Spanning bipartite quadrangulations of even triangulations

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Abstract

A *triangulation* (resp., a *quadrangulation*) on a surface $\mathbb{F}$ is a map of a loopless graph on $\mathbb{F}$ with each face bounded by a closed walk of length three (resp., four). It is easy to see that every triangulation on any surface has a spanning quadrangulation. Kündgen and Thomassen [16] proved that every *even* triangulation G (i.e., each vertex has even degree) on the torus has a spanning non-bipartite quadrangulation, and if G has sufficiently large edge width, then G also has a bipartite one. In this paper, we prove that an even triangulation G on the torus admits a spanning bipartite quadrangulation if and only if G does not have K_7 as a subgraph, and moreover, we give some other results for the problem.

Keywords. triangulation, quadrangulation, bipartite subgraph, locally planar graph

1 Introduction

A *surface* $\mathbb{F}$ is a connected compact 2-dimensional manifold without boundary. By the classification of surfaces, $\mathbb{F}$ is either an orientable surface of genus $g \geq 0$, denoted by $\mathbb{S}_g$, or a nonorientable surface of genus $k > 0$, denoted by $\mathbb{N}_k$. Note that $\mathbb{S}_0$ is the sphere, $\mathbb{N}_1$ is the projective plane and $\mathbb{S}_1$ is the torus. A closed curve γ on $\mathbb{F}$ is *essential* if γ does not bound a 2-cell region on $\mathbb{F}$. Otherwise, γ is *contractible*.

Let G be a graph, and let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. A *k-cycle* is a cycle of length k, and a *k-vertex* is a vertex of degree k. A *vertex k-coloring* is a map $c : V(G) \to \{1, \dots, k\}$, and c is *proper* if $c(x) \neq c(y)$ for any $xy \in E(G)$. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum integer k such that G is k-colorable. An *edge k-coloring* $c : E(G) \to \{1, \dots, k\}$ is defined similarly to a vertex k-coloring. A graph is *k-colorable* if it admits a proper vertex k-coloring.

Let G be a *map* on a surface $\mathbb{F}$, that is, a 2-cell embedding of a loopless graph on $\mathbb{F}$. A cycle C (or a closed walk) of G is *essential* (resp., *contractible*) if C can be regarded as an essential (resp., contractible) closed curve on $\mathbb{F}$. The *link* of a vertex v of G is the boundary walk of the 2-cell region formed by all faces incident to v. We define a *triangulation* as a map with each face bounded by a closed walk of length 3, and a *quadrangulation* as a map with each face bounded by a closed walk of length 4. The *edge width* of G, denoted by $\mathrm{ew}(G)$, is the length of a shortest essential cycle in G. A graph in which all vertices have even degree is *even*.

Let us begin with the following assertion:

Proposition 1. *Every triangulation on any surface admits a spanning quadrangulation.*

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Since the dual map G^* of a triangulation G is easily verified to be a 2-edge-connected cubic graph, G^* has a perfect matching M^*, by Petersen's Theorem [26]. Translating this to G, we see that G has a spanning quadrangulation $Q - M$, where $M \subset E(G)$ is the edge set corresponding to M^*.

Kündgen and Thomassen [16] asked whether the bipartiteness of Q can be controlled in Proposition 1, that is, for a given triangulation G on a surface $\mathbb{F}$,

(i) does G admit a spanning bipartite quadrangulation?

(ii) does G admit a spanning non-bipartite quadrangulation?

(iii) does G admit both of them?

They proved the following result for even triangulations on the torus:

Theorem 2 (Kündgen and Thomassen [16])**.** *Let G be an even triangulation on the torus. Then G has a spanning non-bipartite quadrangulation. Moreover, there exists an integer N such that if the edge width of G is at least N, then G has a spanning bipartite quadrangulation.*

In this paper, we characterize even triangulations on the torus having a spanning bipartite quadrangulation, as follows:

Theorem 3. *An even triangulation G on the torus has a spanning bipartite quadrangulation if and only if G does not have K_7 as a subgraph.*

Since any triangulation of the torus with K_7 as a subgraph has edge width exactly 3, Theorem 3 shows that $N = 4$ suffices in Theorem 2, and it is best possible.

We also give a proof to the following two theorems, by using some known results on colorings of graphs on surfaces. The former has been already proved by Kündgen and Thomassen [16], but our proof strategy is different (and shorter).

Theorem 4 (Kündgen and Thomassen [16])**.** *Let G be an even triangulation on the projective plane. If G is 3-chromatic, then every spanning quadrangulation of G is bipartite. On the other hand, if G is not 3-colorable, then G has both a spanning bipartite quadrangulation and a spanning non-bipartite quadrangulation.*

Theorem 5. *For any non-spherical surface $\mathbb{F}$ other than the projective plane, there exists an integer $N(\mathbb{F})$ such that every even triangulation on $\mathbb{F}$ with edge width at least $N(\mathbb{F})$ has both a spanning bipartite quadrangulation and a spanning non-bipartite quadrangulation.*

In Section 2, we introduce motivations to deal with the problem. In Section 3, we give a proof of Theorem 3. Finally, in Section 4, we give a proof of Theorems 4 and 5.

2 Motivation

In this section, we describe motivation of our research from several points of view.

1. Spanning bipartite subgraphs. For a given graph G, finding a subgraph T of G with certain properties seems to be interesting. One of the typical properties is the bipartiteness with the maximum number of edges. This problem was first considered by Erdős [6] and gave the result that every graph G has a spanning bipartite subgraph H with $|E(H)| \geq \frac{1}{2}|E(G)|$. So, if a triangulation G on a surface $\mathbb{F}$ has a spanning bipartite quadrangulation Q, then we have $|E(Q)| = \frac{2}{3}|E(G)|$, by Euler's formula, which is much better than the general bound.

2. Perfect matchings of 3-regular maps. A spanning quadrangulation Q in a triangulation G corresponds to a perfect matching M^* in the dual G^* of G. More precisely, for a triangulation G, there is a bijection between the spanning quadrangulations in G and the perfect matchings in G^*. It is known that the number of perfect matchings of a 2-edge-connected cubic graph H is exponential in the order of H [7]. Then because of the existence of such a huge number of perfect matchings, we expect that there exists a perfect matching of G^* that corresponds to a spanning bipartite quadrangulation, and respectively a spanning non-bipartite quadrangulation. Actually, this thought is true for many cases (see Theorems 2–5), but surprisingly false for a few cases, namely even triangulations on the torus containing K_7 as a subgraph (Theorem 3) and 3-chromatic even triangulations on the projective plane (Theorem 4).

3. Grünbaum colorings of triangulations. It is known that if a triangulation G has a proper vertex 4-coloring $c : V(G) \to \{1, 2, 3, 4\}$, then the decomposition

$$\{E_{1,2} \cup E_{3,4},\ E_{1,3} \cup E_{2,4},\ E_{1,4} \cup E_{2,3}\}$$

of $E(G)$ gives an edge 3-coloring c_E of G such that every face receives three distinct colors on their boundary edges, where $E_{i,j}$ denotes the set of (i, j)-*edges* for any distinct $i, j \in \{1, 2, 3, 4\}$, i.e., ones whose ends are colored by colors i and j. Such an edge 3-coloring of G is a *Grünbaum coloring* [9]. (See related results [1, 15]). Note that a Grünbaum coloring of a triangulation G can naturally be interpreted into a proper edge 3-coloring c^* of the dual G^* of G.

