

A note on a Brooks' type theorem for DP-coloring

Seog-Jin Kim^{*†} and Kenta Ozeki[‡]

July 13, 2017

Abstract

Dvořák and Postle [8] introduced a *DP-coloring* of a simple graph as a generalization of a list-coloring. They proved a Brooks' type theorem for a DP-coloring, and Bernsheteyn, Kostochka and Pron [5] extended it to a DP-coloring of multigraphs. However, detailed structure when a multigraph does not admit a DP-coloring was not specified in [5]. In this note, we make this point clear and give the complete structure. This is also motivated by the relation to signed coloring of signed graphs.

Keywords: Coloring, list-coloring, DP-coloring, Brooks' type theorem.

1 Introduction

1.1 List-coloring and DP-coloring

We denote by $[k]$ the set of integers from 1 to k . A k -coloring of a graph G is a mapping $f : V(G) \rightarrow [k]$ such that $f(u) \neq f(v)$ for any $uv \in E(G)$. The minimum integer k such that G admits a k -coloring is called the *chromatic number* of G , and denoted by $\chi(G)$.

A *list assignment* $L : V(G) \rightarrow 2^{[k]}$ of G is a mapping that assigns a set of colors to each vertex. A proper coloring $f : V(G) \rightarrow Y$ where Y is a set of colors is called an *L-coloring* of G if $f(u) \in L(u)$ for any $u \in V(G)$. A list assignment L is called a *t-list assignment* if $|L(u)| \geq t$ for any $u \in V(G)$. The *list-chromatic number* or the *choice number* of G , denoted by $\chi_\ell(G)$, is the minimum integer t such that G admits an L -coloring for each t -list assignment L .

Since a k -coloring corresponds to an L -coloring with $L(u) = [k]$ for any $u \in V(G)$, we have $\chi(G) \leq \chi_\ell(G)$. It is well-known that there are infinitely many graphs G satisfying $\chi(G) < \chi_\ell(G)$, and the gap can be arbitrary large: Consider for example, the complete bipartite graph $K_{t,t}$, which satisfies $2 = \chi(K_{t,t}) < \chi_\ell(K_{t,t}) = t + 1$. A list assignment L is called a *degree-list assignment* if $|L(u)| \geq d_G(u)$ for any $u \in V(G)$, where $d_G(u)$ denotes the degree of u in G . A graph G is said to be *degree-choosable* if G admits an L -coloring for degree-list assignment. A Brooks' type theorem for degree-choosability was shown by Borodin [7], and independently Erdős, Rubin, and Taylor [9]. See also [12] for a shorter proof.

Theorem 1 *A connected graph G is not degree-choosable if and only if each block of G is isomorphic to K_n for some integer n or C_n for some odd integer n .*

^{*}Department of Mathematics Education, Konkuk University, Korea. e-mail: skim12@konkuk.ac.kr

[†]This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(NRF-2015R1D1A1A01057008).

[‡]Faculty of Environment and Information Sciences, Yokohama National University, Japan. e-mail: ozeki-kenta-xr@ynu.ac.jp

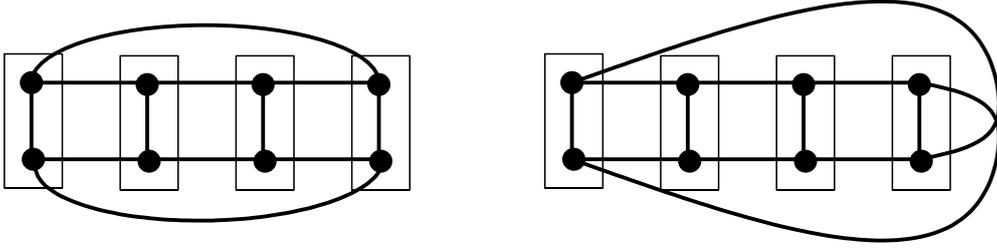


Figure 1: Two examples of the \mathcal{M}_L -cover of C_4 such that $|L(u)| = 2$ for any vertex u . Each thin rectangle represents $\{u\} \times L(u)$ for some vertex u . In fact, the cycle C_4 admits an \mathcal{M}_L -coloring for the left, while does not for the right.

Furthermore, it is known that complete graphs and odd cycles do not have an L -coloring for a degree-list assignment L only when all vertices have same list assignment of size exactly their degree.

In order to consider some problems on list chromatic number, Dvořák and Postle [8] considered a generalization of a list-coloring. They call it a *correspondence coloring*, but we call it a *DP-coloring*, following Bernsheteyn, Kostochka and Pron [5]. It was first proposed for a simple graph, and then extended to a multigraph in [5].

Let G be a multigraph (possibly having multiple edges but no loops) and L be a list assignment of G . For each pair of vertices u and v in G , let $M_{L,uv}$ be the union of $\mu_G(uv)$ matchings between $\{u\} \times L(u)$ and $\{v\} \times L(v)$, where $\mu_G(uv)$ is the multiplicity of uv in G . Note that if u and v are not connected by an edge in G , then $\mu_G(uv) = 0$ and $M_{L,uv}$ is an empty set. With abuse of notation, we sometimes regard $M_{L,uv}$ as a bipartite graph between $\{u\} \times L(u)$ and $\{v\} \times L(v)$ of maximum degree at most $\mu_G(uv)$.

Let $\mathcal{M}_L = \{M_{L,uv} : uv \in E(G)\}$, which is called a *matching assignment over L* . Then a graph H is said to be the \mathcal{M}_L -cover of G if it satisfies all the following conditions:

- (i) The vertex set of H is $\bigcup_{u \in V(G)} (\{u\} \times L(u)) = \{(u, c) : u \in V(G), c \in L(u)\}$.
- (ii) For any $u \in V(G)$, the set $\{u\} \times L(u)$ is a clique in H .
- (iii) For any two vertices u and v in G , $\{u\} \times L(u)$ and $\{v\} \times L(v)$ induce in H the graph obtained from $M_{L,uv}$ by adding those edges defined in (ii).

(See Figure 1 for an example.)

