$[a, b]$-Factors of Graphs on Surfaces

Ryota Matsubara[1], Haruhide Matsuda[1], Nana Matsuo[1],
Kenta Noguchi[2], and Kenta Ozeki[3]

[1] Department of Mathematics, Shibaura Institute of Technology
307 Fukasaku, Saitama, 337-8577 Japan
{ryota, hmatsuda, mf15069}@sic.shibaura-it.ac.jp

[2] Department of Mathematics, Tokyo Denki University
5 Senjuasahicho, Adachi-ku, Tokyo, 120-8551 Japan
noguchi@mail.dendai.ac.jp

[3] Faculty of Environment and Information Sciences,
Yokohama National University
79-7 Tokiwadai, Hodogaya-ku, Yokohama, 240-8501 Japan
ozeki-kenta-xr@ynu.ac.jp

April 24, 2017

Abstract

A well-known conjecture of Grünbaum [2] and Nash-Williams [5] asserts that every 4-connected toroidal graph has a Hamiltonian cycle. Related to this conjecture, Kawarabayashi and Ozeki [3] proved two results on a 2-factor and a 3-factor. In this paper, motivated by these results, we give several sufficient conditions for a graph embedded in a surface to have an $[a, b]$-factor. We also show that several conditions are best possible.

Keywords: graph; surface; factor; Hamiltonian cycle

1 Introduction and main theorems

Tutte [8] proved that every 4-connected planar graph has a Hamiltonian cycle. Thomas and Yu [6] extended the Tutte's theorem to that for projective planar graphs. These results are sharp in the sense that we cannot replace the condition "4-connected" by "3-connected". For 4-connected graphs embedded in a torus, a well-known conjecture of Grünbaum [2] and Nash-Williams [5] asserts that every 4-connected toroidal graph has a Hamiltonian cycle. While this conjecture is still open, Thomas and Yu [7] proved that every 5-connected toroidal graph has a Hamiltonian cycle. Since a Hamiltonian cycle is a special kind of a 2-factor, which is a two-regular spanning subgraph, it is quite natural to consider the existence of a 2-factor. In fact, Dean and Ota [1] showed that every 4-connected graph on a torus has a 2-factor. This is a fundamental research for the above-mentioned conjecture by Grünbaum and Nash-Williams. Note that the connectivity condition for the existence of a 2-factor is also best possible.

Concerning the condition that face-width is sufficiently large, Kawarabayashi and Ozeki [3] proved two results: One is that every 4-connected graph embedded in a surface of Euler genus g with minimum degree at least 5 and face-width at least $4g-12$ has a 2-factor. The other is that every 5-connected graph of even order embedded in a surface of Euler genus g with face-width at least $\max\{44g-117, 5\}$ has a 3-factor, which is a three-regular spanning subgraph. The connectivity conditions of these results are best possible.

Let G be a graph and let a and b be two integers with $1 \le a \le b$. A spanning subgraph F of G each of whose degree is between a and b is called an $[a,b]$*-factor*. In particular, if $a = b = k$, then an $[a,b]$-factor is a k*-factor*, which is a k-regular spanning subgraph.

Motivated by above results, we obtain the following result on an $[a,b]$-factor.

Theorem 1.1. *Let $1 \le a < b$ and $b \ge 3$ be integers and let G be a graph embedded in a surface Σ of Euler genus g. Suppose that $\delta(G) \ge a+2$ and face-width of G is at least*

$$\frac{(b+1)(2g-4)-2}{b-2}.$$

Then G has an $[a,b]$-factor.

In particular, Theorem 1.1 with $a = 2$ and $b = 3$ gives the following corollary.

Corollary 1.2. *Let G be a graph embedded in a surface Σ of Euler genus g. Suppose that $\delta(G) \ge 4$ and face-width of G is at least $8g-18$. Then G has a $[2,3]$-factor.*

We can regard a 2-factor and a 3-factor as an extremal structure of a $[2,3]$-factor, and hence Corollary 1.2 lies between the two results in [3]. Note that we do not need any connectivity condition for the existence of a $[2,3]$-factor, while it is required for the existence of a 2-factor or a 3-factor.

Although Theorem 1.1 does not contain the case $a = 1$ and $b = 2$, we also show the following result.

Theorem 1.3. *Let G be a graph embedded in a surface Σ of Euler genus $g \le 2$. Suppose that $\delta(G) \ge 3$. Then G has a $[1,2]$-factor.*

As we will show in Section 5.1, the condition of Euler genus $g \le 2$ is necessary in Theorem 1.3. In fact, for any surface Σ of Euler genus $g \ge 3$, there exist infinitely many graphs G embedded in Σ such that $\delta(G) \ge 3$ and the face-width of G is sufficiently large, but G has no $[1,2]$-factor.