Hence, taking each color class of c^* as a perfect matching of G^*, we have the following result:

Proposition 6 (Kündgen and Thomassen [16])**.** *Let G be a triangulation on a surface $\mathbb{F}$. Then G has a proper vertex 4-coloring if and only if G has a Grünbaum coloring such that any two colors induce a spanning bipartite quadrangulation of G.*

Hence it seems to be an interesting problem to find a spanning "bipartite" quadrangulation in a non-4-colorable triangulation G.

4. Polychromatic colorings of graphs on surfaces. A *polychromatic k-coloring* of a map G on a surface [2] is a vertex k-coloring c (which is not necessarily proper) such that all k colors appear in the boundary vertices of each face of G. The *polychromatic number* of a map G is the maximum integer k such that G admits a polychromatic k-coloring. For several classes of maps, the polychromatic number are studied [10, 11, 14]. In particular, the authors of the present paper characterized quadrangulations on surfaces with polychromatic number 3 and 4, respectively [23, 24]. However, if we consider the problem for triangulations, then a triangulation with polychromatic number 3 is nothing but a 3-colorable triangulation, and hence an interesting problem for triangulations is to consider a polychromatic 2-coloring. For this problem, we see that the following holds:

Proposition 7. *A triangulation G on a surface $\mathbb{F}$ has a polychromatic 2-coloring if and only if G has a spanning bipartite quadrangulation.*

The "if" part is obvious (since a bipartition can be a polychromatic 2-coloring), and the "only if" part is explained as follows: If G has a polychromatic 2-coloring, then removing all edges whose ends have the same color, we get a spanning bipartite quadrangulation. On the other hand, Kündgen and Ramamurthi [17] presented the following conjecture: For every non-spherical surface $\mathbb{F}$, there exists a constant $N = N(\mathbb{F})$ such that every triangulation on $\mathbb{F}$ with edge width at least N has a polychromatic 2-coloring. Theorem 5 will give the affirmative solution for even triangulations.

5. Quadrangulations extended to an even triangulation. Hoffmann and Kriegel [13] proved that every plane quadrangulation G can be extended to an even triangulation by adding a single edge to each face of G. After that, Zhang and He [29] enumerated the plane even triangulations obtained from a given plane quadrangulation, and gave a lower bound for the number of even triangulations obtained from a quadrangulation on orientable surfaces. For this problem, the authors of the present paper [22] completely gave the exact number for such even triangulations on all surfaces, orientable or nonorientable.

The problem in this paper on a quadrangulation Q obtained from an even triangulation G seems to be an inverse problem of the above-mentioned problem, and it will be meaningful to consider which property of Q can be obtained from G. So we consider the bipartiteness of Q in this paper.

3 The toroidal case

In this section, we prove Theorem 3. We first prove that Theorem 3 holds for all 6-regular maps on the torus in the following subsection, and then all even triangulations by using a generating theorem of them.

3.1 6-regular maps on the torus

In this subsection, we deal with 6-regular maps on the torus. A characterization of those maps was given by Altshuler [3], as follows:

Prepare a grid of m rows and n columns. We label the vertex on the ith row and the jth column by (i, j). The 6-*regular right-diagonal grid* $G[m \times n]$ is the graph with vertex set $V = \{(i, j) | 1 \leq i \leq m, 1 \leq j \leq n\}$, where the neighbors of (i, j) are $(i-1, j-1), (i, j-1), (i-1, j), (i+1, j), (i, j+1), (i+1, j+1)$ with $(0, j) = (m, j)$, $(m+1, j) = (1, j)$, $(i, 0) = (i, n)$ and $(i, n+1) = (i, 1)$. It is obvious that $G[m \times n]$ is 6-regular and can be embedded on the torus. Let $1 \leq k \leq m$ be an integer. The 6-*regular right-diagonal shifted grid* $G[m \times n, k]$ is the twisted $G[m \times n]$, that is, the vertex (i, n) is adjacent to $(i-k+1, 1)$ and $(i-k+2, 1)$ instead of $(i, 1)$ and $(i+1, 1)$ for $i \in \{1, \ldots, m\}$. (See Figure 1. The top and the bottom, the left and the right are, respectively, identified.) Note that $G[m \times n]$ is the same as $G[m \times n, 1]$. It is obvious that for any positive integers m, n and $1 \leq k \leq m$, $G[m \times n, k]$ is 6-regular and can be embedded on the torus.

Theorem 8. (Altshuler [3]) *Every* 6*-regular map on the torus is isomorphic to a* 6*-regular right-diagonal shifted grid* $G[m \times n, k]$ *for some positive integers* m, n *and* k.

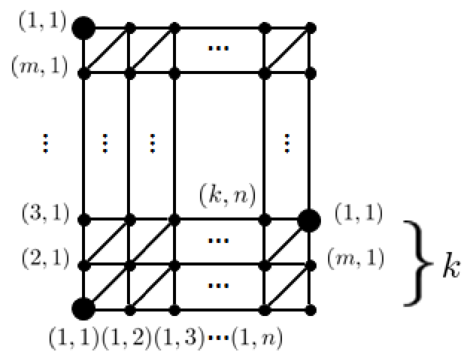

Figure 1: A 6-regular right-diagonal shifted grid $G[m \times n, k]$.

In addition, all 4-colorable 6-regular maps on the torus have been completely determined, as in the following theorem:

Theorem 9. ([5, 27]) *Every* 6*-regular toroidal map is* 4*-colorable, with the following exceptions;*

- $G \in \{G[3\times 3,2], G[3\times 3,3], G[5\times 3,2], G[5\times 3,3], G[5\times 5,3], G[5\times 5,4]\}$.
- $G = G[p\times 2,1]$ *with* p *odd.*
- $G = G[p\times 1, r+2]$ *such that* $p = 2r+2, 2r+3, 3r+1$ *or* $3r+2$ *and* p *is not divisible by* 4.
- $G = G[p\times 1,4]$ *such that* p *is not divisible by* 4.
- $G = G[p\times 1, r+2]$ *with* $(r,p) \in \{(3,13),(3,17),(3,18),(3,25),(4,17),(6,17),(6,25),(6,33),$ $(7,19),(7,25),(7,26),(9,25),(10,25),(10,26),(10,37),(14,33)\}$.

Using Theorems 8 and 9, we prove the following:

Lemma 10. *Every* 6*-regular triangulation on the torus has a spanning non-bipartite quadrangulation. Every* 6*-regular triangulation on the torus, except for* K_7*, has a spanning bipartite quadrangulation.*

Proof. Let G be a 6-regular triangulation. By Theorem 8, it is represented by $G[m\times n,k]$.