An \mathcal{M}_L -coloring of G is an independent set I in the \mathcal{M}_L -cover with $|I| = |V(G)|$. The *DP-chromatic number*, denoted by $\chi_{\text{DP}}(G)$, is the minimum integer t such that G admits an \mathcal{M}_L -coloring for each t -list assignment L and each matching assignment \mathcal{M}_L over L .

Note that when G is a simple graph and

$$M_{L,uv} = \{(u, c)(v, c) : c \in L(u) \cap L(v)\}$$

for any edge uv in G , then G admits an L -coloring if and only if G admits an \mathcal{M}_L -coloring. Furthermore, when $L(u) = [k]$ for each $u \in V(G)$ (that is, when we consider an ordinal k -coloring), then the \mathcal{M}_L -cover of G is isomorphic to the graph $G \square K_k$, which is the *Cartesian product* of G and the complete graph K_k . Recall that the Cartesian product $H_1 \square H_2$ of graphs H_1 and H_2 is the graph with $V(H_1 \square H_2) = V(H_1) \times V(H_2)$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if either $u_1 = v_1$ and $u_2 v_2 \in E(H_2)$ or $u_1 v_1 \in E(H_1)$ and $u_2 = v_2$. In this case, we see that G admits a k -coloring if and only if $G \square K_k$ contains

an independent set of size $|V(G)|$. According to [3], this fact was pointed out by Plesnevič and Vizing [14].

The relation in the previous paragraph implies $\chi_\ell(G) \leq \chi_{\text{DP}}(G)$. There are infinitely many simple graphs G satisfying $\chi_\ell(G) < \chi_{\text{DP}}(G)$: As we will see in Theorems 1 and 4, $\chi(C_n) = \chi_\ell(C_n) = 2 < 3 = \chi_{\text{DP}}(C_n)$ for an even integer n . Furthermore, the gap $\chi_{\text{DP}}(G) - \chi_\ell(G)$ can be arbitrary large. For example, Bernsheteyn [2] showed that for a simple graph G with average degree d , we have $\chi_{\text{DP}}(G) = \Omega(d/\log d)$, while Alon [1] proved that $\chi_\ell(G) = \Omega(\log d)$ and the bound is sharp. See [4] for more detailed results. Recently, there are some works on DP-colorings; see [2, 3, 6, 8].

Bernsheteyn, Kostochka and Pron [5] proved a Brooks' type theorem for DP-coloring of multigraphs. For a multigraph G and an integer t , we denote by G^t the multigraph obtained from G by replacing each edge with a set of t multiples edges. A multigraph G is said to be *degree-DP-colorable* if G admits an \mathcal{M}_L -coloring for each degree-list assignment L and each matching assignment \mathcal{M}_L over L .

The following theorem gives a Brooks' type theorem for a DP-coloring. This is an extension of Theorem 1.

Theorem 2 (Bernsheteyn, Kostochka and Pron [5]) *A connected multigraph G is not degree-DP-colorable if and only if each block of G is K_n^t or C_n^t for some n and t .*

However, Theorem 2 does not explain the \mathcal{M}_L -colorability for a matching assignment \mathcal{M}_L when every block of G is K_n^t or C_n^t for some n and t . The purpose of this paper is to make this point clear and give the complete structure. As explained in the next subsection, it is important work because of the relation to some results on signed colorings of signed graphs.

1.2 Singed colorings of signed graphs

Here, we give a relationship to signed colorings of signed graphs. A *signed graph* (G, σ) is a pair of a multigraph G and a mapping $\sigma : E(G) \rightarrow \{1, -1\}$, which is called a *sign*. For an integer k , let

$$N_k = \begin{cases} \{0, \pm 1, \dots, \pm r\} & \text{if } k \text{ is an odd integer with } k = 2r + 1, \\ \{\pm 1, \dots, \pm r\} & \text{if } k \text{ is an even integer with } k = 2r. \end{cases}$$

A *signed k -coloring* of a signed graph (G, σ) is a mapping $f : V(G) \rightarrow N_k$ such that $f(u) \neq \sigma(uv)f(v)$ for each $uv \in E(G)$. The minimum integer k such that a signed graph (G, σ) admits a signed k -coloring is called the *signed chromatic number* of (G, σ) . This was first defined by Zaslavsky [16] with slightly different form, and then modified by Máčajová, Raspaud, and Škoviera [13] to the above form so that it would be a natural extension of an ordinarily coloring.

We here point out that a signed coloring of a signed graph (G, σ) is a special case of a DP-coloring of G . Let L be the list assignment of G with $L(u) = N_k$ for any vertex u in G . Then for an edge uv in G , let

$$M_{L,uv} = \begin{cases} \{(u, i)(v, i) : i \in N_k\} & \text{if } \sigma(uv) = 1, \\ \{(u, i)(v, -i) : i \in N_k\} & \text{if } \sigma(uv) = -1. \end{cases}$$

With this definition, it is easy to see that the signed graph (G, σ) admits a signed k -coloring if and only if the multigraph G admits an \mathcal{M}_L -coloring.

A Brooks' type theorem for a signed coloring was proven by Máčajová, Raspaud, and Škoviera [13]. Later, Fleiner and Wiener [10] gave a short proof, using a DFS tree.

For a signed graph (G, σ) and a mapping L from $V(G)$ to N_k , a *signed L -coloring* is a signed coloring f of G such that $f(u) \in L(u)$ for each $u \in V(G)$. Some results on signed L -coloring are showed in [11, 15]. In particular, Schweser and Stiebitz [15] gave a Brooks' type theorem for signed list-colorings. In order to explain the exact statement, we here introduce several definitions.

Let (G, σ) be a signed graph. A *switching* at a vertex v is defined as reversing the signs of all edges incident to v . It is not difficult to see that a switching at any vertex does not change the signed chromatic number of (G, σ) . Note that a switching at v in the sense of a signed L -coloring corresponds to taking the mapping with $i \mapsto -i$ on $L(v)$. Two signed graphs or two signs of a multigraph are *signed-equivalent* or simply *equivalent* if one is obtained from the other by a sequence of switchings. A signed graph (G, σ) is *balanced* if σ is equivalent to the sign with all edges positive; otherwise, (G, σ) is *unbalanced*. For a simple graph G , a signed graph (G^2, σ) is *full* if parallel edges with same end vertices have different signs on σ .