We now discuss how sharp the lower bound of $\delta(G)$ in Theorems 1.1 and 1.3 is. Note that for integers a and b with $1 \le a < b$, if a graph G with $\delta(G) \ge a$ contains a vertex v of degree at least $b+1$ and the neighbors of v have degree exactly a, then G clearly has no $[a,b]$-factor. This suggests that the minimum degree condition

$\delta(G) \geq a$ cannot always guarantee the existence of an $[a, b]$-factor for any $1 \leq a < b$. Therefore we need the condition $\delta(G) \geq a + 1$.

Furthermore, for the cases $a \in \{1, 2, 3, 4\}$, the condition $\delta(G) \geq a+2$ in Theorems 1.1 and 1.3 is best possible. We shall show this in Section 5.2. On the other hand, we can lower the minimum degree condition of Theorem 1.1 by one for the case of $a = 5$;

Theorem 1.4. *Let G be a graph embedded in a surface Σ of Euler genus g. Suppose that $\delta(G) \geq 6$ and face-width of G is at least $12g - 24$. Then G has a $[5, 6]$-factor.*

It is well known that any graph G embedded in a surface Σ of Euler genus g with $|V(G)| \geq 3$ satisfies

$$|E(G)| \leq 3|V(G)| + 3g - 6,$$

which is obtained by Lemma 2.1 in Section 2 and hence the average degree is at most $6 + \frac{3g-6}{|V(G)|}$. This implies that for any fixed surface Σ, there are only finitely many graphs embedded in Σ with $\delta(G) \geq 7$. Therefore, it is less motivated to focus on the existence of an $[a, b]$-factor for $6 \leq a < b$.

The remaining part of this paper is organized as follows: In the next section, we prepare some notation and preliminary results that will be used in our proofs. Sections 3 and 4 deal with proofs of Theorems 1.1 and 1.3, respectively. After showing sharpness of some conditions of Theorems 1.1 and 1.3 in Section 5, we prove Theorem 1.4 in Section 6.

2 Notation and preliminary results

For a vertex v of a graph G, we denote by $N_G(v)$ the *neighborhood* of v in G, and we denote by $\deg_G(v)$ the *degree* of v in G. For a subset S of $V(G)$, let $N_G(S) = \bigcup_{v \in S} N_G(v)$ and let $\deg_G(S) = \sum_{v \in S} \deg_G(v)$. We denote by $\delta(G)$ the *minimum degree* of G. For two disjoint sets S and T of $V(G)$, we denote by $E_G(S, T)$ the set of edges in G between S and T, and let $e_G(S, T) := |E_G(S, T)|$.

A *surface* is a compact connected 2-manifold without boundary. We define the *Euler genus* g of a surface Σ as $2 - \chi(\Sigma)$, where $\chi(\Sigma)$ is the Euler characteristic of Σ. A closed curve γ on a surface Σ is *contractible* if γ bounds a disc. A *non-contractible* curve is a closed curve that is not contractible.

A graph G is *embedded* in a surface Σ if the vertices of G are distinct points of Σ and every edge of G is a simple arc such that it connects two points that are its end vertices and the interior is disjoint from other edges and vertices. The *face-width* or the *representativity* of a graph G embedded in a non-spherical surface is the smallest possible cardinality of the intersection of G with a non-contractible curve on the surface. In the case that G is a plane graph, we define its face-width as $+\infty$.

A cycle C of a graph G embedded in a surface Σ is called *non-contractible* if C is non-contractible as a curve on Σ. We say that a subgraph H of G is *flat* if there exists a disk in Σ that bounds H. It is easy to see that if H is not flat, then H contains a non-contractible cycle and furthermore, there exists a non-contractible curve γ in Σ that does not hit $G - V(H)$. In particular, such a non-contractible curve γ can be deformed by homotopic shift so that it hits only $V(H)$.

To show the main theorems, we need the following results.

Theorem 2.1 (Lovász [4]). *Let a and b be two integers with $1 \le a < b$. Then a graph G has an $[a, b]$-factor if and only if for any two disjoint subsets $S, T \subseteq V(G)$,*

$$\lambda_G(S,T) := b|S| + \deg_{G-S}(T) - a|T| \ge 0.$$

For a graph G embedded in a surface Σ, let $F(G)$ be the set of faces in G. For a face f of G, we denote by $\deg_G(f)$ the length of the boundary walk of f. (Note that any edge that is a bridge of G and contained in f is counted twice in $\deg_G(f)$.) Euler's formula states that $|V(G)| - |E(G)| + |F(G)| = 2 - g$ holds if G is connected, where g is the Euler genus of Σ. This gives the next lemmas as easy consequences.

