Pairs of forbidden subgraphs and 2-connected supereulerian graphs

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Abstract

Let G be a 2-connected claw-free graph. We show that

- if G is $N_{1,1,4}$-free or $N_{1,2,2}$-free or Z_5-free or P_8-free, respectively, then G has a spanning eulerian subgraph (i.e. a spanning connected even subgraph) or its closure is the line graph of a graph in a family of well-defined graphs,
- if the minimum degree $\delta(G) \geq 3$ and G is $N_{2,2,5}$-free or Z_9-free, respectively, then G has a spanning eulerian subgraph or its closure is the line graph of a graph in a family of well-defined graphs.

Here Z_i ($N_{i,j,k}$) denotes the graph obtained by attaching a path of length $i \geq 1$ (three vertex-disjoint paths of lengths $i, j, k \geq 1$, respectively) to a triangle.

Combining our results with a result in [Xiong, Discrete Math. 332 (2014) 15-22], we prove that all 2-connected hourglass-free claw-free graphs G with one of the same forbidden subgraphs above (or additionally $\delta(G) \geq 3$) are hamiltonian with the same

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excluded families of graphs. In particular, we prove that every 3-edge-connected claw-free hourglass-free graph that is $N_{2,2,5}$-free or Z_9-free is hamiltonian.
Keywords: claw-free; supereulerian; Hamiltonian cycle; forbidden subgraph.

1 Introduction.

We basically follow common graph-theoretical terminology and notation. For concepts not defined here we refer the reader to [1].

By a *graph* we always mean here a simple finite graph.

We consider a simple finite graph G. We use $N_G(x)$ to denote the neighborhood and $d_G(x)$ to denote the degree of a vertex $x \in V(G)$. A *pendant vertex* is a vertex of degree 1, and a *pendant edge* is an edge having a pendant vertex as an end vertex. We use $\delta(G)$ to denote the minimum degree of G. If F, G are graphs, we write $F \subset G$ if F is a subgraph of G, $F \subseteq G$ if F is an induced subgraph of G, and $F \simeq G$ if F and G are isomorphic. By a *clique* we mean a complete subgraph of G, not necessarily maximal, and we say that a vertex $x \in V(G)$ is *simplicial* if $\langle N_G(x)\rangle_G$ is a clique, where $\langle N_G(x)\rangle_G$ denotes the subgraph of G induced by $N_G(x)$.

If $\mathcal{F}$ is a family of graphs, we say that G is *$\mathcal{F}$-free* if G does not contain an induced subgraph isomorphic to a member of $\mathcal{F}$. The members of $\mathcal{F}$ are in this context referred to as *forbidden (induced) subgraphs.* If $\mathcal{F}$ consists of only one member F, then we write simply *F-free* instead of $\{F\}$-free. In particular, if $F \simeq K_{1,3}$, we say that G is *claw-free.*

Throughout this paper, P_i denotes the path on i vertices. Some graphs used as forbidden induced subgraphs are shown in Fig. 1.

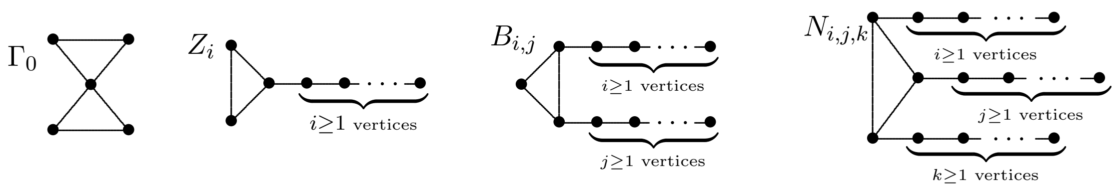

Figure 1: The graphs Γ_0, Z_i, $B_{i,j}$ and $N_{i,j,k}$.

Here the graph Γ_0 is called the *hourglass* (sometimes denoted by H), $B_{i,j}$ the *generalized bull* and $N_{i,j,k}$ the *generalized net.* The complete graph K_3 is called a *triangle.*

Hamiltonian problem is a popular classical research field of graph theory. In particular, related forbidden subgraph conditions revealing local structure of a graph form one of well-studied areas.

Let us start with the following result.

Theorem 1. (*Faudree and Gould, [6]*) *Let X and Y be connected graphs with $X, Y \notin \{P_2, P_3\}$ and let G be a 2-connected graph of order at least* 10*. Then G is $\{X, Y\}$-free implies G is hamiltonian if, and only if, by symmetry, $X = K_{1,3}$ and Y is a subgraph of P_6, Z_3, $B_{1,2}$ or $N_{1,1,1}$.*

In the following, we use $K'_{2,3}$ to denote the graph obtained from the complete bipartite graph $K_{2,3}$ by subdividing exactly one edge once. As in [3], we also use $P_{3,3,3}$ to denote the graph obtained from two vertex-disjoint triangles $x_1x_2x_3$ and $y_1y_2y_3$ by identifying end vertices of three vertex-disjoint paths $z_{i,1}z_{i,2}z_{i,3}$ with three vertices of the two triangles, *i.e.*, by identifying x_i and $z_{i,1}$, and by identifying y_i and $z_{i,3}$, respectively for $i \in \{1, 2, 3\}$.

A graph is said to be *even* if all vertices have even degrees. A connected graph is said to be *supereulerian* if it contains a spanning eulerian subgraph (i.e. a spanning connected even subgraph). The following result characterizes all pairs of forbidden subgraphs for a 2-connected graph to be supereulerian.

Theorem 2. *(Lv and Xiong, [10]) Let X and Y be connected graphs with $X, Y \notin \{P_2, P_3\}$ and let G be a 2-connected graph such that $G \notin \{K_{2,3}, K'_{2,3}, P_{3,3,3}\}$. Then G is $\{X, Y\}$-free implies G is supereulerian if, and only if, either, by symmetry, $X = K_{1,4}$ and Y is an induced subgraph of P_5, or, by symmetry, $X = K_{1,3}$ and Y is an induced subgraph of P_7, Z_4, or $N_{1,1,3}$.*

Note that Liu [9] pointed out that there would be an additional forbidden pair $\{C_4, P_5\}$ in the above theorem, i.e., by symmetry, $X = C_4$ and Y is an induced subgraph of P_5.

In this paper, we study a general problem in which some well-defined families of non-supereulerian graphs are allowed to be excluded.

We need a closure operation on claw-free graphs, invented by Ryjáček in [11]. A vertex $x \in V(G)$ is said to be *eligible* if $\langle N_G(x)\rangle_G$ is a connected noncomplete graph. We will use $V_{EL}(G)$ to denote the set of all eligible vertices of G. For $x \in V(G)$, the *local completion of G at x* is the graph $G^*_x = (V(G), E(G) \cup \{uv|\ u, v \in N_G(x)\})$ (i.e., G^*_x is obtained from G by adding to $\langle N_G(x)\rangle_G$ all missing edges). The *closure* of a claw-free graph G is the graph, denoted by $\mathrm{cl}(G)$, obtained from G by recursively performing the local completion at eligible vertices, as long as this is possible. (To be precise, there is a sequence of graphs $G_1, \ldots, G_k$ such that $G_1 = G$, $G_{i+1} = (G_i)^*_x$ for some vertex $x \in V_{EL}(G_i)$ for $1 \le i \le k-1$, and $V_{EL}(G_k) = \emptyset$, where $G_k = \mathrm{cl}(G)$). We say that G is *closed* if $G = \mathrm{cl}(G)$.

If H is a graph (multigraph), then the *line graph* of H, denoted $L(H)$, is the graph with $E(H)$ as vertex set, in which two vertices are adjacent if and only if the corresponding edges in H have a vertex in common. Recall that every line graph is claw-free. It is well-known that if G is the line graph of a graph, then a graph H such that $G = L(H)$ is uniquely determined (with the only exception of $G = K_3$ for a connected graph G). The graph H for which $L(H) = G$ will be called the *preimage* of G and denoted by $H = L^{-1}(G)$.

The following result summarizes basic properties of the closure operation.

Theorem 3. (Ryjáček, [11]) *Let G be a claw-free graph. Then*

- *$cl(G)$ is uniquely determined,*
- *$c(cl(G)) = c(G)$, where $c(\cdot)$ represents the length of a longest cycle, and*
- *$cl(G)$ is the line graph of a triangle-free graph.*

Using the closure operation, we state our main results for supereulerianicity.