Then we are ready to state a Brooks' type theorem for a signed coloring. This is an extension of Theorem 1.

Theorem 3 (Schweser and Stiebitz [15]) *Let (G, σ) be a signed graph, where G is connected, and let L be a mapping from $V(G)$ to N_k with $|L(u)| \geq d_G(u)$ for each $u \in V(G)$. Then (G, σ) does not admit a signed L -coloring if and only if each block of (G, σ) is one of the following:*

- A balanced K_n for some integer n .
- A balanced C_n for some odd integer n .
- An unbalanced C_n for some even integer n .
- A full K_n^2 for some integer n .
- A full C_n^2 for some odd integer n .

1.3 Degree-DP-colorable graphs

In the previous subsections, we have seen two Brooks' type theorems, namely that for DP-coloring and for signed coloring. As we have seen, a signed coloring can be regarded as a special case of a DP-coloring. Roughly speaking, Theorem 2 is an “almost” improvement of Theorem 3. However, when there is no desired coloring, Theorem 3 completely determines which signs and list assignments forbid to have a desired coloring, while Theorem 2 only gives a structure of graphs. Motivated by this situation, we improve Theorem 2 so that it completely covers Theorem 3.

Before explaining the exact statement of our main theorem, we define three special graphs. For two graphs G and H and a vertex u of G , *blowing up u to H* is the operation to replace u by H so that each vertex of H is joined to every neighbor of u in G . Let n and t be positive integers.

- The graph $H(n, t)$ is defined such that $\{(i, j, k) : i \in [n], j \in [n-1], k \in [t]\}$ is the vertex set and (i, j, k) and (i', j', k') are adjacent if and only if either $i = i'$ or $j = j'$. See Figure 2. Note that the graph $H(n, 1)$ is isomorphic to $K_n \square K_{n-1}$, and $H(n, t)$ is obtained from $H(n, 1)$ by blowing up each vertex to a complete graph K_t .
- A graph H is the *t -fat ladder of length n* if $\{(i, j, k) : i \in [n], j \in \{1, 2\}, k \in [t]\}$ is the vertex set and (i, j, k) and (i', j', k') are adjacent if and only if either $i = i'$, or $i' = i + 1$

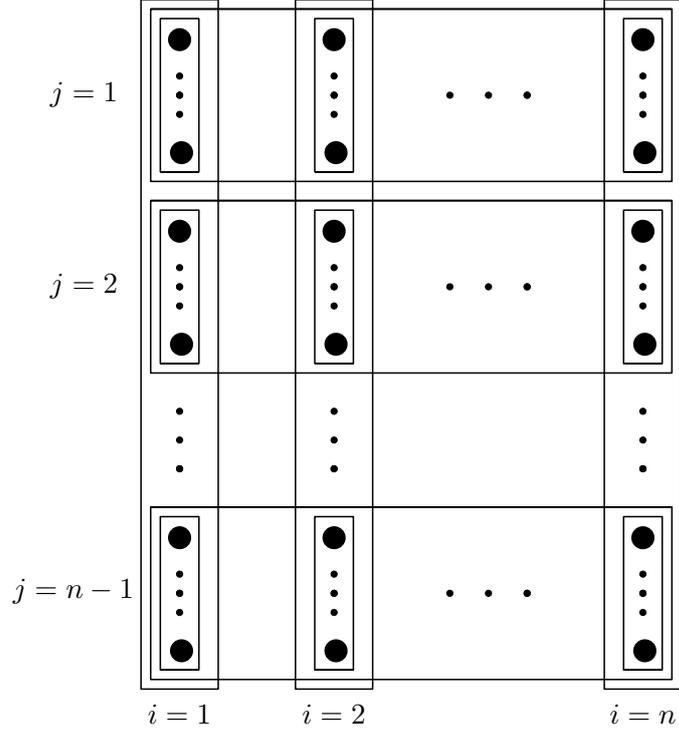


Figure 2: The graph $H(n, t)$. Each thin rectangle represents a clique. In particular, every minimal thin rectangle corresponds to $\{(i, j, k) : k \in [t]\}$ for some $i \in [n]$ and $j \in [n - 1]$, and contains exactly t vertices.

and $j = j'$, where we define $n + 1$ as 1 for the subscript i . In other words, the t -fat ladder of length n is obtained from the ladder of length n (i.e. $C_n \square K_2$) by blowing up each vertex to a complete graph K_t . The left of Figure 1 is a 1-fat ladder of length 4.

- A graph H is the t -fat Möbius ladder of length n if $\{(i, j, k) : i \in [n], j \in \{1, 2\}, k \in [t]\}$ is the vertex set and (i, j, k) and (i', j', k') are adjacent if and only if either
 - $i = i'$, or
 - $i' = i + 1$ and $j = j'$ for $1 \leq i \leq n - 1$, or
 - $i = n, i' = 1$ and $j \neq j'$.

In other words, the t -fat Möbius ladder of length n is obtained from the Möbius ladder of $2n$ vertices by blowing up each vertex to a complete graph K_t . The right of Figure 1 is a 1-fat Möbius ladder of length 4.

For a vertex u in a graph G and a list assignment L of G , we denote $\{u\} \times L(u)$ by $\tilde{L}(u)$ for simplicity. Similarly, $\tilde{L}'(u)$ denotes $\{u\} \times L'(u)$ for $L'(u) \subseteq L(u)$.

Now we are ready to state our main theorem.