Lemma 2.1. *Let G be a connected graph embedded in a surface of Euler genus g. Then the following holds;*

$$\sum_{v \in V(G)} \Big(\deg_G(v) - 6\Big) + \sum_{f \in F(G)} \Big(2\deg_G(f) - 6\Big) = 6g - 12.$$

Lemma 2.2. *Let H be a subgraph of a graph embedded in a surface of Euler genus g. Then the followings holds;*

$$|E(G)| \le \begin{cases} 2|V(H)| - 2 & \textit{if } H \textit{ is flat;} \\ 2|V(H)| + 2g - 4 & \textit{if } H \textit{ is not flat.} \end{cases}$$

Note that even in the case when H is flat in Lemma 2.2, we further have $|E(H)| \le 2|V(H)| - 4$ if $|V(H)| \ge 3$. However, notice that it is not true when $|V(H)| \le 2$.

3 Proof of Theorem 1.1

Suppose that G satisfies all the conditions of Theorem 1.1 but has no $[a, b]$-factor. By Theorem 2.1, there exist two disjoint subsets S, $T \subseteq V(G)$, which satisfy the following inequality;

$$\lambda_G(S,T) := b|S| + \deg_{G-S}(T) - a|T| \le -1. \tag{1}$$

Choose such subsets S and T so that $|T|$ is minimal.

We construct a new bipartite graph H from G by letting $V(H) = S \cup T$ and $E(H) = E_G(S,T)$. Then

$$|V(H)| = |S| + |T|, \quad \text{and} \quad |E(H)| = e_G(S,T).$$

By Lemma 2.2, we have

$$e_G(S,T) \le \begin{cases} 2|S| + 2|T| - 2 & \text{if } H \text{ is flat;} \\ 2|S| + 2|T| + 2g - 4 & \text{if } H \text{ is not flat.} \end{cases} \tag{2}$$

Since $\delta(G) \geq a+2$, we obtain

$$\deg_{G-S}(T) \geq (a+2)|T| - e_G(S, T). \tag{3}$$

It follows from (1) and (3) that

$$e_G(S,T) \geq b|S| + 2|T| + 1.$$

By the above inequality and (2), we obtain

$$(b-2)|S| \leq \begin{cases} -3 & \text{if } H \text{ is flat;} \\ 2g-5 & \text{if } H \text{ is not flat.} \end{cases} \tag{4}$$

In particular, if H is flat or $g \leq 2$, then it deduces a contradiction. Thus, we may assume that H is not flat and $g \geq 3$.

On the other hand, the minimality of T implies the following claim.

Claim 3.1. *For any $x \in T$, $\deg_{G-S}(x) \leq a-1$.*

Proof. Let $T' = T \setminus \{x\}$ for $x \in T$. By the choice of T, we have $\lambda_G(S,T') \geq 0$ and $\lambda_G(S,T) \leq -1$. Thus $1 \leq \lambda_G(S,T') - \lambda_G(S,T) \leq a - \deg_{G-S}(x)$, implying $\deg_{G-S}(x) \leq a-1$. ■

By Claim 3.1 and (3), we obtain

$$(a-1)|T| \geq \deg_{G-S}(T) \geq (a+2)|T| - e_G(S, T)$$

and thus

$$e_G(S, T) \geq 3|T|.$$

Substituting this inequality into (2), since H is not flat, we obtain

$$|T| \leq 2|S| + 2g - 4.$$

Thus, by (4),

$$|S| + |T| \leq 3|S| + 2g - 4 \leq \frac{(b+1)(2g-4)-3}{b-2}. \tag{5}$$

Since H is not flat, there exists a non-contractible curve γ such that $\gamma \cap G = \gamma \cap H \subseteq V(H) = S \cup T$. Using $S \cap T = \emptyset$ and (5), we have

$$|\gamma \cap G| \leq |S \cup T| = |S| + |T| \leq \frac{(b+1)(2g-4)-3}{b-2}.$$

However, this contradicts that the face-width of G is at least $\dfrac{(b+1)(2g-4)-2}{b-2}$. This proves Theorem 1.1. □

4 Proof of Theorem 1.3

The proof is quite similar to that of Theorem 1.1. Assume that G has no $[1, 2]$-factor. By (4) in the proof of Theorem 1.1, we obtain $0 = (b-2)|S| \leq 2g-5$. This contradicts that $g \leq 2$ and proves Theorem 1.3. $\square$

5 Sharpness

In this section, we discuss the sharpness of our results.

5.1 The condition on Euler genus in Theorem 1.3

Let Σ be a surface with Euler genus $g \geq 3$, let H be a triangulation of Σ with sufficiently large face-width and let G be the face subdivision of H, which is obtained by adding a single vertex into each face of H and joining it to all vertices on the corresponding boundary. It is easy to see that the face-width of G is equal to that of H (see Figure 1).