For an integer r with $r \geq 3$, we construct the family $\mathcal{G}_1(r)$ of graphs, which are obtained from the complete bipartite graph $K_{2,r}$ by subdividing those r edges incident with one of the two vertices of degree r once and by adding some pendant edges to the two vertices of degree r in the original graph $K_{2,r}$. For example, see the graph on the left in Figure 2.

Let $\mathcal{H}_0$ be the family of graphs depicted on the right of Figure 2. Note that a graph in $\mathcal{H}_0$ of order n is uniquely determined. In particular, let $\mathcal{H}_0'$ be the family of graphs in $\mathcal{H}_0$ with no pendant edges. Note that $\mathcal{H}_0'$ consists of only one graph, which contains exactly nine vertices.

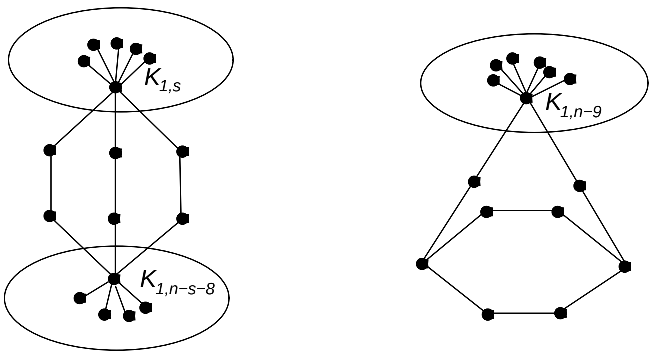

Figure 2: The left graph is in $\mathcal{G}_1(3)$ and the right graph is in $\mathcal{H}_0$ both of which have order n.

Theorem 4. *Let G be a 2-connected claw-free graph. Then all of the following hold.*

(i) If G is $N_{1,1,4}$-free, then G is either supereulerian or its closure cl(G) is the line graph of a graph H with $H \in \mathcal{G}_1(3) \cup \mathcal{H}_0$.

(ii) If G is either $N_{1,2,2}$-free or P_8-free, then G is either supereulerian or its closure cl(G) is the line graph of a graph H with $H \in \mathcal{G}_1(r)$ for some odd integers r with $r \geq 3$.

(iii) If G is Z_5-free, then G is either supereulerian or its closure cl(G) is the line graph of a graph H with $H \in \mathcal{G}_1(r) \cup \mathcal{H}_0'$ for some odd integers r with $r \geq 3$.

In advance of stating our second main result, we need to introduce another class of graphs.

For an integer $r \geq 3$, let $S(K_{2,r})$ be the graph obtained from the complete bipartite graph $K_{2,r}$ by subdividing each edge once. The vertices in $K_{2,r}$ are said to be *original*. For convenience, two vertices of degree r are denoted by u_1, u_2, and r vertices of degree 2 are denoted by $v_1, \ldots, v_r$. Let $\mathcal{G}(r)$ be the family of graphs obtained from $S(K_{2,r})$ by adding some pendant edges to every original vertex of $K_{2,r}$.

We define the family $\mathcal{G}_2(r)$ of graphs as follows. Let m_1, m_2 be two non-negative integers, and let F_i be a graph isomorphic to K_{2,s_i}, where $s_i \geq 2$ for $1 \leq i \leq m_1$, and F_j' be a graph isomorphic to K_{2,t_j}, where $t_j \geq 2$ for $1 \leq j \leq m_2$. Let x_i and y_i be the vertices of degree s_i in F_i (such that x_i and y_i are not adjacent when $s_i = 2$), and let x_j' and y_j' be the vertices of degree t_j in F_j' (such that x_i' and y_i' are not adjacent when $t_j = 2$). Then $\mathcal{G}_2(r)$ is the family of graphs obtained from a graph in $\mathcal{G}(r)$ and $(m_1 + m_2)$ graphs $F_1, \ldots, F_{m_1}, F_1', \ldots, F_{m_2}'$, by identifying all the vertices $x_1, \ldots, x_{m_1}$ to u_1 and all the vertices $x_1', \ldots, x_{m_2}'$ to u_2, and adding some pendant edges to the vertices $y_1, \ldots, y_{m_1}, y_1', \ldots, y_{m_2}'$, respectively. For example, see the left graph depicted in Figure 3.

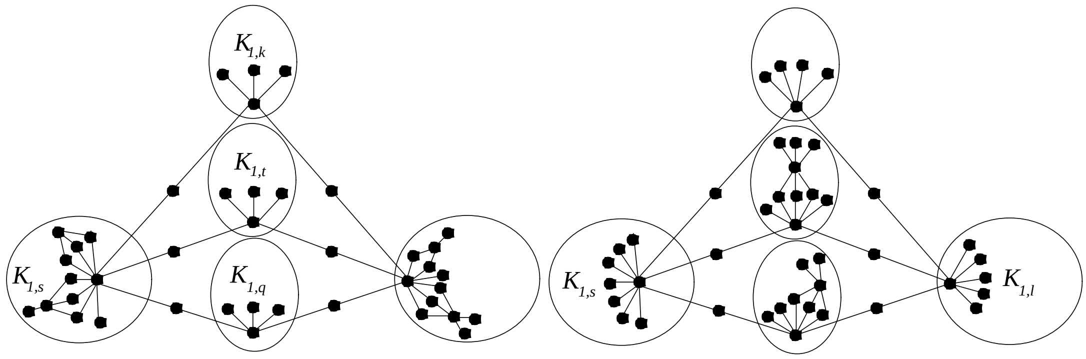

Figure 3: (Left) A graph in $\mathcal{G}_2(3)$, where $m_1 = m_2 = 2$, $s_1 = s_2 = t_2 = 3$ and $t_1 = 2$. (Right) A graph in $\mathcal{G}_3(3)$, where $F_1 \simeq K_1$, $F_2 \simeq K_{2,3}$ and $F_3 \simeq K_{2,3}$.

Next, we define the family $\mathcal{G}_3(r)$ of graphs. For $1 \leq i \leq r$, let F_i be a graph isomorphic to either K_{2,s_i} or K_1, where $s_i \geq 2$. If $F_i \simeq K_{2,s_i}$, then let x_i and y_i be the vertices of degree s_i (such that x_i and y_i are not adjacent when $s_i = 2$). Otherwise, let $x_i = y_i$ be the unique vertex in F_i. Let $\mathcal{G}_3(r)$ be the family of those graphs obtained from a graph in $\mathcal{G}(r)$ and r graphs $F_1, \ldots, F_r$ by identifying x_i with v_i for any i with $1 \leq i \leq r$ and by adding some pendant edges to each vertex in $\{y_1, \ldots, y_r\}$, respectively. For example, see the graph depicted on the right in Figure 3.

A natural problem is to ask whether we may obtain more general results if we increase the minimum degree of a graph, i.e., whether we may characterize all forbidden pairs for those graphs of minimum degree at least three. Up to now, we couldn't completely answer the question. However, we may obtain the following results which may answer this problem partially and which reveal how some forbidden pairs change when we increase slightly the minimum degree. Moreover, in the last section, we have examples to show that the bound "3" of degree is a watershed: there is no change if one increase its minimum degree more.

Theorem 5. *Let G be a 2-connected claw-free graph with $\delta(G) \geq 3$. Then all of the following hold.*

(i) If G is either Z_9-free or P_{11}-free, then G is either supereulerian or its closure $cl(G)$ is the line graph of a graph H with $H \in \mathcal{G}_2(r)$ for some odd integers r with $r \geq 3$.

(ii) If G is $N_{2,2,5}$-free, then G is either supereulerian or its closure $cl(G)$ is the line graph of a graph H with $H \in \mathcal{G}_3(3)$.

The proofs of main results are put in Section 3. Section 2 is devoted to some preparation results used in our proofs. In the last section, we shall give some comments on our results.

2 Preliminary Results

Let H be a graph. For two subgraphs D_1 and D_2 in H, we denote by $D_1 - D_2$ the subgraph of D_1 induced by $V(D_1) - V(D_2)$. Furthermore, $D_1 \cap D_2$ is the graph with $V\big(D_1 \cap D_2\big) = V(D_1) \cap V(D_2)$ and $E\big(D_1 \cap D_2\big) = E(D_1) \cap E(D_2)$, and $D_1 \cup D_2$ is the graph with $V\big(D_1 \cup D_2\big) = V(D_1) \cup V(D_2)$ and $E\big(D_1 \cup D_2\big) = E(D_1) \cup E(D_2)$. We also denote by $D_1 \triangle D_2$ the *symmetric difference of* D_1 *and* D_2, which is the subgraph of H with $V(D_1) \cup V(D_2)$ as the vertex set and $E(D_1) \triangle E(D_2)$ as the edge set. By $d_H(D_1, D_2)$ we denote the length of a shortest path between D_1 and D_2 in H. For $i \in \{1, 2\}$, if D_i consists of only one vertex u_i with no edges, then we may write it by $d_H(u_1, D_2)$ or $d_H(D_1, u_2)$ or $d_H(u_1, u_2)$.

