3-dynamic coloring of planar triangulations

Yoshihiro Asayama[*], Yuki Kawasaki[*], Seog-Jin Kim[†‡], Atsuhiro Nakamoto[§] and Kenta Ozeki[§]

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Abstract

An r-dynamic k-coloring of a graph G is a proper k-coloring such that any vertex v has at least $\min\{r, \deg_G(v)\}$ distinct colors in $N_G(v)$. The *r-dynamic chromatic number* $\chi_r^d(G)$ of a graph G is the least k such that there exists an r-dynamic k-coloring of G.

Loeb et al. [12] showed that if G is a planar graph, then $\chi_3^d(G) \leq 10$, and there is a planar graph G with $\chi_3^d(G) = 7$. Thus finding an optimal upper bound on $\chi_3^d(G)$ for a planar graph G is a natural interesting problem. In this paper, we show that $\chi_r^d(G) \leq 5$ if G is a planar triangulation. The upper bound is sharp.

1 Introduction

Let G be a graph and let $c : V(G) \to \{1, 2, \ldots, k\}$ be a *k-coloring* of G. We say that c is *proper* if for any $xy \in E(G)$, $c(x) \neq c(y)$. The *chromatic number* $\chi(G)$ of a graph G is the least k such that G admits a proper k-coloring. For a positive integer r, an *r-dynamic k-coloring* of a graph G is a proper k-coloring ϕ such that for each vertex $v \in V(G)$, either the number of distinct colors in its neighborhood is at least r or the colors in its neighborhood are all distinct, that is, $|\phi(N_G(v))| \geq \min\{r, \deg_G(v)\}$, where $N_G(v)$ denotes the set of all neighbors of v in G. (Note that 1-dynamic coloring is nothing but a proper coloring.) The *r-dynamic chromatic number* $\chi_r^d(G)$ of a graph G is the least k such that G admits an r-dynamic k-coloring.

When $r = 2$, $\chi_2^d(G)$ is called the *dynamic chromatic number* of G. Dynamic coloring was first introduced in [15], and has been studied actively. Note that $\chi_2^d(G) - \chi(G)$ can be arbitrarily large when G is a subdivision of a complete graph. However, in the case of regular graphs, Montgomery [15] conjected the following.

Conjecture 1 $\chi_2^d(G) \leq \chi(G) + 2$ *if G is a regular graph.*

Conjecture 1 got a lot of attention, and has been studied in many papers. Recently, Conjecture 1 was disproved by Bowler et al. [1]. They showed that for any k, there exist k-regular graphs G satisfying $\chi_2^d(G) = 2\chi(G)$, which implies that $\chi_2^d(G) - \chi(G)$ can be arbitrarily large even for regular graph.

The *square* of a graph G, denoted by G^2, is the graph obtained from G by adding an edge between every pair of vertices of distance 2 in G. Coloring the square of G has recently attracted a lot of attentions, and there are many research in several directions [4, 9, 14]. For any graph G, it is easy to see that

$$\chi(G) = \chi_1^d(G) \leq \chi_2^d(G) \leq \chi_3^d(G) \leq \cdots \leq \chi_{\Delta(G)}(G) = \chi(G^2). \tag{1}$$

Note that r-dynamic coloring was studied in [6, 7], and was also studied in [3, 17, 18] with the name of *r-hued coloring*. Similarly to the Wegner's conjecture [19], the following conjecture on dynamic coloring of planar graphs was proposed in [17].

[*]Graduate School of Environment and Information Sciences, Yokohama National University, Japan. e-mail: asayama-yoshihiro-yd@ynu.jp kawasaki-yuki-xy@ynu.jp

[†]Department of Mathematics Education, Konkuk University, Korea. e-mail: skim12@konkuk.ac.kr

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[§]Faculty of Environment and Information Sciences, Yokohama National University, Japan. e-mail: nakamoto-atsuhiro-hc@ynu.ac.jp ozeki-kenta-xr@ynu.ac.jp

Conjecture 2 *Let G be a planar graph. Then*

$$\chi_r^d(G) \leq \begin{cases} r+3 & \text{if } 1 \leq r \leq 2, \\ r+5 & \text{if } 3 \leq r \leq 7, \\ \lfloor \frac{3r}{2} \rfloor + 1 & \text{if } r \geq 8. \end{cases}$$

Song, Lai, and Wu [18] showed that Conjecture 2 is true for planar graphs with girth at least 6.

