Extension to 3-colorable triangulations

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Abstract

In order to attack some problems in computational geometry, Hoffmann and Kriegel [SIAM J. Discrete Math. 9 (1996) 210–224] considered the problem of whether a plane map can be extended to a 3-colorable triangulation by adding edges. In this paper, we improve their results to maps on non-spherical surfaces, by showing the following two results for a *mosaic*, i.e., a map on a surface each of whose faces is triangular or quadrangular:

- A necessary and sufficient condition for mosaics on a surface to be extended to 3-colorable triangulations. (Theorem 10)
- An explicit formula for calculating the number of distinct 3-colorable triangulations extended from a given mosaic on a surface. (Theorem 11)

These results suggest a significant gap between the planar case and the non-spherical case. We also show that they improve several known results and have an application to polychromatic coloring.

Keywords. 3-colorable triangulation, even triangulation, the fundamental group of a surface, polychromatic coloring

1 Introduction

In this paper, we allow multiple edges and loops for graphs. A *map* on a surface F^2 is a 2-cell embedding of a graph on F^2 without edge intersection. We define a *triangulation* and a *quadrangulation* as a map in which every face is triangular and quadrangular, respectively. It is easy to see that any map G can be *extended* to a triangulation by adding diagonal(s) into every face of G.

Extension of plane maps is known as a powerful tool to solve some problems in computational geometry. One of the most famous and classical example is Fisk's proof [4] of the art-gallery problem, using an extension to a *disk triangulation* (a 2-connected outer plane map in which every inner face is triangular). Note that any disk triangulation is 3-colorable, which is a key fact in Fisk's proof. (Throughout this paper except for the last subsection, a 3*-coloring* of a graph G is a proper vertex-3-coloring of G.) As an analogy to the art-gallery problem, the "prison (yard) problem" is known (see also [5] for example). Hoffmann and Kriegel [8] considered the problem for bipartite plane graphs with the same thought as Fisk's idea. Indeed, they proved the following.

Theorem 1. ([8, Theorem 2.1]) Every bipartite plane graph can be extended to a 3-colorable triangulation.

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Furthermore, Zhang and He [25] established an explicit formula for the number of distinct 3-colorable triangulations extended from a given plane quadrangulation.

In this paper, we generalize those results on plane quadrangulations to *mosaics* on a surface, where a mosaic is a map on a surface in which every face is triangular or quadrangular. Note that both of triangulations and quadrangulations are mosaics. We give the following two results:

- A necessary and sufficient condition for mosaics on a surface to be extended to 3-colorable triangulations. (Theorem 10)
- An explicit formula for calculating the number of distinct 3-colorable triangulations extended from a given mosaic on a surface. (Theorem 11)

As mentioned above, the problem of extending plane quadrangulations to 3-colorable triangulations have been done in [8, 25], but their result heavily depends on the following fact. (See [6, 21] for the proof.) A graph is *even* if all vertices have even degree.

Fact 2. A plane triangulation G is 3-colorable if and only if G is even.

So, for a given plane map G, extending G to an even triangulation is equivalent to that to a 3-colorable triangulation. In fact, as an analogy of the planar case [25], Theorem 1 has been generalized to the result on the extension of quadrangulations on non-spherical surfaces into "even triangulations," as in the following.

Theorem 3 ([18, 25])**.** Every quadrangulation on any surface can be extended to an even triangulation.

Furthermore, the authors [18] established the explicit formula for the number of distinct even triangulations extended from a given quadrangulation on a surface.

In contrast to the planar case, the situation is completely different if we focus on non-spherical surfaces. That is, a 3-colorable triangulation on any surface is always even, but every non-spherical surface admits a non-3-colorable even triangulation. In this paper, we consider the extension of a map on a non-spherical surface to a "3-colorable triangulation."

We emphasize that our problem is much more difficult than that in Theorem 3. For example, given any quadrangulation G on a surface F^2, Theorem 3 guarantees that G can be extended to an even triangulation. However, a quadrangulation G cannot be extended to a 3-colorable triangulation if G is not 3-colorable, since adding edges does not decrease the chromatic number of graphs. Moreover, it is not clear whether every 3-colorable quadrangulation can be extend to a 3-colorable triangulation. What is a characterization of quadrangulations on a surface to be extended to a 3-colorable triangulation? In this paper, we would like to answer those questions completely for not only quadrangulations but also mosaics on surfaces.

The organization of this paper is as follows: We prepare several definitions in Section 2 to state the main theorems, and we describe the theorems in Section 3. Giving lemmas in Section 4, then we prove the main theorems in Section 5. In the last section, we show that our main theorems improve several known results and have an application to polychromatic coloring.

2 Basic definitions

In this section, we give several definitions that play crucial roles in this paper. Those in Subsections 2.2, 2.3, 2.5 and 2.6 also appeared in [18].

2.1 Surfaces and the fundamental group

Let $\mathbb{S}_k$ and $\mathbb{N}_k$ denote the orientable surface of genus k and the nonorientable surface of crosscap number k, respectively. Note that $\mathbb{S}_0$ is the sphere, $\mathbb{S}_1$ is the torus, and $\mathbb{N}_1$ is the projective plane. By the classification theorem, any surface is homeomorphic to $\mathbb{S}_k$ or $\mathbb{N}_k$ for some integer $k \geq 0$. Let $g(F^2)$ denote the Euler genus of a surface F^2, where $g(\mathbb{S}_k) = 2k$ and $g(\mathbb{N}_k) = k$. In this paper, we often write g instead of $g(F^2)$ for the simplicity.

Let F^2 be a surface. A *closed curve on F^2 (with base point x_0)* is the image of a continuous function $\gamma : [0, 1] \to F^2$ with $\gamma(0) = \gamma(1)$ $(= x_0)$. To avoid abuse of notation, we simply write a closed curve γ instead of its image. A closed curve γ on F^2 is *essential* if γ does not bound a 2-cell on F^2, while a closed curve that is not essential is *contractible*. If a tubular neighborhood of γ is a Möbius strip, then it is 1-*sided*; otherwise it is 2-*sided*. Two closed curves γ_1 and γ_2 on F^2 are *homotopic* if there exists a continuous map $\Phi : [0, 1] \times [0, 1] \to F^2$ such that $\Psi(0, x) = \gamma_1(x)$ and $\Psi(1, x) = \gamma_2(x)$ for each $x \in [0, 1]$, where $\Psi(t, 0) = \Psi(t, 1) = x_0$ for each $t \in [0, 1]$.

Now we fix a point x_0 on F^2, and consider the set Γ of all closed curves on F^2 with base point x_0. It is known that being homotopic is an equivalent relation on Γ, called the *homotopy*, and the quotient set of Γ by the homotopy is called the *fundamental group of F^2 with base point x_0*, denoted by $\pi_1(F^2, x_0)$. In fact, $\pi_1(F^2, x_0)$ is a group with the product $*$, where for two homotopy classes $[\gamma_1]$ and $[\gamma_2]$, $[\gamma_1] * [\gamma_2]$ is the homotopy class containing the concatenation $\gamma_1 * \gamma_2$. (Formally it is defined as $(\gamma_1 * \gamma_2)(t) = \gamma_1(2t)$ for $0 \leq t \leq \frac{1}{2}$ and $(\gamma_1 * \gamma_2)(t) = \gamma_2(2t - 1)$ for $\frac{1}{2} < t \leq 1$.) Note that the identity element is the homotopy class corresponding to all contractible closed curves with base point x_0, and the inverse $[\gamma_1]^{-1}$ is defined as the homotopy class containing $-\gamma_1$, where $-\gamma_1(t) = \gamma_1(1 - t)$ for $0 \leq t \leq 1$. It is known that the fundamental group $\pi_1(F^2, x_0)$ does not depend on the base point x_0. Hence we can simply write it by $\pi_1(F^2)$.

Figures 1 and 2 show a *canonical base* of $\pi_1(F^2)$ for a non-spherical surface F^2, when F^2 is orientable and nonorientable, respectively. (To be exact, each closed curve in the figures represents the homotopy class containing it.) It is known that the number of elements in a base is $g = g(F^2)$.

We refer other notations to [20].

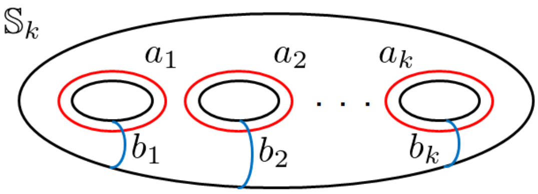

Figure 1: A canonical base of $\pi_1(\mathbb{S}_k)$.

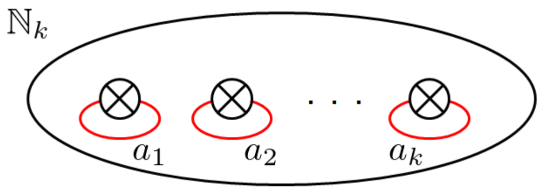

Figure 2: A canonical base of $\pi_1(\mathbb{N}_k)$.

2.2 The $\mathbb{Z}_2$-homology space of a map

An embedding of a graph G on a surface F^2 must be an injective continuous mapping from G to F^2, while a *drawing* of G on F^2 is a continuous one and might not be an injection.

In this paper, we distinguish an *intersection* and a *crossing* of two objects, which are graphs drawn on a surface or closed curves on it. An *intersectiton* is a common point on the surface by the two objects in an ordinary way without considering directions. In contrast, a *crossing* is defined by two objects with directions. The formal definition of a crossing will be given in Subsection 2.7. Throughout this paper, for a drawing of G on F^2, we assume that a closed curve γ on F^2 transversely intersects with G on F^2, and that γ passes through neither a vertex of G nor a intersection point of G.

Let G be a map on a surface F^2. In this paper, we sometimes regard a closed walk in G with fixed direction as a closed curve on F^2. For a closed walk W in G, we denote by $\widetilde{W}$ the (undirected) subgraph of G consisting of the edges that appear in W an odd number of times. Since W is a closed walk, $\widetilde{W}$ is an even subgraph of G.

For two subgraphs H and H' of G, we denote by $H \triangle H'$ the *symmetric difference* of H and H', which is the graph induced by the edges that are contained in exactly one of H and H'. We can easily see that if both H and H' are even graphs, then so is $H \triangle H'$. The set of all even subgraphs of G forms a $\mathbb{Z}_2$-space with the symmetric difference $\triangle$ as a product and an empty graph as identity. It is called the *cycle space of* G. See [3, Page 24] for the details about the cycle space.

Two even subgraphs H and H' of G are $\mathbb{Z}_2$-*homologous (on* G*)* if there exists a set $\{D_1, \ldots, D_r\}$ of facial cycles of G such that $H' = H \triangle D_1 \triangle \cdots \triangle D_r$. Note that the $\mathbb{Z}_2$-homologous relation is an equivalence relation on the cycle space. If two closed walks W and W' in G are homotopic, then $\widetilde{W}$ and $\widetilde{W'}$ are $\mathbb{Z}_2$-homologous, but the converse does not generally hold. The quotient set of the cycle space by the $\mathbb{Z}_2$-homologous relation is the $\mathbb{Z}_2$-*homology space of* G. Let $\mathcal{W} = \{\gamma_1, \ldots, \gamma_g\}$ be a canonical base of the fundamental group $\pi_1(F^2)$ of F^2, see Figures 1 and 2. Since G is 2-cell embedded on F^2, for each γ_i, there exists a closed walk W_i of G such that W_i is homotopic to γ_i. Then the set $\{\widetilde{W_1}, \ldots, \widetilde{W_g}\}$ is a base of the $\mathbb{Z}_2$-homology space of G, called a *canonical base*.