Theorem 4 *Let G be a connected multigraph, L be a degree-list assignment of G , and \mathcal{M}_L be a matching assignment over L . Then G does not admit an \mathcal{M}_L -coloring if and only if each block of G is isomorphic to K_n^t or C_n^t for some integers n and t such that all of the following hold.*

- (I) For each vertex u in G , the list assignment $L(u)$ has a partition

$$\{L_B(u) : B \text{ is a block of } G \text{ containing } u\}$$

such that for any block B containing u ,

$$|L_B(u)| = \begin{cases} t(n-1) & \text{if } B \text{ is isomorphic to } K_n^t, \\ 2t & \text{if } B \text{ is isomorphic to } C_n^t. \end{cases}$$

- (II) If B is a block of G isomorphic to K_n^t for some integers n and t , then $\bigcup_{u \in V(B)} \tilde{L}_B(u)$ induces the graph $H(n, t)$ in the \mathcal{M}_L -cover of G , where each set $\tilde{L}_B(u)$ corresponds to $\{(i_u, j, k) : j \in [n-1], k \in [t]\}$ for some $i_u \in [n]$.
- (III) If B is a block of G isomorphic to C_n^t for some integers n and t with n odd, then $\bigcup_{u \in V(B)} \tilde{L}_B(u)$ induces a t -fat ladder of length n in the \mathcal{M}_L -cover of G , where each set $\tilde{L}_B(u)$ corresponds to $\{(i_u, j, k) : j \in \{1, 2\}, k \in [t]\}$ for some $i_u \in [n]$.
- (IV) If B is a block of G isomorphic to C_n^t for some integers n and t with n even, then $\bigcup_{u \in V(B)} \tilde{L}_B(u)$ induces a t -fat Möbius ladder of length n in the \mathcal{M}_L -cover of G , where each set $\tilde{L}_B(u)$ corresponds to $\{(i_u, j, k) : j \in \{1, 2\}, k \in [t]\}$ for some $i_u \in [n]$.

We see that Theorem 4 is an extension of Theorems 1, 2 and 3. Note that L_B in the “only if” part is a degree-list assignment of B , where B is a block of G . Furthermore, we also see the following, which will be used in our proof.

Remark 5 If each block of a graph G is isomorphic to K_n^t or C_n^t for some integers n and t and (I) holds for a list assignment L of G , then $|L(u)| = d_G(u)$ for each $u \in V(G)$.

2 Proof of Theorem 4

The proof of Theorem 4 uses a lemma, which will be shown in the first subsection, and then we give a proof of Theorem 4.

2.1 Greedy method to find an \mathcal{M}_L -coloring

As described in [8], greedily choice of a color in $L(u)$ gives an \mathcal{M}_L -coloring for a $(k+1)$ -list assignment L of any k -degenerate graph G . We use the same idea in this subsection to obtain a useful lemma.

Let G be a connected multigraph, let L be a list assignment of G , and let \mathcal{M}_L be a matching assignment over L . For $u \in V(G)$ and $c \in L(u)$, let $G^{(u)} := G - u$ and

$$L^{(u,c)}(v) = L(v) - \{c' \in L(v) : (u, c)(v, c') \in M_{L,uv}\}$$

for each $v \in V(G) - \{u\}$. Note that the vertex (u, c) has at most μ neighbors in $\tilde{L}(v)$ and v lost μ edges from G to $G^{(u)}$, where μ is the multiplicity between u and v in G . Thus, if L is a degree-list assignment of G , then $L^{(u,c)}$ is a degree-list assignment of $G^{(u)}$. We naturally denote by $\mathcal{M}_{L^{(u,c)}}$ the restriction of \mathcal{M}_L into $G^{(u)}$ and $L^{(u,c)}$: That is, for each pair of vertices v and w in $G^{(u)}$, $M_{L^{(u,c)},vw}$ is the union of matchings of $M_{L,vw}$ with end vertices contained in $\tilde{L}^{(u,c)}(v)$ and $\tilde{L}^{(u,c)}(w)$. Let $\mathcal{M}_{L^{(u,c)}} = \{M_{L^{(u,c)},vw} : vw \in E(G^{(u)})\}$.

Suppose that $G^{(u)}$ admits an $\mathcal{M}_{L^{(u,c)}}$ -coloring, that is, there is an independent set I_u in the $\mathcal{M}_{L^{(u,c)}}$ -cover of G with $|I_u| = |V(G^{(u)})| = |V(G)| - 1$. For each (v, c_v) in I_u , it follows from the choice of $L^{(u,c)}(v)$ that (v, c_v) is not a neighbor of (u, c) in the \mathcal{M}_L -cover of G . Therefore, $I_u \cup \{(u, c)\}$ is an independent set in the \mathcal{M}_L -cover of $G^{(u)}$ with $|I| = |V(G)|$, and hence G admits an \mathcal{M}_L -coloring. This gives the following lemma.

Lemma 6 *Let G be a connected multigraph, let L be a list assignment of G , and let \mathcal{M}_L be a matching assignment over L . For $u \in V(G)$ and $c \in L(u)$, if $G^{(u)}$ admits an $\mathcal{M}_{L^{(u,c)}}$ -coloring, then G admits an \mathcal{M}_L -coloring.*

2.2 Proof of Theorem 4

It is not difficult to check the “if part” (see [5, Lemmas 7 and 8]), and hence we show the “only if part” by induction on $|V(G)|$. Suppose that G does not admit an \mathcal{M}_L -coloring. By Theorem 2, each block of G is isomorphic to K_n^t or C_n^t for some integers n and t . Let B_0 be an end block of G and let H be the \mathcal{M}_L -cover of G .

Case 1. B_0 is isomorphic to K_n^t for some integers n and t .