Non-bipartite quadrangulations: If m is odd or $n+k$ is even, we get a spanning non-bipartite quadrangulation by deleting all diagonal edges of G. Otherwise, we get a spanning non-bipartite quadrangulation by deleting all but the first column diagonal edges and horizontal edges in the first column of G.

Bipartite quadrangulations: Suppose that G is not isomorphic to K_7. If G is 4-colorable, then G has a spanning bipartite quadrangulation, by Proposition 6. Otherwise, we only have to consider the exceptions in Theorem 9. We first consider the following three cases.

- n is even.
  We get a spanning bipartite quadrangulation by deleting all vertical edges of G.
- n is odd, m is even, and k is odd.
  We get a spanning bipartite quadrangulation by deleting all horizontal edges of G.
- n is odd, m is even, and k is even.
  We get a spanning bipartite quadrangulation by deleting all diagonal edges of G.

Therefore we may assume that both n and m are odd. The remaining cases are listed below.

- $(m,n,k) = (3,3,2), (3,3,3), (5,3,2), (5,3,3), (5,5,3)$ or $(5,5,4)$.
  See Figure 7 in Appendix.
- $n = 1$ and k is odd.
  See the left side of Figure 8 in Appendix.
- $(m,n,k) = (2r+3,1,r+2)$ or $(3r+1,1,r+2)$ for some even integer r.
  If $r = 2$, both are K_7. Otherwise, see the middle and the right side of Figure 8 in Appendix.
- $(m,n,k) = (m,1,4)$ for some odd integer m.
  Let $m \geq 5$ be an odd integer represented by $4l+c$, where c is 1 or 3.
  - If $c = 1$, then let $X = \{4i+2, 4i+3 | 0 \leq i \leq l-1\} \cup \{1\}$ and $Y = \{4i, 4i+1 | 1 \leq i \leq l\}$. We can take a spanning quadrangulation so that its bipartition is X and Y.
  - If $m = 7$ (so $c = 3$ and $l = 1$), then G is isomorphic to K_7.

- If $c = 3$ and $l \geq 2$, then let $X = \{4i, 4i+1 | 2 \leq i \leq l\} \cup \{1,2,3,7\}$ and $Y = \{4i+2, 4i+3 | 2 \leq i \leq l\} \cup \{4,5,6\}$. We can take a spanning quadrangulation so that its bipartition is X and Y.

- $(m,n,k) = (17,1,6), (17,1,8), (25,1,8), (33,1,8), (25,1,12), (37,1,12)$ or $(33,1,16)$. It is easy to see that $G[25 \times 1, 12]$ is isomorphic to $G[5 \times 5, 3]$ and that $G[33 \times 1, 16]$ is isomorphic to $G[11 \times 3, 10]$. Then see Figure 9 in Appendix, and see Figure 10 for other cases.

Hence the proof is completed. □

3.2 A generating theorem for even triangulations on the torus

Let G be an even triangulation on a surface $\mathbb{F}$ and let v be a 4-vertex in G with link $v_1v_2v_3v_4$. A 4-*contraction* of v at $\{v_1, v_3\}$ is to remove v, identify v_1 and v_3, and replace two pairs of double edges by two single edges, respectively. The inverse operation of a 4-contraction is a 4-*splitting*. (See the left side of Figure 2.) Let w be a 2-vertex in G and let w_1 and w_2 be the neighbors of w. A 2-*vertex removal* of w is to remove w and identify the two edges w_1w_2 which bound the two faces incident to w. The inverse operation of a 2-vertex removal is a 2-*vertex addition*. (See the right side of Figure 2.)

The following theorem is a *generating theorem* for even triangulations on the torus, which describes how to generate them.

Theorem 11. (Matsumoto et al. [18]) *Every even triangulation on the torus can be obtained from one of the* 27 *maps in Figure* 3 *or a* 6*-regular triangulation by a sequence of* 4*-splittings and* 2*-vertex additions.*

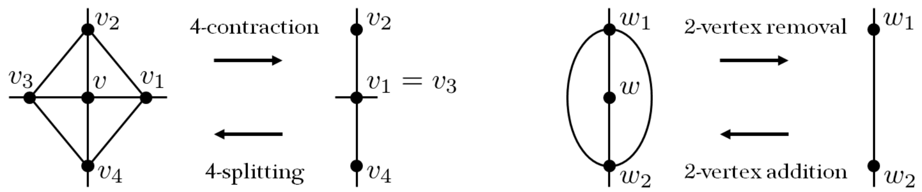

Figure 2: A 4-splitting and a 2-vertex addition.

The existence of spanning bipartite quadrangulations in even triangulations can be preserved by the two operations, as described in the following lemma.

Lemma 12. *Let G be an even triangulation on a surface $\mathbb{F}$, and let G' be an even triangulation on $\mathbb{F}$ obtained from G by either a* 4*-splitting or a* 2*-vertex addition. If G has a spanning bipartite quadrangulation, then so does G'. If G has a non-bipartite one, then so does G'.*

Proof. Let Q be a spanning quadrangulation of G and let $M = E(G) \backslash E(Q)$. We define the edge set $M' \subset E(G')$ such that $Q' = G' - M'$ is a spanning quadrangulation, as follows.

Suppose first that G' is obtained from G by a 4-splitting of a vertex v in G. Let $v_1v_2v_3v_4$ be the link of the new vertex v' in G' such that the 4-contraction of v' at $\{v_2, v_4\}$ yields G. By symmetry, we may assume that one of the following holds, and in each case, define the edge set M' as follows:

- $vv_1, vv_3 \notin M$. Then let $M' = M \cup \{v'v_1, v'v_3\}$.

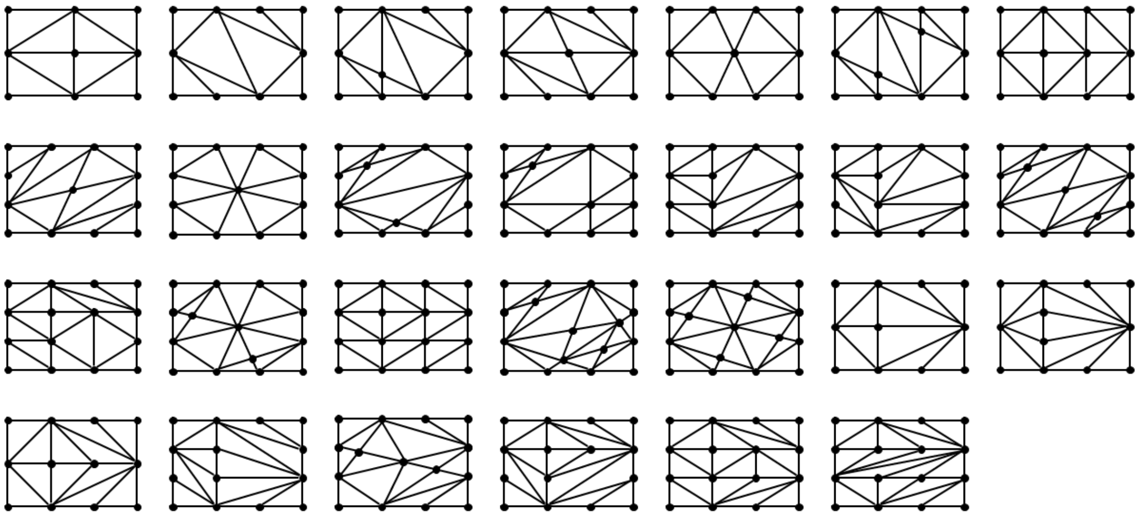

Figure 3: The 27 even triangulations on the torus, where in each map, the top and the bottom, the left and the right are, respectively, identified.