2.3 Dual of a map on a surface

Let G be a mosaic on a surface. For a vertex f^* of degree 4 in the dual G^* of G with four edges e_1^*, e_2^*, e_3^* and e_4^* incident to f^* in this cyclic order (so, f is a quadrangular face of G surrounded by the cycle with edges e_1, e_2, e_3, e_4), we say that e_i^* is *opposite to* e_{i+2}^* *at* v. A walk W of G^* is a *straight walk*, or shortly an *S-walk*, if W satisfies one of the following:

(i) W connects vertices of degree 3 in G^*, and for every internal vertex f^* of W, f^* has degree 4 in G^* and W passes through f^* from one edge to the opposite edge at f^*.

(ii) W is a closed walk using no edges twice, and for every vertex f^* of W, f^* has degree 4 and W passes through f^* from one edge to the opposite edge at f^*.

Note that an S-walk might intersect with itself, and the edge set of G^* is uniquely partitioned into S-walks. The concept of S-walks can be found in [2, 8, 18, 19, 25] (with sometimes different names).

Using the concept of S-walks, we define *the S-walk dual* $S(G)$ of a mosaic G as follows:

$$\begin{aligned} V\big(S(G)\big) &= \{f^* : f \text{ is a triangular face of } G\}, \\ \text{and} \quad E\big(S(G)\big) &= \{W : W \text{ is an S-walk of } G^*\}, \end{aligned}$$

where each S-walk W corresponds to an edge of $S(G)$ connecting two end vertices of W (if W satisfies (i)), or an edge having no vertex (if W satisfies (ii)). In the latter case, we also regard W as a cycle of $S(G)$ of length 0. Note that $S(G)$ is 3-regular and might have multiple edges or loops. When G is a triangulation on a surface, then $S(G) = G^*$, and when G is a quadrangulation on a surface, then $S(G)$ has no vertices and consists of only edges. We can assume that $S(G)$ is drawn on the surface in a natural way as G does. Hence $S(G)$ might have edge intersections, and edges with self-intersection. Note that the S-walk dual is called the *straight walk dual* in [19].

Subdividing each edge of $S(G)$ whenever it intersects with an edge in G, we obtain the *subdivided S-walk dual*, denoted by $\widehat{S}(G)$, of a mosaic G. So, we have

$$V\big(\widehat{S}(G)\big) = V\big(S(G)\big) \cup E(G).$$

Note that the number of components of $\widehat{S}(G)$ is exactly the same as that of $S(G)$. (For example, see Figure 3, where $S(G)$ has two vertices of degree 3 and two components one of which has no vertices.)

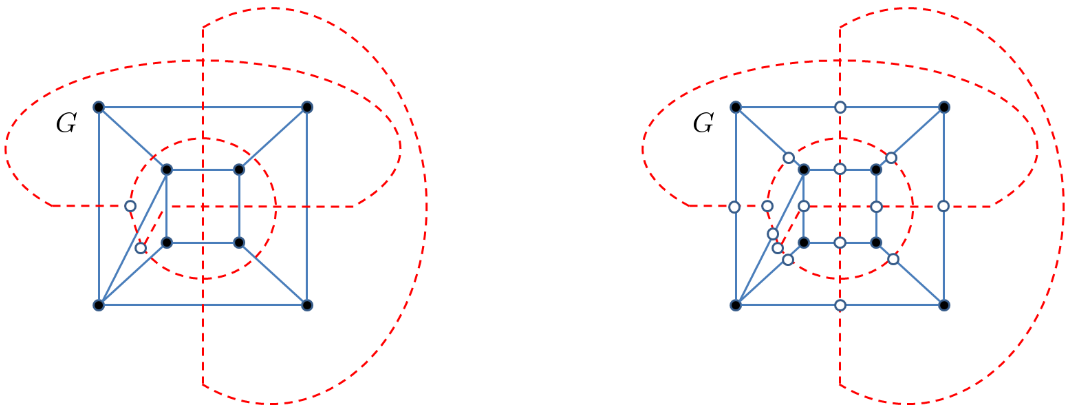

Figure 3: The S-walk dual $S(G)$ in the left side, and the subdivided S-walk dual $\widehat{S}(G)$ in the right side of a plane mosaic G. The dotted lines represent edges of $S(G)$ and those of $\widehat{S}(G)$, respectively. The black circles are vertices of G, while the white circles are vertices of $S(G)$ and those of $\widehat{S}(G)$.

2.4 Face rotation system and orientizing subgraphs

Let G be a map on a surface F^2. We give an arbitrary rotation ξ_f to each face f, clockwise or counter-clockwise, and let $\Xi = \{\xi_f : f \text{ is a face of } G\}$, which is called a *face rotation system* of G. This is the dual of an ordinary rotation system, see [16, Pages 90–94]. Then every edge e in G receives two orientations from both sides of faces by Ξ. If the two directions do not coincide (resp., do coincide), then e is *coherent* (resp., *incoherent*). Note that G has a *consistent* face rotation system, i.e., one such that all edges are coherent if and only if F^2 is orientable. For a given Ξ, the subgraph of G induced by all incoherent edges is called the *orientizing subgraph* of G and denoted by H_Ξ. It is easy to see that each vertex is incident with an even number of incoherent edges. Therefore, any orientizing subgraph of a map G is an even subgraph of G.

Remark 1: (Orientizing subgraph) As mentioned above, any orientizing subgraph H_Ξ is an even subgraph of a map G, and hence we can regard H_Ξ as a set of cycles. We can cut G along those cycles and obtain the map $\overline{G}_\Xi$ on the (possibly disconnected) surface, say F^2_Ξ, obtained from F^2 by cutting along those cycles and pasting a disk to each resultant boundary. Then the graph $\overline{G}_\Xi$ is said to be obtained by *cutting along* H_Ξ. (See [16, Pages 105–106] for the formal definition.) Since cutting along H_Ξ breaks all incoherent edges, F^2_Ξ is orientable. Actually the name of an "orientizing" subgraph comes from this fact.

For an orientizing subgraph, we have the following lemma. Lemma 4 suggests that an orientizing subgraph does not depend on the choice of a face rotation system Ξ, up to the $\mathbb{Z}_2$-homologous relation.

Lemma 4. Let G be a map on a surface F^2.

- If F^2 is orientable, then any orientizing subgraph is $\mathbb{Z}_2$-homologous to an empty graph.
- If F^2 is nonorientable, then any orientizing subgraph is $\mathbb{Z}_2$-homologous to $\widetilde{W}_1 \triangle \ldots \triangle \widetilde{W}_g$, where $\{\widetilde{W}_1, \ldots, \widetilde{W}_g\}$ is a canonical base of the $\mathbb{Z}_2$-homology space of G.

Proof. First, we consider the case when F^2 is orientable. Let Ξ be a face rotation system of G. Since F^2 is orientable, we can give a clockwise rotation ξ^0_f for each face f, and let Ξ^0 be the corresponding face rotation system of G. By this choice, there are no incoherent edges with respect to Ξ^0, and hence the orientizing subgraph H_{Ξ^0} is an empty graph. Let $\mathcal{D}$ be the set of facial cycles of faces f with counter-clockwise rotation by $\xi_f \in \Xi$, that is,

$$\mathcal{D} = \{D : D \text{ is a facial cycle of a face } f \text{ with } \xi_f \neq \xi^0_f\}.$$

By the definition of H_Ξ, it is easy to see that H_Ξ is obtained from H_{Ξ^0} by the symmetric difference of all facial cycles $D \in \mathcal{D}$. This directly implies that H_Ξ and H_{Ξ^0} are $\mathbb{Z}_2$-homologous.

Next, we consider the case when F^2 is nonorientable. Since we obtain the orientable surface F^2_Ξ by cutting along H_Ξ, H_Ξ has to intersect with all 1-sided closed curves on F^2 an odd number of times. It is well known that such even subgraph must be $\mathbb{Z}_2$-homologous to $\widetilde{W_1} \triangle \dots \triangle \widetilde{W_g}$, where $\{\widetilde{W_1}, \dots, \widetilde{W_g}\}$ is a canonical base of the $\mathbb{Z}_2$-homology space of G. □

2.5 An H-subdivided $\mathbb{Z}_3$-orientations

An *orientation* $\mathcal{O}$ of a graph G is an assignment of a direction to each edge of G. The resultant *directed graph* is denoted by $(G, \mathcal{O})$. Since $\mathcal{O}$ also gives an orientation to a subgraph P of G, we simply write $(P, \mathcal{O})$ in a natural way. Note that if we reverse the directions of all edges of $(G, \mathcal{O})$, then we get another orientation, denoted by $\overline{\mathcal{O}}$, of G, where $\mathcal{O}$ and $\overline{\mathcal{O}}$ are called an *orientation pair* of G.

Definition 5. Let H be a subgraph of a mosaic G on a surface F^2. An *H-subdivided $\mathbb{Z}_3$-orientation* of $\widehat{S}(G)$ is an orientation of $\widehat{S}(G)$ satisfying the following three conditions:

(H1) For every vertex f^* of degree 3 in $\widehat{S}(G)$, the out-degree of f^* is 0 or 3. (Hence the in-degree of f^* is also 0 or 3.)

(H2) For every vertex e of degree 2 in $\widehat{S}(G)$, if e does not correspond to an edge of H, then the out-degree of e is exactly 1. (Hence the in-degree of e is also exactly 1.)

(H3) For every vertex e of degree 2 in $\widehat{S}(G)$, if e corresponds to an edge of H, then the out-degree of e is 0 or 2. (Hence the in-degree of e is also 0 or 2.)

In this paper, we mainly consider an H_Ξ-subdivided $\mathbb{Z}_3$-orientation for a face rotation system Ξ of G. We discuss this in the next subsection.

Remark 2: ($\mathbb{Z}_3$-orientation) An H-subdivided $\mathbb{Z}_3$-orientation is a generalization of the ordinary $\mathbb{Z}_3$-*orientation*, which can be regarded as an $\emptyset$-subdivided $\mathbb{Z}_3$-orientation. (See [23, 24] for details on $\mathbb{Z}_3$-orientations.) Indeed, a graph has a $\mathbb{Z}_3$-orientation if and only if it has a nowhere zero 3-flow [13, 14], and a nowhere zero 3-flow is known as the "dual" concept of the 3-colorability for graphs on surfaces [22]. Those facts are behind of the arguments in this paper, while they do not appear explicitly.

2.6 The existence of an H_Ξ-subdivided $\mathbb{Z}_3$-orientation

One of the key tools in this paper is an H_Ξ-subdivided $\mathbb{Z}_3$-orientation for a face rotation system Ξ of G. We here discuss when it exists, using several ideas as in [18]. Note that the authors were concerned mainly with quadrangulations, but as in [18, Remark in Section 3], we can modify the arguments for mosaics, using the cycle space.

