Book embedding of graphs on the projective plane

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Abstract

For a positive integer k, a *book (with k pages)* is a topological space consisting of a *spine*, which is a line, and k *pages*, which are half-planes with the spine as its boundary. We say that a graph G admits a *k-page book embedding* or is *k-page book embeddable* if there exists a linear ordering of the vertices on the spine and one can assign the edges of G to k pages such that no two edges of the same page cross. Yannakakis proved that every plane graph admits a 4-page book embedding, and using it, Nakamoto and Nozawa showed that every graph on the projective plane admits a 9-page book embedding. In this paper, we improve the latter result to 6-page embedding. Furthermore, we also prove that every graph on the projective plane admits a 3-page book embedding if it is 5-connected, and a 5-page book embedding if it is 4-connected. Our idea of the proofs is to use a "*Tutte path*", which is very different from previous ones.

Keywords: Book embedding, graphs on the projective plane, Tutte paths

1 Introduction

For a positive integer k, a *book (with k pages)* is a topological space consisting of a *spine*, which is a line, and k *pages*, which are half-planes with the spine as its boundary. We say that a graph G admits a *k-page book embedding* or is *k-page book embeddable* if there exists a linear ordering of the vertices on the spine and one can assign the edges to k pages such that no two edges of the same page cross. Recall that two edges a_1a_2 and b_1b_2 *cross at a sequence Q* of vertices if the vertices a_1, a_2, b_1 and b_2 are all distinct and they appear on Q in the order $a_i, b_j, a_{3-i}, b_{3-j}$ or $b_j, a_i, b_{3-j}, a_{3-i}$ for some $i, j \in \{1, 2\}$. The *pagenumber* (or sometimes called the *stack number* or the *book thickness*) of a graph G is the minimum of k such that G is k-page book embeddable.

This notion was first introduced by Bernhart and Kainen [2]. Since a book embedding is much concerned with theoretical computer science, such as VLSI design [4, 16], we are interested in bounding the pagenumber. Actually, a number of researchers have established upper bounds of the pagenumber for some graph classes, for example, complete bipartite graphs [8, 14], regular graphs [3, 4], and k-trees [5, 9, 23]. Several algorithms to find an embedding of a given graph into a book with a few pages were also presented [12, 19].

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On the other hand, the pagenumber has been widely studied from the aspect of graphs on surfaces. In fact, a graph G is 1-page embeddable if and only if G is outer planar, and a graph G is 2-page embeddable if and only if G is a subgraph of a Hamiltonian planar graph (see [2]). In this sense, the pagenumber is closely related to graphs on surfaces.

Bernhart and Kainen [2] first conjectured that the pagenumber of planar graphs could be large enough. However, this conjecture was disproved by Buss and Shor [3], who proved that every planar graph is 9-page embeddable. Later this upper bound was improved to seven by Heath [10], and finally, Yannakakis [24] showed that every planar graph has the pagenumber at most four. He [24] also announced that there exists a planar graph which is not 3-page book embeddable, but no proofs appeared yet, and hence we cannot verify that. For graphs on the torus, the algorithm given by Heath and Istrail [11] guaranteed the existence of 13-page book embedding of a toroidal graph. Endo [7] later improved this to 7-page book embedding. For a graph on the orientable surface of genus g, Heath and Istrail [11] proved that its pagenumber is $O(g)$, and later, Melitz [13] improved this result to $O(\sqrt{g})$. Note that there exists a graph of genus g with the pagenumber $\Theta(\sqrt{g})$, see [11].

In contrast with situations above, only few results are known about the pagenumber of graphs on nonorientable surfaces. Heath and Istrail [11] have added a comment, in the paper giving the $O(g)$ bound for the pagenumber of graphs of orientable genus g, that every graph of nonorientable genus k is also $O(k)$-page embeddable. However, they did not describe the details for the nonorientable case, and so we cannot verify whether it is true or not. Nakamoto and Nozawa [15] proved that every graph on the projective plane is 9-page book embeddable. In this paper, we improve their result as follows.

Theorem 1 *Let G be a graph on the projective plane. Then all of the following hold;*

(i) *If G is 5-connected, then G has a 3-page book embedding.*

(ii) *If G is 4-connected, then G has a 5-page book embedding.*

(iii) *G has a 6-page book embedding.*

For the proof of Theorem 1 (iii), we show Lemma 11 in Section 6.1, which implies that "every planar graph admits a 6-page book embedding". This statement is weaker than the one by Yannakakis [24], but the strategy of our proof is very different, using *Tutte paths* (see Section 3.1 for the definition). This is a key idea of this paper, and in fact, this allow us to improve the previous result and obtain Theorem 1. Note that Yannakakis' result is so useful that it has been used for the proofs of several results, e.g. [7, 11, 13, 15] (see also Section 2). We expect that this idea will give better bounds for several other cases.

This paper is organized as follows: In the next section, we give a strategy of the proofs of Theorem 1. In Section 3, we give some terminologies and lemmas used in the proofs of Theorem 1. Then we divide the remaining part into three sections, each of which corresponds to the 5-connected case (Theorem 1 (i)), the 4-connected case (Theorem 1 (ii)), and the case without connectivity condition (Theorem 1 (iii)), respectively. The proofs of Theorem 1 (i)–(iii) and some lemmas are similar, but unfortunately each one has some their own particularities, and because of that, we have not been able to combine them.

2 The strategy of the proofs of Theorem 1

We here consider the ideas of the proofs. In fact, our proof of Theorem 1 is different from the one by Nakamoto and Nozawa [15]. Their method is based on the decomposition of a graph on the projective plane into two planar graphs D_1 and D_2 and a graph B on the Möbious band, and then they used Yannakakis' [24] result to D_1 and D_2, respectively. In fact, we sketchily need four pages to the edges in D_1, four pages to the edges in D_2, together with some edges in B, and one page to almost all edges in B. Then we totally obtain a 9-page book embedding. Note that, because of the topologically property of the projective plane, they needed to decompose a graph into D_1, D_2 and B.

On the other hand, the strategy of our proof is as follows: Similarly to the method by Nakamoto and Nozawa, we first decompose a graph G on the projective plane into a plane graph H and a graph on the Möbious band (Lemma 9). Then, instead of using Yannakakis' result to H, we take a suitable path T, namely a Tutte path, as a "main part" of the spine sequence. (Lemma 5. See Section 3.1 for the definition of a Tutte path.) If G is 5-connected, then T must be a Hamiltonian path, which gives a 2-page book embedding of H. (Recall that a graph admits a 2-page book embedding if and only if it is a subgraph of planar Hamiltonian graph.) Thus, together with an almost 1-page book embedding of the Möbious band part, we obtain a 3-page book embedding of G, so Theorem 1 (i) holds. On the other hand, suppose that the connectivity is not high enough (4-connected or less). In this case, T might not be a Hamiltonian path in H, but the properties of a "Tutte path" guarantee that the vertices in $H - V(T)$ can be decomposed into several parts, namely "*T-bridges*" with at most three attachments. For each T-bridges, we prove the existence of suitable book embeddings (Lemmas 10 and 11), again using a Tutte path and bridges inside. This means that, we obtain a book embedding of T-bridges, using Tutte paths inductively. Then finally we substitute the vertices in each T-bridge to appropriate place of the Tutte path T, and obtain a 6-page book embedding of the graph H. In addition, the edges in $E(G) - E(H)$ can be embedded into appropriate pages, without creating a crossings in each page.

3 Terminology and Lemmas used in our proofs

3.1 Terminologies

Formally, a k-page book embedding Σ of a graph G is defined as a pair $\Sigma = (Q, \mathscr{E})$ of a sequence Q of the vertices of G and a partition $\mathscr{E} = \{E^1, \dots, E^k\}$ of $E(G)$ such that for any i with $1 \leq i \leq k$, any two edges in E^i does not cross at Q. For a book embedding $\Sigma = (Q, \mathscr{E})$ of a graph G, the sequence Q is called the *spine sequence* of Σ.

Let G be a graph, and let Q be a sequence of vertices in G. For $u, v \in V(Q)$, we denote by $Q[u, v]$, the subsequence of Q from u to v. In addition, $Q(u, v]$ is the subsequence of Q obtained from $Q[u, v]$ by deleting the vertex u. Similarly, we define the subsequences $Q[u, v)$ and $Q(u, v)$ of Q. We denote by $\overleftarrow{Q}$ the reverse sequence of Q. For two sequences $Q_1 = x_0x_1 \cdots x_k$ and $Q_2 = y_0y_1 \cdots y_l$ of vertices with $Q_1 \cap Q_2 = \emptyset$, we denote the sequence obtained by the concatenation of Q_1 and Q_2 by Q_1Q_2, that is, $Q_1Q_2 = x_0x_1 \cdots x_ky_0y_1 \cdots y_l$. In this paper, we regard a path in G also as a sequence of vertices. Let T be a path in G. An edge e in $G - E(T)$ is a *chord* of T if e is not an edge in T but connects two vertices in T. Therefore, (with slightly abuse of notation,) an edge e in G is a *chord* of a sequence T if e connects two non-consecutive vertices in T.

For a graph G and a vertex set S of G, the subgraph of G induced by S is denoted by $G[S]$. A path with end vertices a and b is called an *a, b-path*.

Let G be a connected plane graph. The *outer walk* of G is the closed walk bounding the outer face of G. In particular, if the boundary of the outer face is a cycle, then it is called the *outer cycle* of G. Let s and t be two vertices on the outer cycle C of a 2-connected plane graph, and let T be a path in G connecting s and t. Then T divides the disk bounded by C into at least two regions. Now we distinguish the regions bounded by cycles in $C \cup T$ regarding whether they appear on the *left* side or the *right* side along the path T from s to t. Furthermore, all vertices or edges not contained in T are also said to be on the *left* side or the *right* side of T, depending on the property of their regions of $C \cup T$.

