On Homeomorphically Irreducible Spanning Trees in Cubic Graphs

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Abstract

A spanning tree without a vertex of degree two is called a HIST which is an abbreviation for homeomorphically irreducible spanning tree. We provide a necessary condition for the existence of a HIST in a cubic graph. As one consequence, we answer affirmatively an open question on HISTs by Albertson, Berman, Hutchinson and Thomassen. We also show several results on the existence of HISTs in plane and toroidal cubic graphs.

Keywords: HIST, cubic graph, cyclic edge-connectivity, bipartite, spanning tree, fullerene

1 Introduction

All graphs considered here are finite and simple. In a connected graph G, a spanning tree which does not have a vertex of degree two is called a *homeomorphically irreducible spanning tree*, or abbreviated a *HIST*. Several conditions which ensure the existence of a HIST in a graph are known, see for instance [1, 3, 9]. In this paper, we only consider HISTs in cubic graphs. For an integer k, a connected cubic graph G which contains two disjoint cycles is said to be *cyclically k-edge-connected* if deleting any set of at most $k-1$ edges from G does not separate G into two components both of which have a cycle. The following question was asked in [1, p. 253].

Question 1 *Does there exist a cyclically k-edge-connected cubic graph without a HIST for each positive integer k?*

Note that every HIST T in a cubic graph has only vertices of degree one and three. Hence $E(G)$ has a partition into $E(T)$ and the edge set of a union of disjoint cycles.

Let us call a 2-regular subgraph H of a connected graph G *non-separating* if $G-E(H)$ is connected. For a set S of edges in G, we denote by $\langle S\rangle$ the subgraph of G induced by

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the edges in S. So, the vertex set of $\langle S \rangle$ is the set of end vertices of edges in S. We answer Question 1 by applying Corollary 3, a corollary of Theorem 2 which turns out to be useful for proving that certain cubic graphs do not have a HIST.

Theorem 2 *Let G be a cubic graph with a HIST T, and let $H = \langle E(G) - E(T) \rangle$. Then H is a non-separating 2-regular subgraph of G satisfying $|V(H)| = |V(G)|/2 + 1$.*

Proof. Let G be a cubic graph with a HIST T and let $H = \langle E(G) - E(T) \rangle$. Since $V(H)$ is the set of all leaves of T and $G - E(H) = T$, H is a non-separating 2-regular subgraph of G.

Let t_1 be the number of leaves in T, and let t_3 be the number of vertices of degree 3 in T. Since T is a HIST, we have $t_1 + t_3 = |V(G)|$. On the other hand, it is easy to see that $t_1 = t_3 + 2$. (This can be obtained by the Handshaking Lemma or by induction. For example, see [13, Exercise 2.1.23 on p. 70].) Therefore, $|V(G)| = 2t_1 - 2$. Since $V(H)$ is the set of all leaves of T, we have $|V(H)| = t_1$. By using the above equations the proof is completed. $\square$

Corollary 3 *Let G be a bipartite cubic graph. If G has a HIST, then $|V(G)| \equiv 2 \pmod 4$.*

Proof. Let G be a bipartite cubic graph with a HIST. By Theorem 2, $H = \langle E(G) - E(T) \rangle$ is a non-separating 2-regular subgraph of G satisfying $|V(H)| = |V(G)|/2 + 1$. Since G is bipartite, $|V(H)|$ is even and hence $|V(G)| \equiv 2 \pmod 4$. $\square$

Remark: Corollary 3 implies that no bipartite cubic graph G with $|V(G)| \equiv 0 \pmod 4$ has a HIST. However, if G is a bipartite cubic graph with $|V(G)| \equiv 2 \pmod 4$, then G may or may not have a HIST. Both cases could happen, see Section 3.

Now we obtain a positive answer to Question 1 by applying Corollary 3 together with the following proposition.

Proposition 4 *For every positive integer k, there exists a cyclically k-edge-connected bipartite cubic graph G such that $|V(G)| \equiv 0 \pmod 4$.*

Proposition 4 can be directly proved by considering vertex-transitive graphs: it is known that for any positive integer k, there are infinitely many vertex-transitive bipartite cubic graphs G of girth at least k with $|V(G)| \equiv 0 \pmod 4$, see for example [12]. Since the cyclic edge-connectivity of a vertex-transitive graph is equal to its girth (see [11]), Proposition 4 holds. However, since this proof requires several algebraic tools, we prefer to present an elementary proof which also offers a new method to construct cubic bipartite graphs with high cyclic edge-connectivity, see Theorem 7 and Lemma 8 in Section 2.