Theorem 3 ([18]) *If G is a planar graph with girth at least 6, then $\chi_r^d(G) \leq r+5$ for any $r \geq 3$.*

As a starting case of r-dynamic coloring, recently 3-dynamic coloring has been concerned [3, 8, 11, 12]. Loeb, Mahoney, Reiniger, and Wise [12] showed the following result.

Theorem 4 *If G is a toroidal graph, then $\chi_3^d(G) \leq 10$.*

It was noted that Theorem 4 is tight since the Petersen graph P is a toroidal graph and $\chi_3^d(P) = 10$. (In [12], they actually proved a stronger fact on "3-dynamic choice number" and "3-dynamic paint number", see their definitions in [16, 20].) Observe that every planar graph is toroidal, and hence we have the following corollary.

Corollary 5 *If G is a planar graph, then $\chi_3^d(G) \leq 10$.*

It is not yet known whether Corollary 5 is sharp or not. On the other hand, there is a planar graph F such that $\chi_3^d(F) = 7$, see the graph F in Figure 1. Moreover, we can construct infinitely many planar graphs whose 3-dynamic chromatic number is exactly 7 by using the construction in [8].

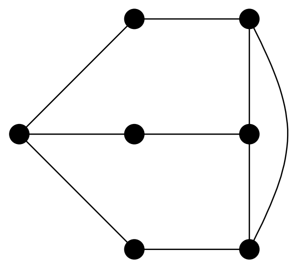

Figure 1: A planar graph F with $\chi_3^d(F) = 7$

In this paper, restricting planar graphs to planar *triangulations*, we would like to consider their r-dynamic chromatic number. Note that four colors are necessary for some planar triangulation to have an ordinary proper coloring (e.g. K_4). Since any planar graph has a proper 4-coloring by Four Color Theorem, the upper bound of the chromatic number for planar triangulations is equal to that for planar graphs. Therefore, one may think that the same situation appears also for dynamic colorings. However, as we will mention below, this is not the case.

Since any vertex in a planar triangulation T belongs to a triangle, every proper coloring is a 2-dynamic coloring, and hence T is 2-dynamically 4-colorable by Four Color Theorem. On the other hand, Kim, Lee, and Park [10] proved that every planar graph G satisfies $\chi_2^d(G) \leq 5$, and this upper bound is attained by C_5. Note that they further proved that C_5 is the only connected planar graph which requires five colors for a 2-dynamic coloring. Thus, in a sense, the gap between planar graphs and planar triangulations is small for a 2-dynamic coloring.

In contrast with a 2-dynamic coloring, there is surprisingly a big gap for a 3-dynamic coloring. As in Figure 1, there is a planar graph F with $\chi_3^d(F) = 7$, while we have the following theorem for planar triangulations.

Theorem 6 *If G is a planar triangulation, then $\chi_3^d(G) \leq 5$. The upper bound is sharp.*

We will prove Theorem 6 in Section 3. By Theorem 6, the upper bound "10" in Corollary 5 can be lowered considerably for planar triangulations. The following seems the reason: For a proper coloring in a planar triangulation,

- any vertex has at least two colors in its neighborhood,
- and furthermore, any vertex of odd degree has at least three colors in its neighborhood.

Thus, it suffices to deal with vertices of even degree. On the other hand, the above two do not necessarily hold for planar graphs without triangles.

Because of the graph F in Figure 1 and Theorem 6, we see that the upper bound of the 3-dynamic chromatic number for planar triangulations is smaller than that for planar graphs. For a further research, we wonder if this is the case for r-dynamic coloring for any $r \geq 2$. Furthermore, if so, can we estimate the gap? We leave these for the readers as open problems.