A 2-connected plane graph G with the outer cycle C is said to be *internally k-connected with respect to C* if for every vertex x in $G - V(C)$, there exist k pairwise internally disjoint paths in G connecting x and C such that the end vertices in C are all distinct. In other words, there exists no vertex set that consists of at most $k-1$ vertices and separates some vertices in $G - V(C)$ from C. We sometimes use the term "internally k-connected" omitting "with respect to C". When $k = 3$, an internally 3-connected graph is also called a *circuit graph*.

A *disk graph* is a 2-connected graph embedded on a disk. A *disk triangulation* is a disk subgraph in which all but the outer cycle are triangular. If the outer cycle of a disk triangulation H is also triangular, then H is indeed a triangulation of the plane.

Let T be a subgraph of a graph G. A connected subgraph B of G is called a *T-bridge*, if either

- B consists of only an edge of $G - E(T)$ with both ends on T, or
- B is the subgraph induced by all edges in a component D of $G - V(T)$ and all edges from D to T.

The former is said to be *trivial*, while the latter is *non-trivial*. For a T-bridge B of G, the vertices in $B \cap T$ are the *attachments* of B (on T). We say that T is a *Tutte subgraph* of G if every T-bridge of G has at most three attachments on T. In addition, for a subgraph C of G, T is a *C-Tutte subgraph* of G if T is a Tutte subgraph of G and every T-bridge of G containing an edge of C has at most two attachments on T. (As such a subgraph C, we usually take the outer cycle.) A *Tutte path* (resp., a *Tutte cycle*) in a graph is a path (resp., a cycle) that is a Tutte subgraph. See [6] for more detail on Tutte subgraphs.

Note that if G is 3-connected internally 4-connected and a C-Tutte path T satisfies $|T| \geq 4$, then T is a Hamiltonian path in G. To see that, suppose that there exists a vertex in $G - V(T)$, which implies the existence of a non-trivial T-bridge B of G. If B does not contain an edge in C, then the attachments of B on T form a cut set of order at most three such that it separates the non-attachments of B from other part of the graph, contradicting the internally 4-connectedness. Thus, we may assume the B contains an edge in C. But in this case, the attachments of B form a cut set of order two, contradicting the 3-connectedness of G.

3.2 Lemmas concerning a book embedding

It is well-known that a graph G admits a 2-page book embedding if and only if G is a subgraph of a Hamiltonian plane graph. The following lemma is a crucial point of the "if" part, which was shown by Bernhart and Kainen [2]. We will use this several times in our proofs.

Lemma 2 *Let G be a 2-connected plane graph with outer cycle C, let $s, t \in V(C)$, let T be a path connecting s and t, and let E_L and E_R be the sets of edges e in $E(G[V(T)]) - E(T)$ such that e is placed on the left side (resp., the right side) of T, respectively. Then no two edges in $E(T) \cup E_L$ (or in $E(T) \cup E_R$) cross at the sequence T.*

In fact, Lemma 2 gives a 2-page book embedding to a subgraph of a Hamiltonian planar graph G: We may first assume that G is a triangulation, and hence G is 3-connected and Hamiltonian itself. (Since otherwise, we can get a plane triangulation $\widetilde{G}$ from G by adding edges suitably. It is easy to see that if $\widetilde{G}$ is 2-page embeddable, then so G is.) Let T' be a Hamiltonian cycle in G. Choosing suitable triangle of G as the outer cycle C, we may further assume that T' contains an edge in C, say st. Then $T = T' - st$ is a Hamiltonian path connecting s and t, and $E(G)$ is partitioned into the three sets $E(T)$, E_L and E_R, where E_L and E_R are defined as in Lemma 2. Therefore, it follows from Lemma 2 that G has a 2-page book embedding $(T, \mathscr{E})$, where $\mathscr{E} = \{E(T) \cup E_L, E_R\}$.

Another basic idea to find a book embedding is the following. This can be proven directly from the definition of crossing of two edges. We will use Lemma 3 several times.

Lemma 3 *Let Q be a sequence of vertices of a graph G, let $u, v \in V(G)$, and let a_1a_2, b_1b_2 be two edges of G. If $a_1, a_2 \in V(Q[u,v])$ and $b_1, b_2 \in V(Q) - V(Q(u,v))$, then the two edges a_1a_2 and b_1b_2 do not cross at Q.*

3.3 Lemmas concerning Tutte paths

As we explained in the last part of Section 1, we will use a Tutte path in order to make it as a "main part" of the spine sequence. In fact, the following two results guarantee the existence of suitable Tutte paths. The first one was shown by Thomas and Yu [20, Theorem (2.7)]. (See also [17, Lemma 3].) The second one is new and we prove it in this section.

Theorem 4 (Thomas and Yu [20]) *Let G be a 2-connected plane graph, let C be the outer cycle of G, and let $e_1, e_2, e_3 \in E(C)$. Then G has a C-Tutte cycle through e_1, e_2, e_3.*

Lemma 5 *Let G be a 2-connected plane graph with outer cycle C, and let s, x, t and y be four distinct vertices on C such that they appear in C in this clockwise order and $xt \in E(G)$. Suppose that G has a path from s to t through x and then y. Then G has a C-Tutte path from s to t through x then y.*

In order to prove Lemma 5, we need the next theorems. They are somewhat technical, but they support several cases when we want to find a Tutte cycle or a Tutte path in certain graphs on a surface. The first one was proved by Sanders [18]. Note that he showed only the 2-connected case, but we can easily show the following, considering a block decomposition. See also [21] by Thomassen. For the second one, see [20, Theorem (2.4)].

Theorem 6 (Sanders [18]) *Let G be a connected plane graph, let C be the outer walk of G, let $x, y \in V(G)$ with $x \neq y$, and let $e \in E(C)$. Assume that G contains a path from x to y through e. Then G has a C-Tutte path from x to y through e.*

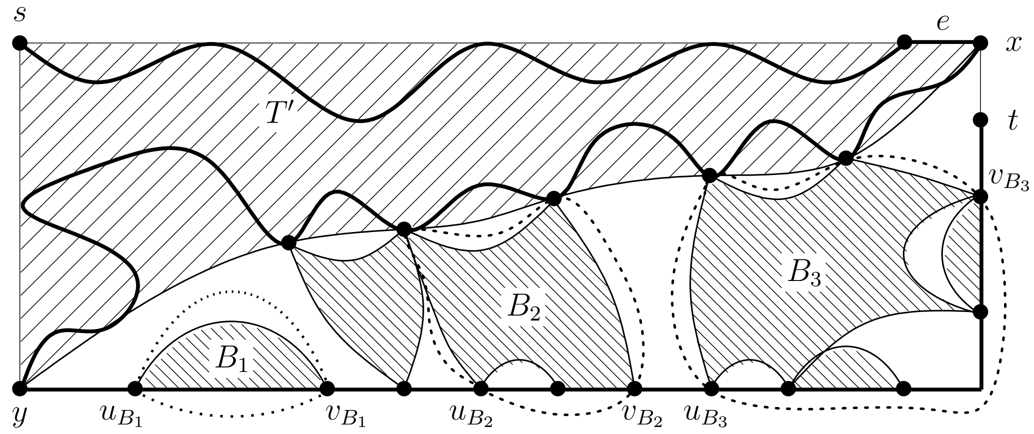

Figure 1: The C'-Tutte path T' from s to y through x, and $B_1, B_2, B_3 \in \widetilde{\mathcal{B}}$.[1]

Theorem 7 (Thomas and Yu [20]) *Let G be a connected plane graph, let C be the outer walk of G, let $x, y \in V(C)$ with $x \neq y$, and let $S \subset V(C)$ with $|S| \leq 2$. Suppose that $V(C[x,y]) \cap S = \emptyset$. Then G has a C-Tutte subgraph consisting of the vertices in S and a path T from x to y with $V(T) \cap S = \emptyset$.*

Proof of Lemma 5. Note that the subpath $\overleftarrow{C}[y,t]$ of C contains neither s nor x. Since G has a path from s to t through x and then y, it follows from the planarity of G that $G' = G - V(\overleftarrow{C}(y,t])$ has a path from s to y through x. Let C' be the outer walk of the component of G' containing s, x and y. By Theorem 6 with specifying an appropriate edge incident with x as e, G' has a C'-Tutte path T' from s to y through x. See Figure 1. Let

$$\begin{aligned}\mathcal{B} = \{B : B \text{ is a non-trivial } \big(T' \cup \overleftarrow{C}[y,t]\big)\text{-bridge of } G \\ \text{having at least two attachments on } \overleftarrow{C}[y,t]\}.\end{aligned}$$

Let $B \in \mathcal{B}$ and let S_B be the set of attachments of B on T'. Since B has an attachment on $\overleftarrow{C}[y,t]$, $B - V\big(\overleftarrow{C}(y,t]\big)$ is either a component of G' or a T'-bridge of G' containing an edge in C'. Since T' is a C'-Tutte path in G', in either case, $B - V\big(\overleftarrow{C}(y,t]\big)$ has at most two attachments on T', that is, $|S_B| \leq 2$. Let u_B and v_B the attachments of B on $\overleftarrow{C}[y,t]$ such that u_B (resp., v_B) is as close to y (resp., t) on $\overleftarrow{C}[y,t]$ as possible. Since B has at least two attachments on $\overleftarrow{C}[y,t]$, we have $u_B \neq v_B$ and $\overleftarrow{C}[u_B, v_B]$ is contained in $\overleftarrow{C}[y,t]$. For $B, B' \in \mathcal{B}$, we write $B' \preceq B$ if either (i) $B = B'$, or (ii) B' is contained in the disk bounded by $P \cup \overleftarrow{C}[u_B, v_B]$, where P is a path in B connecting u_B and v_B. This is well-defined (not depending on the choice of P). Since G is a plane graph and $\overleftarrow{C}[y,t]$ is a subpath of the outer cycle C of G, the binary relation $\preceq$ is a partial order on $\mathcal{B}$. Let $\widetilde{\mathcal{B}}$ be the set of maximal elements of $\mathcal{B}$ with respect to the partial order $\preceq$. Again by the planarity of G, we have the following; For any $B, B' \in \widetilde{\mathcal{B}}$ with $B \neq B'$, $\overleftarrow{C}[u_B, v_B]$ and $\overleftarrow{C}[u_{B'}, v_{B'}]$ are edge-disjoint.