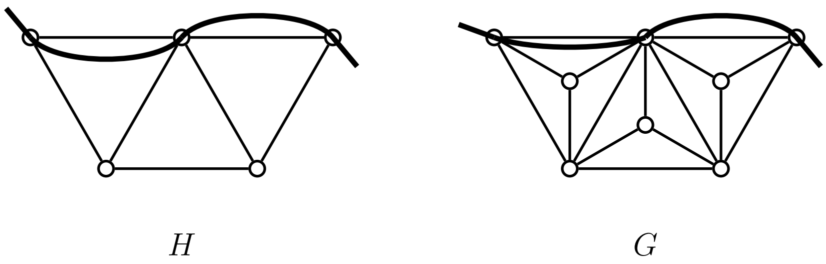

Figure 1: Each bold curve represents a non-contractible curve in Σ.

Since H is a triangulation of Σ, Euler's formula gives

$$|F(H)| = 2|V(H)| + 2g - 4 > 2|V(H)|,$$

because $g \geq 3$. Suppose that G contains a $[1, 2]$-factor R. Since any vertex in $G - V(H)$ has degree at least 1, R must have at least $|F(H)|$ edges between $V(G) - V(H)$ and $V(H)$. On the other hand, R can have at most $2|V(H)|$ edges because any vertex in R has degree at most 2. Thus, we obtain $|F(H)| \leq 2|V(H)|$, which contradicts the above inequality. Therefore, G contains no $[1, 2]$-factor.

5.2 The minimum degree condition in Theorem 1.1

We show that the condition $\delta(G) \geq a + 2$ is sharp for the case of $a = 1, 2, 3$, and 4. Actually, we prove that for any integer $a \in \{1, 2, 3, 4\}$, any integer $b > a$ and any surface Σ, there exist infinitely many graphs G embedded in Σ such that $\delta(G) = a+1$ and the face-width of G is sufficiently large, but G has no $[a, b]$-factor.

Let H be any graph embedded in Σ such that $\delta(H) \geq a + 1$ and the face-width is sufficiently large, and let x and y be any pair of vertices that are incident with the same face. We divide this section into four cases depending on a.

Case 1. $a = 1$.

Let t be an integer with $t > 2b$. Construct a graph G from H and t isolated vertices by joining each of x and y to all t isolated vertices. See Figure 2. Note that $\delta(G) = 2$ and the face-width of G is one more than or equal to that of H. Let $S_1 = \{x, y\}$ and T_1 be the set of t isolated vertices. By $t > 2b$, we obtain

$$\lambda_G(S_1, T_1) = b|S_1| + \deg_{G-S_1}(T_1) - |T_1| = 2b + 0 - t = 2b - t < 0.$$

Hence by Theorem 2.1, G has no $[1, b]$-factor.

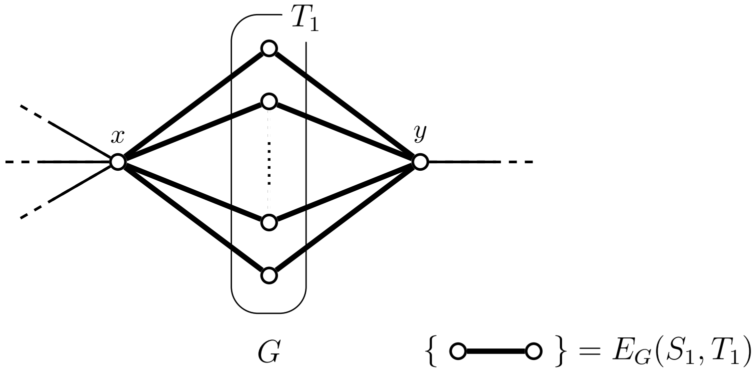

Figure 2: The graph G in the case of $a = 1$.

Case 2. $a = 2$.

Let t be an integer with $t > b$. Denote a path of order two by P_2 and construct a graph G from H and t copies of P_2 by joining each of x and y to all vertices in t copies of P_2. See Figure 3. Note that $\delta(G) = 3$. Let $S_2 = \{x, y\}$ and T_2 be the set of all vertices in t copies of P_2. Since $t > b$ and $\deg_{G-S_2}(T_2) = |T_2| = 2t$, we obtain

$$\lambda_G(S_2, T_2) = b|S_2| + \deg_{G-S_2}(T_2) - 2|T_2| = 2b - 2t < 0.$$

Hence by Theorem 2.1, G has no $[2, b]$-factor.

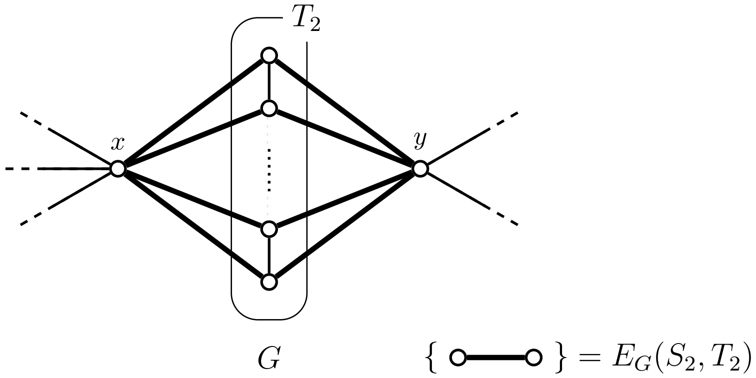

Figure 3: The graph G in the case of $a = 2$.

