A complete bipartite graph without properly colored cycles of length four

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Abstract

A subgraph of an edge-colored graph is said to be *properly colored*, or shortly *PC*, if any two adjacent edges have different colors. Fujita, Li and Zhang gave a decomposition theorem for edge-colorings of complete bipartite graphs without PC C_4. However, their decomposition just focus on only three colors, and does not deal with all of the colors. In this paper, we give a new and detailed decomposition theorem for edge-colorings of complete bipartite graphs without PC C_4. Our decomposition gives a corollary on the existence of a monochromatic star with almost sharp bound.

Keywords: Properly colored cycle, complete bipartite graph, minimum color degree,

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1 Introduction

In this paper, we consider edge-colorings of complete graphs and complete bipartite graphs. Note that such an edge-coloring is not necessarily proper. See [16] for any notation not defined here. The complete graph of order n is denoted by K_n, and the complete bipartite graph with bipartitions of size n and m, respectively, is denoted by $K_{n,m}$. We identify an edge-coloring of a graph with an edge-colored graph. A subgraph of an edge-coloring is said to be *properly colored*, or shortly *PC*, if any two adjacent edges have different colors.

The contribution of this paper is (1) to give a new and detailed decomposition theorem for edge-colorings of complete bipartite graphs without PC C_4, and then (2) to show some application for the existence of a large monochromatic star in edge-colorings of complete bipartite graphs without PC C_4.

1.1 Complete or complete bipartite graphs without certain PC subgraphs

Complete graphs or complete bipartite graphs without certain PC subgraphs have been shown to have particular properties. The most famous one is the characterization due to Gallai [11] for the case of no PC C_3 (or equivalently rainbow C_3, see Section 1.3). Indeed, the characterization gives a special partition of the vertices, which is called a *Gallai partition*.

Theorem 1 (Gallai [11]) *For $n \geq 2$, if G is an edge-coloring of K_n without PC C_3, then there exist at most two colors i and j and a partition of $V(G)$ into at least two parts such that for any two different parts, all edges between them are colored by the same color that is i or j.*

Recently Fujita, Li and Zhang gave a characterization of edge-colorings of complete bipartite graphs without PC C_4. (Because of presentation reason, we change the indices from those in [10].) Since C_4 is a shortest possible cycle in a bipartite graph, in some sense, this is a bipartite analogous to Theorem 1. Before stating their result, we define some terminology.

The *color degree* of a vertex x in an edge-coloring G, denoted by $d^c_G(x)$, is the number of distinct colors appearing in edges incident with x. The *minimum color degree of* G is the minimum of $d^c_G(x)$ over all vertices x in G. For a disjoint vertex subset X and Y of an edge-coloring G of a graph, we denote by $C(X, Y)$ the set of colors that are used for some edge between X and Y. If X consists of only one vertex x, then we abbreviate $C(x, Y)$ instead of $C(\{x\}, Y)$. Similarly, we write $C(X, y)$ when $Y = \{y\}$.

Theorem 2 (Fujita, Li and Zhang [10]) *Let G be an edge-coloring of a complete bipartite graph without PC C_4, and let X and Y be the bipartitions of G. Suppose that the minimum color degree is at least two. Then, for some set of three colors,* *say* $1,2,3$ *by symmetry,* X *and* Y *can be partitioned into* $\{A_1, A_2, A_3, X_1, X_2, X_3, \widehat{X}\}$ *and* $\{B_1, B_2, B_3, Y_1, Y_2, Y_3, \widehat{Y}\}$*, respectively, satisfying the following conditions for any* $i \in \{1,2,3\}$*. (The indices are taken modulo 3.)*

(P1) $A_i, B_i \neq \emptyset$;

(P2) $C(A_i, B_{i-1} \cup B_i) = \{i\}$ *and* $C(A_i, B_{i+1}) = \{i+1\}$;

(P3) $C(A_i, Y_{i-1} \cup Y_i \cup \widehat{Y}) \subseteq \{i\}$ *and* $C(A_i, y_{i+1}) = \{i, i+1\}$ *for each vertex* $y_{i+1} \in Y_{i+1}$;

(P4) $C(X_{i-1} \cup X_i \cup \widehat{X}, B_i) \subseteq \{i\}$ *and* $C(x_{i+1}, B_i) = \{i, i+1\}$ *for each vertex* $x_{i+1} \in X_{i+1}$.

However, they focus on only three colors and do not discuss colors on the edges between $X_1 \cup X_2 \cup X_3 \cup \widehat{X}$ and $Y_1 \cup Y_2 \cup Y_3 \cup \widehat{Y}$. In order to deal with them, one might think that after getting a partition as in Theorem 2, we can apply Theorem 2 again to $G - (A_1 \cup A_2 \cup A_3 \cup B_1 \cup B_2 \cup B_3)$. This gives us a recursive structure, which expresses the colors of all edges. However, the same color may appear several times, and the structure would be complicated. In fact, while Fujita et al. [10] obtained a corollary on a large monochromatic star, their bound was not best possible when there are more than three colors. We will discuss this in Section 5.

Considering the above situation, we give a new and simpler partition considering all colors in this paper. Before stating our main theorem, we give some definition.

Let G be a graph and let k be the number of colors used in G. A mapping $\varphi : V(G) \to \{1, \dots, k\}$ is said to be *good* if for $i, j \in \{1, \dots k\}$, each edge between $\varphi^{-1}(i)$ and $\varphi^{-1}(j)$ is colored by either i or j. Notice that when G is an edge-coloring of a complete bipartite graph, then $\varphi^{-1}(i)$ induces a monochromatic complete bipartite graph of color i. Those sets $\varphi^{-1}(i)$ for $i \in \{1, \dots, k\}$ form indeed a partition of the vertices.