In Section 3, we show another application of Theorem 2 to plane and toroidal cubic graphs.

2 Proof of Proposition 4

In order to prove Proposition 4, we use the following fact which can be proved in several ways, for instance, by the probabilistic method (see [14, Theorems 2.5 and 2.10]) and by the constructive method (see [5]).

Fact 5 *For every positive integer d, there exists a d-connected $4d$-regular graph of girth at least d.*

Then we apply the well known concept of an *inflation* (see for instance [6]):

Definition 6 *Let H be a graph and let G be a cubic graph. Then G is called an inflation of H if G contains a 2-factor F consisting of chordless cycles such that the graph obtained from G by contracting each cycle of F to a vertex is isomorphic to H.*

If the minimum degree of H is at least 3, then obviously an inflation of H exists, since one obtains (informally speaking) an inflation of H by expanding every vertex of H to a cycle. The next theorem guarantees the high cyclic edge-connectivity for each inflation of graphs with high connectivity and girth.

Theorem 7 *Let $k \geq 3$ and let H be a k-connected graph with girth at least k. Then every inflation of H is cyclically k-edge-connected.*

Proof. Let G be an inflation of H. For each vertex $x \in V(H)$, denote the unique cycle of F (as in Definition 6) in G corresponding to x by C_x. We say that a cycle C in G is *transverse* if there are two distinct vertices x_1 and x_2 in H with $V(C_x) \cap V(C) \neq \emptyset$ for each $x \in \{x_1, x_2\}$. Otherwise C is said to be *non-transverse*, that is, $C = C_x$ for some vertex $x \in V(H)$.

Suppose by contradiction that G is not cyclically k-edge-connected. Then G has a set S of edges with $|S| \leq k-1$ such that $G - S$ has precisely two components D_1 and D_2 both having a cycle. By taking such a set S as small as possible, we may assume that S is a matching.

For $i \in \{1, 2\}$, let D_i^H be the subgraph of H induced by the vertex set

$$\{x \in V(H) : V(C_x) \cap V(D_i) \neq \emptyset\}.$$

So, D_1^H is obtained from D_1 in the following way: for each $x \in V(H)$ such that $C_x \cap D_1$ is not the null graph, where $C_x \cap D_1$ is the maximum common subgraph of C_x and D_1, contract $C_x \cap D_1$ into one vertex and delete all resultant loops.

$$\begin{array}{llcl} \text{Let} & S_{\mathrm{V}}^H & = & \{x \in V(H) : E(C_x) \cap S \neq \emptyset\}, \\ \text{and} & S_{\mathrm{E}}^H & = & S \cap E(H) \\ & & = & \{e \in S : e \notin E(C_x) \text{ for any } x \in V(H)\}. \end{array}$$

Note that $|S_V^H| + |S_E^H| \leq |S| \leq k-1$.

Suppose that $V(D_i^H) - S_{\mathrm{V}}^H \neq \emptyset$ for each $i \in \{1, 2\}$. Then $H - S_{\mathrm{V}}^H - S_{\mathrm{E}}^H$ have two components D_1^H and D_2^H. In this case, the number of vertex disjoint paths from a vertex of D_1^H to a vertex of D_2^H is at most $|S_{\mathrm{V}}^H| + |S_{\mathrm{E}}^H| \leq k-1$, which contradicts by Menger's Theorem that H is k-connected.

Therefore, we may assume without loss of generality that $V(D_1^H) - S_{\mathrm{V}}^H = \emptyset$.

Note that D_1 contains by assumption a cycle, say C_1, and C_1 must be transverse (otherwise, $C_1 = C_x$ for some $x \in V(D_1^H) - S_{\mathrm{V}}^H$, but this contradicts that $V(D_1^H) - S_{\mathrm{V}}^H = \emptyset$).