Case 3. $a = 3$.

Let t be an integer with $t > b$ and let O^- be the graph obtained by removing an edge from an octahedral graph. Construct a graph G from H and t copies of O^- by identifying each of x and y with the two vertices whose degrees are three in each copy of O^-, respectively. See Figure 4. Denote $S_3 = \{x, y\}$ and $T_3 = \{v \in V(O^-) \mid \deg_{O^-}(v) = 4\}$. Note that $\delta(G) = 4$ and $\deg_{G-S_3}(T_3) = (2 + 3 + 2 + 3)t = 10t$. By $t > b$ and $|T_3| = 4t$, we obtain

$$\lambda_G(S_3, T_3) = b|S_3| + \deg_{G-S_3}(T_3) - 3|T_3| = 2b + 10t - 12t = 2b - 2t < 0.$$

Hence by Theorem 2.1, G has no $[3, b]$-factor.

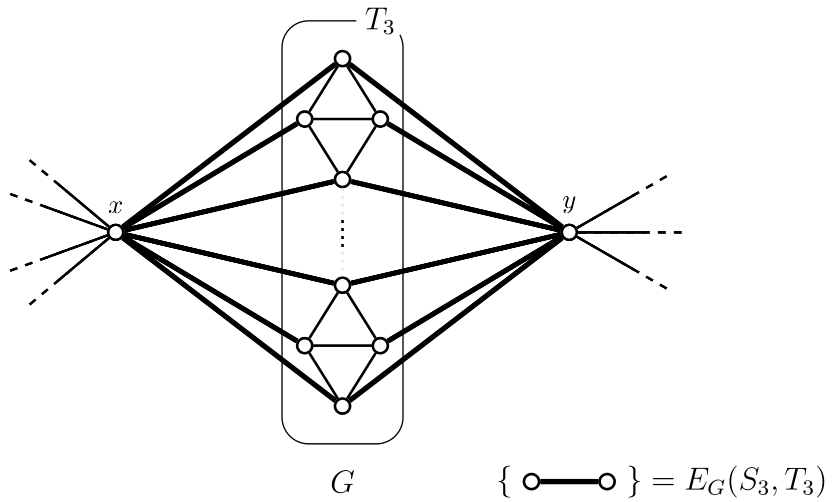

Figure 4: The graph G in the case of $a = 3$.

Case 4. $a = 4$.

Let t be an integer with $t > b$ and let I^- be the graph obtained by removing an edge from an icosahedral graph. Construct a graph G from H and t copies of I^- by identifying each of x and y with the two vertices of degree four in each copy of I^-, respectively. See Figure 5. Let $S_4 = \{x, y\}$ and $T_4 = \{v \in V(I^-) \mid \deg_{I^-}(v) = 5 \text{ and } N_G(v) \cap \{x, y\} \neq \emptyset\}$. Note that $\delta(G) = 5$. By $t > b$, $|T_4| = 6t$ and $e_G(S_4, T_4) = 8t$, we obtain

$$\begin{aligned}\lambda_G(S_4, T_4) &= b|S_4| + \deg_{G-S_4}(T_4) - 4|T_4| \\ &= 2b + \big(5|T_4| - e_G(S_4, T_4)\big) - 4|T_4| \\ &= 2b - 2t < 0.\end{aligned}$$

Hence by Theorem 2.1, G has no $[4, b]$-factor.