Theorem 3 *Let* G *be an edge-coloring of a complete bipartite graph without PC* C_4*, and let* X *and* Y *be the bipartitions of* G*. Then* G *admits a good mapping from* $V(G)$ *to* $\{1, \dots, k\}$*. Furthermore, if the minimum color degree is at least two, then the following holds. For some set of three colors, say* $1,2,3$ *by symmetry, there are six vertices* $\widetilde{x}_i \in \varphi^{-1}(i) \cap X$ *and* $\widetilde{y}_i \in \varphi^{-1}(i) \cap Y$ *satisfying the following condition for* $i \in \{1,2,3\}$*; (The indices are taken modulo 3.)*

$$C\Big(\widetilde{x}_i, Y - \varphi^{-1}(i+1)\Big) = \{i\} \quad \textit{and} \quad C\Big(X - \varphi^{-1}(i+1), \widetilde{y}_i\Big) = \{i\}.$$

After preliminary section, in Section 3, we prove that Theorem 3 implies Theorem 2. The proof of Theorem 3 will appear in Section 4. Then we will discuss a corollary on a monochromatic star in Section 5.

1.2 Color degree condition

Erdős and Tuza [7] showed that using a main idea of Theorem 1, an edge-coloring of K_n admits a PC C_3 if the minimum color degree is at least $\log_2 n + O(1)$. As pointed out by Axenovich, Jiang and Tuza in [4, Proposition 4.3], the bound is best possible, except for the $O(1)$ part. They also proved the following for the existence of a PC C_4.

Theorem 4 (Axenovich, Jiang and Tuza [4, Theorem 4.10]) Let G be an edge-coloring of the complete graph K_n. If the minimum color degree is at least three, then G contains a PC C_4.

As in [10], Theorem 2 (and also Theorem 3) directly gives a bipartite analogue of Theorem 4.

Theorem 5 (Fujita, Li and Zhang, [10]) *Let G be an edge-coloring of a complete bipartite graph. If the minimum color degree of G is at least three, then G admits a PC C_4.*

It was mentioned in [10] that the condition "the minimum color degree of G is at least three" in Theorem 5 is best possible. However, they did not explicitly mention that they need at least three colors in their example. In fact, we can improve this when there are only two colors, see Lemma 7 in Section 2.

We introduce more results on PC cycles. A PC cycle in an edge-coloring of (non-complete) graphs with color degree conditions were considered in several papers [1, 14, 15]. Chen and Daykin [5] showed that for each integers t, n, ℓ with $t \geq 2$, $n \geq 25t$ and $n \geq \ell \geq 2$, an edge-coloring of the complete bipartite graph $K_{n,n}$ admits a PC $C_{2\ell}$ if no vertex is incident to t edges of the same color. More related results can be found in surveys [8, 9].

1.3 Rainbow cycles

A subgraph of an edge-coloring is said to be *rainbow* if no two edges in the subgraph have the same color. Note that a rainbow C_3 is nothing but a PC C_3, but for many other graphs, such as C_4, those two concepts do not coincide. In fact, the properties of graphs without rainbow C_4 are very different from those without PC C_4.

For example, in contrast to Theorem 4, Axenovich, Jiang and Tuza proved that an edge-coloring of the complete graph K_n contains a rainbow C_4 if the minimum color degree is at least $4n^{2/3}$ ([4, Theorem 4.7]), and there is an edge-coloring of the complete graph K_n of minimum color degree $\log_2 n - 1$, but without rainbow C_4 ([4, Proposition 4.3]). In some sense, this gap shows the difficulty of dealing with a rainbow C_4, compared with a PC C_4. It might be interesting to consider the counterpart of Theorem 5 as follows.

Problem 6 *Determine the sharp function $f(n)$ such that an edge-coloring of the complete bipartite graph $K_{n,n}$ contains a rainbow C_4 if the minimum color degree at least $f(n)$.*

The proof of [4, Theorem 4.7] can be adapted to show that an edge-coloring of the complete bipartite graph $K_{n,m}$ with $n \geq m$ contains a rainbow C_4 if the minimum color degree is at least $4(nm)^{1/3}$. This directly gives $f(n) = O(n^{2/3})$. On the other hand, the same construction in the proof of [4, Proposition 4.3] shows $f(n) = \Omega(\log_2 n)$. There is still a huge gap between known upper bound and lower bound of $f(n)$, similarly to the case of the complete graph.

Several researchers have been interested in the condition on the number of colors that forces the existence of a rainbow C_4, which is called an *anti-Ramsey theory.* Alon [2] proved that "an edge-coloring of K_n with at least $\lfloor 4n/3 \rfloor$ colors admits a rainbow C_4", which was conjectured by Erdős, Simonovits and Sós [6]. (Note that their conjecture is also for rainbow C_k ($k \geq 4$), which was proved by Montellano-Ballesteros and Neumann-Lara [12].) Axenovich, Jiang, and Kündgen [3], and independently Mubayi and West [13] considered the bipartite analogous, showing that "an edge-coloring of $K_{n,m}$ with at least $m + n$ colors admits a rainbow C_4".

2 Preliminaries

Let G be an edge-coloring of a graph. Recall that for a disjoint vertex subset X and Y of G, we denote by $C(X, Y)$ the set of colors that are used for some edge between X and Y. With abuse of notation, for an edge e of G, we denote the color of e by $c(e)$. The *color sequence* of a cycle or a path in an edge-coloring is a sequence of colors of the edges along its order.