- $vv_1 \notin M$ and $vv_3 \in M$. Then let $M' = \big(M \backslash \{vv_3\}\big) \cup \{v'v_1, v_2v_3, v_3v_4\}$.
- $vv_1, vv_3 \in M$. Then let $M' = \big(M \backslash \{vv_1, vv_3\}\big) \cup \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$.

Next suppose that G' is obtained from G by a 2-vertex addition in G. Let v_1v_2 be the neighbors of the new vertex v' in G', and let e_1, e_2 be the two multiple edges forming the link of v'. Then one of the following holds, and in each case, define M' as follows:

- $v_1v_2 \notin M$. Then let $M' = M \cup \{v'v_1\}$.
- $v_1v_2 \in M$. Then let $M' = \big(M \backslash \{v_1v_2\}\big) \cup \{e_1, e_2\}$.

In this way, we can construct a spanning quadrangulation $Q' = G' - M'$ in G' from Q in G. It is easy to see that the bipartiteness is preserved in the process to obtain Q' from Q. □

Lemma 13. *All even triangulations shown in Figure 3 have both a spanning bipartite quadrangulation and a spanning non-bipartite quadrangulation.*

Proof. See Figures 5 and 6 in Appendix. □

By using the reductions (4-contractions and 2-vertex removals), we obtain another proof of the first assertion in Theorem 2, that is, every even triangulation on the torus has a spanning non-bipartite quadrangulation; First, all "minimal" even triangulations (the 27 ones in Figure 3 and 6-regular ones) have it, as shown in Lemmas 10 and 13. Then by Theorem 11, all other even triangulations can be constructed from them by a sequence of 4-splittings and 2-vertex additions, through which the existence of a spanning non-bipartite quadrangulation is preserved by Lemma 12.

3.3 Lemmas for the proof of Theorem 3

Before proceeding to a proof of Theorem 3, we put two more lemmas and their proofs in this subsection.

Lemma 14. *Let G be an even triangulation on a surface $\mathbb{F}$. Then for any vertex 2-coloring $c : V(G) \to \{1, 2\}$, G has an even number of* monochromatic *faces (i.e., a face receiving only one color on their boundary vertices).*

Proof. We first consider the case when all vertices are colored by 1. Since $|F(G)| = \frac{2}{3}|E(G)|$, we have that $|F(G)|$ is even, and the number of monochromatic faces is also even. Secondly, we switch the color of a vertex x with link $v_1 \cdots v_k$, where $k = \deg_G(x)$. Since k is even, the number of monochromatic edges in the link is even. Then the parity of the number of monochromatic faces does not change after switching the color of x. So the lemma follows. □

Lemma 15. *Let G be an even triangulation on a surface $\mathbb{F}$ and let $C = uvw$ be a contractible 3-cycle in G which does not bound a face. Then the plane triangulation H consisting of C and the vertices and edges in the interior of C is even.*

Proof. If H has an odd degree vertex, then it coincides with one of u, v or w, since each inner vertex of H must have an even degree. Since the number of odd vertices is even, they must be consecutive two vertices on C. However, it is known [8] that there is no plane triangulation with two adjacent vertices of odd degree and all other vertices of even degree, a contradiction. Therefore, H must be even. □

3.4 Proof of Theorem 3

Proof of Theorem 3. We show the "only if" part. First, suppose $G = K_7$. Since the number of faces of K_7 is 14, we can delete exactly seven edges to obtain a spanning quadrangulation Q, and hence $|E(Q)| = 21 - 7 = 14$. If Q is bipartite, then Q is a spanning subgraph of one of the complete bipartite graphs $K_{3,4}, K_{2,5}$ and $K_{1,6}$ with seven vertices. However, they have at most 12 edges and hence this case does not happen. Therefore Q is non-bipartite.

Secondly suppose that G contains K_7 as a subgraph. If G has a spanning bipartite quadrangulation, then G admits a polychromatic 2-coloring c, by Proposition 7. By Lemmas 14 and 15, every contractible 3-cycle receives both two colors. (For otherwise, i.e., if a 3-cycle C bounding a 2-cell region R receives only one color in its boundary vertices, then the plane even triangulation induced by $V(C)$ and all inner vertices in R has at least one monochromatic face, except the outer face. Hence c is not a polychromatic 2-coloring, a contradiction.) Hence c gives a polychromatic 2-coloring of K_7, but this contradicts the first case.

Then we next show the "if" part. Suppose that G does not contain K_7 as a subgraph. If G has a spanning bipartite quadrangulation, then we are done. So let G be a counterexample of the assertion, that is, G is an even triangulation on the torus which does not contain K_7 as a subgraph, but has no spanning bipartite quadrangulation. Moreover, we suppose that G has the smallest number of vertices among all such counterexamples.

Observe that $G \neq K_7$ by the assumption, and that G is not isomorphic to any one of the 27 triangulations shown in Figure 3 nor G is not 6-regular, by Lemmas 10 and 13, since G is a counterexample of the assertion. Hence, by Theorem 11, we can apply either a 4-contraction or a 2-vertex removal to obtain a smaller even triangulation G'. If G' does not have K_7 as a subgraph, then G' has a spanning bipartite quadrangulation, by the minimality of G. Hence, by Lemma 12, G also has a spanning bipartite quadrangulation, a contradiction. Therefore we may suppose that G' has K_7 as a subgraph. Observe that G is obtained from G' by a 4-splitting, since a 2-vertex addition preserves the existence of K_7 as a subgraph.

If $G' = K_7$, then G is isomorphic to the triangulation X, which is shown in Figure 4. However, the figure shows a polychromatic 2-coloring of G, and hence has a spanning bipartite quadrangulation, by Proposition 7. This contradicts that G is a counterexample.

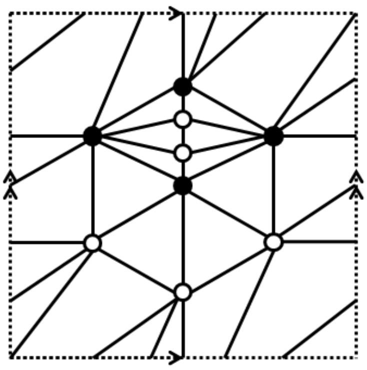

Figure 4: The even triangulation X.