Take a vertex u in B_0 that is not a cut vertex of G . For an element $c \in L(u)$, consider the multigraph $G^{(u)} = G - u$ together with the list assignment $L^{(u,c)}$ of $G^{(u)}$ and the matching assignment $\mathcal{M}_{L^{(u,c)}}$. Note that each block of $G^{(u)}$ is isomorphic to $K_{n'}^{t'}$ or $C_{n'}^{t'}$ for some integers n' and t' . If some of (I)–(IV) do not hold for $G^{(u)}$, $L^{(u,c)}$ and $\mathcal{M}_{L^{(u,c)}}$, then by induction hypothesis $G^{(u)}$ admits an $\mathcal{M}_{L^{(u,c)}}$ -coloring. However, Lemma 6 gives an \mathcal{M}_L -coloring in G , a contradiction. Therefore, we may assume that all of (I)–(IV) hold: That is, (I) for each vertex v in $G^{(u)}$, the list assignment $L^{(u,c)}(v)$ has a partition $\{L'_B(v) : B \text{ is a block of } G^{(u)} \text{ containing } v\}$ such that for any block B containing v ,

$$|L'_B(v)| = \begin{cases} t'(n' - 1) & \text{if } B \text{ is isomorphic to } K_{n'}^{t'}, \\ 2t' & \text{if } B \text{ is isomorphic to } C_{n'}^{t'}, \end{cases}$$

and (II)–(IV) holds. In particular, Remark 5 implies that $|L^{(u,c)}(v)| = d_{G^{(u)}}(v)$ for each $v \in V(G^{(u)})$. Let B'_0 be the subgraph of $G^{(u)}$ induced by $V(B_0) - \{u\}$. Since u is not a cut vertex of G , B'_0 is a block of $G^{(u)}$ that is isomorphic to K_{n-1}^t .

By (II), $\bigcup_{v \in V(B'_0)} \tilde{L}'_{B'_0}(v)$ induces the graph $H(n-1, t)$, where each set $\tilde{L}'_{B'_0}(v)$ corresponds to $\{(i_v, j, k) : j \in [n-2], k \in [t]\}$ for some $i_v \in [n-1]$. For $v \in V(B'_0)$ and $j \in [n-2]$, let $L_j(v)$ be the set of elements $c_v \in L^{(u,c)}(v)$ such that (v, c_v) corresponds to a vertex in $\{(i_v, j, k) : k \in [t]\}$, and let

$$L_{n-1}(v) = \{c_v \in L(v) : (u, c)(v, c_v) \in M_{L,uv}\}.$$

Note that $|L_j(v)| = t$ for any $j \in [n-2]$. So, if v is not a cut vertex of G , then $L(v) = \bigcup_{j=1}^{n-1} L_j(v)$ and $|L_{n-1}(v)| = |L(v)| - t(n-2) \geq t$. In particular, we have $|L_{n-1}(v)| = t$. Similarly, we obtain the same equality even if v is a cut vertex of G . Let

$$L_B(v) = \begin{cases} L'_B(v) & \text{if } v \in V(G) - V(B_0) \text{ or } B \neq B_0, \\ L(u) & \text{if } v = u \text{ and } B = B_0, \\ L'_{B'_0}(v) \cup L_{n-1}(v) & \text{if } v \in V(B_0) - \{u\} \text{ and } B = B_0. \end{cases}$$

Note that this satisfies (I), and also (II)–(IV) for all blocks B with $B \neq B_0$. Since B_0 is isomorphic to K_n^t , it suffices to show (II) for B_0 .

Note that for any two vertices v and w in $B_0 - \{u\}$, since $M_{L,vw}$ is the union of at most t matchings and $\tilde{L}_j(v)$ and $\tilde{L}_j(w)$ are all adjacent for $j \in [n-2]$, there is no edge between $\tilde{L}_j(v)$ and $\tilde{L}_{j'}(w)$ if $j \neq j'$.

Next, we show the following claim.

Claim 1 For $j \neq j'$ with $j, j' \in [n-1]$, there is no element $c' \in L(u)$ such that (u, c') has a neighbor both in $\tilde{L}_j(v)$ and in $\tilde{L}_{j'}(w)$ for some $v, w \in V(B_0) - \{u\}$.

Proof. Suppose that there exists an element $c' \in L(u)$ such that (u, c') has a neighbor both in $\tilde{L}_j(v)$ and in $\tilde{L}_{j'}(w)$ for some $v, w \in V(B_0) - \{u\}$ and $j, j' \in [n-1]$ with $j \neq j'$. Note that in this case, we have $n \geq 3$. Then consider the list assignment $L^{(u, c')}$ of $G^{(u)}$ and the matching assignment $\mathcal{M}_{L^{(u, c')}}$.

- Assume $v \neq w$. By symmetry between v and w , we may assume that w is not a cut vertex of G . Since (u, c') has a neighbor in $\tilde{L}_{j'}(w)$, there is an element $c_w \in L_j(w)$ that still remains in $L^{(u, c')}(w)$. For $j \in [n-1]$, since each vertex in $\tilde{L}_j(w)$ does not have a neighbor in $\tilde{L}(v) - \tilde{L}_j(v)$, the existence of a neighbor of (u, c') in $\tilde{L}_j(v)$ implies that the vertex (w, c_w) has at most $t-1$ neighbors in $\tilde{L}^{(u, c')}(v)$. Thus, $\tilde{L}^{(u, c')}(w)$ cannot be the set $\{(i, j, k) : j \in [n-2], k \in [t]\}$ in $H(n-1, t)$. Hence the $\mathcal{M}_{L^{(u, c')}}$ -cover of B'_0 is not isomorphic to $H(n-1, t)$. By induction hypothesis $G^{(u)}$ admits an $\mathcal{M}_{L^{(u, c')}}$ -coloring, but Lemma 6 gives an \mathcal{M}_L -coloring in G , a contradiction.
- Assume $v = w$. Then consider a vertex $z \neq v$ and an element c_z in $L_j(z) \cup L_{j'}(z)$ that remains in $L^{(u, c')}(z)$. (Such an element c_z exists, since $|L_j(z)| + |L_{j'}(z)| = 2t$ and (u, c') has at most t neighbors in $\tilde{L}_j(z) \cup \tilde{L}_{j'}(z)$.) By symmetry, we may assume that $c_z \in L_j(z)$. Since (u, c') has a neighbor in $\tilde{L}_j(v)$ in H , the vertex (z, c_z) has at most $t-1$ neighbors in $\tilde{L}^{(u, c')}(v)$. Therefore, by the same reason as above, we see that the $\mathcal{M}_{L^{(u, c')}}$ -cover of B'_0 is not isomorphic to $H(n-1, t)$, and hence the induction hypothesis and Lemma 6 give a contradiction, again.