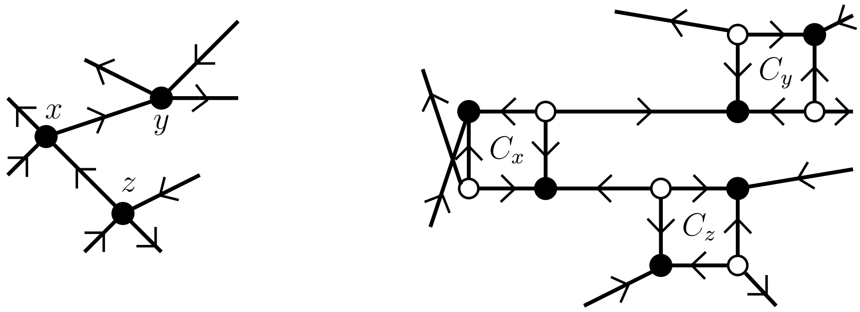

Figure 1: A graph H with Eulerian orientation (the left side) and the bipartite graph G obtained by an inflation of H (the right side) for the case $k = 2$. In G, the vertices with outdegree 3 are represented by white circles, while the vertices with indegree 3 are represented by black circles.

Thus, C_1 corresponds to a closed trail in D_1^H, say C_1^H. Since the girth of H is at least k and every closed trail contains a cycle, we have $|V(C_1^H)| \geq k$, which is a contradiction to the fact that $V(C_1^H) \subseteq V(D_1^H) \subseteq S_{\mathrm{V}}^H$ and $|S_{\mathrm{V}}^H| \leq k-1$. $\square$

Note that the statement of the above theorem does not hold if H is only demanded to be k-edge-connected.

Lemma 8 *Let $k \geq 2$ and let H be a $2k$-regular graph. Then there exists a bipartite cubic inflation of H with $2k|V(H)|$ vertices.*

Proof. Since every inflation of H has $2k|V(H)|$ vertices, it suffices to show that H has a bipartite inflation.

Since each component of H is Eulerian, it has an Eulerian orientation, that is, the indegree equals the outdegree for every vertex of H. Then we can expand every vertex x of H to a cycle C_x to obtain an inflation G with the property that the oriented edges incident with the vertices of C_x are alternately directed towards and away from C_x. See Figure 1. Furthermore, it is possible to extend this partial orientation to an orientation of G (by orienting the edges of each cycle C_x) such that every vertex of G has then either outdegree 3 or indegree 3. This shows a 2-coloring of G, and hence G is bipartite. $\square$

Proof of Proposition 4. Let k be a positive integer. By Fact 5 there exists a k-connected $4k$-regular graph H of girth at least k. Since H is $4k$-regular, it follows from Lemma 8 that there exists a bipartite cubic inflation G with $4k|V(H)|$ vertices. Since H is k-connected and has girth at least k, it follows from Theorem 7 that G is cyclically k-edge-connected, which completes the proof. $\square$

3 HISTs in plane and toroidal cubic graphs

Let us call a plane cubic graph with a HIST in short a *pcH-graph*. A pcH-graph is by its definition a generalization of a cubic Halin graph (defined in [8]) which is a pcH-graph with a HIST such that all the leaves of the HIST induce precisely one cycle. It is easy to see that any cubic Halin graph contains a triangle. In contrast to cubic Halin graphs, pcH-graphs can have girth 4 or even 5, see Figure 2. Note that it is NP-complete to determine whether a plane cubic graph has a HIST, see [4]. (To be exact, Douglas [4] proved that only for plane graphs of maximum degree at most 3. However, replacing each vertex of degree at most 2 with a certain gadget, we can easily modify the proof to show the NP-completeness of the HIST problem for plane cubic graphs.) Since any non-facial cycle of a cubic plane graph is separating, by restricting Theorem 2 to the planar case we obtain:

Corollary 9 *Let G be a plane cubic graph with a HIST. Then G contains a non-separating 2-regular subgraph H consisting of facial cycles such that $|V(H)| = |V(G)|/2 + 1$.*