A vertex of degree exactly two is called a *2-vertex*. A vertex on a path is said to be *inner* if it is not an end vertex. For a subgraph H_0 in a graph H, a path in H is called an H_0*-path* if both end vertices are contained in H_0 but no edges and no inner vertices are contained in H_0. A *branch* B of a graph H is a path whose end vertices are not 2-vertices and all inner vertices (if there are some) are 2-vertices of H. In particular, if B contains at least one inner vertex, we say that B is a *long* branch.

2.1 Stable properties under Ryjáček's closure

In order to prove our results, we need the following basic properties of the closure.

Theorem 6. *(Xiong, [14] and Xiong and Li, [15]) Let G be a claw-free graph. Then G is supereulerian if and only if $cl(G)$ is supereulerian.*

Theorem 7. [4, 5] *Let G be a $\{K_{1,3}, X\}$-free graph, where $X \in \{P_i, Z_i, N_{i,j,k}, \Gamma_0\}$ for some $i, j, k \geq 1$. Then $cl(G)$ is $\{K_{1,3}, X\}$-free (with X fixed).*

2.2 Line and preimage graphs

By using the results in the previous sections, it suffices to focus on a closed graph only, which is the line graph of a triangle-free graph. In our proofs, we will proceed the arguments in the preimage graphs, and hence we need the following well-known relations between line graphs and preimage graphs.

Recall that a graph is said to be *even* if all vertices have even degree. A *dominating* subgraph D of a graph H is one such that all edges in H have an end vertex contained in D. Harary and Nash-Williams [8] showed that for a graph H with $|E(H)| \geq 3$, $L(H)$ is hamiltonian if and only if H has a dominating connected even subgraph. Similarly to this, there is a close relationship between dominating even subgraphs in a graph and the property of being supereulerian for its line graph.

Theorem 8. *(Xiong, [13]; Xiong, Liu and Yi, [16]) Let H be a graph with $|E(H)| \geq 3$. Then $L(H)$ is supereulerian if and only if H has a dominating subgraph D satisfying the following three conditions.*

(D1) *D is an even graph.*

(D2) *Each isolated vertex (a trivial component) of D has degree at least three in H and every vertex of degree at least three in H is contained in D.*

(D3) $d_H(F, D - F) = 1$ *for any subgraph* $F \subseteq D$.

In this paper, instead of condition (D3), we use the following equivalent condition (D3′), which is more suitable for our proofs.

(D3′) The graph $\langle V(D) \rangle_H$ is connected.

Note that by condition (D2), the graph $\langle V(D) \rangle_H$ is obtained from H by deleting all pendant vertices and by deleting 2-vertices of H not contained in D. From this fact, it is easy to check that (D3) and (D3') are indeed equivalent.

The graph $T_{i,j,k}$ is obtained from three vertex-disjoint paths of length $i, j, k \geq 1$ by identifying exactly one end vertex of each of them. The vertex of degree three is called a *center*.

Observation 9. Let i, j, k be positive integers. The line graph $L(H)$ is $N_{i,j,k}$-free if and only if H has no (not necessarily induced) subgraph isomorphic to $T_{i+1,j+1,k+1}$. Similarly, the line graph $L(H)$ is P_i-free if and only if H has no (not necessarily induced) subgraph isomorphic to P_{i+1}.

An edge-cut C of a graph H is said to be *essential* if $H - C$ has exactly two components each of them of order at least 2. For a positive integer k, a connected graph H is called *essentially k-edge-connected* if H has no essential edge-cut C with $|C| \leq k - 1$. It is well-known that a graph H is essentially k-edge-connected if and only if the line graph of H is k-connected (see Section 3 in [2]).

2.3 Other lemma

The following lemma is useful to identify a graph in $\mathcal{G}_2(r) \cup \mathcal{G}_3(r)$.

Lemma 10. Let H be an essentially 2-edge-connected triangle-free graph, and let u be a vertex of degree at least two in H. Suppose that there is no path of length at least four starting at u. Then H is constructed in the following way. Let m be a non-negative integer, let F_i be the graph isomorphic to K_{2,s_i}, where $1 \leq i \leq m$ and $s_i \geq 2$, and let x_i and y_i be two vertices of degree s_i in F_i (such that x_i and y_i are not adjacent when $s_i = 2$). Then H is obtained from m graphs $F_1, \ldots, F_m$ by identifying all the vertices $x_1, \ldots, x_m$ to u and adding some pendant edges to the vertices $y_1, \ldots, y_m$ and u.

Proof of Lemma 10. Partition $V(H) - \{u\}$ depending on the distance from u. Namely, let $S = N_H(u)$, $Y = \{y \in V(H) : d_H(y, u) = 2\}$, and $W = \{w \in V(H) : d_H(w, u) = 3\}$. Since there is no path of length at least four from u, $\{S, Y, W\}$ is indeed a partition of $V(H) - \{u\}$.

Since H is triangle-free, we see that S is an independent set in H. Let S_0 be the set of pendant vertices adjacent to u, and let $Y = \{y_1, \ldots, y_m\}$, where $m = |Y|$. For $y_i \in Y$, let

$$S_i = \{v \in S : v \in N_H(y_i) \cap N_H(u)\}.$$

Note that $S = S_0 \cup \bigcup_{i=1}^{m} S_i$. Let $v \in S - S_0$. If $\{uv\}$ is an edge-cut of H, then it is essential because $v \notin S_0$ and u has degree at least two, contradicting that H is essentially 2-edge-connected. Therefore, there is a path J from v to u in $H - \{uv\}$. Since H is triangle-free, the length of J is at least three. Thus, by the assumption, the length of J is exactly three. This means that J contains some vertices y_i in Y and some vertex in $S_i - \{v\}$ as inner vertices, and hence $|S_i| \geq 2$. Furthermore, if $d_H(v) \geq 3$, say $vy_j \in E(H)$ with $j \neq i$, then the path $J \cup \{vy_j\}$ is a path of length at least four from u, a contradiction. Therefore, any vertex in S_i has degree exactly two in H. This also implies that for any i, j with $i \neq j$, S_i and S_j are disjoint.

Note that $y_i y_j \notin E(H)$ for any $y_i, y_j \in Y$ with $y_i \neq y_j$, since otherwise u, v, y_i, y_j, v' is a path of length four from u, where $v \in S_i$ and $v' \in S_j$, a contradiction.

For $i \in \{1, \ldots, m\}$, let F_i be the subgraph of H induced by $S_i \cup \{u, y_i\}$. Then $F_i \simeq K_{2,s_i}$, where $s_i = |S_i| \geq 2$. Note that F_i and F_j only share the vertex u for any i, j with $i \neq j$. In particular, $\langle \{u\} \cup S \cup Y \rangle_H$ is the graph obtained from m graphs $F_1, \ldots, F_m$ by sharing the vertex u and adding some pendant edges to u.

Since H is connected, any vertex w in W has a neighbor in Y, say $y_i \in Y$. If there is another neighbor of w, say w', then for $v \in S_i$, we can find the path u, v, y_i, w, w' of length four, a contradiction. Therefore, any vertex in W has degree exactly one, i.e. w is a pendant vertex. This completes the proof of Lemma 10. □

3 Proofs of main results

3.1 Proof of Theorem 4

We assume that G is (i) $N_{1,1,4}$-free, (ii) $N_{1,2,2}$-free, P_8-free or (iii) Z_5-free. By Theorems 3, 6 and 7, we may assume that G is closed and there is a triangle-free graph H with $G = L(H)$. Note that H is essentially 2-edge-connected. By Observation 9, H does not contain (i) $T_{2,2,5}$, (ii) $T_{2,3,3}$, P_9, or (iii) $T_{1,1,6}$, respectively. We shall prove that G is supereulerian, unless H is the excluded graph in each statement.