The following lemma is used several times in our proofs. It gives a counterpart of the "best possibility" of Theorem 5 for the case where only two colors appear. (Note that the contraposition of Lemma 7 states that "every edge-coloring of a complete bipartite graph with two colors admits a PC C_4 if the minimum color degree is two".)

Lemma 7 *Let G be an edge-coloring of a complete bipartite graph with at most two colors, say colors 1 and 2, and let X and Y be bipartitions of G. If G admits no PC C_4, then either there exists a vertex x in X such that $c(x, Y) = \{1\}$ or there exists a vertex y in Y such that $c(X, y) = \{2\}$.*

Proof of Lemma 7. Suppose that every vertex in Y is incident with an edge of color 1. Let $x \in X$ be such that the number of edges incident with x of color 1 is maximum over all vertices in X. Suppose further that there exists an edge xy in G with $c(xy) = 2$. By the assumption, y is incident with an edge of color 1, say $x'y$. Then if $c(xy') = 1$ for $y' \in Y$, then $c(x'y') = 1$. Otherwise the cycle $xyx'y'x$ has the color sequence 2121, a contradiction. However, this implies that x' is incident with more edges of color 1 than x, contradicting the choice of x. Therefore, all edges incident with x are of color 1, and we are done. $\square$

3 Proof of Theorem 2 using Theorem 3

Let G be an edge-coloring of a complete bipartite graph without PC C_4, and let X and Y be the bipartitions of G. Suppose that the minimum color degree is at least two. Then by Theorem 3, G admits a good mapping φ from $V(G)$ to $\{1, \dots, k\}$, and for some set of three colors, say $1, 2, 3$ by symmetry, there are six vertices $\widetilde{x}_i \in \varphi^{-1}(i) \cap X$ and $\widetilde{y}_i \in \varphi^{-1}(i) \cap Y$ for $i \in \{1, 2, 3\}$ as in Theorem 3. Let W be the set of vertices of color degree exactly two, that is, $W = \{x \in V(G) : d^c_G(x) = 2\}$, and for $i \in \{1, 2, \dots, k\}$, let

$$A_i = \varphi^{-1}(i) \cap W \cap X \quad \text{and} \quad B_i = \varphi^{-1}(i) \cap W \cap Y.$$

By the second part of Theorem 3, each vertex x_j in X_j for $j \in \{4, \dots, k\}$ is incident to $\widetilde{y}_i$ for $i \in \{1, 2, 3\}$ with $c(x_j \widetilde{y}_i) = i$, and hence $x_j \notin W$. This implies that $A_j = \emptyset$ for $j \in \{4, \dots, k\}$, and by symmetry, $B_j = \emptyset$. We remark that $\{A_1, A_2, A_3, B_1, B_2, B_3\}$ is a partition of W and all other vertices have color degree at least three.

We take $\widehat{X}$ and $\widehat{Y}$ as large as possible subject to the property that

- any vertex $x \in \widehat{X}$ satisfies $C(x, B_i) = \{i\}$ for $i \in \{1, 2, 3\}$, and
- any vertex $y \in \widehat{Y}$ satisfies $C(y, A_i) = \{i\}$ for $i \in \{1, 2, 3\}$.

For any $i \in \{1, 2, 3\}$, let

$$X_i = \varphi^{-1}(i) \cap X - A_i - \widehat{X} \quad \text{and} \quad Y_i = \varphi^{-1}(i) \cap Y - B_i - \widehat{Y}.$$

Then we will show that those sets satisfy conditions (P1)–(P4). By symmetry, it suffices to check only the case $i = 1$. Furthermore, by the symmetry between X and Y, conditions (P3) and (P4) are symmetric, and hence we will omit to check condition (P4).

Since φ is a good mapping, $C\big(\widetilde{x}_1, \varphi^{-1}(2) \cap Y\big) \subseteq \{1, 2\}$. Thus, the property $C\big(\widetilde{x}_1, Y - \varphi^{-1}(2)\big) = \{1\}$ implies that $\widetilde{x}_1 \in A_1$. Similarly, we have $\widetilde{y}_1 \in B_1$, and hence condition (P1) is satisfied.

Since $A_1, B_1, Y_1 \subseteq \varphi^{-1}(1)$ and φ is a good mapping, we have $C(A_1, B_1 \cup Y_1) = \{1\}$. By the property $C\big(X - \varphi^{-1}(3), \widetilde{y}_2\big) = \{2\}$ and $A_1 \subseteq X - \varphi^{-1}(3)$, we have $C(A_1, \widetilde{y}_2) = \{2\}$. Thus, since each vertex $a_1 \in A_1$ has the color degree exactly two, we see $C(A_1, Y) = \{1, 2\}$ and $C(A_1, B_3 \cup Y_3) = \{1\}$. Note that $C(A_1, \widehat{Y}) \subseteq \{1\}$ by the definition of $\widehat{Y}$. Thus, the first part of conditions (P2) and (P3) holds.

Suppose that $C(A_1, B_2) \neq \{2\}$. Then there exists an edge $a_1 b_2$ with $a_1 \in A_1$, $b_2 \in B_2$ and $c(a_1 b_2) \neq 2$. Since $A_1 \subseteq \varphi^{-1}(1)$, we have $c(a_1 b_2) = 1$. Since the first part of condition (P2) for $i = 2$ and 3 implies $C(A_2, B_2) = \{2\}$ and $C(A_3, B_2) = \{3\}$, the vertex b_2 has color degree at least three, contradicting $b_2 \in B_2 \subseteq W$. Therefore, condition (P2) is also satisfied.