Now suppose that G' has K_7 as a subgraph but has at least eight vertices. Let v' be a 4-vertex of G where we applied a 4-contraction, and let $e_1 = uv$ and $e_2 = vw$ be the edges of G' obtained by a 4-contraction. If both e_1 and e_2 are contained in $K_7 \subset G'$, then G has the map X as a subgraph. Then G has a spanning bipartite quadrangulation, since attaching a plane even triangulation to a face of X preserves the existence of a spanning bipartite quadrangulation. Hence we may suppose that at least one of e_1 and e_2, say e_1, does not belong to K_7. In this case, since u is not contained in K_7, the identification of u and w by a 4-contraction of v' makes no loop. Hence we can apply a 4-contraction of v' at $\{u, w\}$ to get another even triangulation, say G''. Note that the embedding of K_7 on the torus is uniquely represented by $G[7 \times 1, 4]$. (For example, see [25].) Then it is not difficult to see that G'' does not contain K_7 as a subgraph. Since $|V(G'')| < |V(G)|$, G'' satisfies the assertion, by the minimality of G. Hence G has a spanning bipartite quadrangulation, by Lemma 12, a contradiction.

Therefore, such a counterexample does not exist, and we are done. □

4 Proofs of Theorems 4 and 5

In this section, we prove Theorems 4 and 5. For proving them, the notions "face subdivision" and "color factor" play important roles.

An *even embedding* on a surface $\mathbb{F}$ is a map such that each face is bounded by a closed walk of even length. Put a new vertex in each face of an even embedding H and join it to all vertices on the corresponding facial walk. Then we see that the resulting map G on $\mathbb{F}$ is an even triangulation on $\mathbb{F}$, which is the *face subdivision* of H and denoted by $G = \mathrm{FS}(H)$. The vertex set $U = V(G) \backslash V(H)$ is the *color factor* of G.

Theorem 16. *The following results hold:*

(i) Every non-bipartite quadrangulation on the projective plane is 4*-chromatic [28].*

(ii) Every even triangulation G on the projective plane is the face subdivision of some even embedding H. Moreover, $\chi(G) \geq 4$ if and only if H is non-bipartite. Such an even embedding H is uniquely taken in G. [19]

(iii) For any non-spherical surface $\mathbb{F}$, there exists an integer $N(\mathbb{F})$ such that

- *if $\mathbb{F}$ is orientable, then every even triangulation G on $\mathbb{F}$ with $\mathrm{ew}(G) \geq N(\mathbb{F})$ is* 4*-colorable. [12]*

– *if $\mathbb{F}$ is nonorientable, then every even triangulation G on $\mathbb{F}$ with* $\mathrm{ew}(G) \geq N(\mathbb{F})$ *is* 5*-colorable. In particular, G is* 5*-chromatic if and only if G is the face subdivision of some even embedding H including a* 4*-chromatic quadrangulation H' as a subgraph. [4, 20]*

Proof of Theorem 4. Let G be an even triangulation on the projective plane. Suppose that a spanning quadrangulation Q is non-bipartite. By Theorem 16(i), Q is not 3-colorable. Then G is not 3-colorable either. This shows the first assertion of Theorem 4.

Suppose that G is not 3-colorable. By Theorem 16(ii), G is the face subdivision of some non-bipartite even embedding H with color factor U. Then we see that $G - E(H)$ is a spanning bipartite quadrangulation with bipartition U and $V(H)$. Furthermore, we obtain a spanning non-bipartite quadrangulation of G by deleting every other edge of G incident to each vertex v in U, since H is non-bipartite □

Before proceeding to the proof of Theorem 5, we note that the assumption for the edge width in the theorem cannot be omitted in general for a spanning *bipartite* quadrangulation. Let G be an even triangulation on a surface $\mathbb{F}$ containing K_m as a triangular embedding for some odd integer $m \geq 5$, where the Euler characteristic $\chi(\mathbb{F})$ of $\mathbb{F}$ and the integer m must satisfy the equation $m^2 - 7m + 6\chi(\mathbb{F}) = 0$. Similarly to K_7 in Theorem 3, we see that G has edge width exactly 3 and admits no spanning bipartite quadrangulation, as in the following. Therefore we must have $N(\mathbb{F}) \geq 4$ in Theorem 5.

If $G = K_m$, then $|E(G)| = \frac{1}{2}m(m-1)$. If G has a spanning bipartite quadrangulation B, then we have

$$|E(B)| = \frac{2}{3}|E(G)| = \frac{2}{3} \cdot \frac{m(m-1)}{2} = \frac{1}{3}m(m-1).$$

Since B is bipartite, B is a spanning subgraph of some complete bipartite graph of m vertices, which is isomorphic to K_{m_1,m_2} with $m_1 + m_2 = m$ and satisfies $|E(K_{m_1,m_2})| = m_1 m_2$. Hence

$$|E(B)| \leq \max\{m_1 m_2 : m_1 + m_2 = m\} \leq \frac{1}{4}m^2,$$

and consequently we have

$$\frac{1}{3}m(m-1) \leq \frac{1}{4}m^2,$$

which is however impossible when $m \geq 5$. Also in the case when G contains a triangular embedding of K_m as a proper subgraph, we can do similarly as in the proof of Theorem 3.

Now we prove Theorem 5.

Proof of Theorem 5. Let $\mathbb{F}$ be a non-spherical surface other than the projective plane, let $N(\mathbb{F})$ be the integer as in Theorem 16(iii), and let G be an even triangulation on $\mathbb{F}$. We prove the existence of a spanning bipartite quadrangulation and that of a spanning non-bipartite quadrangulation in G separately.

Bipartite quadrangulations: If $\chi(G) \leq 4$, then Proposition 6 implies that G has a spanning bipartite quadrangulation. Otherwise, by Theorems 16(iii), the surface is nonorientable, and G is the face subdivision of some non-bipartite even embedding H with color factor U. Then $G - E(H)$ is a spanning bipartite quadrangulation with bipartition U and $V(H)$.

Non-bipartite quadrangulations: Suppose first hat G is 5-chromatic. By Theorem 16(iii), $G = \mathrm{FS}(H)$ for some non-bipartite even embedding H. Then H can be extended to a spanning non-bipartite quadrangulation of G by adding every other edge incident to each vertex in the color factor U.

Suppose next that G is 4-colorable. We first consider the case when $\mathbb{F}$ is an orientable surface. It was implicitly proved in [12] that if the edge width of G is large enough, then G has four disjoint homotopic non-separating cycles $D_1, \ldots, D_4$ lying in this order on the surface satisfying the following;

(i) The annulus triangulation $A = A[D_1, D_4]$ of G bounded by D_1 and D_4 is 3-colorable, where an *annulus triangulation bounded by D and D'* is a plane graph in which all but the two faces D and D' are triangular.

(ii) Any proper vertex 3-coloring of A can be extended to a proper vertex 4-coloring of G.

Hence G has a proper vertex 4-coloring $c : V(G) \to \{1, 2, 3, 4\}$ such that the vertices of A are 3-colored by c with the colors $1, 2$ and 3. An *(i, j)-edge* in G is an edge whose ends are colored by i and j by c, and an *(i, j)-cycle* (or an *(i, j)-closed walk*) is one consisting only of (i, j)-edges. Let $A[D_i, D_j]$ be the annulus triangulation of G bounded by D_i and D_j for $1 \leq i < j \leq 4$. A *face path* $W = f_0 f_1 \cdots f_k$ is a sequence of faces $f_0, \ldots, f_k$ of G such that for each i, the faces f_i and f_{i+1} share a single edge. When $f_0 = f_k$, W is a *face cycle*. In W, an edge shared by two adjacent faces is an *inner edge* of W, and others *outer edges*.