We give an example showing sharpness of the upper bound in Theorem 6. A *k-cycle* is one of length k. For an integer $n \geq 3$, we denote by DW_n the planar graph consisting of an n-cycle $C = v_1 v_2 \cdots v_n$ by adding two vertices x and y into two faces, respectively, and connecting them to all vertices of the n-cycle. The graph DW_n is sometimes called a *double wheel*, and in particular, DW_4 is the octahedron.

Proposition 7 *$\chi_3^d(DW_n) = 5$, if $n \not\equiv 0 \pmod 3$.*

Proposition 7 proves the sharpness of Theorem 6, and can be easily proved, as follows. Suppose that DW_n admits a 3-dynamic 4-coloring c. For the vertices x and y to have three colors in their neighborhoods, exactly three colors must appear in the n-cycle C, and hence the colors of x and y coincide. Since $n \not\equiv 0 \pmod 3$, C contains three consecutive vertices, say v_1, v_2 and v_3, with $c(v_1) = c(v_3) \neq c(v_2)$. However, v_2 has only two colors in the neighborhood, a contradiction. Thus, we have $\chi_3^d(DW_n) \geq 5$ if $n \not\equiv 0 \pmod 3$.

On the other hand, putting three colors on the vertices of C together with two more colors on the vertices x and y respectively, we can construct a 3-dynamic 5-coloring of DW_n. Thus, $\chi_3^d(DW_n) \leq 5$, and this completes the proof of Proposition 7.

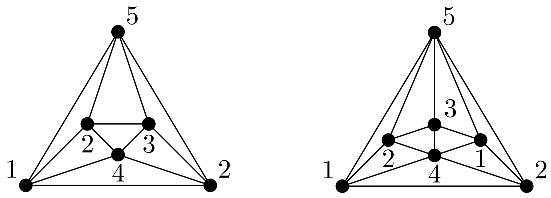

Figure 2: A 3-dynamic 5-coloring of DW_4 and DW_5

2 Lemmas

In order to prove Theorem 6, we introduce some lemmas. We begin with a lemma which is shown by discharging method. A vertex of degree i in a graph G is called an *i-vertex*, and an edge connecting an i-vertex and a j-vertex is an *i-j-edge*. The following is easily obtained from Euler's formula.

Lemma 8 *Let G be a planar triangulation. Then*

$$\sum_{v \in V(G)} \Big(6 - \deg_G(v)\Big) = 12.$$

Lemma 8 and the discharging method give the next lemma.

Lemma 9 *A planar triangulation has either a 3-vertex, a 5-vertex, a 4-4-edge, a 4-6-edge, or a 4-7-edge.*

Proof. For a contradiction, we suppose that a planar triangulation G has neither a 3-vertex, a 5-vertex, a 4-4-edge, a 4-6-edge nor a 4-7-edge. Suppose that each i-vertex has an initial charge $6 - i$. Note that the sum of initial charges over all vertices in G is 12 by Lemma 8.

We move charge $\frac{1}{2}$ from each 4-vertex to all its neighbors. Then each 4-vertex in G now have charge 0, since there does not exist a 4-4-edge in G. Note that, for $i \geq 8$, each i-vertex has at most $\lfloor \frac{1}{2} i \rfloor$ 4-vertices in its neighborhood, since G is a planar triangulation and there does not exist a 4-4-edge. Thus, for $i \geq 8$, each i-vertex has charge at most $(6-i)+\frac{1}{2}\times\lfloor\frac{1}{2}i\rfloor \leq 0$. Moreover, the charge of a 6-vertex or a 7-vertex was not changed, since there exist neither a 4-6-edge nor a 4-7-edge. Therefore, since there are no 3- nor 5-vertices, all vertices have the current charge at most 0, which contradicts the assumption. ■

Next, we consider the contraction of certain subgraphs in planar triangulations. For a vertex v of a planar graph G, the *link* of v is the boundary walk of the region consisting of all faces of G incident to v.