Let G be a mosaic on a surface F^2, let $\widetilde{\mathcal{W}} = \{\widetilde{W_1}, \dots, \widetilde{W_g}\}$ be a canonical base of the $\mathbb{Z}_2$-homology space of G, and let $\mathscr{C} = \{C_1, \dots, C_\ell\}$ be a base of the cycle space of $\widehat{S}(G)$. The *intersection matrix of G with respect to $\mathscr{C}$ and $\widetilde{\mathcal{W}}$*, denoted by $\mathrm{IM}(G)$, is an $(\ell \times g)$-matrix on $\mathbb{Z}_2$ such that its (i, j)-entry w_{ij} is the number of intersections by C_i and $\widetilde{W_j}$ with modulo 2. This is defined as follows:

$$\mathrm{IM}(G) \;=\; \begin{array}{c} \\ C_1 \\ C_2 \\ \vdots \\ C_\ell \end{array} \!\!\begin{array}{c} \begin{array}{cccc} \widetilde{W_1} & \widetilde{W_2} & \cdots & \widetilde{W_g} \end{array} \\ \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1g} \\ w_{21} & w_{22} & \dots & w_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ w_{\ell 1} & w_{\ell 2} & \dots & w_{\ell g} \end{pmatrix} \end{array}.$$

When F^2 is the sphere, the intersection matrix $\mathrm{IM}(G)$ of G is an $(\ell \times 0)$-matrix, which is degenerate.

Let $\vec{c} = (c_1, \dots, c_\ell)^T$ be the column vector, where c_i is the number of triangular faces contained in C_i with modulo 2. Then for a column vector $\vec{x} = (x_1, \dots, x_g)^T$ with $\mathbb{Z}_2$ elements, we set the following system of linear equations, called the *extended intersection system* of G with respect to $\mathscr{C}$ and $\widetilde{\mathcal{W}}$:

$$\mathrm{IM}(G)\ \vec{x} = \vec{c}. \tag{1}$$

Using the extended intersection system (1) of G, the authors gave a necessary and sufficient condition for mosaics to be extended to even triangulations.

Theorem 6. [18, Theorem 13] Let G be a mosaic on a surface F^2. Then G can be extended to an even triangulation if and only if the extended intersection system (1) has a solution.

Example: By an example, we show what the intersection system means. A mosaic G in Figure 4 has a vertex of degree 3 surrounded by three triangular faces, and G cannot be extended to an even triangulation. Let $C_1 = e_1 f_1 e_2 f_2 e_3 f_3$ be the cycle as in Figure 4, and let $\mathscr{C}$ be a base of the cycle space of $\widehat{S}(G)$ containing C_1 as an element. Since C_1 contains exactly three triangular faces, we have $c_1 = 1$. Furthermore, since C_1 is contractible, every $\widetilde{W_j}$ intersects with C_1 an even number of times. This means that $w_{1j} = 0$ for $j \in \{1, \dots, g\}$. Thus, the first equation of the extended intersection system (1) is

$$\vec{0} \cdot \vec{x} = 1$$

and hence (1) has no solution. This corresponds to the fact that G cannot be extended to an even triangulation in terms of Theorem 6.

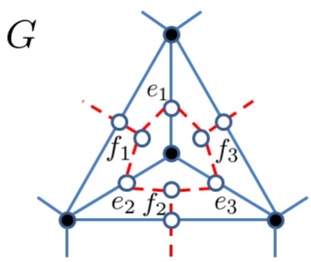

Figure 4: A mosaic G that cannot be extended to an even triangulation.

For the proof of Theorem 6 in [18], the authors gave a connection between a solution $\vec{x}$ of the extended intersection system (1) of G and the existence of an H-subdivided $\mathbb{Z}_3$-orientation. Actually, any solution $\vec{x}$ of the extended intersection system (1) indicates the subgraph $H_{\vec{x}}$ of G as follows. Let $\vec{x} = (x_1, \dots, x_g)^T$ be a column vector with $\mathbb{Z}_2$ elements, and let $i_1, \dots, i_p$ be the indices such that $x_r = 1$ if and only if $r \in \{i_1, \dots, i_p\}$. Then define $H_{\vec{x}}$ as the empty graph if $\vec{x} = \vec{0} = (0, \dots, 0)^T$; otherwise let

$$H_{\vec{x}} = \widetilde{W_{i_1}} \triangle \widetilde{W_{i_2}} \triangle \cdots \triangle \widetilde{W_{i_p}}.$$

We say that the subgraph $H_{\vec{x}}$ is *produced by* $\vec{x}$.

If G is a quadrangulation on F^2, then the following four were shown in [18]. See [18, Definition 4] for the definition of the *parity condition*.

(i) **[18, Lemma 5]** The subdivided S-walk dual $\widehat{S}(G)$ has an H-subdivided $\mathbb{Z}_3$-orientation if and only if H satisfies the parity condition.

(ii) **[18, Lemma 15 (i)]** For any solution $\vec{x}$ of the extended intersection system (1) of G, the subgraph $H_{\vec{x}}$ produced by $\vec{x}$ is an even subgraph of G satisfying the parity condition.

(iii) **[18, Lemma 15 (iii)]** For any even subgraph H of G satisfying the parity condition, there exists a solution $\vec{x}$ of the extended intersection system (1) of G such that H is $\mathbb{Z}_2$-homologous to $H_{\vec{x}}$.

(iv) **A direct consequence of [18, Lemma 19 (i)]** For any two even subgraphs H and H' that are $\mathbb{Z}_2$-homologous, if there is an H-subdivided $\mathbb{Z}_3$-orientation in $\widehat{S}(G)$, then there is also an H'-subdivided $\mathbb{Z}_3$-orientation in $\widehat{S}(G)$.

Then by suitable modification to mosaics, we obtain the next lemma. We leave the detail of the proof to the readers. In fact, the combination of (i), (ii) and (iv) gives a proof of the "if" part, while the combination of (i) and (iii) gives a proof of the "only if" part.

Lemma 7. Let G be a mosaic on a surface F^2 and let H be a subgraph of G. Then $\widehat{S}(G)$ has an H-subdivided $\mathbb{Z}_3$-orientation if and only if there exists a solution $\vec{x}$ of the extended intersection system (1) of G such that H is $\mathbb{Z}_2$-homologous to $H_{\vec{x}}$.

We say that a mosaic G on a surface F^2 satisfies the *orientizing condition* if G satisfies the following:

- If F^2 is orientable, then the vector $\vec{0} = (0, \ldots, 0)^T$ is a solution of the extended intersection system (1) of G with respect to a base $\mathscr{C}$ of the cycle space of $\widehat{S}(G)$ and a canonical base $\widetilde{\mathcal{W}}$ of the $\mathbb{Z}_2$-homology space of G. Note that this condition is equivalent to $\vec{c} = \vec{0}$, and furthermore to the condition that any cycle of $\widehat{S}(G)$ contains an even number of triangular faces of G, i.e. $S(G)$ is bipartite.
- If F^2 is nonorientable, then the vector $\vec{1} = (1, \ldots, 1)^T$ is a solution of the extended intersection system (1) of G with respect to a base $\mathscr{C}$ of the cycle space of $\widehat{S}(G)$ and a canonical base $\widetilde{\mathcal{W}}$ of the $\mathbb{Z}_2$-homology space of G.

Note that $H_{\vec{0}}$ is an empty graph. On the other hand, $H_{\vec{1}}$ is $\mathbb{Z}_2$-homologous to $\widetilde{W_1} \triangle \ldots \triangle \widetilde{W_g}$, where $\{\widetilde{W_1}, \ldots, \widetilde{W_g}\}$ is a canonical base of the $\mathbb{Z}_2$-homology space of G. Therefore, together with Lemmas 4 and 7, these imply the following lemma.

Lemma 8. Let G be a mosaic on a surface F^2, and let Ξ be a face rotation system of G. Then $\widehat{S}(G)$ has an H_Ξ-subdivided $\mathbb{Z}_3$-orientation if and only if G satisfies the orientizing condition.

2.7 The crossing number

Let G be a mosaic on a surface F^2, let Ξ be a face rotation system of G and let $\mathcal{O}$ be an orientation of $\widehat{S}(G)$. For a closed curve γ on F^2 and a subgraph P of $\widehat{S}(G)$, the *crossing number* of γ and $(P, \mathcal{O})$ with respect to Ξ, denoted by $\mathrm{cr}_\Xi\big(\gamma, (P, \mathcal{O})\big)$, is defined as follows. For each crossing point p of γ and $(P, \mathcal{O})$, we may regard that a portion of γ, denoted by $\vec{r}$, and a directed edge of $(P, \mathcal{O})$, denoted by $\vec{e}$, cross orthogonally at p. Let f be the face of G containing p in the interior and let $\xi_f \in \Xi$ be the face rotation of f. Then define

$$\mathrm{cr}_\Xi\big(\gamma, (P, \mathcal{O}); p\big) = \begin{cases} +1 & \text{if } \vec{e} \text{ matches } \vec{r} \text{ after rotating by } \frac{\pi}{2} \text{ in the forward direction of } \xi_f, \\ -1 & \text{if } \vec{e} \text{ matches } \vec{r} \text{ after rotating by } \frac{\pi}{2} \text{ in the backward direction of } \xi_f, \end{cases}$$

(see Figure 5 for example) and define

$$\mathrm{cr}_\Xi\big(\gamma, (P, \mathcal{O})\big) = \sum_p \mathrm{cr}_\Xi\big(\gamma, (P, \mathcal{O}); p\big),$$

where the sum is taken over all crossing points p of γ and $(P, \mathcal{O})$. It is clear from the definition that $\mathrm{cr}_\Xi\big(\gamma, (P, \overline{\mathcal{O}})\big) = -\mathrm{cr}_\Xi\big(\gamma, (P, \mathcal{O})\big)$. For the crossing number, we have the following lemma.

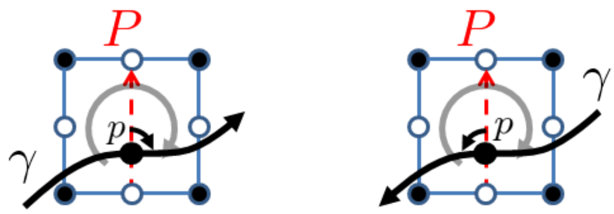

Figure 5: The situations for $\mathrm{cr}_\Xi\big(\gamma, (P, \mathcal{O}); p\big) = +1$ (left) and -1 (right). The gray circle arcs represent $\xi_f \in \Xi$ for the corresponding face, the black thick curves represent γ, and the red dotted lines represent a direct edge $\vec{e}$ in $(P, \mathcal{O})$, respectively.

Lemma 9. Let G be a mosaic on a surface F^2, let Ξ be a face rotation system of G and let P be a component of $\widehat{S}(G)$. Suppose that $\widehat{S}(G)$ has an H_Ξ-subdivided $\mathbb{Z}_3$-orientation $\mathcal{O}$. Then for any two homotopic closed curves γ and γ' on F^2, we have

$$\mathrm{cr}_\Xi\big(\gamma, (P, \mathcal{O})\big) \equiv \mathrm{cr}_\Xi\big(\gamma', (P, \mathcal{O})\big) \pmod 3.$$

In particular, if γ is contractible, then $\mathrm{cr}_\Xi\big(\gamma, (P, \mathcal{O})\big) \equiv 0 \pmod 3$.

Proof. Look at four situations (A)–(D) shown in Figure 6, each of which represents a part of two homotopic closed curves γ and γ', and suppose that the remaining parts of γ and γ' are exactly the same.

