Types of triangle and the impact on domination and k-walks

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1 Abstract

We investigate the minimum number of triangles in hamiltonian triangulations that do not contain an edge of a hamiltonian cycle. We prove upper and lower bounds on the maximum of these numbers for all triangulations with a given number of triangles and present results linking this number to the length of 3-walks in a class of triangulation and the domination number.

2 Introduction

In this article all triangulations are simple triangulations of the plane. For a triangulation G with a hamiltonian cycle C of G, a *type-i triangle* with $i \in \{0, 1, 2\}$ is defined as a facial triangle of G which shares i edges with C. We define $t_i(G, C)$ as the number of type-i triangles. If the triangulation and hamiltonian cycle are clear from the context, we will also just write t_i.

For $i \in \{0,2\}$ we further define

$$t_i(G) = \min\{t_i(G,C) | C \text{ is hamiltonian cycle in } G\},$$

and for even $t \geq 4$

$$t_i(t) = \max\{t_i(G) | G \text{ is a triangulation with } t \text{ triangles}\}.$$

In some cases we might want to restrict the class to 4- or 5-connected triangulations, so for $j = 4$ and $j = 5$ and even $t \geq 8$ resp. $t \geq 20$ we define

$$t_i^j(t) = \max\{t_i(G) | G \text{ is a } j\text{-connected triangulation with } t \text{ triangles}\}.$$

As will be shown in the next section, also the number of type-i triangles on one side of a hamiltonian cycle is relevant, so we also define $\bar{t}_i(G,C)$ as the number of type-i on the side of the hamiltonian cycle with fewer type-i triangles. The numbers $\bar{t}_i(G)$, $\bar{t}_i(t)$, and $\bar{t}_i^j(t)$ are defined correspondingly. By definition $\bar{t}_i(t) \leq t_i(t)/2$ and $\bar{t}_i^j(t) \leq t_i^j(t)/2$ for $i \in \{0,2\}$ and $j \in \{4,5\}$.

The inner dual of either side of a hamiltonian cycle in a triangulation is a subcubic tree in which the vertices of degree $3-i$ correspond to type-i triangles of the triangulation. Using this relation, we get

$$\bar{t}_2 = \bar{t}_0 + 2 \text{ and } t_2 = t_0 + 4.$$

If we denote the total number of vertices in a triangulation with t (so $t = t_0 + t_1 + t_2$), then we have

$$\bar{t}_1 = t/2 - 2\bar{t}_0 - 2 \text{ and } t_1 = t - 2t_0 - 4.$$

So finding the minimum value for t_0 is equivalent to finding the minimum value for t_2, and finding the maximum value for t_1.

Let G be a triangulation and let C be a hamiltonian cycle in G. We say that two type-i triangles are *adjacent* if they share an edge. An *(i,j)-pair* ($i, j \in \{1,2\}$) is defined as a pair of adjacent triangles consisting of a type-i triangle and a type-j triangle such that the common edge is contained in C. Note that each type-1 triangle is contained in at most one (1,2)-pair.

3 Motivation

Type-0 triangles are of their own interest as they are the problematic case for the extendability of partial hamiltonian cycles to the inside of separating triangles (see e.g. [6]), but the number $t_0(G)$ has also an impact for invariants that are not that obviously related to hamiltonian cycles. In this section, we describe the impact of the value of $t_0(G)$ on other results in graph theory, which formed one of the main motivations for this research.

3.1 The domination number of a triangulation

A vertex subset S of a graph G is said to be *dominating* if every vertex in $G-S$ has a neighbour in S. The cardinality of a minimum dominating set of G is called the *domination number* of G and is denoted by $\gamma(G)$. For a triangulation G, Matheson and Tarjan [10] proved that $\gamma(G) \le \frac{|G|}{3}$ (with $|G|$ the number of vertices of G), and they conjectured that $\gamma(G) \le \frac{|G|}{4}$. This conjecture is still open, even when restricted to 4- or 5-connected triangulations.

Plummer, Ye and Zha [12] proved that $\gamma(G) \le \min\left\{\lceil \frac{2|G|}{7} \rceil, \lfloor \frac{5|G|}{16} \rfloor\right\}$ for any 4-connected triangulation G. This is the currently best approach towards the Matheson-Tarjan conjecture. The idea of their inductive proof is to find a hamiltonian cycle with certain properties of type-2 triangles and to use these for reduction of the graph.