Let G be a planar triangulation with a 5-vertex v, and let $v_1v_2v_3v_4v_5$ be the link of v. The 5-*vertex contraction* of v at $\{v_2, v_5\}$ is the operation to remove v, identify v_2 and v_5, and replace a pair $\{v_1v_2, v_1v_5\}$ of multiple edges by a single edge, as shown in Figure 3. We cannot apply it if the resulting graph has multiple edges or loops.

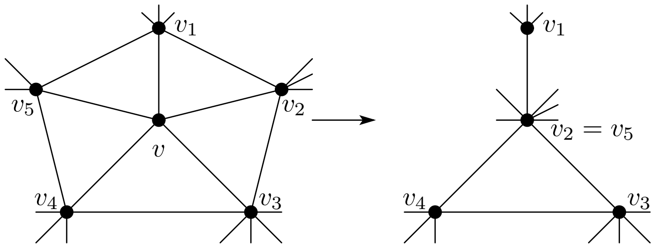

Figure 3: A 5-*vertex contraction of* v *at* $\{v_2, v_5\}$

Lemma 10 *Let G be a planar triangulation with a 5-vertex, and H be the planar triangulation obtained from G by the 5-vertex contraction. If H is 3-dynamically 5-colorable, then so is G.*

Proof. Let v be a 5-vertex in G with link $v_1v_2v_3v_4v_5$, and suppose that H is obtained from G by a 5-vertex contraction of v at $\{v_2, v_5\}$. We let $v' \in V(H)$ be the image of the identification $v_2 = v_5$.

Let $\phi' : V(H) \to \{1,2,3,4,5\}$ be a 3-dynamic 5-coloring of H. By symmetry, we may assume $\phi'(v') = 1, \phi'(v_3) = 2$ and $\phi'(v_4) = 3$. We define the partial coloring ϕ of G as $\phi(w) = \phi'(w)$ for any $w \in V(G) - \{v, v_2, v_5\}$, and $\phi(v_2) = \phi(v_5) = \phi'(v') = 1$. Then, for any $w \in V(G) - \{v, v_1, v_2, v_3, v_4, v_5\}$, we have $N_G(w) = N_H(w)$. In addition, neither v_1, v_3 nor v_4 lose any colors in their neighborhoods, and v has three distinct colors, $1, 2, 3$, in its neighborhood. Hence, it suffices to determine $\phi(v)$ so that ϕ is proper and both of v_2 and v_5 have three distinct colors in their neighborhoods in G.

If $\phi(v_1) = 4$ or 5 (in other words, if $\phi(v_1)$ is neither $\phi(v_3)$ nor $\phi(v_4)$), say $\phi(v_1) = 4$, we can color v as $\phi(v) = 5$. Then we see that ϕ is a 3-dynamic 5-coloring of G.

Therefore, we suppose $\phi(v_1) = \phi(v_3)$ or $\phi(v_1) = \phi(v_4)$, say $\phi(v_1) = \phi(v_3) = 2$. Then we can color v as $\phi(v) = 4$ or 5, and in either case, v_5 has three distinct colors in its neighborhood. So, let first $\phi(v) = 4$, and if $\phi(N_G(v_2)) = \{2, 4\}$, then we recolor v as $\phi(v) = 5$. We see that ϕ is a 3-dynamic 5-coloring of G. ■

Let G be a planar triangulation with 4-4-edge xy, and let $v_1yv_3v_4$ and $v_1v_2v_3x$ be the links of x and y, respectively. The 4-4-*edge contraction* α of xy at $\{v_2, v_4\}$ is the operation to remove x and y, identify v_2 and v_4, and replace two pairs $\{v_1v_2, v_1v_4\}$ and $\{v_2v_3, v_3v_4\}$ of multiple edges by two single edges respectively, as shown in the left of Figure 4. The 4-4-*edge contraction* β of xy at $\{v_1, v_3\}$ is the operation to remove x and y, identify v_1 and v_3, and replace two pairs $\{v_1v_2, v_2v_3\}$ and $\{v_1v_4, v_3v_4\}$ of multiple edges by two single edges respectively, as shown in the right of Figure 4. We cannot apply them if the resulting graph has multiple edges or loops.