Suppose that the second part of condition (P3) does not hold, that is, there is a vertex $y_2 \in Y_2$ such that $C(A_1, y_2) \neq \{1, 2\}$. Since $A_1 \subseteq \varphi^{-1}(1)$ and $y_2 \in Y_2 \subseteq \varphi^{-1}(2)$, either $C(A_1, y_2) = \{1\}$ or $C(A_1, y_2) = \{2\}$. Suppose $C(A_1, y_2) = \{1\}$. By the first part of condition (P3) for $i = 2$ and 3, we have $C(A_2, y_2) = \{2\}$ and $C(A_3, y_2) = \{3\}$.

This shows that moving y_2 from Y_2 to $\widehat{Y}$ leads to a contradiction with the maximality of $\widehat{Y}$. Therefore, we may assume $C(A_1, y_2) = \{2\}$. Recall that $C(A_2, y_2) = \{2\}$ and $C(A_3, y_2) = \{3\}$. Since the color degree of y_2 is at least three, there is a vertex x in $X - \big(A_1 \cup A_2 \cup A_3\big)$ with $c(xy_2) \notin \{2, 3\}$. Since $y_2 \in Y_2 \subseteq \varphi^{-1}(2)$, $X_2 \subseteq \varphi^{-1}(2)$ and $X_3 \subseteq \varphi^{-1}(3)$, we see $x \notin X_2 \cup X_3$. If $x \in \widehat{X}$, then the definition of $\widehat{X}$ implies that the cycle $a_1y_2xb_3a_1$ is a PC C_4 with color sequence $2j31$, where $a_1 \in A_1$, $b_3 \in B_3$ and $j = c(xy_2)$, a contradiction. Thus, we further have $x \notin \widehat{X}$. Therefore, it remains only the case $x \in X_1$. In this case, $c(xy_2) = 1$.

If $c(xb_3) = 3$ for some $b_3 \in B_3$, then the cycle $a_1y_2xb_3a_1$ is a PC C_4 with color sequence 2131, where $a_1 \in A_1$, a contradiction. Thus, we have $C(x, B_3) = \{1\}$. Note that $C(x, B_1) = \{1\}$ and $C(x, B_2) = \{2\}$ by the first part of condition (P3) for $i = 1$ and 2. Since the color degree of x is at least three, there is a vertex y in $Y - \big(B_1 \cup B_2 \cup B_3\big)$ with $c(xy) \notin \{1, 2\}$. However, we obtain a contradiction, as follows;

- Since $x \in X_1$ and φ is a good mapping, we have $y \notin Y_1 \cup Y_2$.
- If $y \in Y_3 \cup \widehat{Y}$, then the cycle $a_1y_2xya_1$ is a PC C_4 with color sequence $21j1$, where $a_1 \in A_1$ and $j = c(xy) \notin \{1, 2\}$.

This contradiction shows the second part of condition (P3), which completes the proof of Theorem 2. □

4 Proof of Theorem 3

4.1 Proof of the first part

We first prove the existence of a good mapping by induction on the number of vertices. If one of the bipartitions of G is empty, then the assertion trivially holds. Thus, we may assume that both bipartitions have at least one vertex.

Suppose that G does not admit a good mapping. Let X and Y be the bipartitions of G, let $x_0 \in X$, let $G' = G - x_0$ and let $X' = X - \{x_0\}$. By the induction hypothesis, G' admits a good mapping $\varphi : V(G') \to \{1, \ldots, k\}$. We call a vertex y in Y *irregular* if $c(x_0y) \neq \varphi(y)$. We take a good mapping φ of G' so that

(X1) the number of irregular vertices is as small as possible.

We now focus on the properties of irregular vertices, showing the following three claims.

Claim 1 *For any irregular vertex y, there exists a vertex $x \in X'$ such that $\varphi(x) \neq \varphi(y) = c(xy)$.*

Proof. Suppose, to the contrary, that there exists an irregular vertex y_0 such that for every $x \in X'$ with $\varphi(x) \neq \varphi(y_0)$, we have $c(xy_0) \neq \varphi(y_0)$. Since φ is a good mapping, we have $c(xy_0) = \varphi(x)$ for such a vertex $x \in X'$. In this case, let $\varphi'(y_0) = c(x_0y_0)$ and

$\varphi'(v) = \varphi(v)$ for all $v \in X' \cup Y - \{y_0\}$. Then we see that φ' is a good mapping of G' for which y_0 is not irregular, which contradicts condition (X1). ■

Claim 2 *There are two irregular vertices y_1 and y_2 such that $c(x_0y_1) \neq c(x_0y_2)$.*

Suppose that there is a color i such that all irregular vertices y satisfy $c(x_0y) = i$. Then setting $\varphi(x_0) = i$, we can obtain a good mapping of G, a contradiction. Therefore, there are two irregular vertices y_1 and y_2 such that $c(x_0y_1) \neq c(x_0y_2)$. ■

Claim 3 *For each pair of two irregular vertices y and y', we have $\varphi(y) = \varphi(y')$.*

Proof. Suppose that there are two irregular vertices y and y' with $\varphi(y) \neq \varphi(y')$.

If $c(x_0y) = c(x_0y')$, then for irregular vertices y_1 and y_2 as in Claim 2, we have $c(x_0y) = c(x_0y') \neq c(x_0y_i)$ for some $i \in \{1, 2\}$. Furthermore, either $\varphi(y) \neq \varphi(y_i)$ or $\varphi(y') \neq \varphi(y_i)$. Thus, by changing the name of y_i with y' or y, we may assume that $c(x_0y) \neq c(x_0y')$.