We choose a subgraph D of H with conditions (D1) and (D2) in Theorem 8 such that

(a) the number of components of the graph $\langle V(D) \rangle_H$ is as small as possible, and

(b) D dominates as many edges as possible, subject to (a).

Then we obtain the following facts according to whether G satisfies the hypothesis (i), (ii) or (iii).

Claim 1. *The graph $\langle V(D) \rangle_H$ is connected, unless G is (i) $N_{1,1,4}$-free and $H \in \mathcal{H}_0$.*

Proof of Claim 1. Suppose tothe contrary that, and let F_0 be a component of $\langle V(D) \rangle_H$. Since H is connected, it follows from condition (D2) for D that there exists a branch B of H connecting F_0 and $D - F_0$. Since F_0 is a component of $\langle V(D) \rangle_H$, any branch between F_0 and $D - F_0$ has length at least two. In particular, B is a long branch of H.

Consider the graph H' obtained from H by deleting all pendant vertices and contracting each component of D into one vertex. Since H is essentially 2-edge-connected, H' is 2-edge-connected. So, an edge in B is contained in a cycle (say) C' in H'. Since all inner vertices of B are 2-vertices, C' contains all edges in B. Let L be a component of D such that C' passes through the vertex obtained by the contraction of L. Then C' uses two edges in H' incident with L, and let v, w be the end vertices of those edges in L. (Possibly $v = w$. In particular, this must occur when L consists of an isolated vertex in D.) Since D is an even graph, the edges of D can be decomposed into edge-disjoint cycles, say *fundamental* cycles of D. Since L is connected, there exists a sequence of fundamental cycles $L_1, \ldots, L_\ell$ with $\ell \geq 1$ such that $v \in V(L_1) - V(L_2)$, $w \in V(L_\ell) - V(L_{\ell-1})$ and L_i and L_{i+1} share a vertex, say v_i, for $1 \leq i \leq \ell - 1$. Let $v_0 = v$ and $v_\ell = w$. For each i with $1 \leq i \leq \ell$, there exist two paths in L_i connecting v_{i-1} and v_i. Then choose a shorter one, say Q_{L_i}. By combining Q_{L_i} for all $1 \leq i \leq \ell$, we obtain a trail in L connecting v and w. Using such a trail for all components L of D such that C' passes through the vertex obtained by the contraction of L, we can extend C' to a cycle (say) C in H.

Suppose that for all fundamental cycles L_i of D with $E(L_i) \cap E(C) \neq \emptyset$, the path Q_{L_i} defined above has length at most two. In this case, consider the subgraph D' of H obtained from $D \triangle C$ by deleting all isolated 2-vertices. Note that D' satisfies conditions (D1) and (D2) in Theorem 8. Notice also that all edges in $L_i - E(Q_{L_i})$ still belong to D'. So, since the path Q_{L_i} has length at most two, all vertices in L_i, except for the inner vertex of Q_{L_i} (if it exists and is an isolated 2-vertex), are contained in the same component of $\langle V(D')\rangle_H$. Since this holds for all fundamental cycles L_i of D with $E(L_i) \cap E(C) \neq \emptyset$, all vertices in a same component of $\langle V(D)\rangle_H$ are contained in the same component of $\langle V(D')\rangle_H$. Furthermore, since B is contained in D', all vertices in F_0 and all vertices in some component of D are connected by B in $\langle V(D')\rangle_H$. This directly contradicts condition (a) for the choice of D.

Therefore, there exists a fundamental cycle L_i of D such that Q_{L_i} has length at least three. For simplicity, we let $Q = Q_{L_i}$, and let u_1 and u_2 be the two end vertices of Q. Note that u_1 and u_2 are connected by a path (say) Q' in $L_i - E(Q)$. By the choice of Q, the path Q' must have length at least three and $Q \cup Q' = L_i$ is a cycle in D.

Since C passes through both F_0 and $D - F_0$, there is a vertex, say v_1, in C with $d_H(v_1) \geq 3$ such that one of v_1 and Q belongs to F_0 and the other belongs to $D - F_0$. For $i \in \{1, 2\}$, let R_i be the subpath of $C - E(Q)$ between u_i and v_1. Since both R_1 and R_2 contain a branch between F_0 and $D - F_0$, they contain at least two edges. Let H_0 be the subgraph of H consisting of Q, Q', R_1, R_2 and all edges in H incident with v_1, and let $T = V(H_0) - V\big(Q \cup Q' \cup R_1 \cup R_2\big)$. Note that all vertices in T are adjacent to v_1 and $T \neq \emptyset$.

Then we may find the following subgraphs in H_0, and hence in H;

- (ii) $T_{2,3,3}$ with center u_1, which is contained in the union of the three paths $Q - u_2$, $R_1 \cup (R_2 - u_2)$ and Q'.
- (ii) P_9, which is contained in $(R_1 - u_1) \cup R_2 \cup Q \cup (Q' - u_2)$.
- (iii) $T_{1,1,6}$ with center v_1, which is contained in the union of the edge v_1x for $x \in T$, the path $R_1 - u_1$ and the path $R_2 \cup Q \cup (Q' - u_2)$.

So it suffices to show (i) the $N_{1,1,4}$-free case. Then we have the following two facts.

- Both Q and Q' have length exactly three. Otherwise we can find $T_{2,2,5}$ with center u_1, which is contained in the union of the three consecutive vertices in Q from u_1, the three consecutive vertices in Q' from u_1, and $R_1 \cup R_2$ together with remaining vertices in Q or in Q'.

- Both R_1 and R_2 have length exactly two. Otherwise we can find $T_{2,2,5}$ with center u_1, which is contained in the union of $Q - u_2$, $Q' - u_2$, and $R_1 \cup R_2$.

In particular, we have $H_0 \in \mathcal{H}_0$. Then we will show that $H = H_0$.

Suppose that $V(H) - V(H_0) \neq \emptyset$. Since H is connected, there is an edge wv in H with $w \in V(H) - V(H_0)$ and $v \in V(H_0)$. By the definition of H_0, we have $v \neq v_1$. If $v \notin T$, we can easily find $T_{2,2,5}$ with center u_1 or u_2 in $\langle V(H_0) \cup \{w\}\rangle_H$. So, we may assume $v \in T$. If $\{v_1 v\}$ is an edge-cut of H, then it is essential because of the edge vw, contradicting that H is essentially 2-edge-connected. Therefore, there is an H_0-path J from v in $H - \{v_1 v\}$. Similarly to the previous arguments, J must reach a vertex in $T - \{v\}$. Since H is triangle-free, the length of J is at least two. However, this guarantees the existence of $T_{2,2,5}$ with center u_1, which is the union of the path $Q_1 - u_2$, the path $Q_2 - u_2$ and the path $R_1 \cup \{v_1 v\} \cup J$, a contradiction. This implies $V(H) = V(H_0)$.

Suppose next that there is an edge in $E(H) - E(H_0)$. Then since H is triangle-free, we can easily check that there is a connected even subgraph D' of H containing all vertices in $H - T$, possibly except for the inner vertex of R_i for some $i \in \{1, 2\}$. Since H is triangle-free, there are no edges connecting two vertices of $H - V(D')$. This means that D' is a dominating subgraph in H satisfying conditions (D1)–(D3$'$) in Theorem 8, contradicting condition (a). Therefore, we have $E(H) = E(H_0)$, which means $H = H_0 \in \mathcal{H}_0$. The proof of Claim 1 is completed.

Claim 2. *D is dominating, unless G is either (i) $N_{1,1,4}$-free and $H \in \mathcal{G}_1(3) \cup \mathcal{H}_0$, or (ii) $N_{1,2,2}$-free or P_8-free and $H \in \mathcal{G}_1(r)$ for some odd integers r with $r \geq 3$, or (iii) Z_5-free and $H \in \mathcal{G}_1(r) \cup \mathcal{H}_0'$, for some odd integers r with $r \geq 3$.*

Proof of Claim 2. Suppose that there exists an edge xy such that $\{x, y\} \cap V(D) = \emptyset$. Then since every vertex of degree at least three belongs to D, we have $d_H(x) = d_H(y) = 2$, and hence there must be a branch B of H containing x and y. Note that the length of B is at least three.