Lemma 11 *Let G be a planar triangulation with a 4-4-edge, and H be the graph obtained from G by the 4-4-edge contraction α. If H is 3-dynamically 5-colorable, then so is G.*

Proof. Let xy be a 4-4-edge in G, let $v_1yv_3v_4$ be the link of x, and let $v_1v_2v_3x$ be the link of y. Suppose that H is obtained from G by a 4-4-edge contraction α of xy at $\{v_2, v_4\}$. We let $v' \in V(H)$ be the image of the identification $v_2 = v_4$.

Let $\phi' : V(H) \to \{1,2,3,4,5\}$ be a 3-dynamic 5-coloring of H. By symmetry, we may assume $\phi'(v') = 1$ and $\phi'(v_1) = 2$. We define the partial coloring ϕ of G as $\phi(w) = \phi'(w)$ for any $w \in V(G) - \{x, y, v_2, v_4\}$, and $\phi(v_2) = \phi(v_4) = \phi'(v') = 1$. Then, for any $w \in V(G) - \{x, y, v_1, v_2, v_3, v_4\}$, we have $N_G(w) = N_H(w)$.

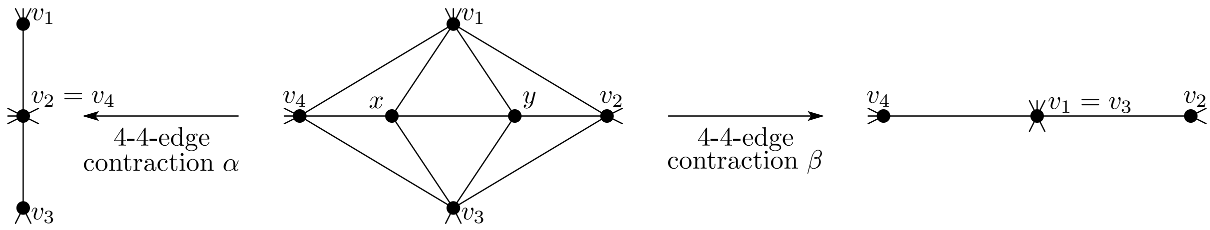

Figure 4: A 4-4-*edge contraction* α and β of xy

In addition, neither v_1 nor v_3 loses any colors in their neighborhoods. Hence, it suffices to determine $\phi(x)$ and $\phi(y)$ so that ϕ is proper and all of x, y, v_2 and v_4 have three distinct colors in their neighborhoods in G.

If $\phi(v_3) \neq \phi(v_1)$, say $\phi(v_3) = 3$, we color x and y as $\phi(x) = 4$ and $\phi(y) = 5$, and we see that ϕ is a 3-dynamic 5-coloring of G. Thus, we may assume $\phi(v_3) = \phi(v_1) = 2$. In this case, we can color both x and y by the color $3, 4$ or 5 to obtain a proper 5-coloring. Now, as in the last part of the proof of Lemma 10, we first choose the color of x so that $|\phi_G(N(v_4))| \geq 3$, and then the color of y so that $|\phi_G(N(v_2))| \geq 3$. This gives a 3-dynamic 5-coloring of G. ■

Lemma 12 *Let G be a planar triangulation with a 4-4-edge, and H be the graph obtained from G by the 4-4-edge contraction β. If H is 3-dynamically 5-colorable, then so is G.*

Proof. Let xy be a 4-4-edge in G, let $v_1yv_3v_4$ be the link of x, and let $v_1v_2v_3x$ be the link of y. Suppose that H is obtained from G by a 4-4-edge contraction β of xy at $\{v_1, v_3\}$. We let $v' \in V(H)$ be the image of the identification $v_1 = v_3$.