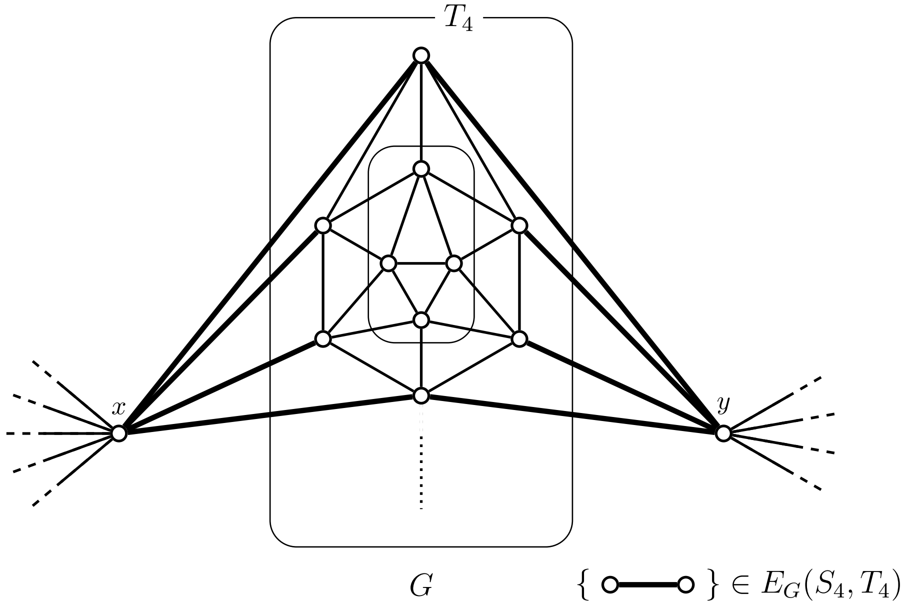

Figure 5: The graph G in the case of $a = 4$.

6 Proof of Theorem 1.4

The *edge-width* of a graph G embedded in a surface is defined as the length of a shortest non-contractible cycle in G; if G contains no non-contractible cycle, then we define its edge-width as $+\infty$. Note that if G is 2-cell embedded in a non-spherical surface, then G always contains a non-contractible cycle.

To prove Theorem 1.4, we consider the edge-width condition instead of the face-width one. Actually, we prove the following theorem.

Theorem 6.1. *Let G be a graph embedded in a surface Σ of Euler genus g. Suppose that $\delta(G) \geq 6$ and edge-width of G is at least $12g - 24$. Then G has a $[5, 6]$-factor.*

It is easy to see that the edge-width of any graph is greater than or equal to its face-width. (Note that if G is a triangulation, then they coincide.) Therefore, Theorem 6.1 directly implies Theorem 1.4.

The usefulness of the edge-width relies on the property that for a graph G embedded in a surface and an edge e in G, the edge-width of $G - e$ is greater than or equal to that of G, while this does not hold for the face-width. This property allows us to restrict ourselves to only "edge minimal graphs" and to use the condition that $V_{\geq 7}$ is independent in the proof.

The proof of Theorem 6.1 depends on the following theorem.

Theorem 6.2 (Generalized Marriage Theorem)**.** *Let H be a bipartite graph with two partite sets X and Y, and let $f : X \to \mathbb{N}$ be a function. Then H has a spanning* *subgraph* R *such that*

$$\begin{aligned} & \deg_R(x) = f(x) \quad \textit{for all } x \in X \\ \textit{and} \quad & \deg_R(y) \leq 1 \qquad \textit{for all } y \in Y \end{aligned}$$

if and only if

$$|N_H(S)| \geq \sum_{x \in S} f(x) \quad \textit{for all } S \subseteq X.$$

Proof of Theorem 6.1. Suppose that G satisfies all the conditions of Theorem 6.1, but has no $[5, 6]$-factor. We choose such a graph G so that $|E(G)|$ is minimal. This choice gives the property that $V_{\geq 7} = \{x \in V(G) \mid \deg_G(x) \geq 7\}$ is an independent set of G; if there exists an edge x_1x_2 in G with $x_1, x_2 \in V_{\geq 7}$, then $G - x_1x_2$ also satisfies all the conditions of Theorem 6.1 and has no desired factor. However, this contradicts the minimality of $|E(G)|$.

Since $\delta(G) \geq 6$, any vertex in $V(G) \setminus V_{\geq 7}$ has degree exactly six in G. By Lemma 2.1 and the fact $\deg_G(f) \geq 3$ for any face f, we obtain

$$|V_{\geq 7}| \leq \sum_{x \in V_{\geq 7}} \big(\deg_G(x) - 6\big) \leq 6g - 12. \tag{6}$$

Let H be a bipartite graph with two partite sets $V_{\geq 7}$ and $N_G(V_{\geq 7})$ such that $E(H)$ is the set of edges in G joining $V_{\geq 7}$ and $N_G(V_{\geq 7})$, that is, $E(H) = E_G\big(V_{\geq 7}, N_G(V_{\geq 7})\big)$. Since $V_{\geq 7}$ is independent in G, we have $\deg_H(x) = \deg_G(x) \geq 7$ for any $x \in V_{\geq 7}$.