By symmetry, we may also assume that $\varphi(y) = 1$ and $\varphi(y') = 2$. Let i and j be the colors with $c(x_0y) = i$ and $c(x_0y') = j$. Note that $1 \neq i \neq j \neq 2$. By Claim 1, there exist vertices $x, x' \in X'$ such that $\varphi(x) \neq \varphi(y) = 1 = c(xy)$ and $\varphi(x') \neq \varphi(y') = 2 = c(x'y')$. If $x = x'$, then the cycle $x_0yxy'x_0$ is a PC C_4 with color sequence $i12j$, a contradiction. Thus, we have $x \neq x'$.

Let $p = \varphi(x)$. Since φ is a good mapping and $\varphi(y') = 2$, the color $c(xy')$ is either p or 2. Then consider the cycle $x_0yxy'x_0$ whose color sequence is either $i1pj$ or $i12j$. Since it is not a PC C_4, we must have $c(xy') = p = \varphi(x) = j$. By the same argument, we also obtain $c(x'y) = \varphi(x') = i$. Therefore, the cycle $xyx'y'x$ is a PC C_4 with color sequence $1i2j$, a contradiction. ■

By Claim 3 and symmetry, we may assume that $\varphi(y) = 1$ for each irregular vertex y. Let $X'_1 = \varphi^{-1}(1) \cap X'$. We now take a good mapping φ of G' so that

(X2) X'_1 is as large as possible, subject to condition (X1).

Consider the complete bipartite subgraph induced by $X' - X'_1$ and S, where S is the set of all irregular vertices. By Lemma 7 with regarding the colors $2, \ldots, k$ as one color, either there exists a vertex $\widetilde{x}$ in $X' - X'_1$ such that $C(\widetilde{x}, S) = \{1\}$ or there exists a vertex $\widetilde{y}$ in S such that $1 \notin C(X' - X'_1, \widetilde{y})$. By Claim 1, the latter cannot occur. Therefore, there exists a vertex $\widetilde{x}$ in $X' - X'_1$ such that $c(\widetilde{x}y) = 1$ for all irregular vertices y.

If $c(\widetilde{x}y) = \varphi(y)$ for all vertices $y \in Y - S$, then the mapping φ' is also good, where $\varphi'(\widetilde{x}) = 1$ and $\varphi'(v) = \varphi(v)$ for every $v \in X' \cup Y - \{\widetilde{x}\}$, which contradicts condition (X2). Therefore, there exists a vertex $y_0 \in Y - S$ such that $c(\widetilde{x}y_0) \neq \varphi(y_0)$. Since φ is a good mapping, we have $c(\widetilde{x}y_0) = \varphi(\widetilde{x}) \neq 1$. Since y_0 is not irregular, we have $c(x_0y_0) = \varphi(y_0) \neq \varphi(\widetilde{x})$. By Claim 2, there are two irregular vertices y_1 and y_2 such that $c(x_0y_1) \neq c(x_0y_2)$. By symmetry, we may assume that $c(x_0y_1) \neq \varphi(y_0)$. Since

y_1 is irregular, we have $c(x_0y_1) \neq 1$. Then the cycle $x_0y_1\widetilde{x}y_0x_0$ has the color sequence $p1\varphi(\widetilde{x})\varphi(y_0)$, where $p = c(x_0y_1)$, a contradiction. This proves the first part of the statement.

4.2 Proof of the second part

We now have that G admits a good mapping from $V(G)$ to $\{1, \ldots, k\}$, where k is the number of colors. We may assume that all of the k colors appear in G. For $i \in \{1, \ldots, k\}$, let $X_i = X \cap \varphi^{-1}(i)$ and $Y_i = Y \cap \varphi^{-1}(i)$. Notice that X_i or Y_i may possibly be an empty set, but at least one of them is non-empty. Assume that all vertices have the color degree at least two.

Now we construct a directed bipartite graph H as follows. The bipartition of H, denoted by C_x and C_y, are copies of the color set $\{1, \ldots, k\}$, where we let $C_x = \{1_x, \ldots, k_x\}$, $C_y = \{1_y, \ldots, k_y\}$ and i_x and i_y both correspond to the color i. Let $i_x \in C_x$ and $j_y \in C_y$. If either $i = j$ or $X_i = Y_j = \emptyset$, then we do not put an edge between i_x and j_y in H. Suppose that $i \neq j$ and $X_i \cup Y_j \neq \emptyset$. Since φ is a good mapping, the subgraph of G induced by $X_i \cup Y_j$ is a complete bipartite graph with at most two colors, which are i and j. Then it follows from Lemma 7 that either

(i) there exists a vertex x in X_i such that $C(x, Y_j) = \{i\}$ or

(ii) there exists a vertex y in Y_j such that $C(X_i, y) = \{j\}$.

Since $i \neq j$, (i) and (ii) do not occur at that same time. Note that when $X_i \neq \emptyset$ and $Y_j = \emptyset$, then (i) must occur, and when $X_i = \emptyset$ and $Y_j \neq \emptyset$, then (ii) must occur. If (i) occurs, then we put a directed edge in H from i_x to j_y; otherwise, we put a directed edge with opposite direction. By the definition, if there is an edge out-going from the vertex i_x (resp. j_y) in H, then $X_i \neq \emptyset$ (resp. $Y_j \neq \emptyset$).

We give the following two claims.

