i$ and $L_v(e) = i$. By the choice of k , there is guaranteed to be a conflict $\{L_v\}$ -colouring c . Due to the maximum separation property of L and the definition of $\{L_v\}$, the colouring c is proper. \square

We remark that Kratochvíl, Tuza and Voigt [12] proved that $\text{ch}_{\text{sep}}(K_n) \sim \sqrt{n}$ as $n \rightarrow \infty$ by the use of affine planes. This is enough to certify sharpness of our Theorems 1 and 4 each up to a logarithmic factor (and Proposition 5 up to a constant factor) for simple graphs.

We also note that Škrekovski [16] conjectured that every planar graph has separation choosability at most 3, but this remains open to the best of our knowledge. If true, it would imply that separation choosability and adaptable choosability can be distinct for some planar graphs.

3 Degree

In this section, we for completeness give the proofs of Propositions 5 and 6. These results closely relate conflict choosability to the maximum and average degrees, respectively, of the multigraph.

The following proof is analogous to proofs for separation and adaptable choosability [12, 11].

Proof of Proposition 5. Let $G = (V, E)$ be a multigraph of maximum degree Δ and fix $k = \lceil \sqrt{e(2\Delta - 1)} \rceil$. Let $\{L_v\}$ be a local k -partition of G . Consider a random colouring $c : V \rightarrow [k]$ where each vertex is given an independent

uniform choice. For each edge $e = uv \in E$, let A_e be the event that $c(u) = L_u(e)$ and $c(v) = L_v(e)$. For all $e \in E$, $\mathbb{P}(A_e) = 1/k^2$ and A_e is mutually independent of all but at most $2\Delta - 2$ other events A_f . Observe that c is a conflict $\{L_v\}$ -colouring if and only if all the events A_e do not occur. The Lovász Local Lemma guarantees with positive probability a conflict $\{L_v\}$ -colouring if $e(2\Delta - 1)/k^2 < 1$, which follows from the choice of k . \square

Note that the bound $\sqrt{e(2\Delta - 1)}$ in Proposition 5 can be slightly improved to $2\sqrt{\Delta}$ using the Local Cut Lemma [3, Theorem 3.1] instead of the Lovász Local Lemma, using the same set of bad events. We have deliberately chosen to present the simpler, weaker bound.

The following proof is analogous to one in [2].

Proof of Proposition 6. Let $G = (V, E)$ be a multigraph of average degree $d = 2m/n$, where $n = |V|$ and $m = |E|$. Let $k = \lfloor \sqrt{d/\log d} \rfloor$ and consider a random local k -partition $\{L_v\}$ of G where, for each edge $e = uv \in E$, the pair $(L_u(e), L_v(e))$ is independently, uniformly chosen from pairs in $[k]^2$. For any fixed $c : V \rightarrow [k]$, c is a conflict $\{L_v\}$ -colouring with probability $(1 - 1/k^2)^m$. By the union bound and Markov's inequality, the probability that G is conflict $\{L_v\}$ -colourable is at most $k^n(1 - 1/k^2)^m \leq k^n \exp(-m/k^2)$. Since G has average degree d , we have by the choice of k that $k^2 \log k = d \log k / \log d < d/2 = m/n$. This implies $n \log k - m/k^2 < 0$ and so $k^n \exp(-m/k^2) < 1$. We have thus shown that with positive probability there is a local k -partition $\{L_v\}$ for which G is not conflict $\{L_v\}$ -colourable. \square

We remark that since $(1 + o(1)) \log_2 d \leq \text{ch}_{\text{sep}}(K_{d,d}) \leq \text{ch}_a(K_{d,d}) \leq \text{ch}(K_{d,d}) \leq (1 + o(1)) \log_2 d$ as $d \rightarrow \infty$ [5, 8], Proposition 6 implies that the ratio between conflict choosability and choosability or adaptable choosability or separation choosability can be arbitrarily large even for bipartite graphs.

4 Proof of Theorem 1

As a warm up to the main proof, we show the following result, an adaptable choosability analogue of Theorem 4.