[1]The region with diagonals from the right top to the left bottom represents the graph G', while the ones with diagonals from the left top to the right bottom represent non-trivial $\big(T' \cup \overleftarrow{C}[y,t]\big)$-bridges of G, including B_1, B_2 and B_3 that belong to $\widetilde{B}$. The bold curve and lines represent the path $T' \cup \overleftarrow{C}[y,t]$, and the dashed curves bound the graphs B_1^*, B_2^* and B_3^*, respectively.

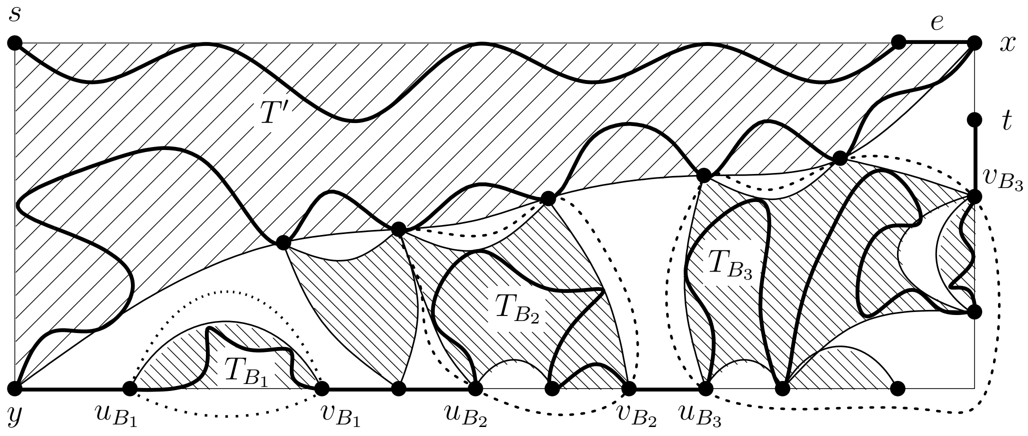

Figure 2: The C-Tutte path T from s to t through x and then y.

For $B \in \widetilde{\mathcal{B}}$, let B^* be the subgraph of G induced by the union of all elements $B' \in \mathcal{B}$ such that $B' \preceq B$, together with $\overleftarrow{C}[u_B, v_B]$, and let C_B be the outer walk of B^*. Note that C_B contains $\overleftarrow{C}[u_B, v_B]$ and S_B, and $V\big(\overleftarrow{C}[u_B, v_B]\big) \cap S_B = \emptyset$. By Theorem 7, B^* has a C_B-Tutte subgraph consisting of the vertices in S_B and a path T_B from u_B to v_B with $V(T_B) \cap S_B = \emptyset$.

Let T be a subgraph of G induced by

$$E(T') \cup \Big(E\big(\overleftarrow{C}[y,t]\big) - \bigcup_{B \in \widetilde{\mathcal{B}}} E\big(\overleftarrow{C}[u_B, v_B]\big)\Big) \cup \bigcup_{B \in \widetilde{\mathcal{B}}} E(T_B).$$

See Figure 2. By the construction, T is a path in G from s to t through x then y. Let D be a T-bridge of G. Then it is easy to see that D is either (i) a $\big(T_B \cup S_B\big)$-bridge of B^* for some $B \in \widetilde{\mathcal{B}}$, or (ii) a $\big(T' \cup \overleftarrow{C}[y,t]\big)$-bridge having at most one attachment on $\overleftarrow{C}[y,t]$. If D satisfies (i), then since $T_B \cup S_B$ is a C_B-Tutte subgraph in B^*, D has at most three attachments and at most two attachments if D contains an edge in $\overleftarrow{C}[u_B, v_B]$. Note that $E(C) \cap E(B^*) \subseteq E\big(\overleftarrow{C}[u_B, v_B]\big)$.

Suppose that D satisfies (ii). If D has no attachment on $\overleftarrow{C}[y,t]$, then D is a T'-bridge of G', and hence D has at most three attachments and at most two attachments if D contains an edge in C'. Note that $E(C) \cap E(G') \subseteq E(C')$. On the other hand, suppose that D has an attachment on $\overleftarrow{C}[y,t]$. Since $D - V\big(\overleftarrow{C}(y,t]\big)$ is a T'-bridge containing an edge in C', $D - V\big(\overleftarrow{C}(y,t]\big)$ has at most two attachments on T'. Therefore, D has at most three attachments on T in total. Suppose further that D has exactly three attachments on T and contains an edge in C. This implies that D has at least two attachments on T that are contained in C. Since D has an attachment on $\overleftarrow{C}[y,t]$ and $x, y \in V(T)$, it follows from the planarity that D has at least two attachments on $\overleftarrow{C}[y,x]$. Recall that D has at most one attachment on $\overleftarrow{C}[y,t]$, and hence D has exactly two attachments on $\overleftarrow{C}[y,x]$ one of which is x and the other is some vertex in $\overleftarrow{C}[y,t]$. However, since $t \in V(T)$ and $xt \in E(C)$, D cannot contain an edge in C, a contradiction.

Therefore, in either case, D has at most three attachments and at most two attachments if D contains an edge in C. Hence T is a C-Tutte path in G, and this completes the proof of Lemma 5. $\square$

3.4 A lemma concerning spanning disk triangulations in triangulations on the projective plane

The following was proven by Nakamoto and Nozawa [15].

Lemma 8 (Nakamoto and Nozawa [15]) *Let G be a triangulation on the projective plane. Then G has two vertices x_0 and y_0 and three internally disjoint x_0, y_0-paths P, L and R such that $x_0y_0 \in E(G)$, $L \cup R$ bounds a disk triangulation of G containing P, and $P \cup x_0y_0$ is a noncontractible cycle on the projective plane.*

Note that the last two conditions in Lemma 8 implies that $x_0y_0 \notin E(P) \cup E(L) \cup E(R)$. Using Lemma 8, we show the following, which will give an efficient partition of edges into the disk graph (namely H) and the graph on the Möbious band (namely $G - E(H)$). This will be the first step to find a book embedding of graphs on the projective plane.

Lemma 9 *Let G be a triangulation of the projective plane. Then G has a spanning disk triangulation H with outer cycle C satisfying the following;*

(G1) *There exist four distinct vertices s, x, t and y on C such that they appear on C in that order, $xt \in E(C)$, $st \in E(G) - E(H)$, and all vertices in $C[y, s]$ are neighbors of x in $G - E(H)$.*

(G2) *H has a path from s to t through x and then y.*

(G3) *Any edges in $G - E(H) - E_x$ connect a vertex in $C[s, x]$ and a vertex in $\overleftarrow{C}[y, t]$, where $E_x = \{xz : z \in V\big(C(y, s]\big)\}$. Furthermore, no two edges in $G - E(H) - E_x$ cross at the sequence $C[s, x]\ \overleftarrow{C}[y, t]$.*

(G4) *If G is 5-connected, then H is 3-connected and internally 5-connected, $sy \in E(G)$, and $E_x = \{xs\}$.*

(G5) *If G is 4-connected, then H is an internally 4-connected disk triangulation, and in addition, either $\{u, v\} \cap V\big(C[s, x]\big) \neq \emptyset$ for any 2-cut $\{u, v\}$ of H, or $\{u, v\} \cap V\big(\overleftarrow{C}[y, t]\big) \neq \emptyset$ for any 2-cut $\{u, v\}$ of H.*

Proof. Let G be a triangulation of the projective plane. It follows from Lemma 8 that G has two vertices x_0 and y_0 and three internally disjoint x_0, y_0-paths P, L and R such that $x_0y_0 \in E(G)$, $L \cup R$ bounds a disk triangulation, say H_0, of G containing P, and $P \cup x_0y_0$ is a noncontractible cycle on the projective plane. Note that $L \cup R$ is the outer cycle of H_0. We take such three internally disjoint x_0, y_0-paths P, L and R so that neither L nor R do not have a chord. (If, for example, L has a chord uv, then we can detour the path L through the edge uv.) Since G is a triangulation, we see that H_0 is internally 3-connected (and internally 4- or 5-connected if G is 4- or 5-connected, respectively). Suppose that H_0 has a 2-cut $\{u, v\}$. Then since H_0 is a disk triangulation with the outer cycle $L \cup R$, we have $u, v \in V\big(L \cup R\big)$ and $uv \in E(H_0)$. However, this contradicts that neither L nor R do not have a chord (when $u, v \in V(L)$ or $u, v \in V(R)$), or the x_0, y_0-path P is contained in H_0 and internally disjoint from L and R (when either $u \in V(L)$ and $v \in V(R)$, or $u \in V(R)$ and $v \in V(L)$). Therefore, we have that H_0 is 3-connected.

Now we take a disk triangulation H of G with the outer cycle C such that

(I) P is contained in H and $V(P) \cap V(C) = \{x_0, y_0\}$. (Therefore, $x_0y_0 \notin E(H)$ since H is a disk triangulation.)

(II) If G is 5-connected, then H is 3-connected and internally 5-connected.

(III) If G is 4-connected, then H is an internally 4-connected graph such that $\{u, v\} \subseteq V\big(C[y_0, x_0]\big)$ for any 2-cut $\{u, v\}$ of H.

(IV) $|E(H)|$ is as large as possible, subject to (I)–(III).