Claim 4 *The directed bipartite graph H does not contain a directed C_4.*

Proof. Suppose that H contains a directed C_4, say $i_xj_yp_xq_yi_x$, where $i_x, p_x \in C_x$ and $j_y, q_y \in C_y$. By the definition of H, we have $i \neq j \neq p \neq q \neq i$. Since H contains the directed edge from i_x to j_y, there exists a vertex x in X_i such that $C(x, Y_j) = \{i\}$. Similarly, there exist vertices $y \in Y_j$, $x' \in X_p$ and $y' \in Y_q$ such that $C(X_p, y) = \{j\}$, $C(x', Y_q) = \{p\}$ and $C(X_i, y') = \{q\}$. Then the cycle $xyx'y'x$ in G is a PC C_4 with color sequence $ijpq$, a contradiction. ■

Claim 5 *We may assume that each i_x in C_x with $X_i \neq \emptyset$ has the in-degree at least one in H, and so is each j_y in C_y with $Y_j \neq \emptyset$.*

Proof. Suppose that the first statement does not hold. By symmetry, we may assume that $X_1 \neq \emptyset$ and the vertex 1_x in C_x has the in-degree 0 in H. Since $X_1 \neq \emptyset$, all edges in $E_H(1_x, C_y - \{1_y\})$ exist and are outgoing from 1_x.

Consider the complete bipartite subgraph of G induced by $X_1 \cup \overline{Y}$, where $\overline{Y} = \bigcup_{j=2}^{k} Y_j = Y - Y_1$. By Lemma 7 with regarding the colors $2, \ldots, k$ as one color, either there exists a vertex $\widetilde{x}$ in X_1 such that $c(\widetilde{x}, \overline{Y}) = \{1\}$, or there exists a vertex $\overline{y}$ in $\overline{Y}$ such that $1 \notin c(X_1, \overline{y})$. Since the edge $1_x j_y$ in H is directed from 1_x to j_y for all j with $2 \leq j \leq k$, the latter cannot occur, and hence the former holds. Since φ is a good mapping, $c(\widetilde{x}, Y_1) = \{1\}$. Therefore, the vertex $\widetilde{x}$ has color degree one in G. contradicting the condition that minimum color degree is at least two.

The second statement can be shown in a symmetrical way. This completes the proof of Claim 5. ■

Using the claims obtained, we also give the next claim.

Claim 6 *There exists a color $i \in \{1, \ldots, k\}$ such that the in-degree of i_x or i_y is exactly one.*

Proof. Suppose that any vertex in H has the in-degree at least two. Within the proof of this claim, we call this assumption *condition* $(*)$. Now take a vertex in H with maximum out-degree. By symmetry we may assume that it is 1_x in C_x. We denote the set of all out-neighbors of i_x by $N_H^+(i_x)$. Then by condition $(*)$, there exist two vertices i_y and j_y in C_y with directed edges from i_y to 1_x and from j_y to 1_x, respectively. Note that $1 \neq i \neq j \neq 1$, $Y_i \neq \emptyset$ and $Y_j \neq \emptyset$.

By condition $(*)$ for i_y, there exists a vertex p_x with $p \neq j$ and with a directed edge from p_x to i_y. Note again that $p \neq i$ and $X_p \neq \emptyset$. By Claim 4 and by the directed path $p_x i_y 1_x$, for all $r_y \in N_H^+(1_x)$ with $r \neq p$, the edge $p_x r_y$ exists and is directed from p_x to r_y. Thus, $N_H^+(1_x) - \{p_y\} \subseteq N_H^+(p_x)$. Since $i_y \in N_H^+(p_x) - N_H^+(1_x)$, it follows from the choice of i_x that $p_y \in N_H^+(1_x)$ and $N_H^+(1_x) - \{p_y\} = N_H^+(p_x) - \{i_y\}$. In particular, we have $j_y \notin N_H^+(p_x)$, which implies that the edge $j_y p_x$ is directed from j_y to p_x.

Again by condition $(*)$ for j_y, there exists a vertex q_x with $q \neq i$ and with a directed edge from q_x to j_y. Since the edge $p_x j_y$ is directed from j_y to p_x, we have $q \neq p$. By the same argument as in the previous paragraph, we see that the edge $i_y q_x$ is directed from i_y to q_x. However, this implies that $p_x i_y q_x j_y p_x$ is a directed C_4, contradicting Claim 4. ■

By Claim 6 and symmetry, we may assume that the in-degree of 1_x is exactly one. Again by symmetry, we may also assume that

- there are a directed edge from 2_y to 1_x and directed edges from 1_x to j_y in H for any $j \in \{3, \ldots, k\}$.

By Claim 5 to 2_y, there is a direct edge from i_x to 2_y for some $i \in \{3, \ldots, k\}$, say $i = 3$ by symmetry. If there is a direct edge from j_y to 3_x for some j with $j \neq 1, 2, 3$, then $1_x j_y 3_x 2_y 1_x$ is a directed C_4, a contradiction. Thus, by Claim 5 to 3_x,

- there are a directed edge from 1_y to 3_x and directed edges from 3_x to j_y in H for any $j \in \{2, \ldots, k\}$ with $j \neq 3$.

Suppose that there is a directed edge from i_x to 1_y for some $i \neq 2$. Because of the directed path $i_x 1_y 3_x 2_y$, there is not a directed edge from 2_y to i_x, and hence there is a directed edge from i_x to 2_y. Then for any $j \in \{3, \ldots, k\}$, because of the directed path $i_x 2_y 1_x j_y$, there is a directed edge from i_x to j_y. However, these imply that the in-degree of i_x is 0, contradicting Claim 5. Therefore, by Claim 5 to 1_y,

- there are a directed edge from 2_x to 1_y and directed edges from 1_y to i_x in H for any $i \in \{3, \ldots, k\}$.

Suppose that there is a directed edge from j_y to 2_x for some $j \in \{4, \ldots, k\}$. Then because of the directed path $j_y 2_x 1_y 3_x$, there is a directed edge from j_y to 3_x. However, $1_x j_y 3_x 2_y 1_x$ is a directed C_4, a contradiction. Therefore, by Claim 5 to 2_x,

- there are a directed edge from 3_y to 2_x and directed edges from 2_x to j_y in H for any $j \in \{1, \ldots, k\}$ with $j \neq 2, 3$.