Theorem 11. *If G is a multigraph with $m \geq 2^{16}$ edges and maximum multiplicity $\mu \geq 1$, then $\text{ch}_a(G) \leq 2^{11/4} \sqrt{e}(\mu m)^{1/4}$.*

The proof of Theorem 11 can be viewed as a simplified version of the proof of Theorem 4. Afterwards, we show how the following result, an adaptable choosability analogue of Theorem 1, is a consequence of Theorem 11. (At the same time, we also show how Theorem 4 implies Theorem 1.)

Theorem 12. *For some constant $C_3 > 0$, if G is a multigraph of maximum multiplicity $\mu \geq 1$ that is embeddable on a surface of Euler genus g , then $\text{ch}_a(G) \leq C_3 \sqrt{\mu}(g + 1)^{1/4}$.*

Theorems 11 and 12 imply the same bounds for separation choosability, and both are sharp up to the choice of C_3 due to the complete graphs with uniform edge multiplicity μ [12]. Let us mention that the question of whether graphs of Euler genus g have adaptable chromatic and choice numbers at most of order $g^{1/4}$ was first raised in December of 2007 during the *Graph Theory 2007* meeting in Fredericia, Denmark.

Proof of Theorem 11. Let $G = (V, E)$ be a multigraph with $|E| = m$ and maximum multiplicity μ . Let $k = 2^{11/4}\sqrt{e}(\mu m)^{1/4}$, let L be a k -list-assignment, and consider any labelling ℓ of the edges of G . We want to prove that there is an L -colouring of G that is adapted to ℓ . We can assume that G is connected (or else we consider each component separately), and in particular G has $n \leq m + 1$ vertices.

Let $X = \bigcup_{v \in V} L(v)$, and let $X_1 \subseteq X$ be chosen uniformly at random. Set $X_2 = X \setminus X_1$. For any $i \in \{1, 2\}$ and $v \in V$, $|L(v) \cap X_i|$ is binomially distributed with parameter $1/2$. The Chernoff Bound implies that $|L(v) \cap X_i| \leq k/4$ with probability at most $\exp(-k/24) \leq \frac{1}{2m+5} < \frac{1}{2n}$, where the first inequality uses $m \geq 2^{16}$. By a union bound, there is a bipartition $X = X_1 \cup X_2$ such that $|L(v) \cap X_i| \geq k/4$ for any $i \in \{1, 2\}$ and $v \in V$.

Let A be the set of vertices of degree at least $\sqrt{2\mu m}$ in G and let $B = V \setminus A$. Since $|E| = m$, A has most $2m/\sqrt{2\mu m} = \sqrt{2m/\mu}$ vertices, and thus $G[A]$ has maximum degree at most $\mu\sqrt{2m/\mu} = \sqrt{2\mu m}$. By definition, $G[B]$ also has maximum degree at most $\sqrt{2\mu m}$. We remove all the colours of X_1 from $L(v)$ for each $v \in A$, and all the colours of X_2 from $L(v)$ for each $v \in B$. After this operation, each list has at least $k/4$ colours left. Since $k/4 = \sqrt{2e\sqrt{2\mu m}}$, it follows from Proposition 5 that $G[A]$ has an L -colouring adapted to ℓ using only colours from X_2 while $G[B]$ has an L -colouring adapted to ℓ using only colours from X_1 . Since X_1 and X_2 are disjoint, we obtain an L -colouring of G adapted to ℓ , as desired. \square

Let us now see that Theorems 1 and 12 follow from Theorems 4 and 11, respectively.

Proofs of Theorems 1 and 12. Assume for a contradiction that there is a counterexample G to Theorem 1 or 12. Take G in such way that g is minimised, and subject to this the number n of vertices of G is minimised. We can assume that G is connected (or else we consider each component separately). Let \tilde{G} be the simple graph underlying G . By the minimality of g , \tilde{G} has no embedding on a surface of smaller Euler genus, and thus has a cellular embedding on a surface Σ of Euler genus g . It follows from Euler's Formula that \tilde{G} has $\tilde{m} \leq 3n + 3g - 6$ edges, and so G has $m \leq \mu(3n + 3g - 6)$ edges. Let $k = \max\{\lceil C_1\sqrt{\mu}(g+1)^{1/4} \log(\mu^2(g+2)) \rceil, 8\mu\}$ (for Theorem 1) or $k = \lceil C_3\sqrt{\mu}(g+1)^{1/4} \rceil$ (for Theorem 12), and assume that each vertex has k local colours. If G has a vertex v of degree less than k , then remove v . By

the minimality of n , we can colour $G - v$ and then find a suitable colour for v (since v has at least k local colours and fewer than k neighbours in G). Thus, we can assume that G has minimum degree at least k , and thus at least $\frac{1}{2}nk$ edges. Consequently, $nk/2 \leq \mu(3n + 3g - 6)$.