Let $D_2 = u_1 \cdots u_l$. We modify D_2 into a $(1, 2)$-cycle, as follows: If $c(u_i) = 3$ for some i, then we replace the two edges $u_{i-1}u_i$ and u_iu_{i+1} with the path P_{u_i} between u_{i-1} and u_{i+1} through the neighbors of u_i contained in $A[D_2, D_3]$. Since A is 3-colored, the vertices of P_{u_i} are colored 1 and 2 alternately. Repeating these procedures for each u_i with $c(u_i) = 3$, we get a $(1, 2)$-closed walk C' contained in $A[D_2, D_3]$. Let C be the shortest essential $(1, 2)$-cycle such that $E(C) \subset E(C')$. Since C is 2-colored, the length of C is even, and hence we let $C = v_1 \cdots v_{2m}$. Observe that the right-hand side and the left-hand side of C can be defined, since C is 2-sided. For each odd integer i, let $R_i^1 \cdots R_i^{p_i}$ be the face path consisting of the faces incident to v_i in the right-hand side of C, and for each even integer i, let $L_i^1 \cdots L_i^{q_i}$ be the face path consisting of the faces incident to v_i in the left-hand side of C. Then let

$$X = R_1^1 \cdots R_1^{p_1}\ L_2^1 \cdots L_2^{q_2}\ R_3^1 \cdots R_3^{p_3}\ L_4^1 \cdots L_4^{q_4}\ \cdots\ R_{2m-1}^1 \cdots R_{2m-1}^{p_{2m-1}}\ L_{2m}^1 \cdots L_{2m}^{q_{2m}}$$

be the face cycle contained in $A[D_1, D_4]$. Note that $(1, 2)$-edges appear in X as inner edges alternately, and that all outer edges of W' are $(2, 3)$-edges and $(1, 3)$-edges.

Removing all $(1, 2)$-edges and $(3, 4)$-edges from G, we get a spanning bipartite quadrangulation Q on $\mathbb{S}_g$ (see Proposition 6). Let Q' be the quadrangulation on $\mathbb{S}_g$ obtained from Q by adding all inner $(1, 2)$-edges in X, and removing all inner $(1, 3)$-edges and inner $(2, 3)$-edges in X. Since the vertices in Q colored by 1 and 2 are contained in the same partite set of the bipartition of Q, the new quadrangulation Q' must be non-bipartite. Hence we are done.

Finally, we consider the case when $\mathbb{F}$ is a nonorientable surface other than the projective plane. (Note that the assertion does not hold for the projective plane.) Suppose that $\mathbb{F}$ is the Klein bottle. By a similar method as in [21], if the edge width of G is large enough, then every even triangulation G on the Klein bottle satisfies one of the following:

(I) G is the face subdivision of some non-bipartite even embedding H.

(II) G has four disjoint homotopic non-separating cycles $D_1, \ldots, D_4$ lying in this order on a handle such that the annulus triangulation $A = A[D_1, D_4]$ is 3-colorable, and any proper vertex 3-coloring of A can be extended to a proper vertex 4-coloring of G.

(III) G has an essential separating cycle C such that

- G is separated by cutting along C into two triangulations M_1 and M_2 on Möbius bands with boundary cycle C, and

– for $i = 1, 2$, the map G_i on the projective plane obtained from M_i by pasting a 2-cell D to the boundary C and adding a single vertex v_i on D and joining v_i to all vertices on C is a non-3-colorable even triangulation.

In the case (I) and (II), we can do as in the previous cases. In the case (III), applying Theorem 16(ii) to each G_i, we can take a non-bipartite even embedding H_i with $\mathrm{FS}(H_i) = G_i$, for $i = 1, 2$. Combining H_1 and H_2 suitably in G, we can construct a spanning non-bipartite quadrangulation of G, but we omit a detailed argument.

For nonorientable surface other than the projective plane nor the Klein bottle, we can do similarly to the above arguments. We leave this case for the readers, see [21]. □

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References

[1] M.O. Albertson, H. Alpert, s.-m. belcastro, and R. Haas, Grünbaum colorings of toroidal triangulations, *J. Graph Theory* **63** (2010), 63–81.

[2] N. Alon, R. Berke, K. Buchin, M. Buchin, P. Csorba, S. Shannigrahi, B. Speckmann, and P. Zumstein, Polychromatic colorings of plane graphs, *Discrete Comput. Geom.* **42** (2009), 421–442.

[3] A. Altshuler, Construction and enumeration of regular maps on the torus, *Discrete Math.* **4** (1973), 201–217.

[4] D. Archdeacon, J.P. Hutchinson, A. Nakamoto, S. Negami and K. Ota, Chromatic numbers of quadrangulations on closed surfaces, *J. Graph Theory* **37** (1999), 100–114.

[5] K.L. Collins, and J.P. Hutchinson, Four-coloring six-regular graphs on the torus, in: P.Hanssen, O. Marcotte (Eds.), Graph Colouring and Applications, CRM Proceedings and Lecture Notes, Vol. 23, 1999, pp. 21–34.

[6] P. Erdős, On some extremal problems in graph theory, *Israel J. Math.* **3** (1965), 113–116.

[7] L. Esperet, F. Kardoš, A.D. King, D. Král, and S. Norine, Exponentially many perfect matchings in cubic graphs, *Adv. in Math.* **227** (2011), 1646–1664

[8] S. Fisk, The nonexistence of colorings, *J. Comb. Theory Ser. B* **24** (1978), 247–248.

[9] B. Grünbaum, Conjecture 6, in W. T. Tutte (ed.) Recent Progress in Combinatorics, Proceedings of the Third Waterloo Conference on Combinatorics, May 1968, Aca- demic Press, New York, (1969), 343.

[10] E. Horev, and R. Krakovski, Polychromatic colorings of bounded degree plane graphs, *J. Graph Theory* **60** (2009), 269–283.

[11] E. Horev, M. J. Katz, R. Krakovski, and A. Nakamoto, Polychromatic 4-coloring of cubic bipartite plane graphs, *Discrete Math.* **312** (2012), 715–719.

[12] J.P. Hutchinson, R.B. Richter, and P. Seymour, Colouring Eulerian triangulations, *J. Combin. Theory Ser. B* **84** (2002), 225–239.

[13] F. Hoffmann, and K. Kriegel, A graph coloring result and its consequences for polygon-guarding problems, *SIAM J. Discrete Math.* **9** (1996), 210–224.

[14] M. Kobayashi, A. Nakamoto, and T. Yamaguchi, Polychromatic 4-coloring of cubic even embeddings on the projective plane, *Discrete Math.* **313** (2013), 2423–2431.

[15] M. Kotrbčík, N. Matsumoto, B. Mohar, A. Nakamoto, K. Noguchi, K. Ozeki, and A. Vodopivec, Grünbaum colorings of even triangulations on surfaces, submitted.