In the situation (A), there is a vertex f^* of degree 3 in P, and let e_1, e_2, and e_3 be three vertices of P adjacent with f^*. Since $\mathcal{O}$ satisfies (H1) in Definition 5, we see that the crossing number $\mathrm{cr}_\Xi\big(\gamma, (P, \mathcal{O})\big)$ of γ and $(P, \mathcal{O})$ changes to $\mathrm{cr}_\Xi\big(\gamma', (P, \mathcal{O})\big)$ as $(-(+1)+(-2)) \equiv 0$ or $(-(-1)+(+2)) \equiv 0 \pmod 3$. Then $\mathrm{cr}_\Xi\big(\gamma, (P, \mathcal{O})\big) \equiv \mathrm{cr}_\Xi\big(\gamma', (P, \mathcal{O})\big) \pmod 3$. Similarly, in the situation (B), since $\mathrm{cr}_\Xi\big(\gamma', (P, \mathcal{O}); p\big) = -\mathrm{cr}_\Xi\big(\gamma', (P, \mathcal{O}); p'\big)$, where p and p' are two crossing points of γ' and $(P, \mathcal{O})$, we have $\mathrm{cr}_\Xi\big(\gamma, (P, \mathcal{O})\big) = \mathrm{cr}_\Xi\big(\gamma', (P, \mathcal{O})\big)$. On the other hand, in the situation (C), since H_Ξ is an even subgraph of G, the remaining parts of γ and γ' have the same consistent rotation with respect to Ξ. Thus, we see that $\mathrm{cr}_\Xi\big(\gamma, (P, \mathcal{O})\big) = \mathrm{cr}_\Xi\big(\gamma', (P, \mathcal{O})\big)$. This is the same for the situation (D). Therefore, the homotopic shifts in those four situations (A)–(D) in Figure 6 do not change the crossing number with modulo 3. Note that for any two homotopic closed curves, one can be obtained from the other by a sequence of homotopic shifts in the situations (A)–(D). This proves the first statement in Lemma 9.

For the second statement, consider a contractible closed curve γ_0 on F^2 that does not intersect with $(P, \mathcal{O})$. Trivially, $\mathrm{cr}_\Xi\big(\gamma_0, (P, \mathcal{O})\big) = 0$. Since any contractible closed curve γ is homotopic to γ_0, we have

$$\mathrm{cr}_\Xi\big(\gamma, (P, \mathcal{O})\big) \equiv \mathrm{cr}_\Xi\big(\gamma_0, (P, \mathcal{O})\big) = 0 \pmod 3.$$

□

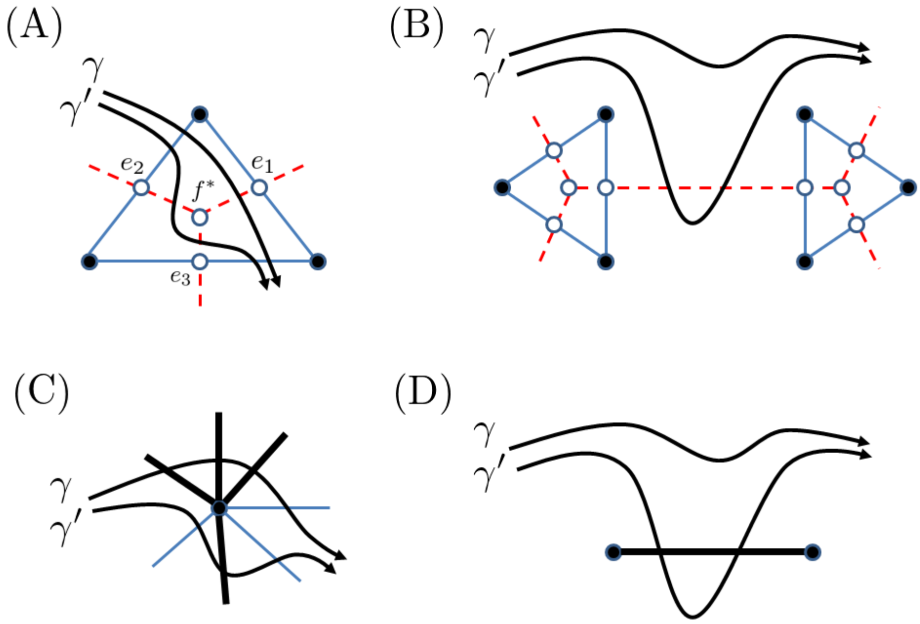

Figure 6: Two homotopic closed curves γ and γ' crossing $(P, \mathcal{O})$ or edges in H_Ξ. In (C) and (D), the bold lines represent edges in H_Ξ.

2.8 The crossing matrix and the crossing system

Let G be a mosaic on a surface F^2, let Ξ be a face rotation system of G, let $\mathcal{W} = \{\gamma_1, \ldots, \gamma_g\}$ be a base of the fundamental group $\pi_1(F^2)$ of F^2 and let $\{P_1, \ldots, P_m\}$ be the set of all components of $\widehat{S}(G)$. Suppose that $\widehat{S}(G)$ has an H_Ξ-subdivided $\mathbb{Z}_3$-orientation $\mathcal{O}_0$. Then the *crossing matrix* of G (with respect to Ξ, $\mathcal{W}$ and $\mathcal{O}_0$), denoted by $\mathrm{CM}(G)$, is a $(g \times m)$-matrix with entry $u_{ij} \in \mathbb{Z}_3$ for $1 \leq i \leq g$ and $1 \leq j \leq m$, where $u_{ij} \equiv \mathrm{cr}_\Xi\big(\gamma_i, (P_j, \mathcal{O}_0)\big) \pmod 3$. That is, it is defined as follows:

$$\mathrm{CM}(G) \;=\; \begin{array}{c} \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_g \end{array} \!\!\begin{array}{c} \begin{array}{cccc} P_1 & P_2 & \cdots & P_m \end{array} \\ \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1m} \\ u_{21} & u_{22} & \cdots & u_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ u_{g1} & u_{g2} & \cdots & u_{gm} \end{pmatrix} \end{array}.$$

Note that when F^2 is the sphere, $\mathrm{CM}(G)$ is a $(0 \times m)$-matrix, which is degenerate.

For a column vector $\vec{x} = (x_1, \ldots, x_m)^T$ with $\mathbb{Z}_3$ elements, we set the following system of linear equations, and call it the *crossing system* of G (with respect to Ξ, $\mathcal{W}$ and $\mathcal{O}_0$):

$$\mathrm{CM}(G)\, \vec{x} = \vec{0}. \tag{2}$$

We say that a vector $\vec{x}$ is *good* if $x_i \in \{1, 2\}$ for $1 \leq i \leq m$.

Remark 3: (The crossing system of G) Observe that the definition of the crossing system (2) of G depends on a face rotation system Ξ, a base $\mathcal{W}$ of the fundamental group $\pi_1(F^2)$ of a surface F^2, and an H_Ξ-subdivided $\mathbb{Z}_3$-orientation $\mathcal{O}_0$. However, the number of good solutions is independent of the choice of those. Actually, we can check this fact for Ξ, $\mathcal{W}$ and $\mathcal{O}_0$ by (iv) before Lemma 7 ([18, Lemma 19]), by Lemma 9, and by Fact 13 in Subsection 4.1, respectively. Since we are only concerned with the number of good solutions, we sometimes take the crossing system (2) of G without specifying Ξ, $\mathcal{W}$ and $\mathcal{O}_0$.

3 Main theorems

We are ready to state the main theorems. The proofs of them will appear in Section 5, after some lemmas in the next section.

Theorem 10. Let G be a mosaic on a surface F^2. Then G can be extended to a 3-colorable triangulation if and only if (I) G satisfies the orientizing condition and (II) the crossing system (2) of G has a good solution.

Theorem 11. Let G be a mosaic on a surface F^2. Suppose that G satisfies the orientizing condition. If the crossing system (2) of G has exactly n good solutions, then G can be extended to exactly $n/2$ distinct 3-colorable triangulations.

Therefore, for the extension of given mosaics, we need two conditions (I) and (II) as in Theorem 10. Recall that any 3-colorable triangulation is an even triangulation (while the converse does not generally hold). Therefore, for mosaics to be extended to 3-colorable triangulations, it must be extended to even triangulations. Theorem 6 suggests that (I) the orientizing condition controls that, and then (II) does 3-colorability.

Remark 4: (Monodoromy) There is some tool, called a *monodoromy*, to "measure" how different from 3-colorability an even triangulation is.

Let T be an even triangulation on a closed surface F^2, and let $W = f_0 f_1 \cdots f_k$ be a sequence of faces of T, called a *face walk*, such that f_i and f_{i+1} share an edge for $i = 0, 1, \ldots, k-1$. Let $f = f_0$, and let $W^i = f_0 \cdots f_i$ for $i \in \{0, 1, \ldots, k\}$. Then define the bijection $\sigma_{T,W^i,f} : V(f_0) \to V(f_i)$ recursively, as follows. For $i = 0$, $\sigma_{T,W^0,f} = \mathrm{id}$, where id represents the identity mapping. For $i > 0$, define $\sigma_{T,W^i,f}$ so that $\sigma_{T,W^i,f}$ and $\sigma_{T,W^{i-1},f}$ coincide on $V(f_{i-1}) \cap V(f_i)$. Let $\sigma_{T,W,f} = \sigma_{T,W^k,f}$. If W is *closed* (i.e., $k > 0$ and $f_0 = f_k$), then $\sigma_{T,W,f}$ determines a unique element in the symmetric group $\mathcal{S}_3$ of degree 3. See [7] for more detailed definition.

It is easy to see that if two closed face walks W_1 and W_2 of T containing f are homotopic, then we have $\sigma_{T,W_1,f} = \sigma_{T,W_2,f}$, and that if W is contractible on F^2, then $\sigma_{T,W,f} = \mathrm{id}$. So, by $\sigma_{T,W,f}$ for each closed face walk W containing f, we can define a homomorphism $\sigma_{T,f} : \pi_1(F^2, x_0) \to \mathcal{S}_3$, where x_0 is a point on F^2 corresponding to f^* of G^*.

Two homomorphisms $\sigma, \sigma' : \pi_1(F^2, x_0) \to \mathcal{S}_3$ are *equivalent* if there exists $s \in \mathcal{S}_3$ with $\sigma s = s\sigma'$. Consequently, the equivalence class of $\sigma_{T,f}$ is independent of the choice of f, and hence we may regard $\sigma_{T,f}$ as a homomorphism from $\pi(F^2)$ to $\mathcal{S}_3$. Then it is called a *monodromy* of T and denoted by σ_T.

By the definition, T is 3-colorable if and only if σ_T is *trivial*, i.e., $\sigma_T(\gamma) = \mathrm{id}$ for any $\gamma \in \pi_1(F^2)$. Recall that the extended intersection system (1) and the crossing system (2) are considered with modulo 2 and modulo 3, respectively. Actually, if a mosaic G is extended to an even triangulation T, then for the 3-colorability of T, condition (I) checks the order 2 subgroup of the image of σ_T, while condition (II) does the order 3 one. In fact, we consider an H_Ξ-face-2-coloring of an even triangulation for the "order 2 subgroup," see Subsections 4.3 and 4.4.

4 Lemmas needed for the proofs

In this section, using the notations defined in Section 2, we discuss how to construct 3-colorable triangulations extended from a given mosaic G by a face rotation system Ξ and an H_Ξ-subdivided $\mathbb{Z}_3$-orientation of $\widehat{S}(G)$.

4.1 Good vectors and orientations of $\widehat{S}(G)$

Let G be a mosaic on a surface F^2, let Ξ be a face rotation system of G, let $\mathcal{W} = \{\gamma_1, \ldots, \gamma_g\}$ be a base of the fundamental group $\pi_1(F^2)$ of F^2 and let $\{P_1, \ldots, P_m\}$ be the set of all components of $\widehat{S}(G)$. Suppose that $\widehat{S}(G)$ has an H_Ξ-subdivided $\mathbb{Z}_3$-orientation $\mathcal{O}_0$. Then consider the crossing system (2) of G with respect to Ξ, $\mathcal{W}$ and $\mathcal{O}_0$. For each good vector $\vec{x} = (x_1, \dots, x_m)^T$, we define the H_Ξ-subdivided $\mathbb{Z}_3$-orientation of $\widehat{S}(G)$, as follows: For $1 \le i \le m$, the orientation of all edges in P_i are the same as (resp. different from) $\mathcal{O}_0$ if $x_i = 1$ (resp. $x_i = 2$). We denote the obtained H_Ξ-subdivided $\mathbb{Z}_3$-orientation by $\mathcal{O}_{\vec{x}}$.