Thus, the claim holds. ■

Claim 1 directly implies that $L_{B_0}(u)$ can be divided into $n-1$ sets $L_1(u), \dots, L_{n-1}(u)$ such that for each $j \in [n-1]$ and $c' \in L_j(u)$, the vertex (u, c') has neighbors only in $\tilde{L}_{B_0}(u) \cup \bigcup_{v \in V(B'_0)} \tilde{L}_j(v)$ in H .

Next, we will prove the following Claim.

Claim 2 (1) For each $j \in [n-1]$ and $c' \in L_j(u)$, every vertex in $\bigcup_{v \in V(B'_0)} \tilde{L}_j(v)$ is a neighbor of (u, c') in H .

(2) $\tilde{L}_{n-1}(v)$ and $\tilde{L}_{n-1}(w)$ are all adjacent in H for any $v, w \in V(B_0) - \{u\}$.

Proof. (1) If for some $j \in [n-1]$ and $c' \in L_j(u)$, some vertex in $\bigcup_{v \in V(B'_0)} \tilde{L}_j(v)$ is not a neighbor of the vertex (u, c') in H , then $|L^{(u, c')}(v)| \geq |L(v)| - (t-1) \geq d_{G^{(u)}}(v) + 1$, and hence the $\mathcal{M}_{L^{(u, c')}}$ -cover of B'_0 is not isomorphic to $H(n-1, t)$. (See Remark 5.) By the induction hypothesis $G^{(u)}$ admits an $\mathcal{M}_{L^{(u, c')}}$ -coloring, but Lemma 6 gives an \mathcal{M}_L -coloring in G , a contradiction. Thus, (1) holds. ■

(2) If some vertex in $\tilde{L}_{n-1}(v)$ and some vertex in $\tilde{L}_{n-1}(w)$ are not adjacent in H for $v, w \in V(B_0) - \{u\}$, then taking $c' \in L_j(u)$ with $j \in [n-2]$, the vertex in $\tilde{L}_{n-1}(v)$ has at most $t-1$ neighbors in $\tilde{L}_{n-1}(w)$. This implies again that the $\mathcal{M}_{L^{(u, c')}}$ -cover of B'_0 is not isomorphic to $H(n-1, t)$. the induction hypothesis and Lemma 6 give a contradiction, again. This completes the proof of Claim 2. ■

Claim 2 implies that $\bigcup_{v \in V(B_0)} \tilde{L}_{B_0}(v)$ induces the graph $H(n, t)$, where each set $\tilde{L}_{B_0}(v)$ corresponds to $\{(i_v, j, k) : j \in [n-1], k \in [t]\}$ for some $i_v \in [n]$. This shows that B_0 also satisfies (II), and completes the proof of Case 1.

Case 2. B_0 is isomorphic to C_n^t for some integers n and t .

Since C_3^t is isomorphic to K_3^t , we may assume that $n \geq 4$. Let u_{n-1}, u_n, u_1 be the three consecutive vertices of C_n^t such that u_n is not a cut vertex of G . Let $c_n \in L(u_n)$.

Suppose first that the vertex (u_n, c_n) in H has at most $t - 1$ neighbors in $\tilde{L}(u_1)$. In this case, consider the graph $G^{(u_n)} = G - u_n$, the list assignment $L^{(u_n, c_n)}$ of $G^{(u_n)}$ and the matching assignment $\mathcal{M}_{L^{(u_n, c_n)}}$ as in Subsection 2.1. Since the vertex (u_n, c_n) has at most $t - 1$ neighbors in $\tilde{L}(u_1)$, we see that

$$|L^{(u_n, c_n)}(u_1)| \geq |L(u_1)| - (t - 1) \geq d_G(u_1) - (t - 1) = d_{G^{(u_n)}}(u_1) + 1.$$

Thus, by Remark 5, (I) does not hold for $G^{(u_n)}$, $L^{(u_n, c_n)}$ and $\mathcal{M}_{L^{(u_n, c_n)}}$, and hence by induction hypothesis $G^{(u_n)}$ admits an $\mathcal{M}_{L^{(u_n, c_n)}}$ -coloring. However, Lemma 6 gives an \mathcal{M}_L -coloring in G , a contradiction. Therefore, the vertex (u_n, c_n) in H has exactly t neighbors in $\tilde{L}(u_1)$. By the same argument, the vertex (u_n, c_n) in H has exactly t neighbors in $\tilde{L}(u_{n-1})$. Similarly, we can prove that $|L(u_1)| = |L(u_{n-1})| = 2t$.

For $i = 1, n - 1$, let $L_1(u_i)$ be the set of elements $c_i \in L(u_i)$ such that (u_i, c_i) is a neighbor of (u_n, c_n) in H , and let $L_2(u_i) = L(u_i) - L_1(u_i)$. Since $|L_1(u_i)| = t$ and $|L(u_i)| = 2t$, we have $|L_2(u_i)| = |L(u_i)| - |L_1(u_i)| = t$. Then we construct the graph G' from $G - u_n$ by adding t multiple edges connecting u_{n-1} and u_1 . Let B'_0 be the subgraph of G' induced by $V(B_0) - \{u_n\}$. Since $n \geq 4$ and u_n is not a cut vertex of G , B'_0 is a block of G' that is isomorphic to C_{n-1}^t . Let L' be the restriction of L into $V(G')$, let $M'_{L',vw} = M_{L,vw}$ for $vw \in E(G') - \{u_{n-1}u_1\}$, let $M'_{L',u_{n-1}u_1}$ be the set of all possible edges between $\tilde{L}_j(u_{n-1})$ and $\tilde{L}_{j'}(u_1)$ for $\{j, j'\} = \{1, 2\}$, and let $\mathcal{M}'_{L'} = \{M'_{L',vw} : vw \in E(G')\}$.