If we can find a hamiltonian cycle with few type-2 triangles, then (as implicitly used in [12]) we obtain a dominating set of bounded size, as follows: Let C be a hamiltonian cycle. By symmetry we can assume that the number of type-2 triangles on the inside of C is at most as large as on the outside of C. Let G' be the maximal outer plane graph consisting of the inside of C together with C. Note that G' contains $\bar{t}_2(G,C)$ type-2 triangles. It is shown in [3, 15] that any maximal outer plane graph H satisfies $\gamma(H) \le \frac{|H|+k(H)}{4}$, where $k(H)$ denotes the number of vertices of degree 2 in H. Any vertex of degree two in G' is the common end vertex of two edges of C in a type-2 triangle. Thus, we have $k(G') = \bar{t}_2(G,C)$. Since $\bar{t}_2(G,C) = \bar{t}_0(G,C) + 2$, we obtain

$$\gamma(G) \le \gamma(G') \le \frac{|G| + k(G')}{4} = \frac{|G| + \bar{t}_2(G,C)}{4} = \frac{|G| + \bar{t}_0(G,C) + 2}{4} \le \frac{2|G| + t_0(G,C) + 4}{8}.$$

So a hamiltonian cycle C with few type-0 triangles possibly gives a good upper bound on the domination number in a triangulation.

3.2 3-walks with few vertices visited more than once

A *k-tree* of a graph G, is a spanning tree of G in which every vertex has degree at most k. A *k-walk* is a spanning closed walk that visits every vertex at most k times. It is well-known that a graph that contains a k-walk also contains a $(k+1)$-tree (see [5]), but the converse does not hold in general. In this proof, the vertices visited k times in the k-walk are vertices of degree $k+1$ in the $(k+1)$-tree that is constructed.

Every 3-connected planar graph admits a 3-tree [1] and a 2-walk [7]. The result about 3-trees was strengthened in [11] where it is shown that every 3-connected planar graph G admits a 3-tree with at most $\frac{|G|-7}{3}$ vertices of degree 3.

As in the construction of 3-trees from 2-walks in [5] vertices visited twice in a 2-walk are vertices of degree 3 in the 3-tree, it was natural to consider the following problem, which was already mentioned in [11].

Problem 1 *Is there for every 3-connected planar graph G a 2-walk such that the number of vertices visited twice is at most $\frac{|G|}{3} - c$ for a constant c?*

Note that for a 2-walk in a graph G, the number of vertices visited twice is at most t if and only if its length is at most $|G|+t$. With this formulation of the problem in mind, the result that every 3-connected planar graph G contains a spanning closed walk of length at most $\frac{4|G|-4}{3}$ (proven in [9]) can be considered a first step towards the solution of Problem 1. However, a spanning closed walk constructed in there may visit a vertex many times, so that Problem 1 is still open.

In this section we describe a further step towards the solution of Problem 1, by limiting the number of times a vertex is visited to 3. The class for which the result is proven is a subclass of all triangulations, but in fact a class containing extremal cases for Problem 1. Type-0 triangles play an important role in the construction of the walks.

In the language of [6] the class of triangulations we will describe now is the class of triangulations where the decomposition tree is a star. In order not to refer the reader to [6] and to fix notation, we will give an independent description of the class here. To simplify notation, we consider K_4 also as a 4-connected graph in this section. Let $\mathcal{K}$ be the set of all graphs G that can be constructed as follows: Take any 4-connected triangulation H and let F be a subset of triangles of H. For each triangle $f = xyz \in F$, take a 4-connected plane graph G_f and let x_f, y_f and z_f be three vertices in G_f that are contained in the same facial cycle. We may assume that the facial cycle containing those three vertices is the outer one. Then G is obtained from H by adding G_f inside f for $f \in F$, so that x, y, z are identified with x_f, y_f, z_f, respectively. For each $G \in \mathcal{K}$ – except for the case when G has exactly one separating triangle – the graph H is uniquely defined and we write $H(G)$ for it. In the case of one separating triangle there are two possible candidates for H and $H(G)$ denotes an arbitrary one of them.

For example, the face subdivision of a 4-connected triangulation belongs to $\mathcal{K}$. In the definition above, F is the set of all triangles of H and for any face f we have $G_f \simeq K_4$. As in [11, Section 2], the face subdivision of a 4-connected triangulation shows that we cannot decrease the coefficient $\frac{1}{3}$ of $|G|$ in Problem 1. So, in this sense, some graphs in $\mathcal{K}$ belong to the most difficult ones for Problem 1.

The following result shows that hamiltonian cycles C in a 4-connected triangulation T with small $t_0(T, C)$ give a 3-walk of short length for the graphs $G \in \mathcal{K}$ with $H(G) = T$. Using Theorem 8 we then get a general upper bound depending only on the number of vertices in G.

Theorem 2 *Let $G \in \mathcal{K}$ be given and C a hamiltonian cycle in $H = H(G)$. We write $t'_0(H, C)$ (or short t'_0) for the number of those type-0 triangles of H that are no faces in G. Then we have:*

(a) *G contains a 3-walk of length at most $\frac{4|G|+t'_0-4}{3}$ which visits each vertex not in H exactly once.*

(b) *G contains a 3-walk of length at most $\frac{22|G|-34}{15}$.*

Proof: Let F, H, G_f, x_f, y_f and z_f for each triangle $f \in F$ be as in the definition of $\mathcal{K}$. We denote the length of a walk W by $l(W)$, the number of triangles of a triangulation G by $t(G)$, and the number of interior vertices of a triangulation G' as $|G'|_i$, so $|G'|_i = |G'| - 3$. With this notation we have $|G| = |H| + \sum_{f \in F} |G_f|_i$.