The next two facts for a mosaic G on F^2 with a face rotation system Ξ directly follow from the definitions.

Fact 12. Let $\vec{x} = (x_1, \dots, x_m)^T$ be a good vector. Suppose that $\widehat{S}(G)$ has an H_Ξ-subdivided $\mathbb{Z}_3$-orientation $\mathcal{O}_0$. Then

$$\mathcal{O}_{\vec{x}} = \mathcal{O}_0 \text{ if and only if } \vec{x} = (1, \dots, 1)^T, \text{ and } \mathcal{O}_{\vec{x}} = \overline{\mathcal{O}_0} \text{ if and only if } \vec{x} = (2, \dots, 2)^T.$$

Furthermore, $\mathcal{O}_{\vec{x}}$ and $\mathcal{O}_{-\vec{x}}$ form an orientation pair of $\widehat{S}(G)$.

Fact 13. Let $\mathcal{W} = \{\gamma_1, \dots, \gamma_g\}$ be a base of the fundamental group $\pi_1(F^2)$ of a surface F^2. Suppose that $\widehat{S}(G)$ has an H_Ξ-subdivided $\mathbb{Z}_3$-orientation $\mathcal{O}_0$. Then

$\vec{x}$ is a good solution of the crossing system (2) of G if and only if for $1 \le i \le g$,

$$\mathrm{cr}_\Xi\big(\gamma_i, (\widehat{S}(G), \mathcal{O}_{\vec{x}})\big) \equiv 0 \pmod 3.$$

4.2 H-subdivided $\mathbb{Z}_3$-orientations and even triangulations

The contents in this section were first mentioned in [25, Section 3], and then appeared in [18, Subsection 4.2]. For the extension of a mosaic G to a triangulation, we need to add a diagonal in every quadrangular face of G, from two choices of adding diagonals. In this section, we consider how to choose it, from a given H-subdivided $\mathbb{Z}_3$-orientation of $\widehat{S}(G)$.

Let G be a mosaic on a surface F^2, let $\mathcal{O}$ be an orientation of $\widehat{S}(G)$, let f be a quadrangular face of G, and let $v_1v_2v_3v_4$ be the facial walk of f. For $i \in \{1, 2, 3, 4\}$, let $e_i = v_iv_{i+1}$, where $v_5 = v_1$. Suppose that in $\widehat{S}(G)$, the edge e_1e_3 (resp. e_2e_4) has the direction from e_1 to e_3 (resp. from e_2 to e_4) by $\mathcal{O}$. In this case, the diagonal of f connecting v_1 and v_3 is called the *$\mathcal{O}$-primary diagonal* at f. Figure 7 is an example of the $\mathcal{O}$-primary diagonal. Adding the $\mathcal{O}$-primary diagonal to every quadrangular face of G, we get a triangulation T, which is *induced by the orientation $\mathcal{O}$* of $\widehat{S}(G)$. For example, see Figure 8.

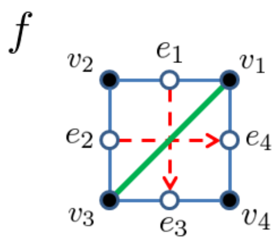

Figure 7: The $\mathcal{O}$-primary diagonal at a quadrangular face f.

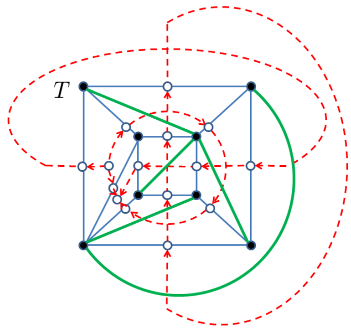

Figure 8: The triangulation T induced by the $\emptyset$-subdivided $\mathbb{Z}_3$-orientation $\mathcal{O}$.

It was shown in [18, Lemma 18] that "an orientation $\mathcal{O}$ of $\widehat{S}(G)$ induces an even triangulation if and only if $\mathcal{O}$ is an H-subdivided $\mathbb{Z}_3$-orientation for some even subgraph H of G." Therefore, an H-subdivided $\mathbb{Z}_3$-orientation also seems to be a good tool for our purpose. The following lemma was proved only for quadrangulations, however it is not difficult to modify them for mosaics. We leave the proof to the readers.

Lemma 14. ([18, Lemma 16]) Let G be a mosaic on a surface F^2 and let H be a subgraph of G. Suppose that both $\mathcal{O}$ and $\mathcal{O}'$ are H-subdivided $\mathbb{Z}_3$-orientations of $\widehat{S}(G)$. Then $\mathcal{O}$ and $\mathcal{O}'$ induce the same triangulation if and only if $\mathcal{O}' = \mathcal{O}$ or $\mathcal{O}' = \overline{\mathcal{O}}$.

4.3 H-face-2-colorings

Let T be a triangulation on a surface F^2 and let H be a subgraph of T. An *H-face-2-coloring* of T is an assignment of two colors, black or white, to each face of T such that two faces sharing an edge e in T receive the same color if and only if $e \in E(H)$. Note that a proper face-2-coloring of T is an H-face-2-coloring for $H = \emptyset$.

4.4 A lemma on mosaics extendable to a 3-colorable triangulation

Lemma 15. Let G be a mosaic on a surface F^2 that can be extended to a 3-colorable triangulation T. Then both of the following hold:

(i) For any face rotation system Ξ of G, there is an H_Ξ-subdivided $\mathbb{Z}_3$-orientation that induces T.

(ii) Furthermore, for any H_Ξ-subdivided $\mathbb{Z}_3$-orientation $\mathcal{O}$ that induces T and for any closed curve γ on F^2, we have $\mathrm{cr}_\Xi\big(\gamma, (\widehat{S}(G), \mathcal{O})\big) \equiv 0 \pmod 3$.

Proof. (i) Let Ξ be a face rotation system of G. We first construct the face rotation system Ξ_T of T from Ξ in the following natural way: Let f be a face of G. If f is triangular, then it is also a face in T, and hence we give the same rotation ξ_f. Otherwise, i.e., if f is quadrangular in G, then f is divided into two faces, say h_1 and h_2, sharing the added diagonal, say e. Then we can give the rotations ξ_{h_1} and ξ_{h_2} to h_1 and h_2, respectively, so that both $f \cap h_1$ and $f \cap h_2$ meet ξ_f. Then the orientations of e along ξ_{h_1} and ξ_{h_2} do not coincide, and hence e is coherent in the new face rotation system. In particular, we have $H_\Xi = H_{\Xi_T}$.

Let $c : V(T) \to \{0, 1, 2\}$ be a 3-coloring of T. Now we define the assignment φ_c of two colors, black or white, to the faces in T, as follows: Note that each face h in T has the face rotation $\xi_h \in \Xi_T$. Let $\varphi_c(h)$ be black (resp. white) if the three colors $0, 1, 2$ (resp. $0, 2, 1$) by c appear on h in that order along ξ_h. Then φ_c is an H_{Ξ_T}-face-2-coloring of T, as explained below. Let h_1 and h_2 be two adjacent faces of T and let e be an edge in T shared by them. If $e \in E(H_{\Xi_T})$, then by the definition of H_{Ξ_T}, the face rotations ξ_{h_1} and ξ_{h_2} coincide at e. So, the order of colors $0, 1, 2$ along $\xi_{h_1} \in \Xi_T$ is the same as that along $\xi_{h_2} \in \Xi_T$. Then by the definition of φ_c, h_1 and h_2 receive the same color. By a similar argument, we can deal with the case when $e \notin E(H_{\Xi_T})$.

Second we define an orientation $\mathcal{O}'$ of $\widehat{S}(G)$ as follows:

- Let f be a triangular face of G bounded by three edges e_1, e_2, e_3. Then orient the three edges fe_1, fe_2, fe_3 in $\widehat{S}(G)$ to be outgoing (resp., incoming) if f is colored black (resp., white) by φ_c.
- Let f be a quadrangular face of G. Then f is divided into two triangular faces, say h_1 and h_2, in T by an added diagonal, say e. Since H_Ξ is a subgraph of G, $e \notin E(H_\Xi)$. Therefore, the colors of h_1 and h_2 by φ_c are different. By symmetry, we may assume that h_1 and h_2 are colored by black and white, respectively. Then we orient two edges of $\widehat{S}(G)$ contained in f so that both are directed from h_2 to h_1. (That is, the black triangular face h_1 includes two heads, while the white triangular face h_2 includes two tails.)

(See Figure 9 for an example on the Klein bottle.) By definition, we see that $\mathcal{O}'$ is an H_Ξ-subdivided $\mathbb{Z}_3$-orientation and induces T. This completes the proof of (i).

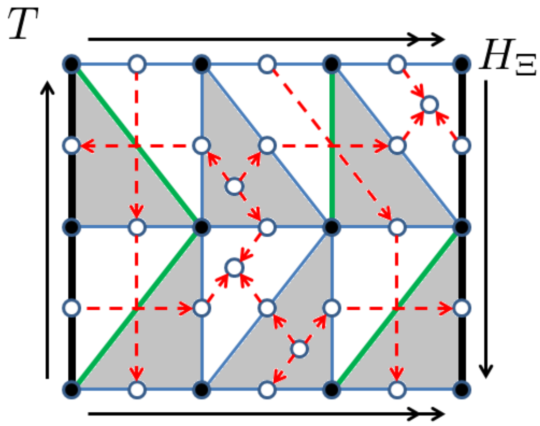

Figure 9: An H_Ξ-subdivided $\mathbb{Z}_3$-orientation induced by an H_Ξ-face-2-coloring of T. The bold line of the left and right side coincide and correspond to H_Ξ.

(ii). Let $\mathcal{O}$ be an H_Ξ-subdivided $\mathbb{Z}_3$-orientation that induces T. Since the orientation $\mathcal{O}'$ defined in the proof of (i) also induces T, it follows from Lemma 14 that $\mathcal{O}' = \mathcal{O}$ or $\mathcal{O}' = \overline{\mathcal{O}}$. By symmetry, we may assume that $\mathcal{O}' = \mathcal{O}$. Let $\widehat{G^*}$ be the map obtained from G^* by subdividing each edge of G^* whenever it meets an edge in G. Note that $\widehat{G^*}$ is also obtained from $\widehat{S}(G)$ by adding a new vertex in G^* in each intersection of edges at a quadrangular face. Therefore, $\widehat{G^*} = \widehat{S}(G)$ if G is a triangulation. It is easy to see that $\widehat{G^*}$ is a bipartite map with bipartition $V(G^*)$ and $E(G)$. We may regard that edges in $\widehat{G^*}$ are naturally oriented by $\mathcal{O}$.

Let γ be a closed curve on F^2, and we consider the crossing number of γ and $(\widehat{S}(G), \mathcal{O})$. Note that it is equivalent to $\mathrm{cr}_\Xi\big(\gamma, (\widehat{G^*}, \mathcal{O})\big)$ since we assumed that any closed curve does not pass through vertices in G^*. Recall that c is a 3-coloring of T. So c naturally gives a color to each face of G^*, and hence to each face of $\widehat{G^*}$. Whenever γ crosses $\widehat{G^*}$, the colors of faces of $\widehat{G^*}$ by c are changed. The next statement is crucial.