Suppose that H has a spanning subgraph R satisfying

$$\begin{aligned} & \deg_R(x) = \deg_H(x) - 6 \quad \text{for all } x \in V_{\geq 7} \\ \text{and} \quad & \deg_R(y) \leq 1 \quad \text{for all } y \in V(G) \setminus V_{\geq 7}. \end{aligned} \tag{7}$$

Then $\overline{R} := G - E(R)$ is a $[5, 6]$-factor of G since for $x \in V_{\geq 7}$ and $y \in V(G) \setminus V_{\geq 7}$,

$$\begin{aligned} & \deg_{\overline{R}}(x) = \deg_G(x) - \deg_R(x) = \deg_H(x) - (\deg_H(x) - 6) = 6, \\ \text{and} \quad & 6 = \deg_G(y) \geq \deg_{\overline{R}}(y) = \deg_G(y) - \deg_R(y) \geq 6 - 1 = 5. \end{aligned}$$

This contradicts that G has no $[5, 6]$-factor and thus such a spanning subgraph R does not exist in H. By Theorem 6.2 and (7), there exists $S \subseteq V_{\geq 7}$ satisfying the inequality

$$|N_H(S)| < \sum_{x \in S} \Big(\deg_H(x) - 6\Big). \tag{8}$$

Let G_S be the subgraph of G induced by $S \cup N_H(S)$. Then we may assume that G_S is connected; otherwise, some subset S_1 of S, together with $N_H(S_1)$, forms a component of G_S but it also satisfies the condition corresponding to (8), and hence we can use S_1 instead of S.

Then we have the following claim.

Claim 6.1. *G_S is flat.*

Proof. If G_S is not flat, then G_S contains an essential cycle C. Thus, by (6) and (8), we have

$$|C| \leq |S| + |N_H(S)| \leq |V_{\geq 7}| + |N_H(S)| < 2(6g - 12) = 12g - 24.$$

This contradicts that the edge-width of G is at least $12g-24$. Hence the claim holds. ■

Since G_S is flat, there exists a subgraph $\widetilde{G_S}$ of G such that $\widetilde{G_S}$ contains all vertices in $S \cup N_H(S)$ and is induced by the vertices contained in a disk on Σ. Take such a subgraph $\widetilde{G_S}$ as small as possible. Let f_0 be the outer face of $\widetilde{G_S}$, and let B be the set of vertices y in the boundary of f_0 with $y \notin S$. Since $N_H(S) \subseteq V(\widetilde{G_S})$, all neighrbors of any vertex in $\widetilde{G_S} - B$ are contained in $\widetilde{G_S}$. This implies that $\deg_{\widetilde{G_S}}(y) = \deg_G(y) \geq 6$ for all $y \in V(\widetilde{G_S}) \setminus B$.

Suppose that $y \in B \setminus N_H(S)$ exists. Since G_S is connected, there exists a component of $\widetilde{G_S} - \{y\}$ that contains all vertices in $S \cup N_H(S)$. However, such a component is induced by the vertices contained in some disk on Σ, contradicting the minimality of $\widetilde{G_S}$. Therefore, we have $B \subseteq N_H(S)$.

By Lemma 2.1, we obtain

$$\sum_{v \in V(\widetilde{G_S})} \Big(\deg_{\widetilde{G_S}}(v) - 6\Big) + \sum_{f \in F(\widetilde{G_S})} \Big(2\deg_{\widetilde{G_S}}(f) - 6\Big) = -12.$$

Since $\deg_{\widetilde{G_S}}(y) \geq 6$ for all $y \in V(\widetilde{G_S}) \setminus B$ and $S \cap B = \emptyset$, we obtain

$$\sum_{v \in V(\widetilde{G_S})} \Big(\deg_{\widetilde{G_S}}(v) - 6\Big) \geq \sum_{x \in S} \Big(\deg_{\widetilde{G_S}}(x) - 6\Big) + \sum_{y \in B} \Big(\deg_{\widetilde{G_S}}(y) - 6\Big).$$

By $\deg_{\widetilde{G_S}}(f) \geq 3$ for all $f \in F(\widetilde{G_S})$, we obtain

$$\sum_{f \in F(\widetilde{G_S})} \Big(2\deg_{\widetilde{G_S}}(f) - 6\Big) \geq 2\deg_{\widetilde{G_S}}(f_0) - 6.$$

Hence we have

$$\sum_{x \in S} \Big(\deg_{\widetilde{G_S}}(x) - 6\Big) + \sum_{y \in B} \Big(\deg_{\widetilde{G_S}}(y) - 6\Big) + \Big(2\deg_{\widetilde{G_S}}(f_0) - 6\Big) \leq -12.$$

By (8), we obtain

$$\begin{aligned} |N_H(S)| &< \sum_{x \in S} \Big(\deg_H(x) - 6\Big) = \sum_{x \in S} \Big(\deg_{\widetilde{G_S}}(x) - 6\Big) \\ &\leq \sum_{y \in B} \Big(6 - \deg_{\widetilde{G_S}}(y)\Big) - 2\deg_{\widetilde{G_S}}(f_0) - 6. \end{aligned} \tag{9}$$

For a positive integer i, denote $B_i = \{y \in B \mid \deg_{\widetilde{G_S}}(y) = i\}$ and $B_{\geq i} = \{y \in B \mid \deg_{\widetilde{G_S}}(y) \geq i\}$. We obtain the following claim.