The arguments in this paragraph are applied in a similar way to the proof of Claim 1. Consider the graph H' obtained from H by deleting all pendant vertices and contracting each component of D into one vertex. Since H' is 2-edge-connected, there exists a cycle C' containing an edge in B. For each component L of D such that C' passes through the vertex obtained by the contraction of L, we take a sequence of fundamental cycles of D connecting two end vertices of the edges in C' incident to L. By choosing an appropriate path Q_{L_i} for each fundamental cycle L_i, we can extend C' to a cycle (say) C in H. Suppose that for all fundamental cycles L_i of D with $E(L_i) \cap E(C) \neq \emptyset$, the path Q_{L_i} defined above has length at most two. Then consider the subgraph D' obtained from $D \triangle C$ by deleting all isolated 2-vertices. Then D' satisfies conditions (D1) and (D2) in Theorem 8. In addition, since all paths Q_{L_i} have length at most two and the path in $L_i - E(Q_{L_i})$ is contained in D', $\langle V(D')\rangle_H$ is still connected. However, D' dominates more edges than D (because of the edge xy), contradicting condition (b). Therefore, there exists a fundamental cycle L_i of D such that the length of

Q_{L_i} is at least three. For simplicity, we let $Q = Q_{L_i}$. Let u_1 and u_2 be the two end vertices of Q, and let Q' be the path in $L_i - E(Q)$ connecting u_1 and u_2. By the choice of Q, the path Q' must have length at least three. Let R be the path in C connecting u_1 and u_2 and containing B, and let H_0 be the subgraph of H consisting of Q, Q' and R. We have the following subclaim.

Subclaim 2.1. All of the paths Q, Q' and R contain exactly three edges, unless G is either (i) $N_{1,1,4}$-free and $H \in \mathcal{H}_0$, or (iii) Z_5-free and $H \in \mathcal{H}'_0$.

If at least two of the paths Q, Q' and R contain more than three edges, or one of them contains more than four edges, then it is easy to see that H_0 contains (i) $T_{2,2,5}$ with center u_1, (ii) $T_{2,3,3}$ with center u_1, P_9, and (iii) $T_{1,1,6}$ with center u_1. Therefore, we may assume that two of Q, Q' and R contain exactly three edges, and the remaining one contains at most four edges.

Suppose that one of the paths Q, Q' and R contains exactly four edges. We here assume that R is the one, but the other cases can be shown in the same way. In this case, H_0 contains (ii) $T_{2,3,3}$ with center u_1 and P_9, and hence it suffices to consider the case of (i) $T_{2,2,5}$ and (iii) $T_{1,1,6}$. Suppose that $V(H) = V(H_0)$. If there are no edges in $E(H) - E(H_0)$, then $H = H_0 \in \mathcal{H}'_0 \subseteq \mathcal{H}_0$. Otherwise, it is easy to see that there is a connected dominating subgraph in H satisfying conditions (D1)–(D3$'$) in Theorem 8, contradicting condition (b) for the choice of D.

Therefore, we may assume that there is an edge wv in H with $w \in V(H) - V(H_0)$ and $v \in V(H_0)$. In this case, we can easily find (iii) $T_{1,1,6}$ with center v in $\langle V(H_0) \cup \{w\}\rangle_H$. For the case of (i) $T_{2,2,5}$, let v_1 be the inner vertex in R such that the distance from u_i to v_1 on R is exactly two for $i \in \{1, 2\}$. If $v \neq v_1$, then we can also find $T_{2,2,5}$ with center u_1 or u_2. Therefore, we may assume that all vertices in $H - V(H_0)$ are adjacent to v_1, and in the same way as in the last paragraph in the proof of Claim 1, we obtain $H \in \mathcal{H}_0$.

This completes the proof of Subclaim 2.1. ■

By Subclaim 2.1, we may assume that all of the paths Q, Q' and R contain exactly three edges. In particular, $R = B$ and u_1, x, y and u_2 are the vertices of it. We here prove that both Q and Q' are branches of H. Suppose that there is a vertex z in Q with $z \neq u_1, u_2$ and $d_H(z) \geq 3$. Let z' be the unique vertex in Q with $z' \neq u_1, u_2, z$. Then consider the graph D' obtained from D by adding all edges in R, deleting all edges in Q (but keeping the vertices), and deleting the vertex z' if z' is a 2-vertex of H. It is easy to see that D' satisfies conditions (D1)–(D3$'$) in Theorem 8 and dominates more edges than D (because of the edge xy), contradicting condition (b). Therefore, both Q and Q' are branches of H. Similarly, we see the following;

Subclaim 2.2. Any path in $H - \{x, y\}$ connecting u_1 and u_2 has length at least three.

Now we take a subgraph H^*_0 of H such that

(H1) H^*_0 contains H_0,

(H2) $H^*_0 \in \mathcal{G}_1(r)$ for some $r \geq 3$, and

(H3) H_0^* contains as many edges as possible, subject to the above two conditions.

Since H_0 is isomorphic to a graph in $\mathcal{G}_1(3)$ with no pendant edges, such a subgraph H_0^* exists. For the proof of Claim 2, it suffices to show that $H = H_0^*$, r is an odd integer, and in particular $r = 3$ for (i) the $N_{1,1,4}$-free case.

Suppose that $V(H) - V(H_0^*) \neq \emptyset$. Since H is connected, there is an edge wv in H with $w \in V(H) - V(H_0^*)$ and $v \in V(H_0^*)$. We have three cases depending on v. For $i \in \{1, 2\}$, let S_i be the set of pendant vertices in H_0^* adjacent to u_i.

- If $v = u_i$ for some $i \in \{1, 2\}$, then we can add w to H_0^* by keeping conditions (H1) and (H2), contradicting condition (H3).

- Suppose that v is a vertex of degree two in H_0^*. Since Q, Q' and R are branches of H, v cannot be contained in them. Therefore, we have $r \geq 4$. By symmetry, we may assume that v is a neighbor of u_1. Then we can easily find (i) $T_{2,2,5}$ with center u_1, (ii) $T_{2,3,3}$ with center u_2, P_9, and (iii) $T_{1,1,6}$ with center u_2 in H_0, a contradiction.

- Suppose that $v \in S_i$, say $i = 1$ by symmetry. By Subclaim 2.2, we have $vu_2 \notin E(H)$. If $\{u_1v\}$ is an edge-cut of H, then it is essential because of the edge vw, contradicting that H is essentially 2-edge-connected. Therefore, there is an H_0-path J from v in $H - \{u_1v\}$. Similarly to the previous arguments, J must reach a vertex in $(S_1 - \{v\}) \cup S_2$. If J reaches a vertex in $S_1 - \{v\}$, then since H is triangle-free, the length of J is at least two; Otherwise, by Subclaim 2.2, the length of J is again at least two. Then $\langle V(H_0^*) \cup V(J) \rangle_H$ contains (i) $T_{2,2,5}$ with center u_2, (ii) $T_{2,3,3}$ with center u_1, P_9, and (iii) $T_{1,1,6}$ with center u_2, a contradiction.

These facts imply that $V(H) = V(H_0^*)$, i.e. H_0^* is a spanning subgraph of H.

Suppose next that there is an edge in $E(H) - E(H_0^*)$ or r is an even integer. In this case, we can easily find a connected even subgraph D' of H containing all vertices in $H - S_1 - S_2$. If there is an edge w_1w_2 in H with $w_1 \in S_1$ and $w_2 \in S_2$, then we can add the path $u_1w_1w_2u_2$ to H_0^* keeping conditions (H1) and (H2), contradicting condition (H3). Since H is triangle-free, there is no edge connecting two vertices of S_i for any $i \in \{1, 2\}$. Therefore, D' is a dominating subgraph in H satisfying conditions (D1)–(D3$'$) in Theorem 8, contradicting condition (b) for the choice of D.

Therefore, we have $E(H) = E(H_0^*)$ and r is an odd integer. In particular, $H = H_0^* \in \mathcal{G}_1(r)$. In addition, since all graphs in $\mathcal{G}_1(r')$ for any $r' \geq 4$ contain $T_{2,2,5}$ with center u_1, we see that $r = 3$ for (i) the $N_{1,1,4}$-free case. Then the proof of Claim 2 is completed.

By Claims 1 and 2, D is a dominating subgraph of H satisfying (D1)-(D3$'$) in Theorem 8. Hence, G is supereulerian by Theorem 8. This completes the proof of Theorem 4. □

3.2 Proof of Theorem 5

Since P_{11} is an induced subgraph of Z_9, it suffices to consider the case that G is (i) Z_9-free or (ii) $N_{2,2,5}$-free. By Theorems 3, 6 and 7, we may assume that G is closed and there is a triangle-free graph H with $G = L(H)$. Note that H is essentially 2-edge-connected. By Observation 9, H does not contain (i) $T_{1,1,10}$, or (ii) $T_{3,3,6}$, respectively.