Claim 15.1. Let p be a crossing point of γ and $(\widehat{S}(G), \mathcal{O})$, let u^* (resp. v^*) be a face of $\widehat{G^*}$ through which γ passes just before (resp. after) p. Then we have

$$c(v^*) \equiv c(u^*) + \mathrm{cr}_\Xi\big(\gamma, (\widehat{S}(G), \mathcal{O}); p\big) \pmod 3.$$

Proof. Recall that T is obtained from G by adding a diagonal in every quadrangular face f. Here we may assume that such a diagonal passes through the vertex f^* in G^*. This assumption implies that each edge in $\widehat{G^*}$ is entirely contained in a triangular face of T. Let e be an edge in $\widehat{G^*}$ containing the crossing point p and let h be the face of T including e. Recall also that $\varphi_c(h)$ is black (resp. white) if the three colors $0, 1, 2$ (resp. $0, 2, 1$) by c appear on h in that order along $\xi_h \in \Xi_T$. We consider only the case where $\varphi_c(h)$ is black, since the other case can be symmetrically shown. Since $\widehat{G^*}$ is bipartite, one of the end vertices, say w, of e is contained in $V(G^*)$ and the other, z, in $E(G)$. By the construction of the H_Ξ-subdivided $\mathbb{Z}_3$-orientation $\mathcal{O}' = \mathcal{O}$, we have the following two cases. (See Figure 10 for example.)

(a) h is also a face of G and the out-degree of h^* is three in $(\widehat{S}(G), \mathcal{O})$.

(b) h is obtained from a quadrangular face of G by adding a diagonal, and includes two heads of edges in $(\widehat{S}(G), \mathcal{O})$.

In either case, we see that the edge e is directed from w to z. Furthermore,

$$\begin{aligned} \text{if } \mathrm{cr}_\Xi\big(\gamma, (\widehat{S}(G), \mathcal{O}); p\big) = +1, &\quad \text{then } c(v^*) \equiv c(u^*) + 1 \pmod 3, \\ \text{and if } \mathrm{cr}_\Xi\big(\gamma, (\widehat{S}(G), \mathcal{O}); p\big) = -1, &\quad \text{then } c(v^*) \equiv c(u^*) - 1 \pmod 3. \end{aligned}$$

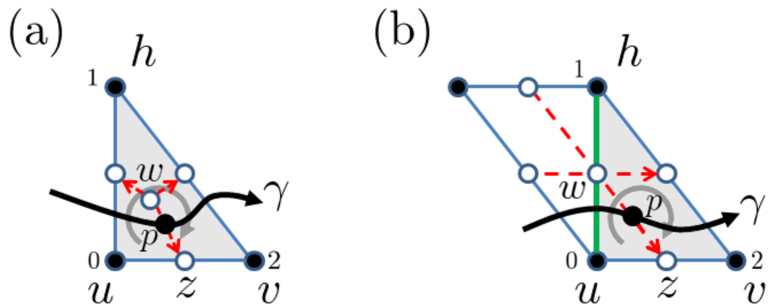

Figure 10: The cases (a) and (b) with $\mathrm{cr}_\Xi\big(\gamma, (P, \mathcal{O}); p\big) = -1$.

This implies the conclusion of Claim 15.1. □

Let $u_1^*, u_2^*, \ldots, u_t^*, u_{t+1}^*$ be the sequence of faces of $\widehat{G^*}$ such that γ passes through them in that order, where $u_{t+1}^* = u_1^*$. For $1 \le i \le t$, let p_i be the crossing point of γ and $(\widehat{S}(G), \mathcal{O})$ such that p_i appears on the boundaries of u_i^* and u_{i+1}^*. By Claim 15.1, we have

$$c(u_{i+1}^*) \;\equiv\; c(u_i^*) + \mathrm{cr}_\Xi\big(\gamma, (\widehat{S}(G), \mathcal{O}); p_i\big) \pmod 3.$$

Therefore, it follows from (2) in Subsection 2.7 that

$$\begin{aligned} c(u_1^*) \;=\; c(u_{t+1}^*) &\;\equiv\; c(u_1^*) + \sum_{i=1}^{t} \mathrm{cr}_\Xi\big(\gamma, (\widehat{S}(G), \mathcal{O}); p_i\big) \\ &\;=\; c(u_1^*) + \mathrm{cr}_\Xi\big(\gamma, (\widehat{S}(G), \mathcal{O})\big) \pmod 3. \end{aligned}$$

This implies that $\mathrm{cr}_\Xi\big(\gamma, (\widehat{S}(G), \mathcal{O})\big) \equiv 0 \pmod 3$, which completes the proof. □

4.5 A lemma on an H_Ξ-subdivided $\mathbb{Z}_3$-orientation

Lemma 16. Let G be a mosaic on a surface F^2, let Ξ be a face rotation system of G and let $\mathcal{W} = \{\gamma_1, \ldots, \gamma_g\}$ be a base of the fundamental group $\pi_1(F^2)$ of F^2. Suppose that $\widehat{S}(G)$ has an H_Ξ-subdivided $\mathbb{Z}_3$-orientation $\mathcal{O}$. If

$$\mathrm{cr}_\Xi\big(\gamma_i, (\widehat{S}(G), \mathcal{O})\big) \;\equiv\; 0 \pmod 3$$

for $1 \le i \le g$, then $\mathcal{O}$ induces a 3-colorable triangulation on F^2.

Proof. Let T be the triangulation on F^2 induced by $\mathcal{O}$. We regard each edge uv in T as two directed edges (u, v) and (v, u). Then define the mapping $\beta : \big\{(u, v) : uv \in E(T)\big\} \to \{1, -1\}$ such that

$$\beta\big((v, u)\big) \;=\; -\beta\big((u, v)\big)$$

for any $uv \in E(T)$, as follows:

(a) Suppose that $uv \in E(T) - E(G)$. Then uv is the $\mathcal{O}$-primary diagonal at some quadrangular face f of G, say, bounded by $uxvy$, where we suppose that this ordering of the four vertices is along the rotation ξ_f. If the directed edges of $\big(\widehat{S}(G), \mathcal{O}\big)$ are from ux to vy and from vx to uy, then let $\beta\big((u, v)\big) = +1$ and $\beta\big((v, u)\big) = -1$; if the directions are from vy to ux and from uy to vx, then let $\beta\big((u, v)\big) = -1$ and $\beta\big((v, u)\big) = +1$.

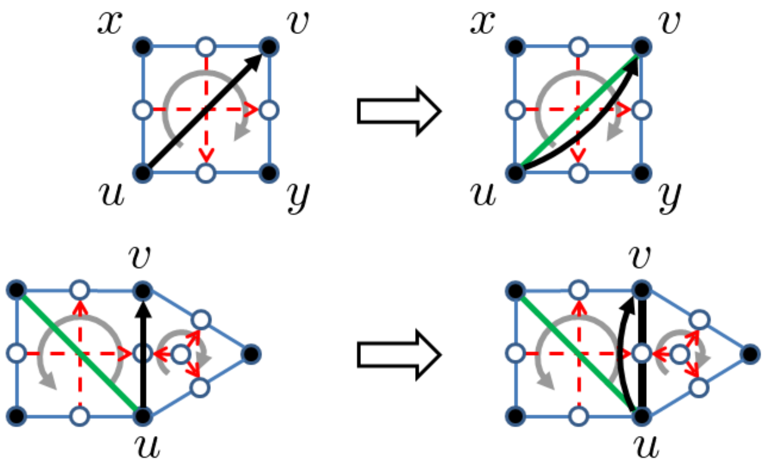

Figure 11: Modifying (u, v) in the cases (a) and (c). The thick black arrows represent directed edges (u, v) with $\beta\big((u, v)\big) = +1$.

(b) Suppose that $uv \in E(G) - E(H_\Xi)$. Let f_1 and f_2 be the two faces in G sharing the edge uv. Since $uv \notin E(H_\Xi)$, the face rotations ξ_{f_1} and ξ_{f_2} do not coincide at uv. By symmetry, we may assume that the direction (u, v) coincides with ξ_{f_1}. (So, the direction (v, u) coincides ξ_{f_2}.) Since $\mathcal{O}$ is an H_Ξ-subdivided $\mathbb{Z}_3$-orientation, it follows from (H2) in Definition 5 that the in-degree and the out-degree of uv in $\big(\widehat{S}(G), \mathcal{O}\big)$ is both 1. If the directed edge of $\big(\widehat{S}(G), \mathcal{O}\big)$ incoming to uv is contained in the face f_1 (that is, the directed edge outgoing from uv is contained in the face f_2), then let $\beta\big((u, v)\big) = +1$ and $\beta\big((v, u)\big) = -1$; if the directed edge of $\big(\widehat{S}(G), \mathcal{O}\big)$ outgoing from uv is contained in the face f_1 (that is, the directed edge incoming to uv is contained in the face f_2), then let $\beta\big((u, v)\big) = -1$ and $\beta\big((v, u)\big) = +1$.

(c) Suppose that $uv \in E(H_\Xi)$. Then it follows from (H3) in Definition 5 that the out-degree of uv in $\big(\widehat{S}(G), \mathcal{O}\big)$ is 0 or 2. Let f_1, f_2 be the two faces of G sharing uv. Since $uv \in E(H_\Xi)$, the two orientations ξ_{f_1} and ξ_{f_2} coincide at the edge uv. By symmetry, we may assume that it is directed from u to v. If the out-degree of uv in $\big(\widehat{S}(G), \mathcal{O}\big)$ is 0, then let $\beta\big((u, v)\big) = +1$ and $\beta\big((v, u)\big) = -1$; if the out-degree of uv in $\big(\widehat{S}(G), \mathcal{O}\big)$ is 2, then let $\beta\big((u, v)\big) = -1$ and $\beta\big((v, u)\big) = +1$.

Note that the directed edge (u, v) passes through a crossing point of two edges of $(\widehat{S}(G), \mathcal{O})$ (in the case (a)) or a vertex of $\widehat{S}(G)$ (in the cases (b) and (c)). By this reason, we cannot directly apply the definition of the crossing number of (u, v) from $(\widehat{S}(G), \mathcal{O})$ as in Subsection 2.7. However, modifying (u, v) slightly by homotopic shift, we are allowed to do that. (See Figure 11 for example in the cases (a) and (c).) Notice that in either case, the choice of a face to which (u, v) moves does not change the crossing number. By the definition, we have

$$\beta\big((u, v)\big) \equiv \mathrm{cr}_\Xi\big((u, v), (\widehat{S}(G), \mathcal{O})\big) \pmod 3 \quad \text{for any edge } uv \text{ in } T. \tag{3}$$

Using the mapping β, we define the assignment $c : V(T) \to \{0, 1, 2\}$ inductively as follows:

(i) Let u_0 be a fixed vertex in T. Then set $c(u_0) = 0$.

(ii) Let v be a vertex in T which has not yet been assigned a color by c, but a neighbor u of v already has a color. Then define

$$c(v) \equiv c(u) + \beta\big((u, v)\big) \pmod 3.$$

Since T is connected, the above definition of c gives the assignment to all vertices in T. We prove that c is actually a 3-coloring of T. Before that, we show the next claim, which is crucial in our proof.