(a)

Claim 1 *For G a 4-connected triangulation (including K_4) with boundary vertices x, y, z and $a, b \in \{x, y, z\}$ (with possibly $a = b$), there is a (possibly closed) walk $P_{G,a,b}$ of length $|G|_i + 1$ from a to b in G visiting exactly all vertices in G except those in $\{x, y, z\} \setminus \{a, b\}$ and visiting vertices not in the boundary exactly once.*

Proof: The case $G = K_4$ can be easily checked by hand, so assume that G is not K_4.

If $a = b$ (w.l.o.g. $a = b = x$) then according to [14, (3.4)] there exists a hamiltonian cycle in $G - \{y, z\}$, which is a closed walk with the given properties starting and ending in a.

If $a \neq b$ (w.l.o.g. $a = x$, $b = y$), due to Corollary 2 in [13] there is a hamiltonian cycle C in G through $\{a, z\}$ and $\{b, z\}$. $C - \{\{a, z\}, \{b, z\}\}$ is the walk $P_{G,a,b}$.

□

Fix a vertex c_1 and a direction of C, so that we get a linear order $c_1 < c_2 < \cdots < c_n$ of the vertices of C. For each triangle f of H we fix the notation of x_f, y_f, z_f so that $x_f < y_f < z_f$.

With this notation we have:

Claim 2 *For any two triangles f and f' that belong to the same side of C we have $y_f \neq y_{f'}$.*

Proof: Assume $x_f \leq x_{f'}$. C is divided into three segments by the vertices x_f, y_f and z_f and – as $x_{f'}, y_{f'}$ and $z_{f'}$ are all at least x_f and smaller than c_n they occur in one of these segments in the order $x_{f'}, y_{f'}, z_{f'}$. This implies that only $x_{f'}$ and $z_{f'}$ can be one of the boundary vertices of the segment and $y_{f'}$ is in fact different from each of x_f, y_f and z_f.

□

We consider the following spanning subgraph H^*_C of the dual of H: The vertex set of H^*_C is the set of triangles of H, and two faces are adjacent in H^*_C if and only if they share an edge in C. Note that for $i \in \{0, 1, 2\}$, a type-i triangle has degree exactly i in H^*_C. In particular, each component of H^*_C is an isolated vertex, a path or a cycle. Note that we can give an orientation to the edges of such a component P^*, so that each vertex in P^*, except for isolated vertices and one of the end vertices when P^* is a path, has out-degree one. In cases where only one end vertex v of such a path P^* belongs to F, we choose v to have out-degree one.

We can partition F into two sets F_0 and F_1:

$$F_i = \{f \in F : f \text{ has out-degree exactly } i\}$$

for $i \in \{0, 1\}$.

Our construction gives $|F_0| \leq t'_0 + \frac{t'_1}{2}$.

Now we modify C so that for each triangle $f \in F$ it visits each vertex inside G_f exactly once:

- Suppose that $f \in F_0$. Then we add the walk P_{G_f,y_f,y_f} to C. This increases the length of C by $|G_f|_i + 1$.
- Suppose that $f \in F_1$. Let f' be the out-neighbour of f, and let $\{a, b\}$ be the edge in C that is shared by f and f'.

  Then we replace $\{a, b\}$ in C by $P_{G_f,a,b}$. This increases the length of C by only $|G_f|_i$ as one edge in C is also deleted.

The resulting walk C' is a 3-walk as due to Claim 2 the number of time a vertex is visited is at most increased by 1 for each side of C.

We will first give some equations we will use to compute the length of C'. For the given hamiltonian cycle C we denote $t_0(H, C)$, $t_1(H, C)$ and $t_2(H, C)$, by t_0, t_1, t_2, respectively.

As $t_0+t_1+t_2 = t(H) = 2|H|-4$ and $t_2 = t_0+4$ we get $|H| = \frac{2t_0+t_1}{2}+4 \geq \frac{2t'_0+t'_1}{2}+4$.

As in each face of F at least one vertex is inserted, we get $|G| \geq |H| + t'_0 + t'_1$, so with the previous equation $|G| \geq \frac{4t'_0+3t'_1}{2} + 4 = \frac{6t'_0+3t'_1}{2} + 4 - t'_0$ which can be rewritten as $t'_0 + \frac{t'_1}{2} \leq \frac{|G|+t'_0-4}{3}$.

Computing the length of C' we get

$$\begin{aligned} l(C') &= l(C) + \sum_{f \in F_1} |G_f|_i + \sum_{f \in F_0} (|G_f|_i + 1) = l(C) + \sum_{f \in F} |G_f|_i + |F_0| \\ &= |H| + \sum_{f \in F} |G_f|_i + |F_0| = |G| + |F_0| \leq |G| + t'_0 + \frac{t'_1}{2} \\ &\leq |G| + \frac{|G| + t'_0 - 4}{3} = \frac{4|G| + t'_0 - 4}{3} \end{aligned}$$

This completes the proof of part (a) of Theorem 2.