[16] A. Kündgen, and C. Thomassen, Spanning quadrangulations of triangulated surfaces, submitted.

[17] A. Kündgen, and R. Ramamurthi, Coloring face-hypergraphs of graphs on surfaces, *J. Combin. Theory Ser. B* **85** (2002), 307–337.

[18] N. Matsumoto, A. Nakamoto, and T. Yamaguchi, Generating even triangulations of the torus, submitted.

[19] B. Mohar, Coloring Eulerian triangulations of the projective plane, *Discrete Math.* **244** (2002), 339–343.

[20] A. Nakamoto, 5-chromatic even triangulations on surfaces, *Discrete Math.* **308** (2008), 2571–2580.

[21] A. Nakamoto, 5-chromatic even triangulations on the Klein bottle, *Discrete Math.* **303** (2005) 154–166.

[22] A. Nakamoto, K. Noguchi, and K. Ozeki, Extension to even triangulations, *SIAM J. Discrete Math.* **29** (2015), 2075–2087.

[23] A. Nakamoto, K. Noguchi, and K. Ozeki, Cyclic 4-colorings of graphs on surfaces, *J. Graph Theory*, DOI 10.1002/jgt.21900.

[24] A. Nakamoto, K. Noguchi, and K. Ozeki, Extension to 3-colorable triangulations, in preparation.

[25] S. Negami, Uniqueness and faithfulness of embeddings of toroidal graphs, *Discrete Math.* **44** (1983), 161–180.

[26] J. Petersen, Die Theorie der regulären graphs, *Acta Math.* **15** (1891), 193–220.

[27] H.G. Yeh, and X. Zhu, 4-colorable 6-regular toroidal graphs, *Discrete Math.* **274** (2003), 261–274.

[28] D.A. Youngs, 4-chromatic projective graphs, *J. Graph Theory* **21** (1996), 219–227.

[29] H. Zhang, and X. He, On even triangulations of 2-connected embedded graphs, *SIAM J. Comput.* **34** (2005), 683–696.

Appendix

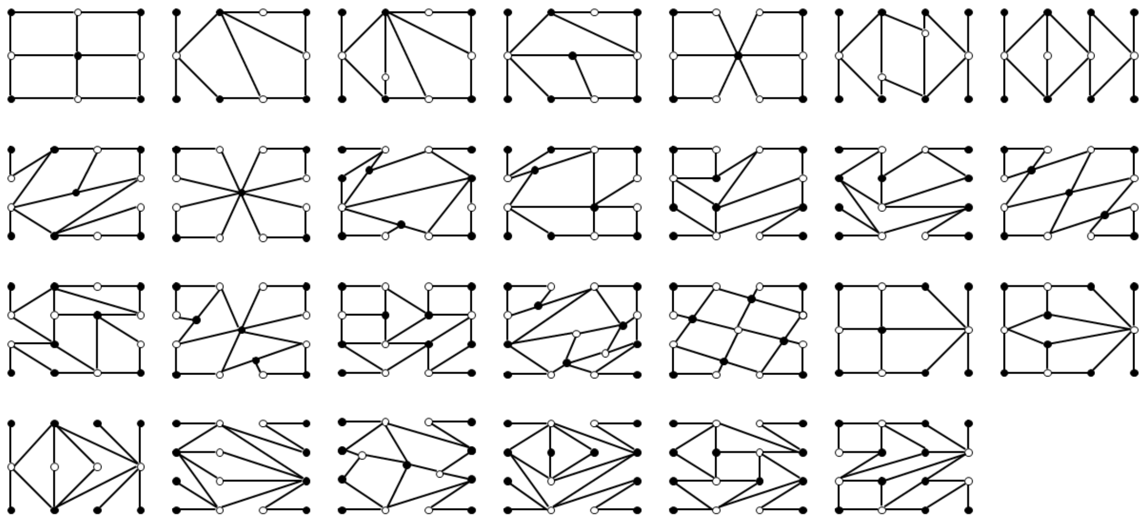

Figure 5: Spanning bipartite quadrangulations.

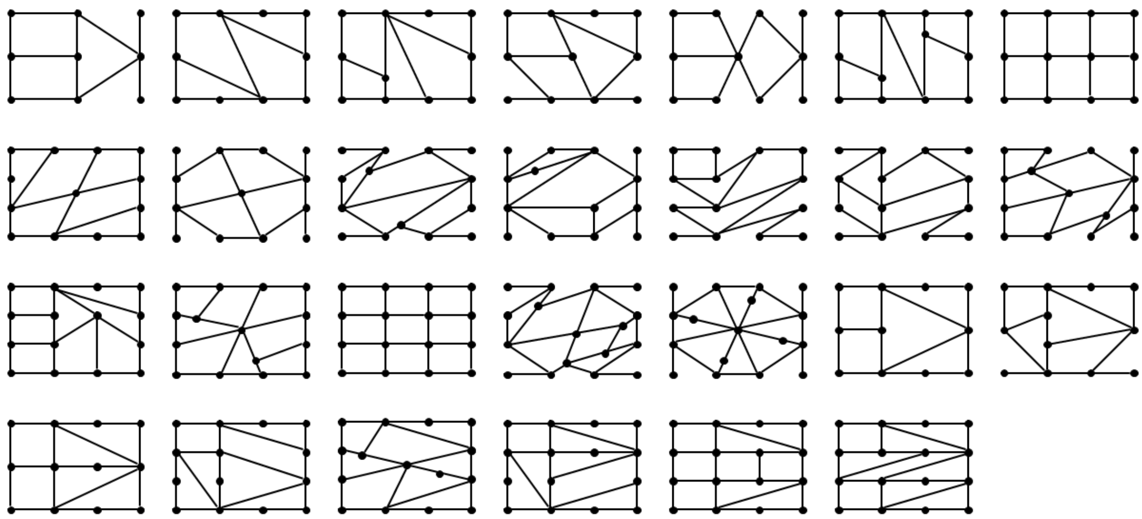

Figure 6: Spanning non-bipartite quadrangulations.

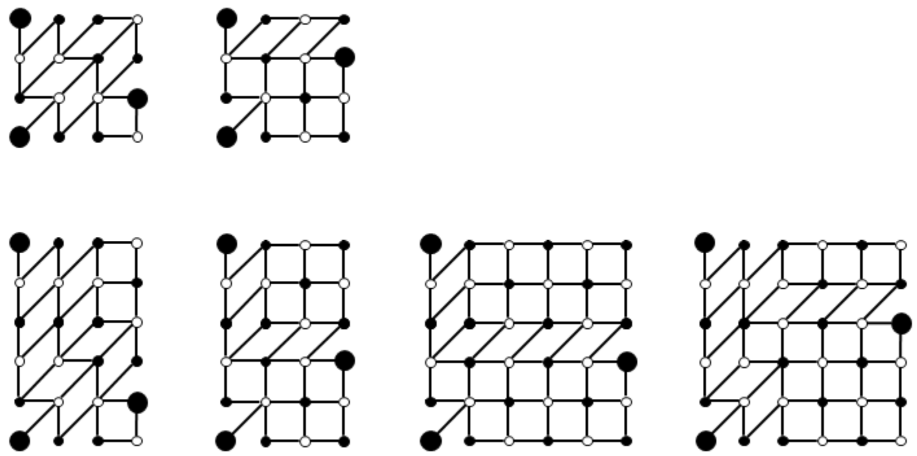

Figure 7: Spanning bipartite quadrangulations of $G[3\times3,2], G[3\times3,3], G[5\times3,2], G[5\times3,3], G[5\times 5,3], G[5\times5,4]$.