For Theorem 1, since $k/(2\mu) \geq 4$, we have $n \leq 3g - 6$ and $m \leq \mu(12g - 24)$. It then follows from Theorem 4 and a large enough choice of constant that G has conflict choosability strictly smaller than k , which is a contradiction.

For Theorem 12, observe that not only G , but also \tilde{G} has minimum degree at least k . Thus $nk/2 \leq \tilde{m} \leq 3n + 3g - 6$. For a large enough choice of constant C_3 , $k \geq 8$ and thus $n \leq 3g - 6$ and $m \leq \mu(12g - 24)$. It then follows from Theorem 11 and a large enough choice of constant that G has conflict choosability strictly smaller than k , which is a contradiction. \square

To prove Theorem 4, we require the following slightly technical result.

Lemma 13. *For any $d \geq 2^{23}$, let $G = (V, E)$ be a multigraph with a vertex partition $V = A \cup B$ such that*

- (i) *the induced submultigraph $G[A]$ has maximum degree at most d ,*
- (ii) *all vertices in A have maximum degree at most d^2 in G , and*
- (iii) *all vertices in B have maximum degree at most d in G .*

There is a constant $C > 0$ such that for any local k -partition $\{L_v\}$ of G , where $k \geq C\sqrt{d} \log d$, there is a colouring $c : A \rightarrow [k]$ such that c is a conflict $\{L_v\}$ -colouring of $G[A]$ and no vertex $x \in B$ has more than \sqrt{d} incident edges $e = xy$, $y \in A$, such that $c(y) = L_y(e)$.

Proof of Theorem 4. Let $G = (V, E)$ be a multigraph with m edges and maximum multiplicity μ . Let A be the set of vertices of degree at least $\sqrt{2\mu m}$ in G and let $B = V \setminus A$. Since $|E| = m$, A has at most $2m/\sqrt{2\mu m} = \sqrt{2m/\mu}$ vertices, and thus $G[A]$ has maximum degree at most $\mu\sqrt{2m/\mu} = \sqrt{2\mu m}$. It follows from the definition of B that $G[B]$ also has maximum degree at most $\sqrt{2\mu m}$. Note that $\text{ch}_{\leftrightarrow}(G)$ is trivially at most m . So by a large enough fixed choice of C_2 we may assume m is large enough so that the conditions of Lemma 13 are satisfied with $d = \sqrt{2\mu m}$. Let $C > 0$ be the constant associated to the corresponding application of Lemma 13. Let k be an integer at least $\max\{C\sqrt{d} \log d, \lceil \sqrt{e(2d-1)} \rceil + \sqrt{d}\}$ and let $\{L_v\}$ be a local k -partition of G . It follows from an application of Lemma 13 that there is a conflict $\{L_v\}$ -colouring c of $G[A]$. It remains to colour B in such a way that it is compatible with c .

For each vertex $x \in B$, remove from G any edge f incident to x if there exists some incident edge $e = xy$, $y \in A$, such that $c(y) = L_y(e)$ and $L_x(f) = L_x(e)$. We also (locally) remove each of the colours associated to the

edges we removed. By one of the properties of c guaranteed by Lemma 13, this process removes at most \sqrt{d} of the colours incident to each vertex in B . By arbitrarily deleting any excess local colours as well as any of the incident edges with those colours, then relabelling colours, we are left with a local k' -partition $\{L'_v\}$ of a submultigraph of $G[B]$ with maximum degree at most d , where $k' = \lceil \sqrt{e(2d-1)} \rceil$. By Proposition 5, this submultigraph admits a conflict $\{L'_v\}$ -colouring c' . The colour and edge removal process we performed ensures that, by reversing the relabelling, c' corresponds to a conflict $\{L_v\}$ -colouring of $G[B]$ that combines with c to produce a conflict $\{L_v\}$ -colouring of all of G . \square

It remains only to prove Lemma 13. This is done with an application of the General Local Lemma (see Section 1.1).