(b) As each triangle of H that is not a face of G contains at least one vertex, we have that $|G| \geq |H| + t'_0$. Using Theorem 8 we get

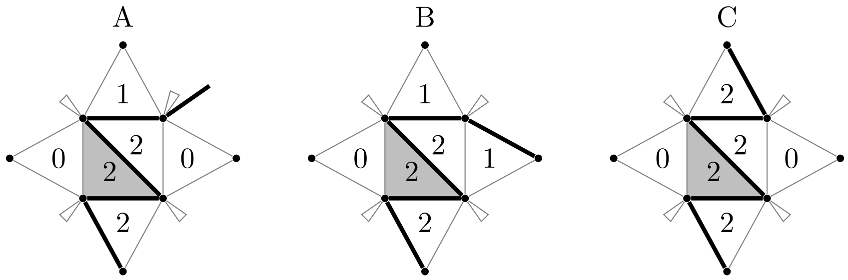


Figure 1: The three possible different neighbourhoods of a type-2 triangle (gray) contained in two $(2,2)$-pairs.

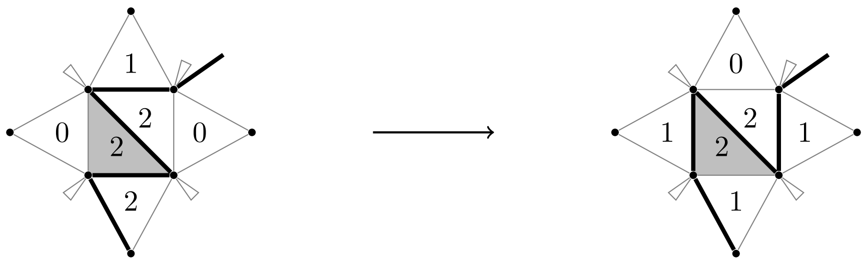


Figure 2: A rerouting strategy for neighbourhoods of type A.

$$t_0' \leq t_0(t(H)) = t_0(2|H|-4) \leq \frac{2|H|-14}{3} \leq \frac{2(|G|-t_0')-14}{3}$$

which implies

$$t_0' \leq \frac{2|G|-14}{5}.$$

Inserting this into the equation given in (a) we get (b).

■

4 Upper bounds for $t_0(t)$, $t_0^4(t)$,and $t_0^5(t)$

Lemma 3 *Let G be a triangulation with a hamiltonian cycle C, but without a vertex of degree $|G|-1$. Then there exists a hamiltonian cycle C' in G such that C' has at least as many $(1,1)$-pairs as C and C' has no triangle of type-2 which is contained in two $(2,2)$-pairs.*

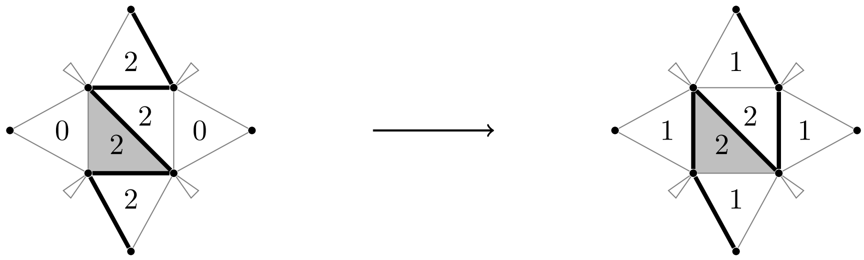
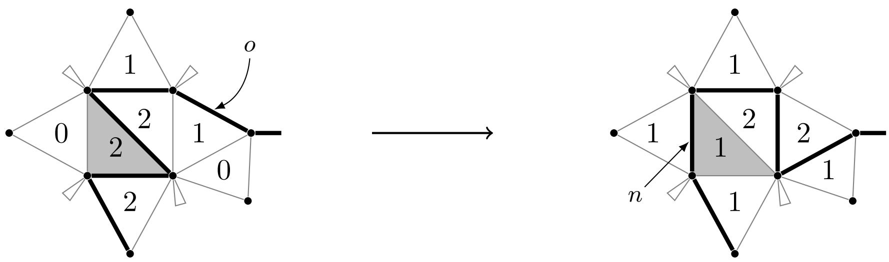
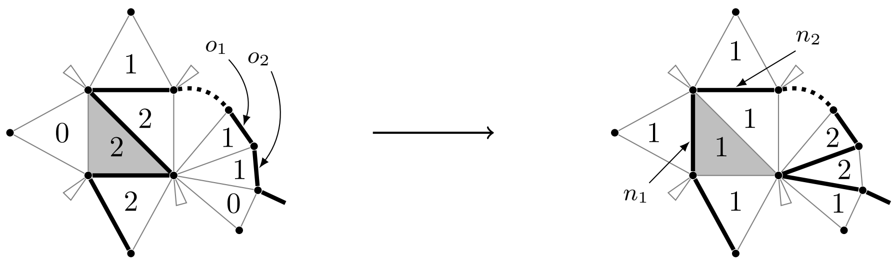


Figure 3: A rerouting strategy for neighbourhoods of type C.