Claim 16.1. For any directed cycle $\overrightarrow{W}$ of T, we have

$$\sum_{(u,v)\in E(\overrightarrow{W})} \beta\big((u,v)\big) \;\equiv\; 0 \pmod 3.$$

Proof. Let $\overrightarrow{W}$ be a directed cycle of T. We can regard $\overrightarrow{W}$ as a closed curve on F^2. So, since $\mathcal{W} = \{\gamma_1, \dots, \gamma_g\}$ is a base of the fundamental group $\pi_1(F^2)$ of F^2, there exists an integer r such that $\overrightarrow{W}$ is homotopic to $\gamma = \eta_1 * \cdots * \eta_r$, where for $1 \le i \le r$, $\eta_i = \gamma_j$ or $\eta_i = -\gamma_j$ for some $1 \le j \le g$. By the assumption of Lemma 16, we have

$$\mathrm{cr}_{\Xi}\big(\eta_i, (\widehat{S}(G), \mathcal{O})\big) \;\equiv\; 0 \pmod 3$$

for $1 \le i \le r$. Therefore,

$$\mathrm{cr}_{\Xi}\big(\gamma, (\widehat{S}(G), \mathcal{O})\big) \;=\; \sum_{i=1}^{r} \mathrm{cr}_{\Xi}\big(\eta_i, (\widehat{S}(G), \mathcal{O})\big) \;\equiv\; 0 \pmod 3.$$

It follows from Lemma 9 and (3) that

$$\begin{aligned}
\sum_{(u,v)\in E(\overrightarrow{W})} \beta\big((u,v)\big) \;&\equiv\; \sum_{(u,v)\in E(\overrightarrow{W})} \mathrm{cr}_{\Xi}\big((u,v), (\widehat{S}(G), \mathcal{O})\big) \\
&=\; \mathrm{cr}_{\Xi}\big(\overrightarrow{W}, (\widehat{S}(G), \mathcal{O})\big) \\
&\equiv\; \mathrm{cr}_{\Xi}\big(\gamma, (\widehat{S}(G), \mathcal{O})\big) \;\equiv\; 0 \pmod 3.
\end{aligned}$$

This completes the proof of Claim 16.1. □

Now we are ready to prove that c is a 3-coloring of T. Suppose contrary that there is an edge uv in T such that $c(u) = c(v)$. Let $u_0, u_1, \dots, u_s$ be the path in T with $u_s = u$ such that the vertex u_{i-1} is used to define $c(u_i)$ for $1 \le i \le s$. Similarly, let $v_0, v_1, \dots, v_t$ be the path in T with $v_t = v$ such that the vertex v_{j-1} is used to define $c(v_j)$ for $1 \le j \le t$, where we choose $v_0 = u_0$. Let r be the maximum integer such that $u_r = v_r$. Since $u_0 = v_0$, such an integer r must exist. It follows from the definition of c that

$$\begin{aligned}
c(u_s) \;&\equiv\; c(u_r) + \sum_{i=r+1}^{s} \beta\big((u_{i-1}, u_i)\big), \\
\text{and}\quad c(v_t) \;&\equiv\; c(v_r) + \sum_{j=r+1}^{t} \beta\big((v_{j-1}, v_j)\big) \pmod 3.
\end{aligned}$$

Therefore, since $c(u_s) = c(u) = c(v) = c(v_t)$ and $c(u_r) = c(v_r)$, we have

$$\begin{aligned}
0 \;=\; c(u_s) - c(v_t) \;&\equiv\; \sum_{i=r+1}^{s} \beta\big((u_{i-1}, u_i)\big) - \sum_{j=r+1}^{t} \beta\big((v_{j-1}, v_j)\big) \\
&\equiv\; \sum_{i=r+1}^{s} \beta\big((u_{i-1}, u_i)\big) + \sum_{j=r+1}^{t} \beta\big((v_j, v_{j-1})\big) \pmod 3. \qquad (4)
\end{aligned}$$

Let $\overrightarrow{W}$ be the directed cycle, defined as

$$\overrightarrow{W} \;=\; u_r, u_{r+1}, \dots, u_s, v_t, v_{t-1}, \dots, v_{r+1}, v_r.$$

Recall that $v_r = u_r$, $u_s = u$, $v_t = v$ and $uv \in E(T)$, and hence $\overrightarrow{W}$ is indeed a directed cycle in T. By Claim 16.1, we have

$$\sum_{(u',v')\in E(\overrightarrow{W})} \beta\big((u',v')\big) \equiv 0 \pmod 3.$$

However, it follows from (4) that

$$\begin{aligned}\beta\big((u_s,v_t)\big) &\equiv \sum_{(u',v')\in E(\overrightarrow{W})} \beta\big((u',v')\big) - \left(\sum_{i=r+1}^{s} \beta\big((u_{i-1},u_i)\big) + \sum_{j=r+1}^{t} \beta\big((v_j,v_{j-1})\big)\right) \\ &\equiv 0 \pmod 3,\end{aligned}$$

which contradicts that $\beta\big((u_s,v_t)\big) = +1$ or -1. Therefore, $c(u) \neq c(v)$ for any edge uv in T, and hence c is a 3-coloring of T. □

Remark 5: For any 3-colorable triangulation, a 3-coloring is unique, up to permutations of colors. Thus, the proof of Lemma 16 also shows that the 3-coloring c is well-defined; it does not depend on the choice of the starting vertex u_0 in (i) and the order of vertices to which we color in (ii).

5 Proofs of Theorems 10 and 11

Proof of Theorem 10. We first prove the "only if" part. Suppose that G is extended to a 3-colorable triangulation T. Let Ξ be a face rotation system of G. By Lemma 15(i), $\widehat{S}(G)$ has an H_Ξ-subdivided $\mathbb{Z}_3$-orientation $\mathcal{O}$ which induces T, and by Lemma 8, G satisfies the orientizing condition. On the other hand, by Fact 13 and Lemma 15(ii), the crossing system (2) of G has a good solution. These complete the proof of the "only if" part.

Next we prove the "if" part. Suppose that G satisfies the orientizing condition. Let Ξ be a face rotation system of G. By Lemma 8, $\widehat{S}(G)$ has an H_Ξ-subdivided $\mathbb{Z}_3$-orientation. Since the crossing system (2) of G has a good solution $\vec{x}$, Fact 13 implies that the H_Ξ-subdivided $\mathbb{Z}_3$-orientation $\mathcal{O}_{\vec{x}}$ satisfies

$$\mathrm{cr}_\Xi\big(\gamma_i, (\widehat{S}(G), \mathcal{O}_{\vec{x}})\big) \equiv 0 \pmod 3$$

for $1 \le i \le g$. Therefore, it follows from Lemma 16 that $\mathcal{O}_{\vec{x}}$ induces a 3-colorable triangulation. This completes the proof of the "if" part. □

Proof of Theorem 11. Suppose that G satisfies the orientizing condition. By Lemma 8, $\widehat{S}(G)$ has an H_Ξ-subdivided $\mathbb{Z}_3$-orientation, say $\mathcal{O}_0$, where Ξ is a face rotation system of G. Furthermore, assume that the crossing system (2) of G with respect to Ξ, $\mathcal{W}$ and $\mathcal{O}_0$ has exactly n good solutions, where $\mathcal{W}$ is a base of the fundamental group $\pi_1(F^2)$ of F^2.

First, we show the upper bound of the number of distinct 3-colorable triangulations extended from G. For that purpose, it suffices to prove that for each 3-colorable triangulation T extended from G, there are two good solutions of the crossing system (2) of G, which corresponds to T. It follows from Lemma 15(i) that there exists an H_Ξ-subdivided $\mathbb{Z}_3$-orientation $\mathcal{O}$ of $\widehat{S}(G)$ that induces T. By Lemma 14, $\overline{\mathcal{O}}$ also induces T. It follows from Lemma 15(ii) that for any closed curve γ on F^2, we have $\mathrm{cr}_\Xi\big(\gamma, (\widehat{S}(G), \mathcal{O})\big) \equiv 0 \pmod 3$. In particular,

$$\mathrm{cr}_\Xi\big(\gamma_i, (\widehat{S}(G), \mathcal{O})\big) \equiv 0 \pmod 3$$

for $1 \le i \le g$. Then by Fact 13, there exists a good solution $\vec{x}$ of the crossing system (2) of G with respect to Ξ, $\mathcal{W}$ and $\mathcal{O}_0$ such that $\mathcal{O} = \mathcal{O}_{\vec{x}}$. The same holds also for $\overline{\mathcal{O}}$, which corresponds to the good solution $-\vec{x}$ by Fact 12. Thus, there are two good solutions of the crossing system (2) of G, which corresponds to T. By Lemma 14, for two distinct 3-colorable triangulations T_1 and T_2 extended from G, the pairs of corresponding two good solutions are disjoint. This shows that the number of distinct 3-colorable triangulations extended from G is at most $n/2$.

Conversely, we show the lower bound. Suppose that the crossing system (2) of G has a good solution $\vec{x}$. By Fact 13, we see that the H_Ξ-subdivided $\mathbb{Z}_3$-orientation $\mathcal{O}_{\vec{x}}$ of $\widehat{S}(G)$ satisfies that

$$\mathrm{cr}_\Xi\big(\gamma_i, (\widehat{S}(G), \mathcal{O}_{\vec{x}})\big) \equiv 0 \pmod 3$$

for $1 \le i \le g$. By Lemma 16, $\mathcal{O}_{\vec{x}}$ induces a 3-colorable triangulation. Suppose that for two good solutions $\vec{x}$ and $\vec{y}$ of the crossing system (2) of G, the H_Ξ-subdivided $\mathbb{Z}_3$-orientations $\mathcal{O}_{\vec{x}}$ and $\mathcal{O}_{\vec{y}}$ of $\widehat{S}(G)$ induce the same 3-colorable triangulation. It follows from Lemma 14 that $\mathcal{O}_{\vec{x}} = \mathcal{O}_{\vec{y}}$ or $\mathcal{O}_{\vec{x}} = \overline{\mathcal{O}_{\vec{y}}}$, and hence $\vec{x} = \vec{y}$ or $\vec{x} = -\vec{y}$ by Fact 12. This shows that exactly two good solutions create the same 3-colorable triangulation extended from G, and hence the number of distinct 3-colorable triangulations extended from G is at least $n/2$.

These complete the proof of Theorem 11. □

6 Remarks and corollaries

In this section, we show that our main theorems improve several known results and have an application to polychromatic coloring.

6.1 The planar case

Let G be a mosaic on the plane. By the definition of the orientizing condition, the extended intersection system (1) of G has a solution if and only if $S(G)$ is bipartite. Furthermore, in the planar case, the crossing matrix $\mathrm{CM}(G)$ of G is degenerate, and hence any good vector is a good solution of the crossing system (2) of G. Therefore, by Theorem 10, a mosaic G on the plane can be extended to 3-colorable triangulation if and only if $S(G)$ is bipartite. (This fact was pointed out by Hoffmann-Ostenhoff [9].)

Furtheremore, there are 2^m good vectors, where m is the number of components of $\widehat{S}(G)$. So, by Theorem 11, G can be extended to exactly $2^m/2 = 2^{m-1}$ distinct 3-colorable triangulations. This is indeed equivalent to [18, Theorem 13] and [25, Theorem 6]. To be exact, those considered the extension to "even" triangulations instead of 3-colorable ones, which are equivalent in the planar case by Fact 2.