Proof of Lemma 13. Let $k = \lceil C\sqrt{d} \log d \rceil$ where C is some constant large enough to guarantee certain properties as specified later in the proof. Let $\{L_v\}$ be a local k -partition of G .

We must do a pruning operation before proceeding—in fact, this is the crucial step in the proof. By taking C large enough, we may assume for each $v \in A$ and each $i \in [k]$ that the number of edges in $L_v^{-1}(i)$ with its other endpoint also in A is at most \sqrt{d} . (We summarily remove all edges associated to every colour not satisfying the property, and since the maximum degree of $G[A]$ is at most d this removes at most \sqrt{d} of the colours around each vertex in A .)

Let $p = 2^{-4}/\sqrt{d}$. Consider a random selection of colours where each of the $|V|k$ local colours is selected according to an independent Bernoulli trial of probability p . With an eye to applying the General Local Lemma, let us define three types of (bad) events.

- I For a vertex $x \in A$, none of the colours around x is selected.
- II For an edge $e = xy \in E$ with $x, y \in A$, $L_x(e)$ and $L_y(e)$ are both selected.
- III For a vertex $x \in B$, there are more than \sqrt{d} edges $e = xy$, $y \in A$, for which $L_y(e)$ is selected.

If we obtain a selection for which none of the above events occurs, then we are done. This is because the deselection of a colour does not introduce any new event of Type II or III. So we can arbitrarily deselect all but one of the colours around each vertex, and the remaining selection induces the desired colouring c , thanks to the fact that no events of Type II or III hold.

For each $x \in A$, the probability of a Type I event is $\mathbb{P}(\text{Bin}(k, p) = 0) = (1-p)^k \leq \exp(-pk) \leq \exp(-2^{-4}C \log d) < 2^{-8}/d$ if C is chosen large enough. For each edge $e = xy \in E$, $x, y \in A$, the probability of a Type II event is $p^2 = 2^{-8}/d$. For each vertex $x \in B$, the probability of a Type III

event is at most $\mathbb{P}(\text{Bin}(d, p) > \sqrt{d}) \leq \mathbb{P}(|\text{Bin}(d, p) - dp| > \sqrt{3} \cdot 2^{-4}\sqrt{d}) < 2 \exp(-2^{-4}\sqrt{d})$ by the Chernoff Bound.

The choice to generate the random colouring according to independent Bernoulli trials rather than a uniform colour per vertex (as in Proposition 5) is important for us in establishing the following bounds on dependence between bad events, especially for Type III events. Each Type I event is mutually independent of all but at most d events of Type I, at most d events of Type II, and at most d^2 events of Type III. Each Type II event is mutually independent of all but at most 2 events of Type I, at most $2d - 1$ events of Type II, and at most $2d^2$ events of Type III. By the pruning operation we did at the beginning, each Type III event is mutually independent of all but at most d events of Type I, at most $d^{3/2}$ events of Type II, and at most $d^{3/2}$ events of Type III. (To be more explicit, each Type III event is determined by up to d independent Bernoulli random variables, each of which corresponds to a local colour of a neighbour. Thanks to the pruning, the number of Type II events, say, that also use this randomness is at most $d^{3/2}$. The Type III event is mutually independent of all other Type II events.)

We associate weight $x_i = 2^{-7}/d$ to each event i of Type I or II, and weight $x_i = 2 \exp(-2^{-6}\sqrt{d})$ to each event i of Type III. By the considerations above, the General Local Lemma guarantees the desired selection of colours with positive probability, provided the following three inequalities hold (where we repeatedly used that $\exp(-x - x^2) \leq 1 - x \leq \exp(-x)$ if $0 < x < 0.69$):

$$1/2 \leq \exp\left(-\frac{1}{2^7} - \frac{1}{2^{14}d} - \frac{1}{2^7} - \frac{1}{2^{14}d} - \frac{2d^2}{\exp(\frac{\sqrt{d}}{2^6})} - \frac{4d^2}{\exp(\frac{\sqrt{d}}{2^5})}\right); \quad (\text{I})$$