(a) A rerouting strategy for neighbourhoods of type B where the distance between T and T_0 is 3.



(b) A rerouting strategy for neighbourhoods of type B where the distance between T and T_0 is greater than 3.

Figure 4: A rerouting strategy for neighbourhoods of type B.

Proof: Figure 1 shows the three possible different neighbourhoods of a type-2 triangle T contained in two $(2,2)$-pairs. A neighbourhood of type A or of type C can be rerouted as shown in Figure 2, resp. Figure 3. Note that all triangles affected by the rerouting are shown. In both cases the original type-2 triangle contained in two $(2,2)$-pairs is no longer contained two $(2,2)$-pairs. The rerouting does not create any new type-2 triangles, so it also does not create any new $(2,2)$-pairs. In case of a neighbourhood of type A, one type-1 triangle is turned into a type-0 triangle, but this original type-1 triangle was contained in a $(1,2)$-pair, so we do not loose any $(1,1)$-pairs in this case. In case of a neighbourhood of type C, the affected neighbourhood does not contain any type-1 triangles, so we also do not loose any $(1,1)$-pairs in this case.

The same rerouting does not work for a neighbourhood of type B, since it would create a new $(2,2)$-pair and also destroys a type-1 triangle that might be contained in a $(1,1)$-pair. We can apply a different rerouting in this case. Let v be the vertex of T that is incident to both edges of T that are contained in C. If v is not incident to a type-0 triangle, we immediately get that v is adjacent to all other vertices of G which contradicts the fact that G has no dominating vertex. So we have that v is incident to at least one type-0 triangle. Consider the first type-0 triangle T_0 from T in clockwise direction in the cyclic order around v. There are two cases which behave slightly different. The first case is that the distance in clockwise direction around v between T and T_0 is 3 (see the left part of Figure 4a). The other case is that this distance is more than 3 (see the left part of Figure 4b).

In the first case, we can reroute C as shown in Figure 4a. The edge o is possibly contained in a $(1,1)$-pair. If so, then that pair is the only pair that gets destroyed by the rerouting, but we also introduce at least one new $(1,1)$-pair at the edge n, so we have that the new hamiltonian cycle has at least as many $(1,1)$-pairs and fewer type-2 triangles contained in two $(2,2)$-pairs.

In the second case, we can reroute C as shown in Figure 4b. The edge o_1 is contained in a $(1,1)$-pair, and the edge o_2 is possibly contained in a $(1,1)$-pair. So, at most two pairs get destroyed by the rerouting, but we also introduce at least two new $(1,1)$-pairs at the edges n_1 and n_2, so we have that the new hamiltonian cycle has at least as many $(1,1)$-pairs and fewer type-2 triangles contained in two $(2,2)$-pairs. ■

Using a result by Whitney [16], we can prove the existence of a hamiltonian cycle with at least one $(1,1)$-pair in a 4-connected triangulation. Below we first give the lemma by Whitney, but use a simplified version of the formulation from [8].

Lemma 4 *Let G be a 4-connected triangulation. Consider a cycle D in G together with the vertices and edges on one side of D (referred to as the outside of D). Let a and b be two vertices of D dividing D into two paths P_1 and P_2 each of which contains both a and b. If*

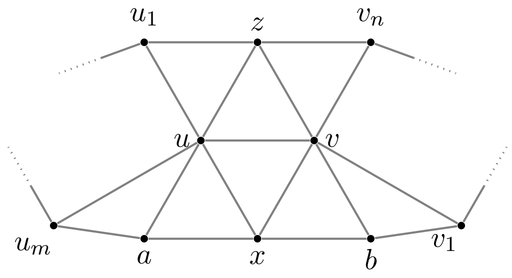


Figure 5: Construction of a hamiltonian cycle with at least one $(1,1)$-pair in a 4-connected triangulation.

- *no two vertices of P_1 are joined by an edge which lies outside of D; and*
- *there is a vertex z (distinct from a and b) dividing P_2 into two paths P_3 and P_4 each of which contains z such that no pair of vertices in P_3 and no pair of vertices in P_4 are joined by an edge which lies outside of D.*

then there is a path from a to b using only edges on and outside of D which passes through every vertex on and outside of D.

Using this lemma, we can give the following result. Note that for triangulations being k-connected is equivalent to having no separating cycles of length shorter than k.

Lemma 5 *Let G be a 4-connected triangulation which is not isomorphic to the octahedron. There exists a hamiltonian cycle C in G such that C has at least one $(1,1)$-pair.*

Proof: Let x be a vertex of G which has degree at least 5, and let uvx be an arbitrary triangle containing x. The edge uv is contained in a second triangle, say uvz. Let the vertices adjacent to u (in counter clockwise order) be $v, z, u_1, \ldots, u_m, a, x$ (note that there are no u_i vertices if u has degree 4), and let the vertices adjacent to v be $u, x, b, v_1, \ldots, v_n, z$ (note that there are no v_i vertices if v has degree 4) (see Figure 5).