Suppose that G' admits an $\mathcal{M}'_{L'}$ -coloring, that is, the $\mathcal{M}'_{L'}$ -cover of G' contains an independent set I' of size $|V(G')| = |V(G)| - 1$. For $i = 1, n - 1$, let $c_i \in L(u_i)$ with $(u_i, c_i) \in I'$. If $c_1 \in L_2(u_1)$ and $c_{n-1} \in L_2(u_{n-1})$, then $I' \cup \{(u_n, c_n)\}$ is an independent set in H of size $|V(G)|$, a contradiction. Thus, we may assume that either $c_1 \in L_1(u_1)$ or $c_{n-1} \in L_1(u_{n-1})$. Since (u_1, c_1) and (u_{n-1}, c_{n-1}) are not adjacent in the $\mathcal{M}'_{L'}$ -cover of G' , the choice of $M'_{L',u_{n-1}u_1}$ implies that both $c_1 \in L_1(u_1)$ and $c_{n-1} \in L_1(u_{n-1})$ hold. Furthermore, for any $c'_n \in L(u_n)$, since $I' \cup \{(u_n, c'_n)\}$ is not an independent set in H , the vertex (u_n, c'_n) must be a neighbor of either (u_1, c_1) or (u_{n-1}, c_{n-1}) . Since (u_n, c_n) is a neighbor of both (u_1, c_1) and (u_{n-1}, c_{n-1}) , there are at least $|L(u_1)| + 1 \geq 2t + 1$ edges in H between $\{(u_1, c_1), (u_{n-1}, c_{n-1})\}$ and $\tilde{L}(u_n)$. This contradicts that both (u_1, c_1) and (u_{n-1}, c_{n-1}) have at most t neighbors in $\tilde{L}(u_n)$. Therefore, we have

(P1): the $\mathcal{M}'_{L'}$ -cover of G' contains no independent set of size $|V(G')|$.

By induction hypothesis, all of (I)–(IV) hold for G' , L' and $\mathcal{M}'_{L'}$: That is, (I) for each vertex v in G' , the list assignment $L'(v)$ has a partition $\{L'_B(v) : B \text{ is a block of } G' \text{ containing } v\}$ such that for any block B containing v ,

$$|L'_B(v)| = \begin{cases} t'(n' - 1) & \text{if } B \text{ is isomorphic to } K_{n'}^{t'}, \\ 2t' & \text{if } B \text{ is isomorphic to } C_{n'}^{t'}, \end{cases}$$

and (II)–(IV) holds. For $v \in V(G)$ and for a block B containing v , let

$$L_B(v) = \begin{cases} L'_B(v) & \text{if } v \neq u_n, \\ L(v) & \text{if } v = u_n \text{ and } B = B_0. \end{cases}$$

Note that this satisfies (I), and also (II)–(IV) for all blocks B with $B \neq B_0$. Since B_0 is isomorphic to C_n^t , it suffices to show (III) and (IV) for B_0 .

Suppose that there are three elements $c'_{n-1} \in L_1(u_{n-1})$, $c'_n \in L(u_n)$ and $c'_1 \in L_2(u_1)$ such that neither (u_{n-1}, c'_{n-1}) nor (u_1, c'_1) is a neighbor of (u_n, c'_n) in H . Let $M_{L',vw}^- = M_{L,vw}$ for $vw \in E(G') - \{u_{n-1}u_1\}$, let

$$M_{L',u_{n-1}u_1}^- = M_{L',u_{n-1}u_1} - \{(u_{n-1}, c'_{n-1})(u_1, c'_1)\},$$

and let $\mathcal{M}_{L'}^- = \{M_{L',vw}^- : vw \in E(G')\}$. Since G' , L' and $\mathcal{M}_{L'}^-$ satisfy (I)–(IV) and the $\mathcal{M}_{L'}^-$ -cover of G' is obtained from the $\mathcal{M}_{L'}'$ -cover of G' by deleting one edge, we see that G' , L' and $\mathcal{M}_{L'}^-$ do not satisfy (III) or (IV). Therefore, by the induction hypothesis, G' admits an $\mathcal{M}_{L'}^-$ -coloring: That is, the $\mathcal{M}_{L'}^-$ -cover of G' contains an independent set I^- of size $|V(G')| = |V(G)| - 1$. If either $(u_{n-1}, c'_{n-1}) \notin I^-$ or $(u_1, c'_1) \notin I^-$, then I^- is also an independent set of the $\mathcal{M}_{L'}'$ -cover of G' , which contradicts (P1). Thus, $(u_{n-1}, c'_{n-1}) \in I^-$ and $(u_1, c'_1) \in I^-$ and hence $I^- \cup \{(u_n, c'_n)\}$ is an independent set in H of size $|V(G)|$, again a contradiction. Therefore, the following holds.

(P2) : There are no three elements $c'_{n-1} \in L_1(u_{n-1})$, $c'_n \in L(u_n)$ and $c'_1 \in L_2(u_1)$ such that neither (u_{n-1}, c'_{n-1}) nor (u_1, c'_1) is a neighbor of (u_n, c'_n) in H .

Furthermore, by symmetry, there are no three elements $c'_{n-1} \in L_2(u_{n-1})$, $c'_n \in L(u_n)$ and $c'_1 \in L_1(u_1)$ such that neither (u_{n-1}, c'_{n-1}) nor (u_1, c'_1) is a neighbor of (u_n, c'_n) in H .

Then, we can show the following property.

(P3) : For any $c'_n \in L(u_n)$, either all vertices in $\tilde{L}_1(u_{n-1})$ or all vertices in $\tilde{L}_2(u_{n-1})$ are neighbors of (u_n, c'_n) .