If H contains adjacent 2-vertices, then the edge connecting them corresponds to a vertex of degree two in G, contradicting the condition $\delta(G) \geq 3$. Therefore, no pair of 2-vertices is adjacent, and hence any branch of H has length at most two. Thus, the following claim holds.

Claim 3. *Any subgraph of H satisfying condition (D2) in Theorem 8 dominates all edges in H.*

We choose a subgraph D of H with conditions (D1) and (D2) in Theorem 8 such that

(a) the number of components of the graph $\langle V(D) \rangle_H$ is as small as possible.

By Claim 3, it suffices to prove the next claim.

Claim 4. *The graph $\langle V(D) \rangle_H$ is connected, unless G is (i) Z_9-free and $H \in \mathcal{G}_2(r)$ for some odd integers r with $r \geq 3$, or (ii) $N_{2,2,5}$-free and $H \in \mathcal{G}_3(3)$.*

Proof of Claim 4. Suppose, by contradiction, that $\langle V(D) \rangle_H$ is not connected, and let F_0 be a component of $\langle V(D) \rangle_H$. Since H is connected, it follows from condition (D2) for D that there exists a branch B of H connecting F_0 and $D - F_0$. Note that the length of B is exactly two.

The arguments in this paragraph are proceeded in a similar way to the proof of Claim 1, but it is slightly different on the length of Q_{L_i}. Consider the graph H' obtained from H by deleting all pendant vertices and contracting each component of D into one vertex. Since H' is 2-edge-connected, there exists a cycle C' containing an edge in B. For each component L of D such that C' passes through the vertex obtained by the contraction of L, we take a sequence of fundamental cycles of D connecting two end vertices of the edges in C' incident to L. By choosing an appropriate path Q_{L_i} for each fundamental cycle L_i, we can extend C' to a cycle (say) C in H. Suppose that for all fundamental cycles L_i of D with $E(L_i) \cap E(C) \neq \emptyset$, the path Q_{L_i} defined above has length at most three. In this case, consider the subgraph D' obtained from $D \triangle C$ by deleting all isolated 2-vertices. Note that D' satisfies conditions (D1) and (D2) in Theorem 8 and is dominating by Claim 3. Since the path Q_{L_i} has length at most three, it contains at most one 2-vertex of H. Thus, since all edges in $L_i - E(Q_{L_i})$ still belong to D', all vertices in L_i, except for the 2-vertex contained in Q_{L_i} (if it exists), are contained in the same component of $\langle V(D') \rangle_H$. Therefore, all vertices in a component of $\langle V(D) \rangle_H$ are contained also in the same component of $\langle V(D') \rangle_H$, and all vertices in F_0 and all vertices in some component of D are connected by B in $\langle V(D') \rangle_H$. This contradicts condition (a) for the choice of D. Therefore, there exists a fundamental cycle L_i of D such that the path Q_{L_i} has length at least four. For simplicity, we let $Q = Q_{L_i}$, and let u_1 and u_2 be the two end vertices of Q. Note that u_1 and u_2 are connected by a path Q' (say) in $L_i - E(Q)$. By the choice of Q, the path Q' must have length at least four and $Q \cup Q' = L_i$ is a cycle in D.

Since C passes through both F_0 and $D - F_0$, there is a vertex, say v_1, in C with $d_H(v_1) \geq 3$ such that one of v_1 and Q belongs to F_0 and the other belongs to $D - F_0$. For $i \in \{1, 2\}$, let R_i be the subpath of $C - E(Q)$ between u_i and v_1. Note that there is no edge between v_1 and $Q \cup Q'$, and both R_1 and R_2 contain a branch between F_0 and $D - F_0$. Thus, each of them contains at least two edges. Let H_0 be the subgraph of H consisting of Q, Q', R_1 and R_2.

Let z, x be neighbors of v_1 in H with $v_1 z \in E(R_1)$ and $x \notin V(H_0)$. Then we first consider the subgraph of H obtained by the union of the edges $v_1 z, v_1 x$, and the path $R_2 \cup Q \cup (Q' - u_2)$. It is $T_{1,1,\ell}$ with center v_1 for some $\ell \geq 9$. In particular, if the length of R_2 is at least three or the length of Q is at least five or the length of Q' is at least five, then we have $\ell \geq 10$, a contradiction for (i) the Z_9-free case. For (ii) the $N_{2,2,5}$-free case, consider the subgraph of H obtained by the union of the path u_1, z, v_1, x, the path $Q - u_2$, and the path $Q' \cup (R_2 - v_1)$. If either the length R_2 is at least three or the length Q' is at least five, then it contains $T_{3,3,6}$, a contradiction. This and the symmetry imply the following two:

- Both Q and Q' have length exactly four, and
- both R_1 and R_2 have length exactly two.

In particular, H_0 is isomorphic to $S(K_{2,3})$. Let v_2 (v_3, respectively) be the inner vertex in Q (in Q', respectively) such that the distance from u_i to v_2 (v_3, respectively) on Q (on Q', respectively) is exactly two for $i \in \{1, 2\}$.

We now claim the following.

Subclaim 4.1. Any path in $H - v_1$ between u_1 and u_2 has length at least four. In addition, if exactly four, then the path consists of two long branches (of length two).

To see the first part of Subclaim 4.1, suppose that there is a path J in $H - v_1$ between u_1 and u_2 of length at most three. Then let $C' = R_1 \cup R_2 \cup J$, which is a cycle in H, and consider the subgraph D' obtained from $D \triangle C'$ by deleting all isolated 2-vertices. Note that D' satisfies conditions (D1) and (D2) in Theorem 8 and is dominating by Claim 3. Since the length of J is at most three, it can contain at most one 2-vertex. Since both R_1 and R_2 are contained in D', u_1, u_2 and all vertices in F_0 are contained in the same component of D'. This directly contradicts condition (a) for the choice of D, and the first part holds.

The other part can be proven in a similar way. ■

Subclaim 4.2. Any H_0-path between v_1 and v_2 of length at least five.

Suppose tothe contrary that there is an H_0-path J between v_1 and v_2 of length at most four. Recall that F_0 is a component in $\langle V(D) \rangle_H$ and one of v_1 and Q belongs to F_0 and the other belongs to $D - F_0$. These imply that J must contain a branch, say B', of H connecting F_0 and $D - F_0$. Note that $E(B') \cap E(D) = \emptyset$. Let $C' = R_1 \cup J \cup \big(Q - \{z_{2,2}, u_2\}\big)$, and let D' be obtained from $D \triangle C'$ by deleting all isolated 2-vertices. (Since $z_{1,2}$ becomes an isolated 2-vertex, it must be deleted.) We see that D' satisfies conditions (D1) and (D2) in Theorem 8 and is dominating by Claim 3. Since B' is contained in D', at most one isolated 2-vertex contained in J is deleted. Therefore, since $R_1 \cup Q' \cup \big(Q - \{u_1, z_{1,2}\}\big) \cup B' \subseteq E(D')$, all vertices in $C' - z_{1,2}$, except for the deleted 2-vertex in J if it exists, are contained in the same component of $\langle V(D') \rangle_H$. This implies that all vertices in F_0 and all vertices in a component of $\langle V(D) \rangle_H$ are contained in the same component of $\langle V(D') \rangle_H$, contradicting condition (a) for the choice of D. ■

Now, we divide the proof, depending on (i) the Z_9-free case and (ii) the $N_{2,2,5}$-free case.

Case (i): In this case, we take a subgraph $\widehat{H_0}$ of H such that

($\widehat{\text{H}}$1) $\widehat{H_0}$ contains H_0 and a pendant edge incident to v_1,

($\widehat{\mathrm{H}}2$) $\widehat{H_0}$ is isomorphic to the graph $S(K_{2,r})$ for some $r \geq 3$ with some pendant edges added to the original vertices of degree 2, and

($\widehat{\mathrm{H}}3$) $\widehat{H_0}$ contains as many edges as possible, subject to the above two conditions.