$$1/2 \leq \left(1 - \frac{1}{2^7d}\right)^2 \exp\left(-\frac{1}{2^6} - \frac{1}{2^{13}d} - \frac{4d^2}{\exp(\frac{\sqrt{d}}{2^6})} - \frac{8d^2}{\exp(\frac{\sqrt{d}}{2^5})}\right); \quad (\text{II})$$

$$-\frac{\sqrt{d}}{2^4} + \frac{\sqrt{d}}{2^6} \leq -\frac{1}{2^7} - \frac{1}{2^{14}d} - \frac{\sqrt{d}}{2^7} - \frac{1}{2^{14}\sqrt{d}} - \frac{2d^{3/2}}{\exp(\frac{\sqrt{d}}{2^6})} - \frac{4d^{3/2}}{\exp(\frac{\sqrt{d}}{2^5})}. \quad (\text{III})$$

It is straightforward to check that $d \geq 2^{23}$ suffices. \square

The above proof can be straightforwardly adapted for the same upper bound (with a larger constant C) on a stronger type of conflict choosability where additionally we must assign $\Omega(\log d)$ distinct colours per vertex. What this then directly implies is that, for any simple graph G that is embeddable on a surface of Euler genus g , the conflict choosability is $O(g^{1/4}(\log g)^{5/4})$ even if we allow $O(\log g)$ conflicts per edge and demand $\Omega(\log d)$ distinct colours per vertex.

References

- [1] N. Alon and M. Tarsi. Colorings and orientations of graphs. *Combinatorica*, 12(2):125–134, 1992.
- [2] A. Bernshteyn. The asymptotic behavior of the correspondence chromatic number. *Discrete Mathematics*, 339(11):2680–2692, 2016.
- [3] A. Bernshteyn. The local cut lemma. *European Journal of Combinatorics*, 63:95–114, 2017.
- [4] Z. Dvořák and L. Postle. Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8. *Journal of Combinatorial Theory, Series B*, 2017.
- [5] P. Erdős, A. L. Rubin, and H. Taylor. Choosability in graphs. In *Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, Calif., 1979)*, Congress. Numer., XXVI, pages 125–157, Winnipeg, Man., 1980. Utilitas Math.
- [6] L. Esperet, R. J. Kang, and S. Thomassé. Separation choosability and dense bipartite induced subgraphs. *ArXiv e-prints*, Feb. 2018.
- [7] P. Fraigniaud, M. Heinrich, and A. Kosowski. Local conflict coloring. In *IEEE 57th Annual Symposium on Foundations of Computer Science, FOCS 2016, 9-11 October 2016, Hyatt Regency, New Brunswick, New Jersey, USA*, pages 625–634, 2016.
- [8] Z. Füredi, A. Kostochka, and M. Kumbhat. Choosability with separation of complete multipartite graphs and hypergraphs. *J. Graph Theory*, 76(2):129–137, 2014.
- [9] P. J. Heawood. Map colour theorem. *Quart. J. Math.*, 24:332–338, 1890.
- [10] P. Hell and X. Zhu. On the adaptable chromatic number of graphs. *European J. Combin.*, 29(4):912–921, 2008.
- [11] A. V. Kostochka and X. Zhu. Adapted list coloring of graphs and hypergraphs. *SIAM J. Discrete Math.*, 22(1):398–408, 2008.
- [12] J. Kratochvíl, Z. Tuza, and M. Voigt. Brooks-type theorems for choosability with separation. *J. Graph Theory*, 27(1):43–49, 1998.
- [13] J. Kratochvíl, Z. Tuza, and M. Voigt. Complexity of choosing subsets from color sets. *Discrete Math.*, 191(1-3):139–148, 1998. Graph theory (Elgersburg, 1996).

- [14] M. Molloy and B. Reed. *Graph colouring and the probabilistic method*, volume 23 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2002.
- [15] V. G. Vizing. Coloring the vertices of a graph in prescribed colors. *Diskret. Analiz*, 29 Metody Diskret. Anal. v Teorii Kodov i Shem:3–10, 101, 1976.
- [16] R. Škrekovski. A note on choosability with separation for planar graphs. *Ars Combin.*, 58:169–174, 2001.