Consider the cycle $D = axbv_1 \cdots v_n z u_1 \cdots u_m a$ in G. The vertices a and b partition D into two paths satisfying the conditions of Lemma 4 with $P_1 = axb$. The path P_2 is divided into P_3 and P_4 by the vertex z. As x has degree at least 5, a and b are not adjacent. All vertices of P_3, resp. P_4, are adjacent to u, resp. v, so any edge which lies outside of D and joins two vertices of P_3 or P_4 would be part of a separating triangle.

Let P be the path from a to b described in Lemma 4. The hamiltonian cycle $C = P \cup auvb$ contains the $(1,1)$-pair (uvx, uvz). ∎

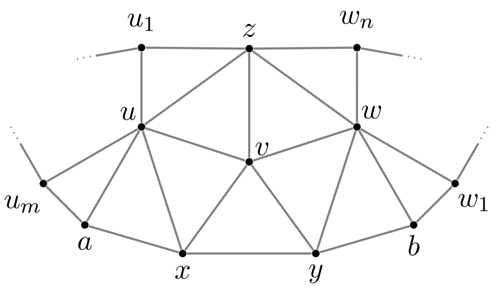


Figure 6: Construction of a hamiltonian cycle with at least two $(1,1)$-pairs in a 5-connected triangulation.

In the case of 5-connected triangulations, we can prove a slightly stronger result.

Lemma 6 *Let G be a 5-connected triangulation. There exists a hamiltonian cycle C in G such that C has at least two $(1,1)$-pairs.*

Proof: Let v be a vertex of G which has degree 5, and let u and w be two neighbouring vertices of v which are not adjacent to each other. Let the vertices adjacent to u be $v, z, u_1, \ldots, u_m, a, x$, and let the vertices adjacent to w be $v, y, b, w_1, \ldots, w_n, z$ (see Figure 6).

Consider the cycle $D = axybw_1 \cdots w_n z u_1 \cdots u_m a$ in G. The vertices a and b partition D into two paths satisfying the conditions of Lemma 4 with $P_1 = axyb$. The path P_2 is divided into P_3 and P_4 by the vertex z. As all vertices have degree at least 5, any edge outside of D connecting two vertices of P_1 is contained in a separating triangle or a separating quadrangle. All vertices of P_3, resp. P_4, are adjacent to u, resp. w, so any edge which lies outside of D and joins two vertices of P_3 or P_4 would be part of a separating triangle.

Let P be the path from a to b described in Lemma 4. The hamiltonian cycle $C = P \cup auvwb$ contains the $(1,1)$-pairs (uvx, uvz) and (vwy, vwz). ■

By combining the results above, we can prove upper bounds for $t_0(t)$, $t_0^4(t)$, and $t_0^5(t)$.

Lemma 7 *Let G be a triangulation with a dominating vertex v and t triangles. Then $t_0(G) < \frac{t}{4} - 1$ if G is not K_4 and $t_0(K_4) = 0$.*

Proof: We can easily check K_4 by hand, so assume that G is not K_4.

$G - \{v\}$ is a triangulation of a disc, so it has a vertex w of degree 2. Let w' be a vertex sharing a boundary edge of $G - \{v\}$ with w and let C be the hamiltonian cycle containing $\{v, w\}$, $\{v, w'\}$ and the boundary cycle of $G - \{v\}$ without the edge $\{w, w'\}$.

Then no type-0 triangle contains v and with t_0^d, t_1^d and t_2^d the number of triangles of type 0, 1 and 2 on the side of the hamiltonian cycle containing the triangle v, w, w' we have: $t_0(G) = t_0^d$, $t_0^d + t_1^d + t_2^d = \frac{t}{2}$. Furthermore (as G is not K_4) we have $t_1^d \geq 1$ (the unique triangle containing w but not v). So

$t_0^d + t_2^d < t_0^d + t_1^d + t_2^d = \frac{t}{2}$. With $t_2^d = t_0^d + 2$ we get

$2t_0^d + 2 = 2t_0 + 2 < \frac{t}{2}$ and finally $t_0(G) < \frac{t}{4} - 1$.

■

Theorem 8

- *For $t \geq 8$ we have $t_0(t) \leq \frac{t-8}{3}$, and for $4 \leq t < 8$ we have $t_0(t) = 0$.*
- *For $t \geq 10$ we have $t_0^4(t) \leq \frac{t-10}{3}$, and for $t = 8$ we have $t_0^4(t) = 0$.*
- *For $t \geq 20$ we have*

$$t_0^5(t) \leq \frac{t-12}{3}.$$

Proof: The values given with equality are computational results. In the proof we will focus on values where just a bound is given. Graphs G with $|G| > 4$ and a vertex of degree $|G| - 1$ have a 3-cut, so they have only to be taken into account for the bounds of $t_0(.)$.