Note that H_0 satisfies conditions (I)–(III), and hence we can take such a disk triangulation H. The condition (I) implies that C is divided into two paths $C[x_0, y_0]$ and $C[y_0, x_0]$. We will show that H is a spanning disk triangulation satisfying all of the conditions (G1)–(G5).

We first show that H is a spanning disk triangulation of G. For the contrary, suppose that H is not spanning, that is, there exists a vertex v in $G - V(H)$.

Suppose that G is 5-connected. Then there exist internally disjoint five paths from v to C, say P_1, P_2, P_3, P_4 and P_5, where each P_i is a v, p_i-path for some $p_i \in V(C)$ and $p_i \neq p_j$ for any $1 \leq i < j \leq 5$. Taking such paths as short as possible, we may assume that none of the paths P_1, P_2, P_3, P_4 and P_5 have a chord. By symmetry, we may further assume that p_1, p_2 and p_3 are distinct vertices on $C[x_0, y_0]$, $C[p_1, p_3]$ is a subpath of $C[x_0, y_0]$, and $p_2 \in V\big(C(p_1, p_3)\big)$. Now if we let H' be the disk triangulation of G bounded by $C' = P_1 \cup P_3 \cup C[p_3, p_1]$, then H' is a disk triangulation of G satisfying condition (I) and $E(H) \subseteq E(H')$. Furthermore, since P_1 and P_3 has no chord, we see that C' has no inner chord, and hence H' is 3-connected. Therefore, H also satisfies conditions (II) and (III), which contradicts the condition (IV) for H.

Suppose next that G is not 5-connected, but 4-connected. In this case, the condition (II) is automatically satisfied. There exist internally disjoint four paths from v to C, say P_1, P_2, P_3 and P_4, where each P_i is a v, p_i-path for some $p_i \in V(C)$ and $p_i \neq p_j$ for any $1 \leq i < j \leq 4$. Taking such paths as short as possible, we may assume that none of the paths P_1, P_2, P_3 and P_4 have a chord. By symmetry, we may further assume that either

(i) p_1, p_2 and p_3 are distinct vertices on $C[x_0, y_0]$, $C[p_1, p_3]$ is a subpath of $C[x_0, y_0]$, and $p_2 \in V\big(C(p_1, p_3)\big)$, or

(ii) p_1 and p_2 are distinct vertices on $C[y_0, x_0]$, and $C[p_1, p_2]$ is a subpath of $C[y_0, x_0]$.

In the case (i), let H' be the disk triangulation of G bounded by $C' = P_1 \cup P_3 \cup C[p_3, p_1]$; Otherwise, let H' be the disk triangulation of G bounded by $C' = P_1 \cup P_2 \cup C[p_2, p_1]$. In either case, H' satisfies condition (I) and $E(H) \subseteq E(H')$. Furthermore, if the case (i) occurs, then the addition of the subgraph bounded by $P_1 \cup P_3 \cup C[p_1, p_3]$ to H does not create new 2-cuts; if the case (ii) occurs, then it might create new 2-cuts, say $\{u, v\}$, but we have $u, v \in V\big(P_1 \cup P_2\big) \subseteq V\big(C'[p_1, p_2]\big) \subseteq V\big(C'[y_0, x_0]\big)$. In either case, H also satisfies condition (III), which contradicts the condition (IV) for H.

Suppose finally that G is not 4-connected. In this case, the conditions (II) and (III) are automatically satisfied. By the 3-connectedness of a triangulation G, there exist internally disjoint three paths from v to C, say P_1, P_2, and P_3, where each P_i is a v, p_i-path for some $p_i \in V(C)$ and $p_i \neq p_j$ for any $1 \leq i < j \leq 3$. By symmetry, we may further assume that p_1 and p_2 are distinct vertices on $C[x_0, y_0]$, and $C[p_1, p_2]$ is a subpath of $C[x_0, y_0]$. Now if we let H' be the disk

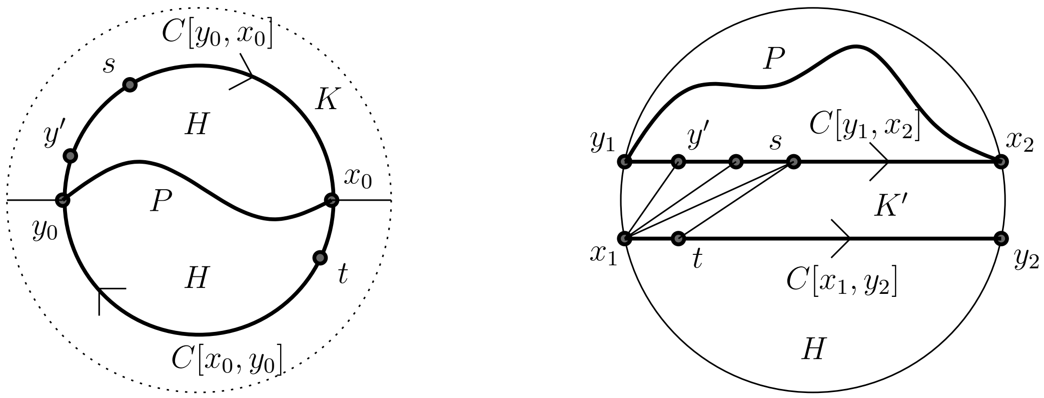

Figure 3: Two figures of a graph G together with the vertices x_0, y_0, s, t, y' on C: In the left side, the spanning disk triangulation H is centered with the outer part K, while the Möbious band including K' is centered in the right side.

triangulation of G bounded by $C' = P_1 \cup P_2 \cup C[p_2, p_1]$, then it follows from the same way as in the previous paragraphs that H' satisfies condition (I) and $E(H) \subseteq E(H')$, contradicting the condition (IV).

This concludes that H is a spanning disk triangulation of G. Then we show that H satisfies the conditions (G1)–(G5) for particular vertices x and y. In fact, the vertices x_0 and y_0 are candidates of those vertices, respectively, but we need to change them when G is 5-connected. (When G is not 5-connected, then as we will see later, we set $x = x_0$ and $y = y_0$.)

By condition (I), we have $x_0y_0 \in E(G) - E(H)$. Let

$$W = C[y_0, x_0] \cup x_0y_0 \cup \overleftarrow{C}[y_0, x_0] \cup x_0y_0$$

be the closed walk in G, and let K be the subgraph of G bounded by W. See Figure 3, where in the right side, we put $W = C[y_1, x_2] \cup x_2y_2 \cup \overleftarrow{C}[y_2, x_1] \cup x_1y_1$ with $x_0 = x_1 = x_2$ and $y_0 = y_1 = y_2$ in G, distinguishing two appearances of x_0 and y_0 in the boundary walk W of K. Note that K has no inner edge $e = uv$ with $u, v \in V\big(C[y_1, x_2]\big)$ but $uv \notin E\big(C[y_1, x_2]\big)$. (For otherwise, replacing the path $C[u, v]$ (or $C[v, u]$) with uv, we can modify H to increase $|E(H)|$, contradicting to the condition (IV).) Similarly, K has no inner edge $e = uv$ with $u, v \in V\big(\overleftarrow{C}[y_2, x_1]\big)$ but $uv \notin E\big(\overleftarrow{C}[y_2, x_1]\big)$. Let $y' \in V\big(C[y_1, x_2]\big)$ and $t \in V\big(\overleftarrow{C}[y_2, x_1]\big)$ such that $y_1y' \in E\big(C[y_1, x_2]\big)$ and $x_1t \in E\big(\overleftarrow{C}[y_2, x_1]\big)$. Note that y', x_0, t, y_0 appear on C in this order. Since G is a triangulation, either $x_1y' \in E(G) - E(H)$ or $y_1t \in E(G) - E(H)$. Here we suppose that the former occurs and proceed the proof, but even when the latter occurs, the same argument can work. (In the case $y_1t \in E(G) - E(H)$, the latter holds in the condition (G4), while the former holds if $x_1y' \in E(G) - E(H)$. This is the only difference between two cases.)

We take a vertex s in $C[y', x_2)$ so that $sx_1 \in E(G) - E(H)$ and $C[s, x_2]$ is as short as possible. This choice implies that $st \in E(G) - E(H)$. Since H is a disk triangulation and $C[y_1, x_2]$ has no chord, we see that all vertices in $C[y_1, s]$ are adjacent with x_1 in $G - E(H)$. Let $x = x_0 = x_1 (= x_2)$. If G is 5-connected, then we let y be the neighbor of s in $C[y_1, s]$; Otherwise let $y = y_0 = y_1 (= y_2)$. Let $E_x = \{xz : z \in V\big(C(y, s]\big)\}$. This choice imply that the conditions (G1) and (G4) are satisfied.

Then we check the condition (G2). Suppose first that G is not 5-connected. Then it follows from the condition (I) and the fact $x = x_0$ and $y = y_0$ that P is a path from x to y in $H - \big(V(C) - \{x, y\}\big)$. Then combining P, $C[s, x]$ and $\overleftarrow{C}[y, t]$, we obtain a path in H from s to t that passes through x and then y. This shows that the condition (G2) is satisfied.

Suppose next that G is 5-connected. It follows from the condition (II) that H is 3-connected, and hence H has internally disjoint three paths from x to y. This, together with the planarity, directly implies that $H - \big(V(C) - \{x, y\}\big)$ has a path from x to y. Thus, by the same way as in the previous paragraph, we can find a path in H from s to t that passes through x and then y. Thus, the condition (G2) is satisfied also in this case.