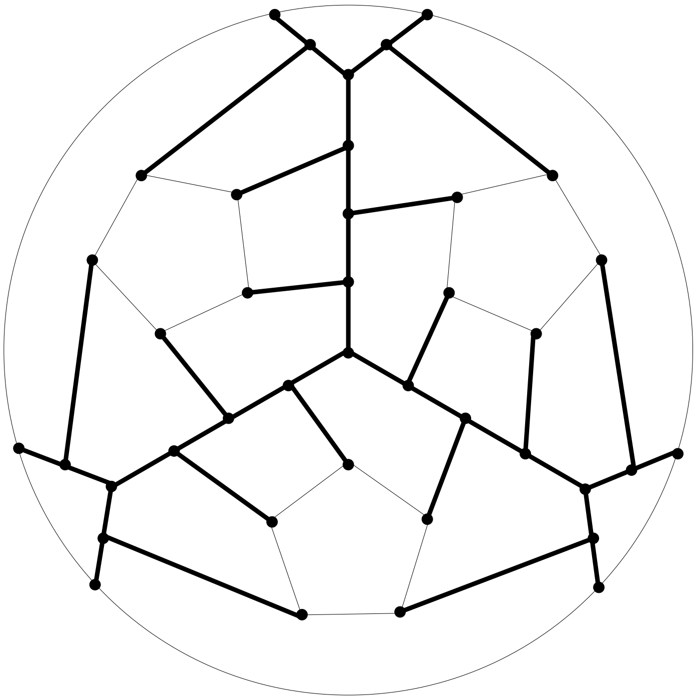

Figure 2: A fullerene graph with a HIST.

By applying Corollary 9, we see for instance that the dodecahedron does not have a HIST, since it has 20 vertices and every facial cycle has length 5. The dodecahedron belongs to the class of *fullerene graphs*, which are plane 3-connected cubic graphs with facial cycles of length 5 and 6 only, see Figure 2 for an example. Using Corollary 9 it is straightforward to prove that other plane cubic graphs, for instance the Buckminster fullerene graph [7, Figure 9.5 on p. 211] and the Grinberg graph [2, Figure 18.9 on p. 480], do not have a HIST. This should illustrate the usefulness of the above corollary. We asked in the first version of this paper whether there are finitely or infinitely many fullerene graphs with a HIST which is answered below.

Theorem 10 *There are infinitely many fullerene graphs with a HIST.*

Proof. Let A, B and H be the plane graphs shown in Figure 3 (every label of a vertex in the figure is shown left above the vertex). By identifying the cycle C_1 of A and the cycle C_2 of B such that u and v are identified, we obtain a fullerene graph with a HIST which is illustrated in bold edges. In order to construct infinitely many fullerene graphs with HISTs, we use the graphs H_i which are defined as follows (during the construction of H_i we keep the bold edges of every copy of H which will then define the edges of the HIST within H_i in the fullerene graph). Firstly, let H_0 be a plane cycle of length 18 and let $H_1 \simeq H$. Then define the graph H_i ($i \geq 2$) recursively, by identifying the outer cycle C_1' in a copy of H and the cycle (18-gon) C_2' in H_{i-1} so that u' in C_1' and v' in C_2' are identified. Now we construct for every nonnegative integer k the fullerene graph F_k with $36k + 46$ vertices. We identify the cycle C_1 in A and the outer cycle C_1' in H_k such that u in C_1 and u' in C_1' are identified. Finally, we identify the cycle C_2' in H_k and the outer cycle C_2 in B such that v' in C_2' and v in C_2 are identified. Note that the 12 shaded faces in Figure 3 are pentagons of F_k. It is not difficult to verify that the bold edges in A, B and the bold edges of H_i induce a HIST in F_k. $\square$

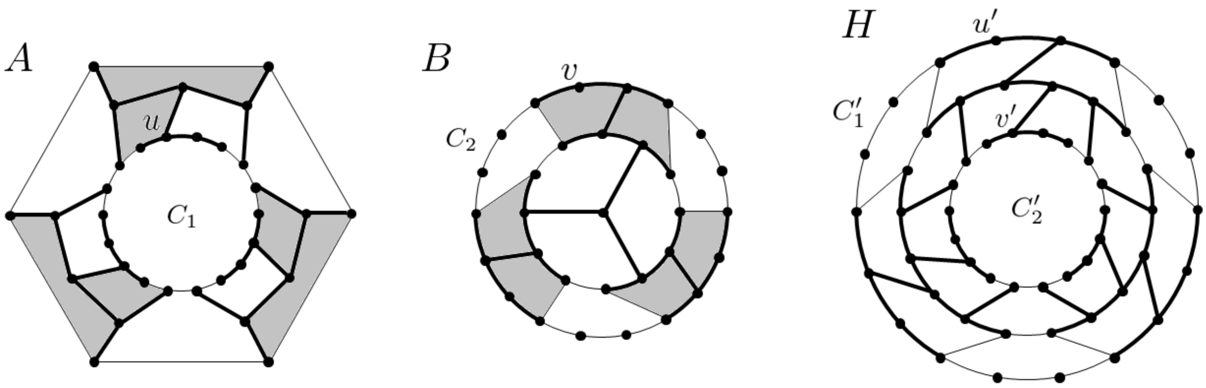

Figure 3: Plane graphs A, B and H. The shaded faces correspond to pentagons in the obtained fullerene graph, and the bold edges represent those in a HIST.