For $t < 20$ we also verified the bounds with a computer. For $t \geq 20$ we have $\frac{t}{4} - 1 \leq \frac{t-8}{3}$ and Lemma 7 implies the result for graphs with a dominating vertex v.

Let now G be a hamiltonian triangulation with $t \geq 20$ triangles, no dominating vertex, and a hamiltonian cycle with p $(1,1)$-pairs. Due to Lemma 3, G contains a hamiltonian cycle C' which has at least p $(1,1)$-pairs and in which each type-2 triangle is contained in at least one $(1,2)$-pair.

This gives us

$$t_2(G, C') \leq t_1(G, C') - 2p.$$

Since $t_2(G, C') = t_0(G, C') + 4$, this can be rewritten as

$$t_1(G, C') \geq t_0(G, C') + 4 + 2p.$$

Combining this with $t = t_0(G, C') + t_1(G, C') + t_2(G, C')$, we get

$$t \geq t_0(G, C') + t_0(G, C') + 4 + 2p + t_0(G, C') + 4.$$

This can be rewritten as

$$t_0(G, C') \leq \frac{t-8-2p}{3},$$

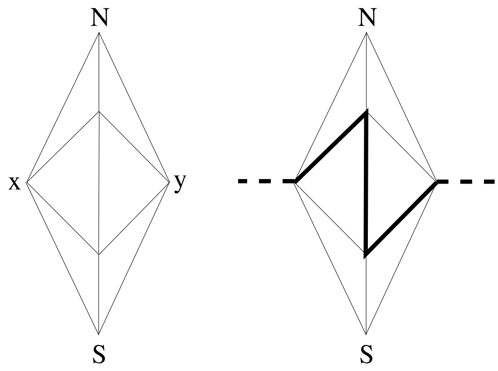


Figure 7: The fragment B used to construct a family of graphs in which each hamiltonian cycle has a lot of type-0 triangle and the most common way for a hamiltonian cycle to pass through this fragment.

and so we also have

$$t_0(G) \leq \frac{t-8-2p}{3}.$$

Using the values for p from Lemma 5 and Lemma 6, we get the given bounds. ■

5 Lower bounds for $t_0(t)$, $t_0^4(t)$ and $t_0^5(t)$

In order to prove lower bounds for $t_0(t)$, $t_0^4(t)$ and $t_0^5(t)$, we will construct families of graphs in which each hamiltonian cycle has many type-0 triangles.

Theorem 9

- *Let $t \geq 15$ be even. Then $t_0(t) \geq \lfloor \frac{t}{3} \rfloor - 5$ and $\bar{t}_0(t) \geq \lfloor \frac{t+2}{6} \rfloor - 3$. We have $t_0(14) = 1$ and $\bar{t}_0(14) = 0$. For $t < 14$ we have $t_0(t) = \bar{t}_0(t) = 0$.*
- *Let $t \geq 18$ be even. Then $t_0^4(t) \geq 2(\lfloor \frac{t}{6} \rfloor - 3)$ and $\bar{t}_0^4(t) \geq \lfloor \frac{t}{6} \rfloor - 3$. For $t < 18$ we have $t_0^4(t) = \bar{t}_0^4(t) = 0$.*
- *Let t be even. Then there exists a constant c so that $t_0^5(t) \geq \frac{t}{6} - c$. For $t \leq 66$ we have that $\bar{t}_0^5(t) = 0$.*

Proof: $\mathbf{t_0^4(t)}$ and $\mathbf{\bar{t}_0^4(t)}$:

First consider the case where t is a multiple of six, and let $k = \frac{t}{6}$. Consider the fragment B shown in the left part of Figure 7. Take k copies $B_0, \ldots, B_{k-1}$ and identify all vertices labelled N and all vertices labelled S (we call the resulting vertices the poles) and for $0 \leq i < k$ identify vertex y in B_i met vertex x in $B_{i+1 (\mathrm{mod}\ k)}$. This graph has $6k$ triangles, and we denote it by G_k.

An edge of the hamiltonian cycle which is incident to a pole is contained in at most two fragments. Since there are two edges incident to each pole, there are at most 8

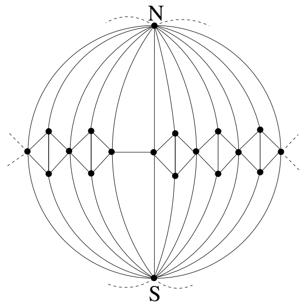


Figure 8: Modifications of the construction for the 4-connected case when t is not a multiple of 6 and for the 3-connected case.