Next, we show that the condition (G3) is satisfied. It is easy to see that each edge in $G - E(H) - E_x$ connects a vertex in $C[s, x_2]$ and a vertex in $\overleftarrow{C}[y_2, t]$, which implies the first part of the condition (G3). Let K' be the disk triangulation of G bounded by the cycle $D' = C[s, x_2] \cup x_2y_2 \cup \overleftarrow{C}[y_2, t] \cup ts$. Note that $D = C[s, x_2] \cup x_2y_2 \cup \overleftarrow{C}[y_2, t]$ is a Hamiltonian path in K' connecting s and t, $s, t \in V(D')$, and all edges in K' appear in the same side of the path D or are contained in D. Therefore, it follows from Lemma 2 that no two edges in K' cross at D. Note that $E(G) - E(H) - E_x \subseteq E(K')$. Therefore, since the sequence $C[s, x]\ \overleftarrow{C}[y, t]$ has the same sequence as the path D, we see that no two edges in $E(G) - E(H) - E_x$ cross at $C[s, x]\ \overleftarrow{C}[y, t]$, and hence the condition (G3) also holds.

So, it only remains to show the condition (G5). Suppose that G is not 5-connected, G is 4-connected and there exists a 2-cut $\{u, v\}$ of H. By the condition (III), we have $\{u, v\} \cap \in V\big(C[y, x]\big) \neq \emptyset$, and by the symmetry between u and v, we may assume that $C[u, v] \subseteq C[y, x]$. To show the condition (G5), suppose contrary that $\{u, v\} \cap \in V\big(C[s, x]\big) = \emptyset$, which directly implies $u, v \in V\big(C[y, s]\big)$. It follows from the condition (G1) that all vertices in $C[u, v]$ are neighbors of x in $G - E(H)$. This implies that u, v and x form a cut-set of G of size at most three separating the non-attachments of B from others, which contradicts the 4-connectedness of G. Therefore, B contains no edge in $C[y, s]$, and hence B contains an edge in $C[s, x]$.

This completes the proof of Lemma 9. $\square$

4 5-connected case

Proof of Theorem 1 (i). Let G be a 5-connected graph on the projective plane. We first show that we may assume that G is a triangulation. Suppose that G is not triangulation. Then G has a facial cycle $v_1v_2 \cdots v_kv_1$ for some $k \geq 4$.

If $k \geq 5$, then let $\widetilde{G}$ be the graph obtained from G by adding a new vertex z inside of the face bounded by $v_1v_2 \cdots v_kv_1$ and joining z to all the vertices $v_1, v_2, \ldots, v_k$. Since $k \geq 5$, it is easy to see that $\widetilde{G}$ is also 5-connected. Furthermore, if $\widetilde{G}$ is 3-page embeddable, then so G is. Therefore, we may assume that G does not has a facial cycle of length at least 5. So, assume that $k = 4$. If G has no edge connecting v_1 and v_3, then we can add the edge v_1v_3 through the face, keeping the embedding on the projective plane. Note that the new graph, say $\widetilde{G}$, is 5-connected, and if $\widetilde{G}$ is 3-page embeddable, then so G is. Thus, we may assume that $v_1v_3 \in E(G)$. By symmetry, we may also assume that $v_2v_4 \in E(G)$. These two imply that G contains a copy of K_4 as a subgraph, and is embedded on the projective plane so that it has only three quadrangular faces (one of which is the face bounded by $v_1v_2v_3v_4v_1$). Since G is 5-connected, the regions corresponding

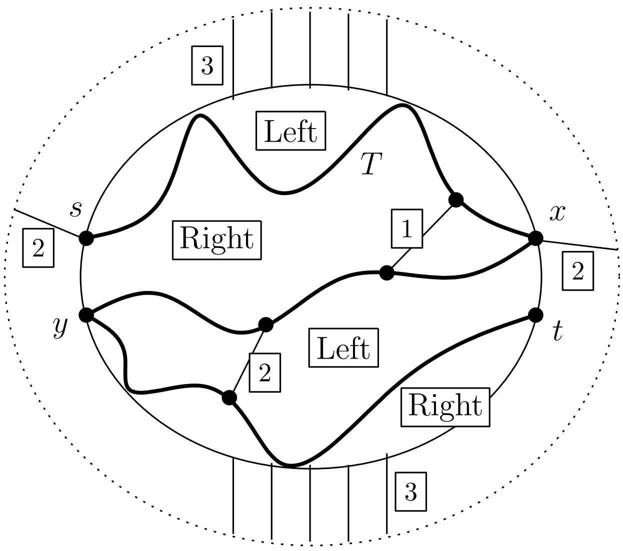

Figure 4: A 3-page book embedding of a 5-connected graph on the projective plane. [2]

those three quadrangular faces cannot contain any vertices in G, which implies that G is itself K_4, a contradiction. Therefore, we may assume that G is a triangulation.

By Lemma 9, G has a spanning disk triangulation H with outer cycle C satisfying the conditions (G1)–(G5).

By condition (G3), H has a path from s to t that passes through x, and then y. Then it follows from Theorem 5 that H has a C-Tutte path T from s to t that passes through x, and then y. Since H is 3-connected and internally 5-connected, T is a Hamiltonian path in H. It follows from the planarity that the vertices $C[s,x]\ \overleftarrow{C}[y,t]$ appear in T in this order. By the symmetry, we may assume that the edges in $C[s,x]$ are contained in T or placed on the left side of T. See Figure 4. Thus, the edges in $\overleftarrow{C}[y,t]$ are contained in T or placed on the right side of T. Let E_L (resp., E_R) be the sets of edges e in $E(H) - E(T)$ such that e is placed in the left side (resp., the right side). Since T is a Hamiltonian path in H, we have $E(H) = E(T) \cup E_L \cup E_R$.

Let $E^1 = E(T) \cup E_R, \qquad E^2 = E_L \cup E_x, \qquad \text{and} \qquad E^3 = E(G) - E(H) - E_x.$

Note that $\mathscr{E} = \{E^1, E^2, E^3\}$ is indeed a partition of $E(G)$. It follows from Lemma 2 and condition (G3) that no two edges in E^1, or in $E^2 - E_x$, or in E^3 cross at T. (Recall that $C[s,x]\overleftarrow{C}[y,t]$ appear in T in that order.) Therefore, since $E_x = \{sx\}$, it suffices to prove that no edge in E_L crosses sx at T.

Let $a_1a_2 \in E_L$. Then we see that either $a_1, a_2 \in V\big(T[s,x]\big)$ or $a_1, a_2 \in V\big(T[x,t]\big)$, which implies that in either case, the edges a_1a_2 and sx do not cross at T by Lemma 3. Therefore, this completes the proof of Theorem 1 (i). $\square$

[2]In all figures in this paper, the number in a square indicates the index of the set E^i containing those edges.

5 4-connected case

5.1 A crucial lemma for 4-connected case

To prove Theorem 1 (ii), we first show the following lemma, which will be used to find a suitable book embedding of inside of each T-bridge.

Lemma 10 *Let H be an internally 4-connected disk triangulation with outer cycle C, and let u, v be two vertices on C with $uv \in E(C)$. We give an orientation to C so that $C[v, u]$ consists of only the edge uv. Then $H - uv$ has a 3-page book embedding $\Sigma = (Q, \mathscr{E})$, where $\mathscr{E} = \{E^1, E^2, E^3\}$, satisfying the following;*

(P1) *u is the first vertex in Q and v is the last vertex,*

(P2) *the vertices on $C[u, v]$ appear in Q in this order, and*

(P3) *all edges connecting $\{u, v\}$ and $V(H) - \{u, v\}$ belong to E^1.*

Proof of Lemma 10. We show Lemma 10 by induction on $|H|$. If $|H| = 3$, then it is easy to see that Lemma 10 holds. Hence we may assume that $|H| \geq 4$. Let e_1 and e_2 be the two edges in C such that e_1 is incident with u and $e_1 \neq uv$, and e_2 is incident with v and $e_2 \neq uv$. If $|C| = 3$, (so C consists of only the three edges uv, e_1 and e_2,) then the internally 4-connectedness of H implies that $V(H) - V(C) = \emptyset$, which contradicts that $|H| \geq 4$. Therefore, we have $|C| \geq 4$.

It follows from Theorem 4 that H has a C-Tutte cycle T' through uv, e_1 and e_2. Let T be the path obtained from T' by deleting the edge uv. By symmetry, we may assume that T has a direction from u to v and the interior of T' is placed on the right side of T. Note that the vertices in $V\big(C[u, v]\big) \cap V(T)$ appear in T in the order of $C[u, v]$ and all the edges in $C[u, v]$ are either contained in T or placed on the left side of T.

Let E_L and E_R be the sets of edges e in $E(H[V(T)]) - E(T)$ such that e is placed on the left side (resp., the right side) of T. Note that $E(H[V(T)]) = E(T) \cup E_L \cup E_R$ and $E(C[u, v]) \subseteq E(T) \cup E_L$. Since $uv, e_1, e_2 \in E(T')$ and the interior of T' is placed on the right side of T, we have the following claim.

Claim 1 *All edges incident with u or v, except for uv, e_1, e_2, are contained in E_R.*

Here we will regard T as a "main part" of a desired sequence Q of $V(H)$, and appropriately insert the vertices in $H - V(T)$. Let

$$\mathcal{B} \;=\; \{B : B \text{ is a non-trivial } T\text{-bridge of } H\}.$$

By the definition of non-trivial T-bridges, every edge in H is either an edge connecting two vertices in T or contained in B for some $B \in \mathcal{B}$, and hence

$$E(H) - E\big(H[V(T)]\big) \;=\; \bigcup_{B \in \mathcal{B}} E(B).$$

Let $B \in \mathcal{B}$. We first claim that B contains an edge in $C[u, v]$ (so, B is placed on the left side) and has exactly two attachments on T. Since T' is a C-Tutte cycle in H, B has at most three attachments on T. So, if B does not contain an edge in $C[u, v]$, then the attachments of B form a cut-set of H of size at most three such that it separates the non-attachments of B from C, which contradicts that H is internally 4-connected. Therefore, B contains an edge in $C[u,v]$. Since T' is a C-Tutte cycle in H, B has at most two attachments on T. Since H is 2-connected, B has exactly two attachments on T. Therefore, the claim holds.