Remark: In the proof of Theorem 10, every facial cycle of the fullerene graph F_k which is edge-disjoint with the defined HIST has length 6. In contrast to F_k, the fullerene graph in Figure 2 has facial cycles of length 5 which are edge-disjoint with the illustrated HIST. By computer search, T. Jatschka [10] showed that there are fullerene graphs with HISTs with 38 vertices and that no fullerene graph with less than 38 vertices has a HIST.

A class of graphs similar to fullerene graphs are cubic hexangulations. Recall that a *hexangulation* of a surface is a 2-connected graph with an embedding on the surface such that every facial cycle has length 6. For example, consider the dual of the triangulation in Figure 4. Using this type of construction, we see that there are infinitely many bipartite cubic hexangulations G of the torus with $|V(G)| \equiv 0 \pmod 4$. Corollary 3 directly shows that such hexangulations G do not contain a HIST. We asked in the first version of this paper whether there are finitely or infinitely many hexangulations of the torus with a HIST. This question is answered in the next theorem.

Theorem 11 *There are infinitely many cubic hexangulations of the torus with a HIST.*

Proof. Let G_0 and G_1 be the hexangulations of the torus shown in the left and the center of Figure 5. (The top and the bottom, the left and the right are identified, respectively.)

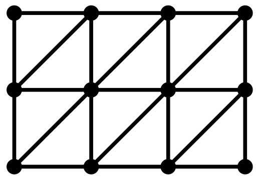

Figure 4: The dual of bipartite cubic hexangulations G of the torus satisfying $|V(G)| \equiv 0 \pmod 4$. The top and the bottom, the left and the right are identified, respectively.

Let T be the hexangulation of the annulus shown in the right of Figure 5. (The top and the bottom are identified.) We construct the cubic hexangulation G_k ($k \geq 2$) of the torus recursively, by (i) cutting G_{k-1} along the cycle $v_1v_2v_3v_4v_5v_6$, and (ii) inserting T with appropriate identification. Then G_k is a cubic hexangulation with $(12k + 10)$ vertices, and has a HIST, which is presented in Figure 5 by the bold edges. □

The length of a shortest non-contractible cycle of a graph embedded on a non-spherical surface is called the *edge-width* of the graph. Note that G_k in the proof of Theorem 11 is bipartite for every $k \geq 0$, and has girth 6 and edge-width exactly 6 for every $k \geq 1$.

After submitting the first version of this paper, the authors were informed that Zhai, Wei, He and Ye [15] also proved Theorem 11, together with the case of the Klein bottle. It was also announced that their constructed hexangulations can have arbitrary large edge-width.

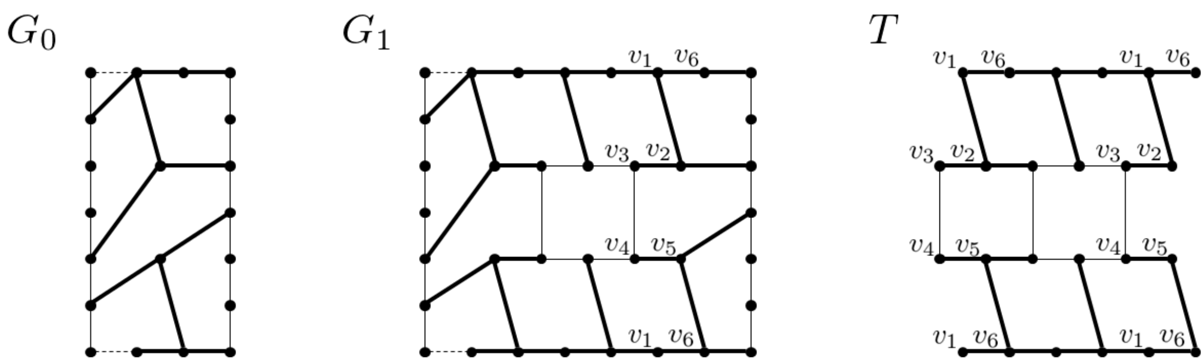

Figure 5: Hexangulations G_0, G_1 of the torus and T of the annulus. In each figure, we obtain a cylinder by identifying the top and the bottom. The bold edges represent edges in a HIST.

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