fragments that contain an edge of the hamiltonian cycle that is incident to a pole. For all other fragments the hamiltonian cycle visits the fragment – up to symmetry – as shown in the right part of Figure 7. This partial hamiltonian cycle produces two type-0 triangles in each fragment – one on each side. So a hamiltonian cycle with a minimum number of type-0 triangles in G_k with $k \geq 9$ can be reduced to a hamiltonian cycle with a minimum number of type-0 triangles in G_{k-1}, and we find that $t_0(G_k) = t_0(G_{k-1}) + 2$ for $k \geq 9$ and $\bar{t}_0(G_k) = \bar{t}_0(G_{k-1}) + 1$ for $k \geq 9$. Computations give that for $3 \leq k \leq 8$ we have $t_0(G_k) = 2k - 6$ and $\bar{t}_0(G_k) = k - 3$. Since G_k contains $6k$ triangles, we can also write this as $t_0(G_k) = \frac{t}{3} - 6$ and $\bar{t}_0(G_k) = \frac{t}{6} - 3$

For the case where t is not a multiple of six, we let $k = \lfloor \frac{t}{6} \rfloor$. We apply the same construction, but for one or two fragments we connect the x- and y-vertex with an edge instead of identifying them – see the left part of Figure 8. This gives 2, resp. 4 extra triangles and for the fragments the argumentation above still holds. The results of the computation showed that even in the case of two edges the relative position did not matter and the results in the lemma were confirmed.

$\mathbf{t_0(t)}$ and $\mathbf{\bar{t}_0(t)}$:

For $t_0(t)$ and $\bar{t}_0(t)$ where 3-cuts are allowed, we use the same fragment and the same constructions as for $t_0^4(t)$ and $\bar{t}_0^4(t)$, but for two fragments we do not identify x and y but instead connect N and S by an edge between these segments – see the right part of Figure 8. This construction with k fragments gives triangulations with $6k + 2$ triangles that can be extended to $6k + 4$ and $6(k + 1)$ by inserting vertices of degree 3 in one or both triangles containing the edge between the poles.

The reduction argument can in this case already be applied for $k > 6$ fragments, as the fragments neighbouring the edge connecting the poles must both contain an edge to a pole that is contained in only one fragment. It turned out that for $t \equiv 4 (\text{mod } 6)$ the largest value for $t_0(t)$ is obtained when the edge is adjacent to a fragment neighbouring the edge connecting the poles. In any case computations for $k \leq 8$ fragments give that $t_0(t) \geq \lfloor \frac{t}{3} \rfloor - 5$ and $\bar{t}_0(t) \geq \lfloor \frac{t+2}{6} \rfloor - 3$.

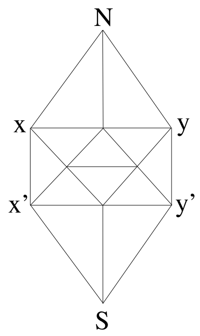


Figure 9: The fragment used for the 5-connected case.

For small values of t a double wheel where triangles are subdivided with a vertex of degree 3 alternatingly on both sides of the rim gives a larger result for $t_0(t)$ and $\bar{t}_0(t)$, but the linear factor is only $\frac{1}{4}$, so that the advantage compared to the sequence described is only for small values.

$\mathbf{t_0^5(t)}$ **and** $\mathbf{\bar{t}_0^5(t)}$**:**

For 5-connected triangulations we have to use a different fragment. As we will not be able to give a good approximation of the additive constant anyway, we will only give the construction for t a multiple of 12. For t not a multiple of 12, fragments with $t' \equiv 2(\text{mod } 12), t' \equiv 4(\text{mod } 12), \dots t' \equiv 10(\text{mod } 12)$ triangles must also be used. By a construction similar to the one used for $t_0^4(t)$ and $\bar{t}_0^4(t)$, but using $k > 8$ fragments like depicted in Figure 9, for each hamiltonian cycle C there are at least $k - 8$ fragments not containing an edge of C to a pole. Checking the different ways how a hamiltonian cycle can pass such a segment and saturate the 4 interior vertices (some boundary vertices can also be saturated from outside the segment), gives that each such segment contains at least 2 type-2 triangles. So for the number n_2 of type-2 triangles and therefore also for the number n_0 of type-0 triangles we have that $n_2 = 2k - c'$, $n_0 = 2k - c''$ for some constants c' and c''. As each segment contains 12 triangles we get $n_0 = \frac{t}{6} - c$ for some constant c – also taking into account triangles in a segment necessary to get the cases when t is not a multiple of 12. In fact the graphs G constructed with only the segments in Figure 9 all allow a hamiltonian cycle with $\bar{t}_0(G) = 0$ and a computer search using graphs constructed by the program plantri [2] showed that in fact all 5-connected triangulations G with up to 66 triangles have $\bar{t}_0(G) = 0$.

■

For $t_0(t)$, $\bar{t}_0(t)$, $t_0^4(t)$, and $\bar{t}_0^4(t)$ the upper and lower bounds differ only by an additive constant.

6 Correctness of the computer programs used

The programs constructing hamiltonian cycles and computing $t_0()$ and $\bar{t}_0()$ are straightforward branch and bound programs that can be obtained from the authors. Two independent programs were developed and implemented and the result was compared for each of the around 150.000.000 triangulations with up to 30 triangles. There was full agreement. The computation of $\bar{t}_0()$ for 5-connected triangulations was done independently up to 60 triangles and for larger values only by the faster of the two programs.

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