Since H_0 is isomorphic to $S(K_{2,3})$ and no neighbors of v_1 are contained in $Q \cup Q'$, such a subgraph $\widehat{H_0}$ exists. Note that v_1, v_2 and v_3 are the original vertices of degree 2, and we denote the remaining $(r-3)$ original vertices of degree 2 by $v_4, \dots, v_r$. For $i \in \{1,2\}$ and $j \in \{1, \dots, r\}$, let $z_{i,j}$ be the vertex of degree two in $\widehat{H_0}$ with $N_{\widehat{H_0}}(z_{i,j}) = \{u_i, v_j\}$. For $j \in \{1, 2, \dots, r\}$, let T_j be the set of pendant vertices adjacent to v_j in $\widehat{H_0}$. Note that $T_1 \neq \emptyset$.

We next claim the following three subclaims.

Subclaim 4.3. There are no edges wv in H with $w \in V(H) - V(\widehat{H_0})$ and $v \in V(\widehat{H_0}) - \{u_1, u_2\}$.

To show Subclaim 4.3, suppose the contrary. We have the following four cases depending on v.

- If $v = v_j$ for some $j \in \{1, \dots, r\}$, then we can add w to T_j, contradicting condition ($\widehat{\mathrm{H}}3$).
- Suppose that $v = z_{i,j}$ for some $i \in \{1,2\}$ and some $j \in \{1, \dots, r\}$. Since both R_1 and R_2 are branches of H, we have $j \neq 1$. Then this contradicts Subclaim 4.1 (for the path $u_1, z_{1,j}, v_j, z_{2,j}, u_2$).
- Suppose $v \in T_1$. By Subclaim 4.1, v_2 is an end vertex of branches of H. Thus, there exists a neighbor x of v_2 with $x \neq z_{1,2}, z_{2,2}$. If $x \in T_1$, then the path v_1, x, v_2 contradicts Subclaim 4.2. Therefore, $x \notin T_1$. However, we can find $T_{1,1,10}$ with center v_2, obtained by the union of the two edges $v_2 z_{1,2}$ and $v_2 x$, and the path $(Q - \{u_1, z_{1,2}\}) \cup Q' \cup R_1 \cup \{v_1 v, vw\}$, a contradiction.
- Suppose $v \in T_j$ for some $j \in \{2, \dots, r\}$. We here assume that $j = 2$, but the other cases $(j \geq 3)$ can be shown similarly. Let $x \in T_1$. Then we can find $T_{1,1,10}$ with center v_1, obtained by the union of the two edges $v_1 z_{1,1}$ and $v_1 x$, and the path $R_2 \cup Q' \cup (Q - \{z_{2,2}, u_2\}) \cup \{v_2 v, vw\}$, a contradiction.

Therefore, Subclaim 4.3 holds. ■

Subclaim 4.4. For $i \in \{1,2\}$, there is no path in $H - \big(V(\widehat{H_0}) - u_i\big)$ starting at u_i of length four.

Suppose tothe contrary that there is a path J in $H - \big(V(\widehat{H_0}) - u_i\big)$ from u_i of length four. By symmetry, we may assume that $i = 1$. For $x \in T_1$, consider the subgraph of H consisting of the two edges $v_1 z_{1,1}$ and $v_1 x$, and the path $R_2 \cup Q \cup J$, a contradiction. Therefore, Subclaim 4.4 holds. ■

Subclaim 4.5. There is no $\widehat{H_0}$-path in H connecting u_1 and u_2.

Suppose contrary that there exists an $\widehat{H_0}$-path J in H connecting u_1 and u_2. By Subclaim 4.1, the length of J is at least four, but by Subclaim 4.4, the length of $J - u_2$ is at most three. Thus, the length of J has to be exactly four. However we can add J to $\widehat{H_0}$, keeping conditions ($\widehat{\mathrm{H}}$1) and ($\widehat{\mathrm{H}}$2) (such that the new subgraph is isomorphic to $S(K_{2,r+1})$ with pendant edges), contradicting condition ($\widehat{\mathrm{H}}$3). Therefore, Subclaim 4.5 holds. ■

Now we divide $V(H) - V(\widehat{H_0})$ into two parts; For $i \in \{1, 2\}$, let

$$W_i \;=\; \{w \in V(H) - V(\widehat{H_0}) : \text{there is a path in } H - \big(V(\widehat{H_0}) - u_i\big) \text{ from } w \text{ to } u_i\}.$$

By Subclaim 4.3, $W_1 \cup W_2 = V(H) - V(\widehat{H_0})$, and by Subclaim 4.5, $W_1 \cap W_2 = \emptyset$. Therefore, $\{W_1, W_2\}$ is indeed a partition of $V(H) - V(\widehat{H_0})$.

For $i \in \{1, 2\}$, let H_i be the subgraph of H induced by $W_i \cup \{u_i\}$. Note that H_i is connected. By Subclaims 4.3 and 4.5, u_i is a cut vertex of H separating W_i from other part (unless the case $W_i = \emptyset$ occurs). Therefore, the graph H_i is essentially 2-edge-connected, and hence H_i is constructed as in Lemma 10 if u_i has degree at least two in H_i. When u_i has degree one in H_i, then H_i is isomorphic to K_2 since H is essentially 2-edge-connected.

This directly implies that the graph $\widehat{H_0} \cup H_1 \cup H_2$ is contained in $\mathcal{G}_2(r)$. Note that it is a spanning subgraph of H. By Subclaims 4.3 and 4.5, any edge in $E(H) - E\big(\widehat{H_0} \cup H_1 \cup H_2\big)$ connects two vertices in $V(\widehat{H_0})$. Therefore, if r is an even integer, or if r is an odd integer and there is an edge in $E(H) - E\big(\widehat{H_0} \cup H_1 \cup H_2\big)$, then we can easily find a connected dominating subgraph in H satisfying conditions (D1)–(D3′) in Theorem 8, contradicting condition (a). Therefore, we have $H = \widehat{H_0} \cup H_1 \cup H_2 \in \mathcal{G}_2(r)$ for some odd integers r with $r \geq 3$.

This proves Claim 4 and Theorem 5 for (i), the Z_9-free case. □

Case (ii): In this case, we take a subgraph $\widehat{H_0'}$ of H, which is slightly different from $\widehat{H_0}$ in Case (i), such that

($\widehat{\mathrm{H}}$1′) $\widehat{H_0'}$ contains H_0,

($\widehat{\mathrm{H}}$2′) $\widehat{H_0'}$ is isomorphic to the graph $S(K_{2,r})$ for some $r \geq 3$ with some pendant edges added to the original vertices of degree r, and

($\widehat{\mathrm{H}}$3′) $\widehat{H_0'}$ contains as many edges as possible, subject to the above two conditions.

Since H_0 is isomorphic to $S(K_{2,3})$, such a subgraph $\widehat{H_0'}$ exists. For $i \in \{1, 2\}$ and $j \in \{1, \ldots, r\}$, let $z_{i,j}$ be the vertex of degree two in $\widehat{H_0'}$ with $N_{\widehat{H_0'}}(z_{i,j}) = \{u_i, v_j\}$. For $i \in \{1, 2\}$, let S_i be the set of pendant vertices adjacent to u_i.

If $r \geq 4$, then we can find $T_{3,3,6}$ with center u_1, which is the union of the path $R_1 \cup (R_2 - u_2)$, the path $Q - u_2$, and the path Q' together with the path $u_2, z_{2,4}, v_4$. Thus, we may assume that $r = 3$.

We next claim the following three subclaims.

Subclaim 4.6. There are no edges wv in H with $w \in V(H) - V(\widehat{H_0'})$ and $v \in V(\widehat{H_0'}) - \{v_1, v_2, v_3\}$.

To show Subclaim 4.6, suppose the contrary. Then we have the following three cases depending on v.

- If $v = u_i$ for some $i \in \{1, 2\}$, then we can add w to S_i, contradicting condition ($\widehat{\mathrm{H}}1'$).
- Suppose that $v = z_{i,j}$ for some $i \in \{1, 2\}$ and some $j \in \{1, 2, 3\}$. Since both R_1 and R_2 are branches of H, we have $j \neq 1$. However, this contradicts Subclaim 4.1 (for the path $u_1, z_{1,j}, v_j, z_{2,j}, u_2$).
- Suppose $v \in S_i$ for some $i \in \{1, 2\}$, say $i = 1$ by symmetry. Then we can find $T_{3,3,6}$ with center u_2, which is the union of the path $(R_1 - u_1) \cup R_2$, the path $Q - u_1$, and the path $Q' \cup \{u_1 v, vw\}$, a contradiction.