Claim 6.2. $\deg_{\widetilde{G_S}}(f_0) \geq 2|B_1| + \frac{3}{2}|B_2| + |B_{\geq 3}|$.

Proof. Consider the boundary walk of f_0 with some direction, and for $y \in B$, let $\mathrm{pre}(y)$ (resp. $\mathrm{suc}(y)$) be the edge incoming to (resp. outgoing from) y along the boundary walk of f_0. (If a vertex $y \in B$ appears twice or more in the boundary walk of f_0, then we choose arbitrary one as $\mathrm{pre}(y)$ and $\mathrm{suc}(y)$.) We count the order

$$\left|\left\{\big(y, \mathrm{pre}(y)\big) : y \in B\right\}\right| + \left|\left\{\big(y, \mathrm{suc}(y)\big) : y \in B\right\}\right|$$

in two different ways: It is definitely equal to $2|B|$ and at most $2\deg_{\widetilde{G_S}}(f_0)$. Furthermore, the latter can be improved, since for an edge e with $e = \mathrm{pre}(y)$ or $e = \mathrm{suc}(y)$ with $y \in B$, if the end vertex x of e other than y is contained in S, then (x, e) appears in neither of the two sets above. Now, we count the number of such edges.

Let $y \in B$. Since $B \subseteq N_H(S)$, there exists a vertex $x \in S$ with $xy \in E(\widetilde{G_S})$. If $y \in B_1$, then the unique neighbor x of y in $\widetilde{G_S}$ is contained in S, and hence $\mathrm{pre}(y)$ and $\mathrm{suc}(y)$ coincide and both are incident with a vertex in S. On the other hand, if $y \in B_2$, then both edges incident with y appear in the boundary walk of f_0, and hence at least one of $\mathrm{pre}(y)$ and $\mathrm{suc}(y)$ is incident with a vertex in S. These imply

$$2|B| = \left|\left\{\big(y, \mathrm{pre}(y)\big) : y \in B\right\}\right| + \left|\left\{\big(y, \mathrm{suc}(y)\big) : y \in B\right\}\right| \leq 2\deg_{\widetilde{G_S}}(f_0) - 2|B_1| - |B_2|.$$

Since $|B| = |B_1| + |B_2| + |B_{\geq 3}|$, we obtain the desired inequality. ■

By (9) and Claim 6.2, we obtain

$$\begin{aligned}
|N_H(S)| &< \sum_{y \in B} \Big(6 - \deg_{\widetilde{G_S}}(y)\Big) - 2\deg_{\widetilde{G_S}}(f_0) - 6 \\
&\leq \sum_{y \in B} \Big(6 - \deg_{\widetilde{G_S}}(y)\Big) - 2\Big(2|B_1| + \frac{3}{2}|B_2| + |B_{\geq 3}|\Big) - 6 \\
&= \sum_{y \in B_1} \Big(2 - \deg_{\widetilde{G_S}}(y)\Big) + \sum_{y \in B_2} \Big(3 - \deg_{\widetilde{G_S}}(y)\Big) + \sum_{y \in B_{\geq 3}} \Big(4 - \deg_{\widetilde{G_S}}(y)\Big) - 6 \\
&\leq |B_1| + |B_2| + |B_{\geq 3}| - 6 \leq |N_H(S)| - 6.
\end{aligned}$$

This is a contradiction. This proves Theorem 6.1. □

References

[1] N. Dean and K. Ota, 2-factors, connectivity and graph minors. *Graph structure theory (Seattle, WA, 1991)*, 381–386, Contemp. Math. **147**, *Amer. Math. Soc., Providence, RI*, 1993.

[2] B. Grünbaum, Polytopes, graphs, and complexes. *Bull. Amer. Math. Soc.* **76** (1970), 1131–1201.

[3] K. Kawarabayashi and K. Ozeki, 2- and 3-factors of graphs on surfaces. *J. Graph Theory.* **67** (2011), 306–315.

[4] L. Lovász, The factorization of graphs. II. *Acta Math. Acad. Sci. Hungar.* **23** (1972), 223–246.

[5] C. St. J. A. Nash-Williams, Unexplored and semi-explored territories in graph theory. *New directions in the theory of graphs (Proc. Third Ann Arbor Conf., Univ. Michigan, Ann Arbor, Mich., 1971)*, 149–186.

[6] R. Thomas and X. Yu, 4-connected projective-planar graphs are Hamiltonian. *J. Combin. Theory Ser. B* **62** (1994), 114–132.

[7] R. Thomas and X. Yu, Five-connected toroidal graphs are Hamiltonian. *J. Combin. Theory Ser. B* **69** (1997), 79–96.

[8] W. T. Tutte, A theorem on planar graphs. *Trans. Amer. Math. Soc.* **82** (1956), 99–116.