6.2 The case of the projective plane

We put a remark to the case of the projective plane $\mathbb{N}_1$, from a topological point of view. Let $[\gamma_1]$ be the non-identity element of $\pi_1(\mathbb{N}_1)$, where γ_1 is homotopic to a_1 in Figure 2. The important property for γ_1 is that "$\gamma_1 * \gamma_1$ is contractible." Therefore, it follows from Lemma 9 that for any H_Ξ-subdivided $\mathbb{Z}_3$-orientation $\mathcal{O}$ of $\widehat{S}(G)$ and any component P_j of $\widehat{S}(G)$, we have

$$2 \cdot \mathrm{cr}_\Xi\big(\gamma_1, (P_j, \mathcal{O})\big) \equiv 0 \pmod 3,$$

which directly implies

$$\mathrm{cr}_\Xi\big(\gamma_1, (P_j, \mathcal{O})\big) \equiv 0 \pmod 3.$$

So, all entries of the crossing matrix $\mathrm{CM}(G)$ of G are zero, and hence any good vector $\vec{x}$ is a good solution of it. Therefore, we have the following corollary of Theorem 10.

Corollary 17. Let G be a mosaic of the projective plane. Then G can be extended to a 3-colorable triangulation if and only if G satisfies (I) the orientizing condition.

Furthermore, Theorem 11 implies that if G satisfies the orientizing condition, then G can be extended to exactly 2^{m-1} distinct 3-colorable triangulations, where m is the number of components of $\widehat{S}(G)$.

Remark 6: The property in this subsection holds also for nonorientable surfaces other than the projective plane. Let $[\gamma_0]$ be the element of $\pi_1(\mathbb{N}_k)$, where γ_0 is homotopic to $a_1 * \ldots a_k$ in Figure 2. (Note that γ_0 corresponds to an orientizing subgraph.) Then it is well-known that "$\gamma_0 * \gamma_0$ is *separating*," i.e. $\mathbb{N}_k \setminus (\gamma_0 * \gamma_0)$ is not arcwise connected. Though we have only shown in Lemma 9 that $\mathrm{cr}_\Xi\big(\gamma, (P_j, \mathcal{O})\big) \equiv 0 \pmod 3$ for any contractible curve γ, the same holds also for any separating curve γ. Therefore, the same argument implies that for any H_Ξ-subdivided $\mathbb{Z}_3$-orientation $\mathcal{O}$ of $\widehat{S}(G)$ and any component P_j of $\widehat{S}(G)$, we have

$$\mathrm{cr}_\Xi\big(\gamma_0, (P_j, \mathcal{O})\big) \equiv 0 \pmod 3.$$

Thus, the rank of the crossing matrix $\mathrm{CM}(G)$ is always at most $k-1$ if G is a mosaic of the nonorientable surface $\mathbb{N}_k$.

6.3 The case of quadrangulations

Now we consider quadrangulations G on a surface F^2. In this case, the S-walk dual $S(G)$ of G consists of no vertices and only S-walks. Therefore, trivially $\vec{c} = \vec{0}$ for any base $\mathscr{C}$ of the cycle space of $\widehat{S}(G)$. In fact, the cycle space of $\widehat{S}(G)$ has the unique base, which is the set of all S-walks of G^*.

Let S be an S-walk of G^* and suppose that S is contractible. Since S is contractible, it intersects with any even subgraph of G even number of times. Furthermore, for any closed curve γ on F^2, the crossing number of γ and $(S, \mathcal{O})$ is zero in total, where $\mathcal{O}$ is an orientation of S.

This fact implies that if all S-walks of G^* is contractible, then all entries of the intersection matrix $\mathrm{IM}(G)$ and the crossing matrix $\mathrm{CM}(G)$ are zero. Thus, any vector is a solution of the extended intersection system (1) of G, and any good vector is a good solution of the crossing system (2) of G. This and Theorem 10 imply the next corollary.

Corollary 18. ([17, Theorem 1]) Let G be a quadrangulation on any surface. If all S-walks are contractible, then G can be extended to a 3-colorable triangulation.

Furthermore, consider the projective planar case. Let G be a quadrangulation on the projective plane $\mathbb{N}_1$ and let Ξ be a face rotation system of G. In this case, the intersection matrix $\mathrm{IM}(G)$ is an $(\ell \times 1)$-matrix, where ℓ is the number of S-walks of G^*. Suppose that G can be extended to a 3-colorable triangulation. By Theorem 10, G satisfies the orientizing condition, that is, the vector $\vec{1} = (1)^T$ is a solution of the extended intersection system (1) of G. This implies that $\mathrm{IM}(G) = \mathrm{IM}(G)\vec{1} = \vec{c} = \vec{0}$, which means that each S-walk of G^* meets $\widetilde{W_1}$ an even number of times, where $\{\widetilde{W_1}\}$ is a base of the $\mathbb{Z}_2$-homology space of G. Recall that $\widetilde{W_1}$ is obtained from a spanning closed walk W_1 of G that is homotopic to an essential curve on $\mathbb{N}_1$. Note that for two closed curves on $\mathbb{N}_1$, they intersect odd number of times if and only if both are essential. Therefore, each S-walk of G^* must be contractible. This gives a proof of the converse of Corollary 18 for the projective planar case. Namely:

Corollary 19. ([17, Theorem 1]) Let G be a quadrangulation on the projective plane. Then G can be extended to a 3-colorable triangulation if and only if each S-walk of G^* is contractible.

6.4 Extension of 3-colorable quadrangulations

In Section 1, we posed the question whether every 3-colorable quadrangulation can be extend to a 3-colorable triangulation. Actually, Theorem 10 gives the answer NO. Consider quadrangulations

G on an orientable surface with only one S-walk of G^* such that the *edge-width* of G is high enough. (See [16, Pages 129–130] for the definition of the edge-width.) By the result by Hutchinson [12], such quadrangulation G is 3-colorable. However, we can see that if the crossing number of γ_i and the unique S-walk with some orientation is not zero modulo 3, where γ_i is a closed curve in a base of the fundamental group $\pi(F^2)$, then trivially the crossing system (2) of G does not have a good solution. Thus, it follows from Theorem 10 that G cannot be extend to a 3-colorable triangulation. On a non-orientable surface, it is known that a quadrangulation G is 3-colorable if the edge-width of G is high enough and the length of an orientizing closed walk of G is even. Then the same kind of examples do exist.

6.5 The case of triangulations

Let G be a triangulation on a surface. In this case, G can be extended to 3-colorable triangulation if and only if G itself is 3-colorable. Therefore, Theorem 10 (and also Theorem 11) gives us how to check whether a triangulation G is 3-colorable. To be exact, this is equivalent to check the monodoromy σ_G of G, see Remark 4 in Section 3.

6.6 Application to polychromatic coloring

A *polychromatic k-coloring* of a map G on a surface is a (not necessarily a proper) k-coloring of G such that all k colors appear on the vertices of the facial cycle of any face of G. Horev and Krakovski [11] showed that every planar subcubic map has a proper polychromatic 3-coloring, except for the complete graph K_4 on 4 vertices and a subdivision of K_4. Alon et al. [1] showed that every planar map with girth at least 6 has a (not necessarily proper) polychromatic 3-coloring. There are also some results of proper polychromatic 4-colorings of maps [10, 15, 19]. It is easy to see that a mosaic G has a proper polychromatic 3-coloring if and only if G can be extended to a 3-colorable triangulation. Therefore, Theorem 10 implies the following corollary.

Corollary 20. Let G be a mosaic on a surface F^2. Then G has a proper polychromatic 3-coloring if and only if G satisfies the orientizing condition and the crossing system (2) of G has a good solution.

Furthermore, any results in this paper can be regarded as those for proper polychromatic 3-colorings.

Acknowledgements

The first author's work (A. Nakamoto) was partially supported by JSPS KAKENHI Grant Number 15K04975. The third author's work (K. Ozeki) was partially supported by JSPS KAKENHI Grant Number 25871053, and by Grant for Basic Science Research Projects from the Sumitomo Foundation.

References

[1] N. Alon, R. Berke, K. Buchin, M. Buchin, P. Csorba, S. Shannigrahi, B. Speckmann, and P. Zumstein, Polychromatic colorings of plane graphs, Discrete Comput. Geom. 42 (2009), 421–442.

[2] K.A. Berman and H. Shank, Full 4-colorings of 4-regular maps, J. Graph Theory 3 (1979), 291–294.

[3] R. Diestel, Graph Theory, 4th edition (Graduate Texts in Mathematics **173**), Springer (2010).

[4] S. Fisk, A short proof of Chvátal's watchman theorem, J. Combin. Theory Ser. B 24 (1978), 374.

[5] Z. Füredi and D.J. Kleitman, The prison yard problem, Combinatorica 14 (1994), 287–300.

[6] P.J. Heawood, On the four-colour map theorem, Quart. J. Pure Appl. Math. 29 (1898), 270–285.

[7] Y. Higuchi, A. Nakamoto, K. Ota and T. Sakuma, N-Flips in even triangulations and Dehn twists on the torus, Discrete Math. 311 (2011), 1128–1135.

[8] F. Hoffmann and K. Kriegel, A graph coloring result and its consequences for polygon-guarding problems, SIAM J. Discrete Math. 9 (1996), 210–224.

[9] A. Hoffmann-Ostenhof, Private communication.

[10] E. Horev, M.J. Katz, R. Krakovski and A. Nakamoto, Polychromatic 4-coloring of cubic bipartite plane graphs, Discrete Math. 312 (2012), 715–719.

[11] E. Horev and R. Krakovski, Polychromatic colorings of bounded degree plane graphs, J. Graph Theory 60 (2009), 269–283.

[12] J.P. Hutchinson, Three-coloring graphs embedded on surfaces with all faces even-sided, J. Combin. Theory Ser. B 65 (1995), 139–155.

[13] F. Jaeger, Nowhere-zero flow problems, in Selected Topics in Graph Theory 3, (L.W. Beineke and R. J. Wilson eds.), Academic Press, London, 1988, 71–95.

[14] F. Jaeger, On circular flows in graphs, Proc. Colloq. Math. Soc. János Bolyai, 37 (1984), 391–402.

[15] M. Kobayashi, A. Nakamoto and T. Yamaguchi, Polychromatic 4-coloring of cubic even embeddings on the projective plane, Discrete Math. 313 (2013), 2423–2431.

[16] B. Mohar and C. Thomassen, Graphs on Surfaces, Johns Hopkins University Press, Baltimore and London, 2001.

[17] R. Mukae, A. Nakamoto, Y. Suzuki, 3-Polychromatic quadrangulations on surfaces, Congr. Numer. 219 (2014), 43–51.

[18] A. Nakamoto, K. Noguchi and K. Ozeki, Extension to even triangulations, SIAM J. Discrete Math. 29 (2015), 2075–2087.

[19] A. Nakamoto, K. Noguchi and K. Ozeki, Cyclic 4-colorings of graphs on surfaces, J. Graph Theory 82 (2016), 265–278.

[20] E.H. Spanier, Algebraic Topology, Springer, 1991.

[21] M.-T. Tsai, D.B. West, A new proof of 3-colorability of Eulerian triangulations, Ars Math. Contemp. 4 (2011), 73–77.

[22] W.T. Tutte, A contribution on the theory of chromatic polynomial, Canad. J. Math. 6 (1954), 80–91.

[23] C.Q. Zhang, Circuit double cover of graphs, Cambridge University Press, Cambridge, United Kingdom, 2012.

[24] C.Q. Zhang, Integer flows and cycle covers of graphs, Marcel Dekker, New York, 1997.

[25] H. Zhang and X. He, On even triangulations of 2-connected embedded graphs, SIAM J. Comput. 34 (2005), 683–696.