Let $\phi' : V(H) \to \{1, 2, 3, 4, 5\}$ be a 3-dynamic 5-coloring of H. By symmetry, we may assume $\phi'(v') = 1$ and $\phi'(v_2) = 2$. We define the partial coloring ϕ of G as $\phi(w) = \phi'(w)$ for any $w \in V(G) - \{x, y, v_1, v_3\}$, and $\phi(v_1) = \phi(v_3) = \phi'(v') = 1$. Then, for any $w \in V(G) - \{x, y, v_1, v_2, v_3, v_4\}$, we have $N_G(w) = N_H(w)$. In addition, neither v_2 nor v_4 loses any colors in their neighborhoods. Hence, it suffices to determine $\phi(x)$ and $\phi(y)$ so that ϕ is proper and all of x, y, v_1 and v_3 have three distinct colors in their neighborhoods in G.

We have either $\phi(v_4) = \phi(v_2) = 2$ or $\phi(v_4) \neq \phi(v_2)$, say $\phi(v_4) = 3$. In either case, by coloring x and y as $\phi(x) = 4$ and $\phi(y) = 5$, we see that ϕ is a 3-dynamic 5-coloring of G. ■

Let G be a planar triangulation with a 4-6-edge xy, and let $ayef$ and $abcdex$ be the links of x and y, respectively. The 4-6-*edge contraction* of xy at $\{a, d\}$ is the operation to remove x and y, and identify a and d, as shown in Figure 5. We cannot apply it if the resulting graph has multiple edges or loops.

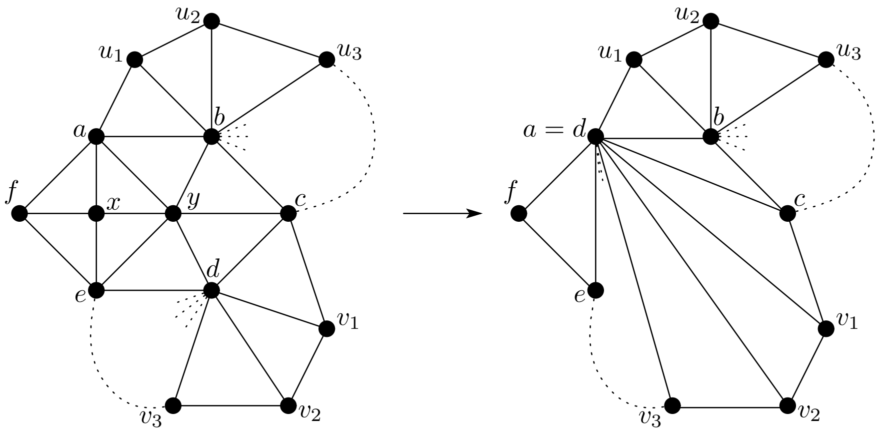

Figure 5: A 4-6-*edge contraction of xy at* $\{a, d\}$

Lemma 13 *Let G be a planar triangulation with a* 4*-*6*-edge, and H be the graph obtained from G by the* 4*-*6*-edge contraction. If H is* 3*-dynamically* 5*-colorable, then so is G.*

Proof. Let xy be a 4-4-edge in G, let $ayef$ be the link of x, and let $abcdex$ be the link of y. Suppose that H is obtained from G by a 4-6-edge contraction of xy at $\{a,d\}$. We let $v' \in V(H)$ be the image of the identification $a = d$.

Let $\phi' : V(H) \to \{1,2,3,4,5\}$ be a 3-dynamic 5-coloring of H. By symmetry, we may assume $\phi'(v') = 1, \phi'(f) = 2$ and $\phi'(e) = 3$. We define the partial coloring ϕ of G as $\phi(w) = \phi'(w)$ for any $w \in V(G) - \{x,y,a,d\}$, and let $\phi(a) = \phi(d) = \phi'(v') = 1$. Then, for any $w \in V(G) - \{x,y,a,b,c,d,e,f\}$, we have $N_G(w) = N_H(w)$. In addition, neither b,c,e nor f loses any colors in their neighborhoods, and x has three distinct colors, $1,2,3$, in its neighborhood. Furthermore, since ϕ' is a proper coloring in H, we see that $1 \neq \phi(b) \neq \phi(c) \neq 1$, and hence y also has three distinct colors in its neighborhood. Therefore, it suffices to determine $\phi(x)$ and $\phi(y)$ so that ϕ is proper and both of a and d have three distinct colors in their neighborhoods in G.