Let u_B and v_B be the two attachments of B on T. Since H is a disk triangulation, we see that $u_Bv_B \in E(H)$ but $u_Bv_B \notin E(B)$. We may assume that u, u_B, v_B and v appear in T in this order. (This choice implies that for any $B_1, B_2 \in \mathcal{B}$ with $B_1 \neq B_2$, we have $u_{B_1} \neq u_{B_2}$.) Note that $B + \{v_Bu_B\}$ is an internally 4-connected disk triangulation with outer cycle $C[u_B, v_B] \cup v_Bu_B$, say C_B. Then by the induction hypothesis to $B + \{v_Bu_B\}$ with u_B, v_B playing the roles of u and v, respectively, B has a 3-page book embedding $(Q_B, \mathscr{E}_B)$, where $\mathscr{E}_B = \{E_B^1, E_B^2, E_B^3\}$, such that (P1) u_B is the first vertex in Q_B and v_B is the last vertex, (P2) the vertices on $C_B[u_B, v_B]$ appear in Q_B in this order, and (P3) all edges connecting $\{u_B, v_B\}$ and $B - \{u_B, v_B\}$ belong to E_B^1.

Now we insert the sequence $Q_B(u_B, v_B)$ to T just after u_B in T. That is, we obtain the new sequence

$$T[u, u_B]\ Q_B(u_B, v_B)\ T(u_B, v].$$

We do the above insertion for all $B \in \mathcal{B}$ independently, and let Q be the obtained sequence of the vertices in H. Note that Q contains all vertices of H, and Q satisfies condition (P1). Since the vertices in $V\big(C[u,v]\big) \cap V(T)$ appear in T in the order of $C[u,v]$, it follows from condition (P2) for each $B \in \mathcal{B}$ and the construction of Q that Q also satisfies condition (P2). Now we partition all edges in $H - \{uv\}$ into three sets as follows; Let

$$E^1 = \big(E_R - \{uv\}\big) \cup \bigcup_{B \in \mathcal{B}} E_B^2, \qquad E^2 = E(T) \cup E_L \cup \bigcup_{B \in \mathcal{B}} E_B^3, \qquad \text{and} \quad E^3 = \bigcup_{B \in \mathcal{B}} E_B^1.$$

See Figure 5. Recall that $E\big(H[V(T)]\big) = E(T) \cup E_L \cup E_R$ and $E(H) - E\big(H[V(T)]\big) = \bigcup_{B \in \mathcal{B}} E(B)$. Since $\{E_B^1, E_B^2, E_B^3\}$ is a partition of $E(B)$, $\{E^1, E^2, E^3\}$ is indeed a partition of $E(H) - \{uv\}$. Furthermore, it follows from Claim 1 that the partition satisfies condition (P3). Thus, it suffices to prove that no two edges in E^i cross at Q for any $1 \leq i \leq 3$. Let $i \in \{1,2,3\}$ and $a_1a_2, b_1b_2 \in E^i$.

Case 1. $i = 1$.

If $a_1a_2, b_1b_2 \in E_R - \{uv\}$, then it follows from Lemma 2 that they do not cross at T, and hence at Q. Therefore, we may assume that a_1a_2 is contained in E_B^2 for some T-bridge B in $\mathcal{B}$. By condition (P3) for B, we see that $\{a_1, a_2\} \cap \{u_B, v_B\} = \emptyset$, and hence $a_1, a_2 \in V\big(Q_B(u_B, v_B)\big)$. Note that $Q_B(u_B, v_B)$ is a subsequence of Q. If $b_1b_2 \in E_B^2$, then since $(Q_B, \mathscr{E}_B)$ is a 3-page book embedding of B, the edges a_1a_2 and b_1b_2 do not cross at Q_B, and hence at Q; Otherwise it follows from the construction of Q that $b_1, b_2 \in V(Q) - V\big(Q_B(u_B, v_B)\big)$, and hence it follows from Lemma 3 that a_1a_2 and b_1b_2 do not cross at Q, neither. In either case, we see that a_1a_2 and b_1b_2 do not cross at Q, and we are done.

Case 2. $i = 2$.

This case can be proven by the same way as Case 1. If $a_1a_2, b_1b_2 \in E(T) \cup E_L$, then it follows from Lemma 2 that they do not cross at T, and hence at Q. Therefore, we may assume that a_1a_2 is contained in E_B^3 for some T-bridge B in $\mathcal{B}$. By condition (P3) for B, we see that

[4] In all remaining figures in this paper, each region with diagonals from the right top to the left bottom represents a non-trivial T-bridge.

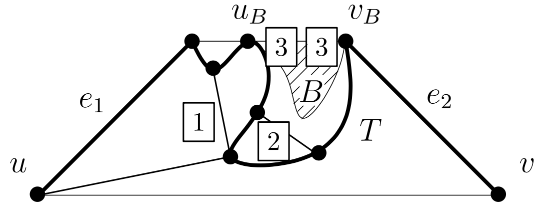

Figure 5: A 3-page book embedding of an internally 4-connected disk triangulation. [4]

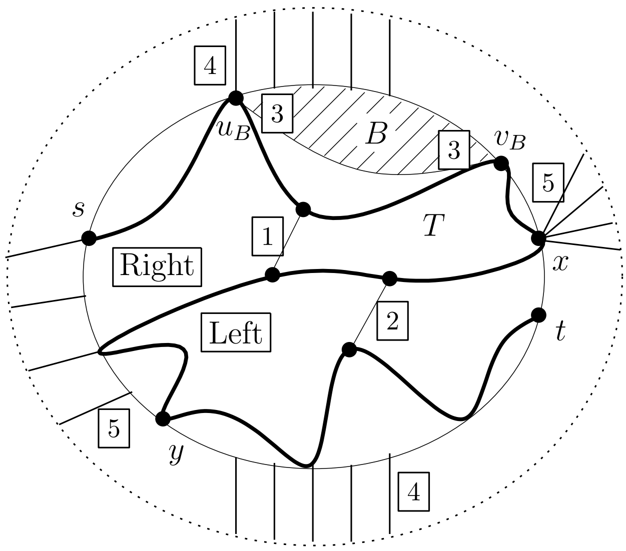

Figure 6: A 5-page book embedding of a 4-connected graph on the projective plane.

$\{a_1, a_2\} \cap \{u_B, v_B\} = \emptyset$, and hence $a_1, a_2 \in V\big(Q_B(u_B, v_B)\big)$. If $b_1 b_2 \in E_B^3$, then since $(Q_B, \mathscr{E}_B)$ is a 3-page book embedding of B, the edges a_1a_2 and b_1b_2 do not cross at Q_B, and hence at Q; Otherwise by the construction of Q, we have $b_1, b_2 \in V(Q) - V\big(Q_B(u_B, v_B)\big)$, and hence it follows from Lemma 3 that a_1a_2 and b_1b_2 do not cross at Q, neither. In either case, we see that a_1a_2 and b_1b_2 do not cross at Q, and we are done.

Case 3. $i = 3$.

Note that a_1a_2 is contained in E_B^1 for some T-bridge B in $\mathfrak{B}$. If $b_1b_2 \in E_B^1$, then since $(Q_B, \mathscr{E}_B)$ is a 3-page book embedding of B, the edges a_1a_2 and b_1b_2 do not cross at Q_B, and hence at Q. Thus, we may assume that $b_1b_2 \notin E_B^1$, and hence it follows from the construction of Q that $b_1, b_2 \in V(Q) - V\big(Q_B(u_B, v_B)\big)$. So, if $a_1 \neq v_B$ and $a_2 \neq v_B$, then $a_1, a_2 \in V\big(Q_B[u_B, v_B)\big)$, and hence it follows from Lemma 3 that a_1a_2 and b_1b_2 do not cross at Q. Therefore, by the symmetry between a_1 and a_2, we may assume that $a_2 = v_B$. By the symmetry between a_1a_2 and b_1b_2, we may also assume that $b_1b_2 \in E_{B'}^1$ and $b_2 = v_{B'}$ for some $B' \in \mathfrak{B}$ with $B' \neq B$. Note that both of the edges u_Bv_B and $u_{B'}v_{B'}$ are contained in $E(T) \cup E_L$, and hence it follows from Lemma 2 that the vertices $u_B, v_B, u_{B'}, v_{B'}$ appear in T in this order or in the order of $u_{B'}, v_{B'}, u_B, v_B$. Note that it follows from construction of Q that the vertices in $Q_B(u_B, v_B)$ (resp., the vertices in $Q_{B'}(u_{B'}, v_{B'})$) appear just after u_B (resp., $u_{B'}$). Since $a_1 \in V\big(Q_B(u_B, v_B)\big)$ and $b_1 \in V\big(Q_{B'}(u_{B'}, v_{B'})\big)$, we see that the edges a_1a_2 and b_1b_2 do not cross at Q. This completes the proof of Case 3, and the proof of Lemma 10. $\square$

5.2 Proof of Theorem 1 (ii)

Let G be a 4-connected graph on the projective plane. To find a book embedding of G, we may assume that G is a triangulation. (If G is not a triangulation, then we can get a 4-connected triangulation $\widetilde{G}$ from G by adding vertices and edges suitably. It is easy to see that if $\widetilde{G}$ is 4-page embeddable, then so G is.) By Lemma 9, G has a spanning disk triangulation H with outer cycle

C satisfying conditions (G1)–(G5).