Therefore, Subclaim 4.6 holds. ■

Subclaim 4.7. For $j \in \{1, 2, 3\}$, the following two facts hold.

(i) There is no path in $H - \big(V(\widehat{H'_0}) - v_j\big)$ from v_j of length four.

(ii) There are no two paths in $H - \big(V(\widehat{H'_0}) - v_j\big)$ from v_j of length three such that they share only v_j.

For (i), Suppose to the contrary that there is a path J in $H - \big(V(\widehat{H'_0}) - v_j\big)$ from v_j of length four. If $j = 1$, we can find $T_{3,3,6}$ with center u_1, which is the union of the path $Q - u_2$, the path $Q' - u_2$, and the path $R_1 \cup J$, a contradiction. Similarly, if $j \in \{2, 3\}$, then we can find $T_{3,3,6}$ with center u_i, a contradiction.

For (ii), Suppose to the contrary that there are two paths J_1 and J_2 in $H - \big(V(\widehat{H'_0}) - v_j\big)$ from v_j of length three such that they share only v_j. If $j = 1$, then we can find $T_{3,3,6}$ with center v_1, which is the union of the path J_1, the path J_2, and the path $R_1 \cup Q$, a contradiction. Similarly, if $j \in \{2, 3\}$, then we can find $T_{3,3,6}$ with center u_i, a contradiction.

Therefore, Subclaim 4.7 holds. ■

Subclaim 4.8. For any $1 \le j < j' \le 3$, there is no $\widehat{H'_0}$-path in H between v_j and $v_{j'}$.

The proof of Subclaim 4.8 can be done in a similar way to the proof of Subclaim 4.2, but we need more arguments. Suppose that there is an $\widehat{H'_0}$-path J in H between v_j and $v_{j'}$. By Subclaim 4.7 (i) for $J - v_{j'}$, the length of J is at most four. Therefore, by Subclaim 4.2, $j = 2$ and $j' = 3$ must hold.

Let $C' = R_1 \cup R_2 \cup (Q - \{u_1, z_{1,2}\}) \cup J \cup (Q' - \{z_{2,3}, u_2\})$, which is a cycle in H. Then consider the subgraph D' obtained from $D \triangle C'$ by deleting all isolated 2-vertices. (Since $z_{2,2}$ and $z_{1,3}$ become isolated 2-vertices, they must be deleted.) We see that D' satisfies conditions (D1) and (D2) in Theorem 8 and is dominating by Claim 3. So, if the number of components of $\langle V(D') \rangle_H$ is less than that of $\langle V(D) \rangle_H$, then this contradicts condition (a) for the choice of D. Since all vertices in $\widehat{H'_0} - \{z_{2,2}, z_{1,3}\}$ are contained in the same component of $\langle V(D') \rangle_H$, this directly implies that the path J is contained in D, has length exactly four, and consists of two long branches. Let w_J be the inner vertex of J such that w_J is an end vertex of the long branches. Note that w_J and $V(\widehat{H'_0}) - \{z_{2,2}, z_{1,3}\}$ must belong to different components of $\langle V(D')\rangle_H$.

Since $E(Q) \cup E(J) \subseteq E(D)$ and D is an even graph, there is a path in $D - \big(E(Q) \cup E(J)\big)$ connecting v_2 and v_3. Note that by Subclaim 4.6, it contains a path, say J', from v_2 to $v_{j''}$ for some $j'' \in \{1, 3\}$. By the same argument as above, J' has length exactly four, and hence by Subclaim 4.2, $j'' = 3$. Since $J' \subseteq E(D')$ and w_J and $V(\widehat{H'_0}) - \{z_{2,2}, z_{1,3}\}$ belong to different components of $\langle V(D')\rangle_H$, J' does not pass through w_J. Therefore, $J - v_3$ and $J' - v_3$ are paths from v_2 of length three sharing only v_2, which contradicts Subclaim 4.7 (ii).

This proves Subclaim 4.8. ■

Now we divide $V(H) - V(\widehat{H'_0})$ into three parts as follows. For $j \in \{1, 2, 3\}$, let

$$W'_j = \{w \in V(H) - V(\widehat{H'_0}) : \text{there is a path in } H - \big(V(\widehat{H'_0}) - v_j\big) \text{ from } w \text{ to } v_j\}.$$

By Subclaim 4.6, $W'_1 \cup W'_2 \cup W'_3 = V(H) - V(\widehat{H'_0})$, and by Subclaim 4.8, $W'_j \cap W'_{j'} = \emptyset$ for any $1 \leq j < j' \leq 3$. Therefore, $\{W'_1, W'_2, W'_3\}$ is indeed a partition of $V(H) - V(\widehat{H'_0})$.

For $j \in \{1, 2, 3\}$, let H'_j be the subgraph of H induced by $W'_j \cup \{v_j\}$. Note that H'_j is connected. By Subclaims 4.6 and 4.8, v_j is a cut vertex of H separating W'_j from other part (unless the case $W'_j = \emptyset$ occurs). Therefore, the graph H'_j is essentially 2-edge-connected, and hence H'_j is constructed as in Lemma 10 if v_j has degree at least two in H'_j. When v_j has degree one in H'_j, then H'_j is isomorphic to K_2 since H is essentially 2-edge-connected.

Suppose that there are two graphs F_1 and F_2 such that for $i \in \{1, 2\}$, F_i is isomorphic to K_{2,s_i} for some $s_i \geq 2$ and F_i is attached to v_j as in Lemma 10. Then for each $i \in \{1, 2\}$, F_i contains a path, say J_i, of length three starting at v_j. However, this contradicts Subclaim 4.8 (ii). Therefore, at most one graph isomorphic to K_{2,s_1} is attached to v_j.

This implies that the graph $\widehat{H'_0} \cup H'_1 \cup H'_2 \cup H'_3$ is contained in $\mathcal{G}_3(3)$. Note that it is a spanning subgraph of H. By Subclaims 4.6 and 4.8, any edge in $E(H) - E\big(\widehat{H'_0} \cup H'_1 \cup H'_2 \cup H'_3\big)$ connects two vertices in $V(\widehat{H'_0})$. Therefore, if there is an edge in $E(H) - E\big(\widehat{H'_0} \cup H'_1 \cup H'_2 \cup H'_3\big)$, then we can find a connected dominating subgraph in H satisfying conditions (D1)–(D3′) in Theorem 8, a contradiction. Therefore, we have $H = \widehat{H'_0} \cup H'_1 \cup H'_2 \cup H'_3 \in \mathcal{G}_3(3)$.

This proves Claim 4 and Theorem 5 for (ii), the $N_{2,2,5}$-free case. □

4 Concluding remarks

We have the following remarks.

- Generally, there is no much difference after we impose the condition that $\delta(G) \geq 3$. However, our results (Theorems 4 and 5) show that there is quite a difference if we impose this condition in our settings. From the example, we have no hope to get a more different situation if we increase $\delta(G)$ any large (it is the same as $\delta(G) \geq 3$). Therefore condition that $\delta(G) \geq 3$ is a watershed for pair of forbidden subgraphs conditions to guarantee a 2-connected graph to be supereulerian.

- Although few triples of forbidden subgraphs for a 2-connected graph to be hamiltonian has been known, the authors [7] characterizes all triples of forbidden subgraphs without a star, while Brousek [3] characterized all triples of forbidden subgraphs which includes the claw. From [13], we know that all the conditions on claw-free graphs guaranteeing the existence of spanning eulerian subgraphs plus the condition of either hourglass-free or the hourglass property and minimum degree at least 4 also guarantee the existence of a hamiltonian cycle. Note that line graphs of graphs in the four families $\mathcal{G}_1(r)$, $\mathcal{H}_0$ depicted in Figure 2, $\mathcal{G}_2(r)$ and $\mathcal{G}_3(r)$ described before Theorem 5 are all hourglass-free and nonhamiltonian. We may obtain some triples of forbidden subgraphs conditions (one of them is a star (claw)) that guarantee a 2-connected graph to be hamiltonian. These triples of forbidden subgraphs are counterparts of the results in [7]

- Note that every 3-edge-connected claw-free hourglass-free graph must be 2-connected and the excluded graphs in Theorem 5 are all 2-edge-connected. From this fact and the result [13] mentioned above, one may obtain the following corresponding result of Theorem 5.

  Corollary 11. Let G be a 3-edge-connected claw-free hourglass-free graph. If G is either $N_{2,2,5}$-free or Z_9-free, then G is hamiltonian.

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