Suppose first $\phi(c) \neq 3$. In this case, we can color y with the color that is neither $1,3,\phi(b)$ nor $\phi(c)$, and d has three distinct colors in its neighborhood. If $\phi(y) \neq 2$, then by coloring x with the color that is neither $1,2,3$ nor $\phi(y)$, we obtain a 3-dynamic 5-coloring of G. On the other hand, if $\phi(y) = 2$, then we can choose the color of x from 4 and 5 so that $|\phi(N_G(a))| \geq 3$, and we also obtain a 3-dynamic 5-coloring of G.

Therefore, we may assume $\phi(c) = 3$. In this case, we can choose the color of y by the color that is neither $1,3$ nor $\phi(b)$ and $|\phi(N_G(d))| \geq 3$. Then by the same way as in the previous case, we can also choose the color of x and obtain a 3-dynamic 5-coloring of G. ■

Finally, we consider a triangulation with a 4-7-edge. In this case, we use an ordinary contraction of an edge in a triangulation.

Lemma 14 *Let G be a planar triangulation on with a* 4*-*7*-edge, and H be the graph obtained from G by the contraction of the* 4*-*7*-edge. If H is* 3*-dynamically* 5*-colorable, then so is G.*

Proof. Let xy be a 4-7-edge in G, let $v_1yv_6v_7$ be the link of x, and let $v_1v_2v_3v_4v_5v_6x$ be the link of y. Suppose that H is obtained from G by a contraction of xy. We let $v \in V(H)$ be the image of the identification $x = y$.

Let $\phi' : V(H) \to \{1,2,3,4,5\}$ be a 3-dynamic 5-coloring of H. By symmetry, we may assume $\phi'(v) = 1, \phi'(v_1) = 2$, and $\phi'(v_7) = 3$. We define the partial coloring ϕ of G as $\phi(w) = \phi'(w)$ for any $w \in V(G) - \{x,y\}$, and $\phi(y) = \phi'(v) = 1$. Then, for any $w \in V(G) - \{x,y,v_1,v_6,v_7\}$, we have $\phi(N_G(w)) = \phi'(N_H(w))$. In addition, neither v_1 nor v_6 lose any colors in their neighborhoods, and x has three distinct colors, $1,2,3$, in its neighborhood. Moreover, if we obtain a proper coloring ϕ in G, then $|\phi(N_G(y))| \geq 3$, since the degree of y is odd in G. Hence, it suffices to determine $\phi(x)$ so that ϕ is proper and v_7 has three distinct colors in its neighborhood. This can be done by the same way as in the proofs of the previous lemmas. ■

3 Proof of Theorem 6

We use the induction on the number of vertices. Suppose that G has a separating 3-cycle C, and let H_1 (resp., H_2) be the subgraphs of G consisting of the vertices in the interior (resp., exterior) of C and these of C. Note that both H_1 and H_2 are planar triangulations that are smaller than G. By the induction hypothesis, both H_1 and H_2 are 3-dynamically 5-colorable. Hence, pasting 3-dynamic 5-colorings of H_1 and H_2 along C, possibly after permuting colors, we see $\chi_3^d(G) \leq 5$. Therefore, we may assume that G has no separating 3-cycle. By Lemma 9, G has either a 3-vertex, a 5-vertex, a 4-4-edge, a 4-6-edge or a 4-7-edge.

If G has a 3-vertex, then $G \cong K_4$, since G has no separating 3-cycle. By proper coloring of K_4, we see that $\chi_3^d(K_4) = 4$, and the theorem holds in this case.