By condition (G2), H has a path from s to t that passes through x, and then y. Then it follows from Theorem 5 that H has a C-Tutte path T from s to t that passes through x, and then y. By the symmetry, we may assume that the edges in $C[s,x]$ are contained in T or placed on the left side of T. Thus, the edges in $\overleftarrow{C}[s,t]$ are contained in T or placed on the right side of T. Let E_L (resp., E_R) be the sets of edges e in $E(H)-E(T)$ such that e is placed in the left side (resp., the right side), respectively. Note that $E(H[V(T)]) = E(T) \cup E_L \cup E_R$.

$$\text{Let} \quad \mathcal{B} \;=\; \{B : B \text{ is a non-trivial } T\text{-bridge of } H\}.$$

By the definition of non-trivial T-bridges, any edge in H is either an edge connecting two vertices in T or contained in B for some $B \in \mathcal{B}$, and hence

$$E(H) - E\big(H[V(T)]\big) \;=\; \bigcup_{B\in\mathcal{B}} E(B).$$

We first claim that either B contains an edge in $C[s,x]$ for any $B \in \mathcal{B}$, or B contains an edge in $\overleftarrow{C}[y,t]$ for any $B \in \mathcal{B}$. Since T is a C-Tutte path in H, B has at most three attachments on T. So, if B does not contain an edge in C, then the attachments of B form a cut-set of G of size at most three such that it separates the non-attachments of B from others, which contradicts the 4-connectedness of G. Therefore, B contains an edge in C. Furthermore, since T is a C-Tutte path in H, B has at most two attachments on T. Since G is 2-connected, B has exactly two attachments on T. It follows from the condition (G5) that B contains an edge in $C[s,x]$ or in $\overleftarrow{C}[y,t]$. Therefore, the claim holds.

Since two cases are symmetric, we may assume that the former holds, that is, B contains an edge in $C[s,x]$ for any $B \in \mathcal{B}$. This implies that any $B \in \mathcal{B}$ has exactly two attachments on $T[s,x]$.

Let u_B and v_B be the two attachments of B. Since H is a disk triangulation, we see that $u_Bv_B \in E(H)$ but $u_Bv_B \notin E(B)$. By the symmetry, we may assume that s, u_B, v_B and x appear in T in this order (Possibly $s = u_B$ and/or $v_B = x$). This choice implies that $C[u_B,v_B] \subseteq C[s,x]$, and for any $B_1, B_2 \in \mathcal{B}$ with $B_1 \neq B_2$, we have $u_{B_1} \neq u_{B_2}$. Note that $B + \{v_Bu_B\}$ is an internally 4-connected disk triangulation with outer cycle $C[u_B,v_B] \cup v_Bu_B$, say C_B. Then it follows from Lemma 10 that B has a 3-page book embedding $(Q_B, \mathscr{E}_B)$, where $\mathscr{E}_B = \{E_B^1, E_B^2, E_B^3\}$ such that (P1) u_B is the first vertex in Q_B and v_B is the last vertex, (P2) the vertices on $C_B[u_B,v_B]$ appear in Q_B in this order, and (P3) all edges connecting $\{u_B,v_B\}$ and $B - \{u_B,v_B\}$ belong to E_B^1.

Now we insert the sequence $Q_B(u_B,v_B)$ to T just after u_B in T. That is, we obtain the new sequence

$$T[s,u_B]\; Q_B(u_B,v_B)\; T(u_B,t].$$

We do the above insertion for all $B \in \mathcal{B}$ independently, and let Q be the obtained sequence of the vertices in G. Note that Q contains all vertices of G. Since the vertices $C[s,x] \cap V(T)$ appear in T in the order, it follows from condition (P2) for each $B \in \mathcal{B}$ and the construction of Q that Q contains the vertices in $C[s,x]\; \overleftarrow{C}[y,t]$ in this order.

Let

$$E^1 = E_R \cup \bigcup_{B \in \mathcal{B}} E^2_B, \qquad E^2 = E(T) \cup E_L \cup \bigcup_{B \in \mathcal{B}} E^3_B,$$

$$E^3 = \bigcup_{B \in \mathcal{B}} E^1_B, \qquad E^4 = E(G) - E(H) - E_x. \quad \text{and} \quad E^5 = E_x.$$

See Figure 6. Recall that $E(H[V(T)]) = E(T) \cup E_L \cup E_R$ and $E(H) - E(H[V(T)]) = \bigcup_{B \in \mathcal{B}} E(B)$. Thus, since $\{E^1_B, E^2_B, E^3_B\}$ is a partition of $E(B)$, $\{E^1, E^2, E^3\}$ is indeed a partition of $E(G)$. So, it suffices to prove that no two edges in E^i cross at Q for any $1 \le i \le 5$. Indeed, we can prove this by the same way as Case 1–3 in the proof of Lemma 10 (for $1 \le i \le 3$), by the condition (G3) (for $i = 4$), and by the fact that all edges in E_x share x as an end vertex (for $i = 5$). This completes the proof Theorem 1 (ii). $\Box$

6 The case without connectivity assumption

6.1 A crucial lemma

To show Theorem 1 (iii), we need the following lemma. (See Figure 7.)

Lemma 11 *Let H be a plane triangulation such that $|H| \ge 4$, let uvw be the outer cycle of H, and let C be the outer cycle of $H - w$. We give an orientation to C so that $C[v, u]$ consists of only the edge uv. Then $H - \{uv, vw, wu\}$ has a 6-page book embedding $\Sigma = (Q, \mathscr{E})$, where $\mathscr{E} = \{E^1, \ldots, E^6\}$, satisfying the following;*

(Q1) *u is the first vertex in Q, v is the second last vertex, and w is the last vertex,*

(Q2) *the vertices on $C[u, v]$ appear in Q in this order,*

(Q3) *all edges connecting u and $V(H) - \{u, v, w\}$ belong to E^1,*

(Q4) *all edges connecting v and $V(H) - \{u, v, w\}$ belong to E^2, and*

(Q5) *all edges connecting w and $V(H) - \{u, v, w\}$ belong to E^3.*

Proof of Lemma 11. We show Lemma 11 by induction on $|H|$. If $|H| = 4$, then it is easy to see that Lemma 11 holds. Hence we may assume that $|H| \ge 5$. Let $H' = H - w$. Since any plane triangulation is 3-connected, H' is 2-connected.

Note that C is the outer cycle of H' with $uv \in E(C)$. Let e_1 and e_2 be the two edges in C such that e_1 is incident with u and $e_1 \neq uv$, and e_2 is incident with v and $e_2 \neq uv$. It follows from Theorem 4 that H' has a C-Tutte cycle T' through uv, e_1 and e_2. By the symmetry, we may assume that the interior of T' is placed on the right side of T'. (See Figure 8.)

We may assume that $T' - \{uv\}$ is a sequence starting from u and ending at v, and let T be the sequence obtained from $T' - \{uv\}$ by adding the vertex w in the last. Note that T satisfies the condition (Q1), and we see that the vertices in $V(C[u, v]) \cap V(T')$ appear in T in the order of $C[u, v]$. Since $uv, e_1, e_2 \in E(T')$ and the interior of T' is placed on the right side of T', the following claim holds.

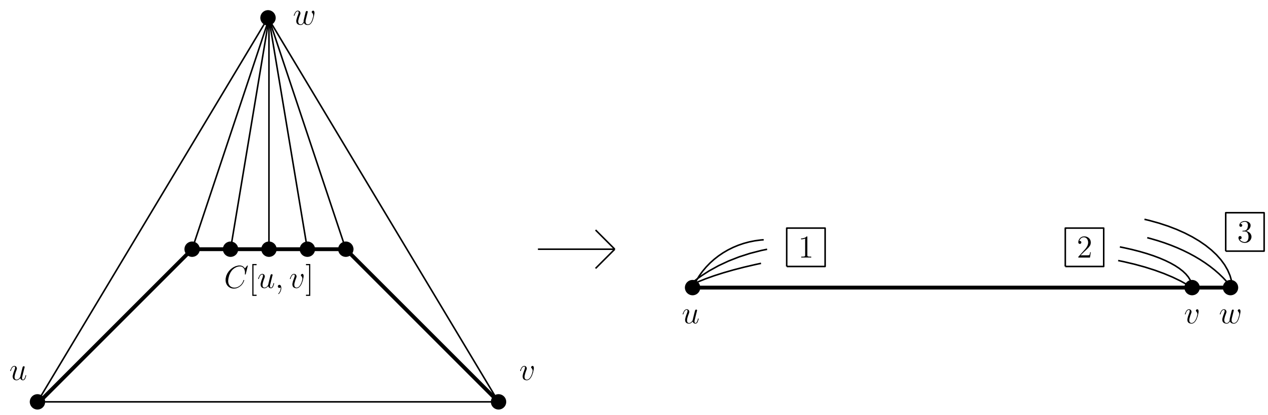

Figure 7: A 6-page book embedding desired in Theorem 11.

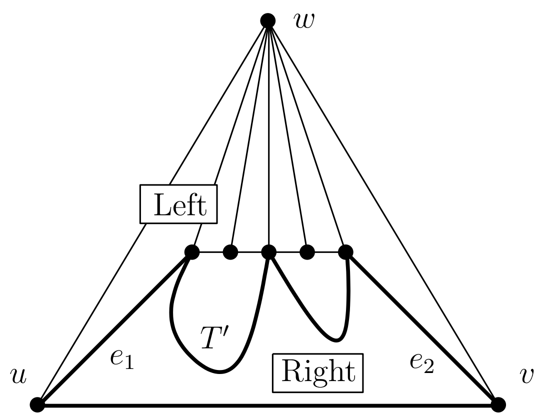

Figure 8: A C-Tutte cycle T' through uv, e_1 and e_2.

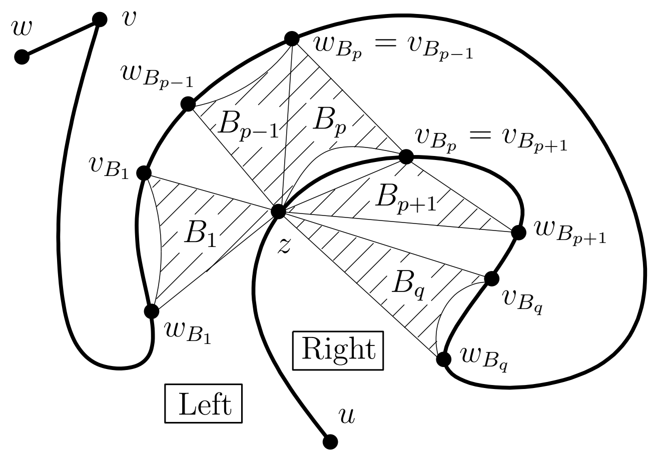

Figure 9: The non-trivial T-bridges $B_1, B_2, \ldots, B_p$ in $\mathcal{B}_z^L$ and $B_{p+1}, \ldots, B_q$ in $\mathcal{B}_z^R$.