Now, by changing the role of 1_x with 2_x and 3_x, we also have

- there are a directed edge from 3_x to 2_y and directed edges from 2_y to i_x in H for any $i \in \{1, \ldots, k\}$ with $i \neq 2, 3$, and

- there are a directed edge from 1_x to 3_y and directed edges from 3_y to i_x in H for any $i \in \{2, \ldots, k\}$ with $i \neq 3$.

Consider the complete bipartite subgraph of G induced by $X_1 \cup \overline{Y_1}$, where $\overline{Y_1} = Y - Y_2$. By Lemma 7 with regarding the colors $3, \ldots, k$ as one color, either there exists a vertex $\widetilde{x}_1$ in X_1 such that $c(\widetilde{x}_1, \overline{Y_1}) = \{1\}$, or there exists a vertex $\overline{y}$ in $\overline{Y_1}$ such that $1 \notin c(X_1, \overline{y})$. If the latter occurs, then it contradicts that the edge $1_x i_y$ in H is directed from 1_x to i_y, where i is a color satisfying $\overline{y} \in Y_i$. Thus, the former holds. By the same way, we can check all required conditions, and we are done. $\square$

5 Monochromatic large star

In this section, we discuss the existence of a large monochromatic star in an edge-coloring of a complete bipartite graph without PC C_4. Fujita, Li and Zhang [10, Theorem 4 (ii) and (iii)] proved the following, using Theorem 2.

Theorem 8 *Let G be an edge-coloring of a complete bipartite graph without a PC C_4. Then G contains a monochromatic star $K_{1,t}$ with $t = \frac{2n}{3}$ where n is the order of the smaller bipartition of G.*

Furthermore, they constructed an edge-coloring of a complete bipartite graph without a PC C_4 nor a monochromatic star $K_{1,t}$ with $t > \frac{2n}{3}$ [10, Remark 2]. However, their example uses only three colors, and it is natural to ask what happens if there are more than three colors. Theorem 3 can answer this question implying the following theorem. As we will show in Proposition 10 later, the conclusion is best possible, except for the constant term, and the constant term cannot be improved by more than $\frac{1}{6}$.

Theorem 9 *Let G be an edge-coloring of a complete bipartite graph without PC C_4, and let k be the number of colors used in G. Then G contains a monochromatic star $K_{1,t}$ with $t = \min\left\{n, \frac{2n}{3} + \frac{k-3}{6}\right\}$ where n is the order of the smaller bipartition of G.*

Proof. Suppose that an edge-coloring G of a complete bipartite graph admits no PC C_4. Let X and Y be the bipartition of G. By symmetry, we may assume that $|X| \geq |Y| = n$. For $i \in \{1, \dots, k\}$, let $X_i = X \cap \varphi^{-1}(i)$ and $Y_i = Y \cap \varphi^{-1}(i)$. When $k \leq 2$, then trivially or by Lemma 7 we are done. Thus, we may assume that $k \geq 3$. Furthermore, we may also assume that the minimum color degree is at least two.

By Theorem 3, G admits a good mapping from $V(G)$ to $\{1, \dots, k\}$ satisfying the conditions in Theorem 3. Let

$$\overline{X_i} \;=\; X - \varphi^{-1}(i+1) \cap X \quad \text{and} \quad \overline{Y_i} \;=\; Y - \varphi^{-1}(i+1) \cap Y$$

for $i \in \{1, 2, 3\}$ with indices taken module 3. Since for $i \in \{1, 2, 3\}$, we have a monochromatic star K_{1,t_i} of color i with center $\widetilde{x}_i$ or $\widetilde{y}_i$ where $t_i = |\overline{Y_i}|$ or $|\overline{X_i}|$, respectively. Note that

$$\begin{aligned} |\overline{X_1}| + |\overline{X_2}| + |\overline{X_3}| &= 3|X| - \big(|X_1| + |X_2| + |X_3|\big) \\ &= 2|X| + \sum_{i=4}^{k} |X_i|. \end{aligned}$$

Similarly, we also obtain

$$|\overline{Y_1}| + |\overline{Y_2}| + |\overline{Y_3}| \;=\; 2|Y| + \sum_{i=4}^{k} |Y_i|.$$

Since all of the k colors appear in G, $X_i \neq \emptyset$ or $Y_i \neq \emptyset$ for any $i \in \{4, \dots, k\}$, and hence $\sum_{i=4}^{k} |X_i| + \sum_{i=4}^{k} |Y_i| \geq k - 3$. This implies that

$$|\overline{X_1}| + |\overline{X_2}| + |\overline{X_3}| + |\overline{Y_1}| + |\overline{Y_2}| + |\overline{Y_3}| \;\geq\; 2|X| + 2|Y| + k - 3 \;\geq\; 4n + (k-3).$$

Therefore, we have either

$$|\overline{X_i}| \geq \frac{2n}{3} + \frac{k-3}{6} \qquad \text{or} \qquad |\overline{Y_i}| \geq \frac{2n}{3} + \frac{k-3}{6}$$

for some $i \in \{1, 2, 3\}$. So, there exists a monochromatic star of color i with center $\widetilde{x}_i$ or $\widetilde{y}_i$ and with desired size. $\square$

Proposition 10 *There exists an edge-coloring G of a complete bipartite graph with exactly k colors such that G admits neither a PC C_4 nor a monochromatic star $K_{1,t}$ with*

$$t = \begin{cases} \frac{2n}{3} + \frac{k-1}{6} & \text{if } k \text{ is even,} \\ \frac{2n}{3} + \frac{k+4}{6} & \text{if } k \text{ is odd,} \end{cases}$$

where n is the order of the smaller bipartition of G.