Suppose that G has a 5-vertex v. If we can apply the 5-vertex contraction of v, then $\chi_3^d(G) \leq 5$, by the induction hypothesis and Lemma 10. Hence, we suppose that we cannot apply the 5-vertex contraction of v. Let $v_1v_2v_3v_4v_5$ be the link of v. Since there is no separating 3-cycle, if there does not exist a vertex u which forms a 4-cycle vv_2uv_5, then we can apply a 5-vertex contraction of v at $\{v_2,v_5\}$. Thus, such a vertex u exists. By symmetry, there exists a vertex u' which forms a 4-cycle $vv_1u'v_3$. Then, Jordan curve theorem implies that $u = u'$. Similarly for u_1 and u_4, we have $uv_i \in E(G)$ for $i \in \{1, 2, 3, 4, 5\}$ with faces $v_iv_{i+1}u$, where the index is taken modulo 5. So $G \cong DW_5$. By Proposition 7, $\chi_3^d(DW_5) = 5$, and the theorem holds.

Suppose that G has a 4-4-edge xy. If we can apply the 4-4-edge contraction α of xy, then $\chi_3^d(G) \leq 5$, by the induction hypothesis and Lemma 11. Hence, we suppose that we cannot apply the 4-4-edge contraction α of xy. Let $v_1yv_3v_4$ be the link of x, and $v_1v_2v_3x$ be the link of y. To forbid the 4-4-edge contraction α, there exists a 4-cycle xyv_2v_4 or a vertex u which forms a 5-cycle xyv_2uv_4.

First, suppose that there exists a 4-cycle xyv_2v_4. Since there is no separating 3-cycle, both of two regions bounded by $v_1v_2v_4$ and $v_2v_4v_3$ are faces, which shows that $G \cong DW_4$. By Proposition 7, $\chi_3^d(DW_4) = 5$, and the theorem holds.

Thus, we may assume that G has a vertex u which forms the 5-cycle xyv_2uv_4. If we can apply the 4-4-edge contraction β of xy at $\{v_1, v_3\}$ to G, then $\chi_3^d(G) \leq 5$, by the induction hypothesis and Lemma 12. Hence, we suppose that it is not the case. Since G has no separating 3-cycle, G has a 4-cycle v_1xv_3u'. By Jordan curve theorem, $u' = u$. Since G again has no separating 3-cycle, all of four regions bounded by v_1v_2u, v_1v_4u, v_2v_3u and v_3v_4u are faces. So, we have $G \cong DW_5$. By Proposition 7, $\chi_3^d(DW_5) = 5$ and the theorem holds.

Suppose that G has a 4-6-edge xy. Let $ayef$ be the link of x, $abcdex$ be the link of y, $ycv_1v_2 \dots v_ke$ be the link of d, and $yau_1u_2 \dots u_lc$ be the link of b. Since G has no separating 3-cycle, we have $ae, ac, ce \notin E(G)$, $a \neq v_i$ and $e \neq u_j$ for any $i \in \{1, 2, \dots, k\}$ and $j \in \{1, 2, \dots, l\}$. We claim that we can always apply a 4-6-edge contraction xy either at $\{a, d\}$ or $\{e, b\}$ to G. To see this, suppose that we cannot apply the 4-6-edge contraction of xy at $\{a, d\}$. Then there exists an edge $av_i \in E(G)$, for some $i \in \{1, 2, \dots, k\}$. By Jordan curve theorem, there does not exist an edge $eu_j \in E(G)$ for any $j \in \{1, 2, \dots, l\}$, and hence we can apply the 4-6-edge contraction of xy at $\{e, b\}$ to G. Thus, the claim holds. By the induction hypothesis and Lemma 13, $\chi_3^d(G) \leq 5$, and the theorem holds.

If G has a 4-7-edge, then a contraction of the 4-7-edge can be always applied since there is no separating 3-cycle. Therefore, by the induction hypothesis and Lemma 14, $\chi_3^d(G) \leq 5$, and the theorem holds in this case.

As above, the theorem has been proved. ■

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