Suppose that for $c'_n \in L(u_n)$, there are a vertex in $\tilde{L}_1(u_{n-1})$ and a vertex in $\tilde{L}_2(u_{n-1})$ neither of which are neighbors of (u_n, c'_n) in H . Since (u_n, c'_n) has at most t neighbors in $\tilde{L}(u_1)$, there exists an element $c'_1 \in L(u_1)$ such that (u_1, c'_1) is not a neighbor of (u_n, c'_n) . We here assume $c'_1 \in L_2(u_1)$, but the other case is symmetric. (Here we do not use c_n). Let $c'_{n-1} \in L_1(u_{n-1})$ such that (u_{n-1}, c'_{n-1}) is not a neighbor of (u_n, c'_n) . Then the three elements $c'_{n-1} \in L_1(u_{n-1})$, $c'_n \in L(u_n)$ and $c'_1 \in L_2(u_1)$ contradict (P2). Thus (P3) holds.

Thus we have followings:

- For any $c'_n \in L(u_n)$, since (u_n, c'_n) has at most t neighbors in $\tilde{L}(u_{n-1})$ and $|\tilde{L}_1(u_n)| = |\tilde{L}_2(u_n)| = t$, (P3) directly implies that the vertex (u_n, c'_n) has no neighbor either in $\tilde{L}_1(u_{n-1})$ or in $\tilde{L}_2(u_{n-1})$.
- Thus, $L(u_n)$ can be divided into two sets $L_1(u_n)$ and $L_2(u_n)$ such that for each $j \in \{1, 2\}$ and $c'_n \in L_j(u_n)$, the set of neighbors of (u_n, c'_n) in $\tilde{L}(u_{n-1})$ is $\tilde{L}_j(u_{n-1})$. Note that $c_n \in L_1(u_n)$.
- Let $c'_n \in L_2(u_n)$. Since (u_n, c'_n) has no neighbors in $\tilde{L}_1(u_{n-1})$, (P2) implies that all vertices in $\tilde{L}_2(u_1)$ are neighbors of (u_n, c'_n) in H . Since (u_n, c'_n) has at most t neighbors in $\tilde{L}(u_1)$, no vertices in $\tilde{L}_1(u_1)$ are neighbors of (u_n, c'_n) in H .
- Similarly, for $c'_n \in L_1(u_n)$, all vertices in $\tilde{L}_1(u_1)$ are neighbors of (u_n, c'_n) in H and no vertices in $\tilde{L}_2(u_1)$ are neighbors of (u_n, c'_n) .

Recall that $M_{L',u_{n-1}u_1}'$ is the set of all possible edges between $\tilde{L}_j(u_{n-1})$ and $\tilde{L}_{j'}(u_1)$ for $\{j, j'\} = \{1, 2\}$. Then the above conditions imply that $M_{L,u_{n-1}u_n}$ is the set of all possible

edges between $\tilde{L}_j(u_{n-1})$ and $\tilde{L}_j(u_n)$ for $j \in \{1, 2\}$, and $M_{L, u_n u_1}$ is the set of all possible edges between $\tilde{L}_j(u_n)$ and $\tilde{L}_j(u_1)$ for $j \in \{1, 2\}$.

When n is odd, it follows from (IV) for B'_0 that $\bigcup_{v \in V(B'_0)} \tilde{L}'_{B'_0}(v)$ induces a t -fat Möbius ladder of length $n-1$ in the $\mathcal{M}'_{L'}$ -cover of G' . Then it is not difficult to see that $\bigcup_{v \in V(B_0)} \tilde{L}_{B_0}(v)$ induces a t -fat ladder of length n . Therefore, (III) and trivially (IV) hold for B_0 .

By the same way, we show that when n is even, $\bigcup_{v \in V(B_0)} \tilde{L}_{B_0}(v)$ induces a t -fat Möbius ladder of length n . Therefore, (IV) and trivially (III) hold for B_0 . This completes the proof of Theorem 4. \square

References

- [1] N. Alon, Degrees and choice numbers, *Random Structures & Algorithms* 16 (2000), 364–368.
- [2] A. Bernshteyn, The asymptotic behavior of the correspondence chromatic number, *Discrete Math.* 339 (2016), 2680–2692.
- [3] A. Bernshteyn, and A. Kostochka, Sharp Dirac’s theorem for DP-critical graphs, arXiv:1609.09122, preprint, 2016.
- [4] A. Bernshteyn, and A. Kostochka, On differences between DP-coloring and list coloring, arXiv:1705.04883, preprint, 2017.
- [5] A. Bernshteyn, A. Kostochka, and S. Pron, On DP-coloring of graphs and multigraphs *Sib. Math. J.* 58 (2017), 28–36.
- [6] A. Bernshteyn, A. Kostochka, and X. Zhu, DP-colorings of graphs with high chromatic number, arXiv:1703.02174, preprint, 2017.
- [7] O.V. Borodin, Criterion of chromaticity of a degree prescription, in abstracts of IV All-Union Conf. on Th. Cybernetics, 1977, 127–128.
- [8] Z. Dvořák, and L. Postle, Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8 arXiv:1508.03437, preprint, 2016.
- [9] P. Erdős, A.L. Rubin, and H. Taylor, Choosability in graphs. in *Proc. West Coast Conf. Combinatorics, Graph Theory and Computing* (Humboldt State Univ., Arcata, Calif., 1979), *Congr. Numer.* 26 (1980), 125–157.
- [10] T. Fleiner, and G. Wiener, Coloring signed graphs using DFS, *Optimization Letters* 10 (2016), 865–869.
- [11] L. Jin, Y. Kang, and E. Steffen, Choosability in signed planar graphs, *Europ. J. Combin.* 52 (2016), 234–243.
- [12] A.V. Kostochka, M. Stiebitz, and B. Wirth, The colour theorems of Brooks and Gallai extended, *Discrete Math.* 162 (1996), 299–303.
- [13] E. Máčajová, A. Raspaud, and M. Škovič, The chromatic number of a signed graph, *the Electron. J Combin.* 23 (2016), #P1.14.

- [14] G.S. Plesnevič, and V.G. Vizing, On the problem of the minimal coloring of the vertices of a graph, *Sib. Math. J.* 6 (1965), 234–236.
- [15] T. Schweser, and M. Stiebitz, Degree choosable signed graphs, *Discrete Math.* 340 (2017), 882–891.
- [16] T. Zaslavsky, Signed graph coloring, *Discrete Math.* 39 (1982), 215–228.