Proof. We here prove the case k is an even integer, but the case k is odd can be shown by suitable modification.

We set pairwise disjoint sets X_i and Y_i for $i \in \{1, \dots, k\}$ satisfying the following conditions for a large integer r;

$$\begin{aligned}
&|X_1| = |X_2| = |X_3| = |Y_1| = |Y_2| = |Y_3| = r, \\
&|X_i| = 1 \text{ and } |Y_i| = 0 \quad \text{if } i \text{ is an even integer with } 4 \le i \le k-2, \\
&|X_i| = 0 \text{ and } |Y_i| = 1 \quad \text{if } i \text{ is an odd integer with } 5 \le i \le k-1, \\
&|X_k| = 1 \text{ and } |Y_k| = 1.
\end{aligned}$$

Now we construct an edge-coloring of the complete bipartite graph with bipartition $\bigcup_{i=1}^{k} X_i$ and $\bigcup_{i=1}^{k} Y_i$. We first put an edge of color k between X_k and Y_k. Then for $i \in \{4, \dots, k-1\}$, we play the following operation iteratively with the reverse order;

- If i is an even integer, then put edges between X_i and $\bigcup_{j=i+1}^{k} Y_j$ of color i;
- If i is an odd integer, then put edges between Y_i and $\bigcup_{j=i+1}^{k} X_j$ of color i.

For $i \in \{1, 2, 3\}$, we connect X_i to $\overline{Y_i}$ and Y_i to $\overline{X_i}$ by edges of color i respectively, where

$$\overline{X_i} \ = \ X - \varphi^{-1}(i+1) \cap X \quad \text{and} \quad \overline{Y_i} \ = \ Y - \varphi^{-1}(i+1) \cap Y$$

for $i \in \{1, 2, 3\}$ with indices taken module 3. Then we obtain an edge-coloring, say G, of the complete bipartite graph with bipartition $X = \bigcup_{i=1}^{k} X_i$ and $Y = \bigcup_{i=1}^{k} Y_i$, and it uses exactly k colors. Let $n = |X| = |Y| = 3r + \frac{k-2}{2}$.

It is easy to see that G contains a monochromatic star of center in X_1 and the leaves $\overline{Y_1}$, and its size is maximum in G. Since $r = \frac{n}{3} - \frac{k-2}{6}$, note that

$$|\overline{Y_1}| = 2r + \frac{k-2}{2} = \frac{2n}{3} + \frac{k-2}{6},$$

which completes the proof. $\square$

References

[1] A. Abouelaoualim, K.Ch. Das, W. Fernandez de la Vega, M. Karpinski, Y. Manoussakis, C.A. Martinhon and R. Saad, Cycles and paths in edge-colored graphs with given degrees, J. Graph Theory **64** (2010) 63–86.

[2] N. Alon, On a conjecture of Erdős, Simonovits and Sós concerning anti-Ramsey theorems, J Graph Theory **7** (1983) 91–94.

[3] M. Axenovich, T. Jiang, and A. Kündgen, Bipartite anti-Ramsey numbers of cycles, J. Graph Theory **47** (2004) 9–28.

[4] M. Axenovich, T. Jiang, and Z. Tuza, Local anti-Ramsey numbers of graphs, Combin. Probab. Comput. **12** (2003) 495–511.

[5] C.C. Chen and D.E. Daykin, Graphs with Hamiltonian cycles having adjacent lines different colors, J. Combin. Theory, Ser. B **21** (1976) 135–139.

[6] P. Erdős, M. Simonovits and V.T. Sós, Anti-Ramsey theorems, Infinite and finite sets (Colloq, Keszthely, 1973; dedicated to P. Erdős on his 60thbirthday), Vol. II, 633–643; Colloq Math Soc Janos Bolyai, Vol. 10, North-Holland, (1975).

[7] P. Erdős and Z. Tuza, Rainbow subgraphs in edge-colorings of complete graphs, Ann. Discrete Math. **55** (1993) 81–88.

[8] S. Fujita, C. Magnant and K. Ozeki, Rainbow generalizations of ramsey theory: A survey, Graphs Combin. **26** (2010) 1–30.

[9] S. Fujita, C. Magnant and K. Ozeki, Rainbow generalizations of ramsey theory - a dynamic survey, Theory and Applications of Graphs, **0** (2014) Article 1.

[10] S. Fujita, R. Li and S. Zhang, Color degree and monochromatic degree conditions for short properly colored cycles in edge]colored graphs, J. Graph Theory **87** (2018) 362–373.

[11] T. Gallai, Transitiv orientierbare graphen, Acta Math. Acad. Sci. Hungar, **18** (1967) 25–66.

[12] J.J. Montellano-Ballesteros and V. Neumann-Lara, An anti-Ramsey theorem on cycles, Graphs Combin. **21** (2005) 343–354.

[13] D. Mubayi and D.B. West, Restricted edge-colorings of bicliques, Discrete Math. **257** (2002) 513–529.

[14] G. Wang and H. Li, Color degree and alternating cycles in edge-colored graphs, Discrete Math. **309** (2009) 4349–4354.

[15] G. Wang, H. Li, Y. Zhu and G. Liu, A note on heterochromatic C_4 in edge colored triangle-free graphs, Graphs Combin. **28** (2012) 901–905.

[16] D.B. West, Introduction to Graph Theory, Second Edition, Prentice-Hall, Inc. (1996).