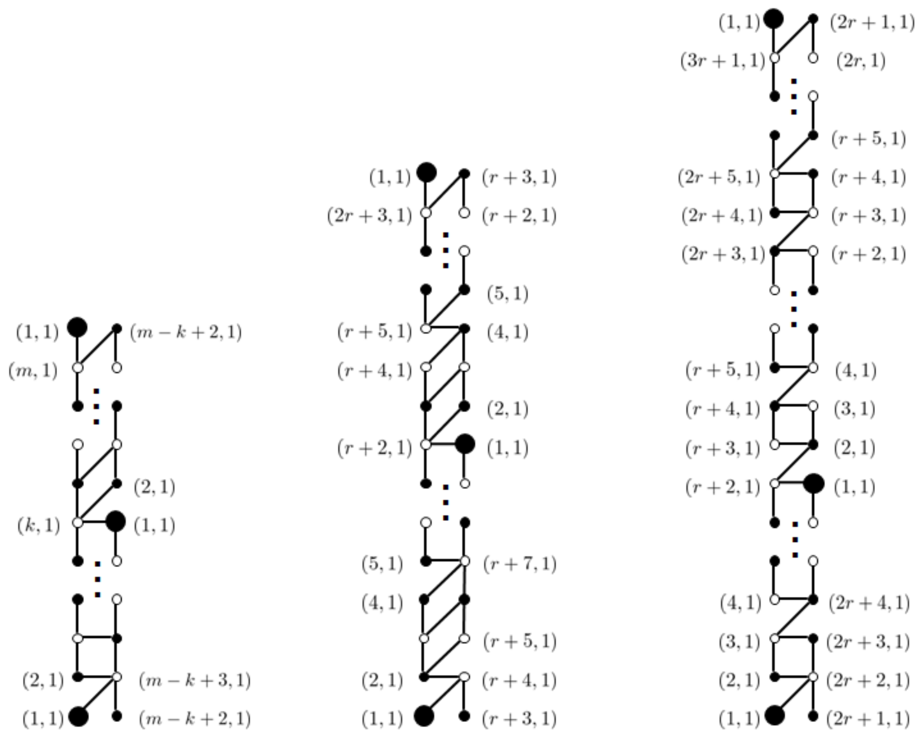

Figure 8: Spanning bipartite quadrangulations of $G[m\times1,k], G[(2r+3)\times1,r+2], G[(3r+1)\times 1,r+2]$.

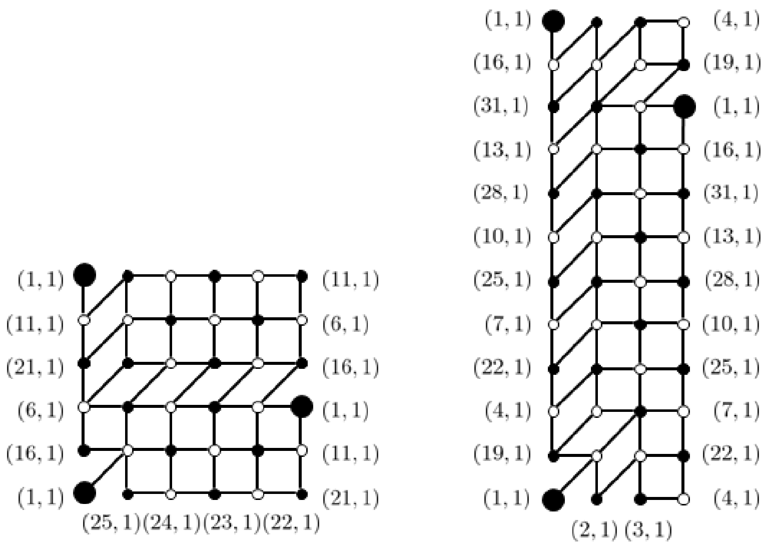

Figure 9: Spanning bipartite quadrangulations of $G[25 \times 1, 12], G[33 \times 1, 16]$.

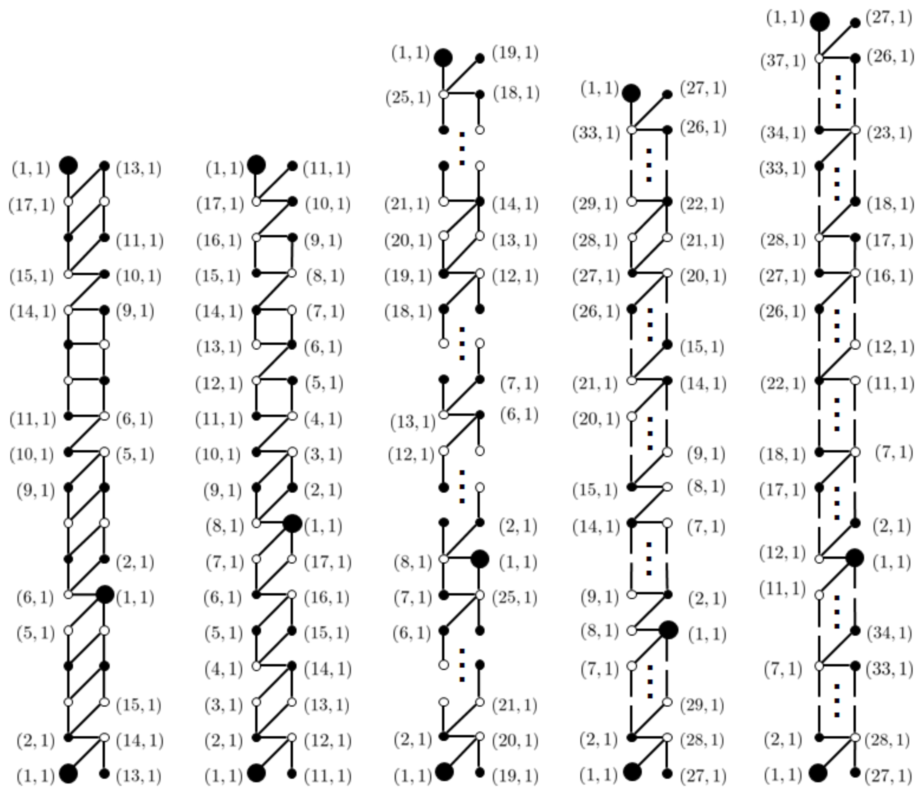

Figure 10: Spanning bipartite quadrangulations of $G[17 \times 1, 6], G[17 \times 1, 8], G[25 \times 1, 8], G[33 \times 1, 8], G[37 \times 1, 12]$.