Claim 2 *All edges incident with u or v, except for uv, uw, wv, e_1, e_2, are placed on the right side of T', and all edges incident with w are placed on the left side of T'.*

Let E_0^1, E_0^2, E_0^3 and E_0^5 be the sets of edges e in $E(H[V(T)]) - \{uv, vw, wu\}$ such that

- if e is incident with v, then $e \in E_0^2$,
- if e is incident with w, then $e \in E_0^3$,
- if e is placed on the right side of T or a chord of T and e is not incident with v, then $e \in E_0^1$, and
- if e is placed on the left side of T and e is not incident with w, then $e \in E_0^5$,

respectively. Note that the set $\{E_0^1, E_0^2, E_0^3, E_0^5\}$ is indeed a partition of $E(H[V(T)]) - E(T) - \{uv, vw, wu\}$. By Claim 2, all edges incident with u are contained in E_0^1, except for uv.

Here we will regard T as a "main part" of a desired sequence Q of $V(H)$, and appropriately insert the vertices in H but not in T. To do that, we need some definitions. Let

$$\mathcal{B} \;=\; \{B : B \text{ is a non-trivial } T\text{-bridge of } H\}.$$

By the definition of non-trivial T-bridges, each edge in H is either an edge connecting two vertices in T or contained in B for some $B \in \mathcal{B}$, and hence

$$E(H) - E\big(H[V(T)]\big) \;=\; \bigcup_{B \in \mathcal{B}} E(B).$$

Let $B \in \mathcal{B}$. Note that B is a T-bridge of H. We first claim that B has exactly three attachments on T. Since H is 3-connected, B has at least three attachments on T. So, if w is not an attachment of B on T, then B has exactly three attachments on T', and hence on T. (Recall that T' is a C-Tutte cycle in H'.) So, suppose that w is an attachment of B on T. Then $B - w$ is a T'-bridge of H' and contains an edge in C. Again since T' is a C-Tutte cycle in H', B has exactly two attachments on T', and hence B has exactly three attachments on T' one of which is w. Therefore, the claim holds.

Let u_B, v_B and w_B be the three attachments of B. Since H is a disk triangulation, we see that $u_Bv_B, v_Bw_B, w_Bu_B \in E(H)$ but $u_Bv_B, v_Bw_B, w_Bu_B \notin E(B)$. We may assume that u_B, v_B and w_B appear in T in this order. This condition and the places of u, v and w in T directly imply that $u_B \neq v, w$, $v_B \neq u, w$ and $w_B \neq u$. Furthermore, if v is an attachment of B, then it follows from Claim 2 that B is placed on the right side, which implies that w is not an attachment of B. Therefore, we also have $v_B \neq v$. This argument implies the following claim.

Claim 3 (1) *If u is an attachment of a non-trivial T-bridge B in $\mathcal{B}$, then $u = u_B$.*

(2) *If v is an attachment of a non-trivial T-bridge B in $\mathcal{B}$, then $v = w_B$.*

(3) *If w is an attachment of a non-trivial T-bridge B in $\mathcal{B}$, then $w = w_B$.*

For $B \in \mathcal{B}$, note that $B + \{u_B v_B, v_B w_B, w_B u_B\}$ is a plane triangulation with outer cycle $u_B v_B w_B$. Let C_B be the outer cycle of $B + \{u_B v_B\} - w$ such that $C_B[v_B, u_B]$ consists of only the edge $u_B v_B$. By the induction hypothesis to $B + \{u_B v_B, v_B w_B, w_B u_B\}$ with u_B, v_B and w_B playing the roles of u, v and w, respectively, B has a 6-page book embedding $(Q_B, \mathscr{E}_B)$, where $\mathscr{E}_B = \{E_B^1, \ldots, E_B^6\}$, such that (Q1) u_B is the first vertex in Q_B, v_B is the second last vertex, and w_B is the last vertex, (Q2) the vertices on $C_B[u_B, v_B]$ appear in Q in this order, (Q3) all edges connecting u_B and $V(B) - \{u_B, v_B, w_B\}$ belong to E_B^1, (Q4) all edges connecting v_B and $V(B) - \{u_B, v_B, w_B\}$ belong to E_B^2, and (Q5) all edges connecting w_B and $V(B) - \{u_B, v_B, w_B\}$ belong to E_B^3.

Here, we say that a non-trivial T-bridge B in $\mathcal{B}$ is a *corner* if $w_B = w$. It is easy to see that B is a corner if and only if $B - \{u_B, v_B, w_B\}$ contains a vertex in $C[u, v]$. Furthermore, the planarity directly implies the following;

Claim 4 *For each corner T-bridge B in $\mathcal{B}$, we have $C_B[u_B, v_B] = C \cap B$, and $u_B \neq u_{B'}$ for any corner T-bridges B' in $\mathcal{B}$ with $B' \neq B$.*

Let $z \in V(T)$. Define $\mathcal{B}_z^L$ (resp., $\mathcal{B}_z^R$) as the set of non-trivial T-bridges B in $\mathcal{B}$ such that $u_B = z$ and B is placed on the left side (resp., the right side) of T'. Let $B_1, \ldots, B_p$ be the elements in $\mathcal{B}_z^L$ along the clockwise order around z, where $p = |\mathcal{B}_z^L|$. (See Figure 9.) It follows from Claim 4 and the planarity that if there exists a corner T-bridge B in $\mathcal{B}$ with $u_B = z$, then $B = B_1$, and furthermore,

$$\text{the vertices } u, z, v_{B_p}, w_{B_p}, v_{B_{p-1}}, \ldots, v_{B_1}, w_{B_1}, w \text{ appear in } T \text{ in this order.} \tag{1}$$

(Possibly, $w_{B_p} = v_{B_{p-1}}$, and/or $\cdots$, and/or $w_{B_1} = w$.) Similarly, let $B_{p+1}, \ldots, B_q$ be the elements in $\mathcal{B}_z^R$ along the clockwise order around z, where $q = p + |\mathcal{B}_z^R|$. (See Figure 9.) Again, it follows from the planarity that

$$\text{the vertices } u, z, v_{B_{p+1}}, w_{B_{p+1}}, v_{B_{p+2}}, \ldots, v_{B_q}, w_{B_q}, v, w \text{ appear in } T \text{ in this order.} \tag{2}$$

(Possibly, $u = z$, and/or $w_{B_{p+1}} = v_{B_{p+2}}$ and/or $\cdots$, and/or $w_{B_q} = v$.) Now we insert the sequence

$$Q_{B_1}(z, v_{B_1})\ Q_{B_2}(z, v_{B_2})\ \cdots\ Q_{B_p}(z, v_{B_p})\ Q_{B_q}(z, v_{B_q})\ Q_{B_{q-1}}(z, v_{B_{q-1}})\ \cdots\ Q_{B_{p+1}}(z, v_{B_{p+1}})$$

to T just after z in T. That is, we obtain the new sequence

$$T[u, z]\ Q_{B_1}(z, v_{B_1})\ \cdots\ Q_{B_p}(z, v_{B_p})\ Q_{B_q}(z, v_{B_q})\ \cdots\ Q_{B_{p+1}}(z, v_{B_{p+1}})\ T(z, w].$$

We do the above insertion for all $z \in V(T)$ independently, and let Q be the obtained sequence of the vertices in H. By Claims 3 (2) and (3), $v, w \neq u_B$ for any $B \in \mathcal{B}$, and hence no vertices are inserted after v. Therefore, Q satisfies the condition (Q1). Furthermore, it follows from Claim 4 and the condition (Q2) for Q_B that Q satisfies the condition (Q2). Now we will partition all edges in $H - \{uv, vw, wu\}$ into six sets so that no two edges in a same set cross at Q. To do that, we first partite all edges in $\bigcup_{B \in \mathcal{B}} B$ into six sets $E_1^1, \ldots, E_1^6$ as follows; Let

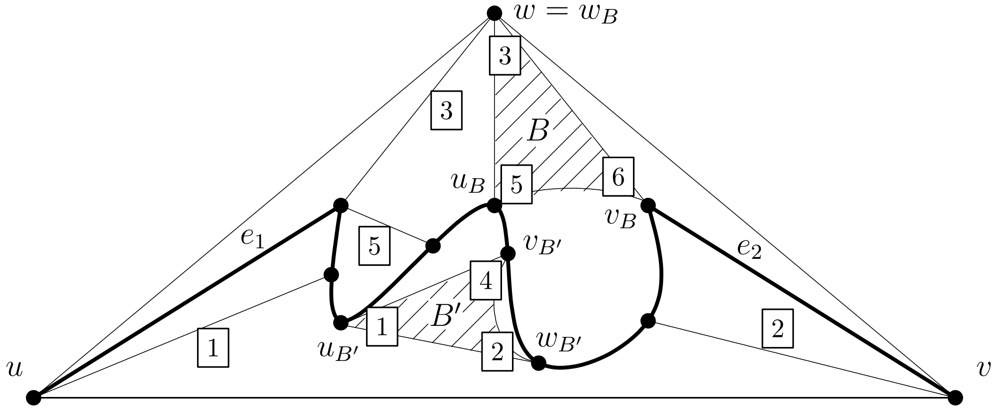

Figure 10: A 6-page book embedding of a plane triangulation. The thick curve represents the path T in $H - w$ from u to v, and $B, B' \in \mathcal{B}$.

$$E_1^1 = \bigcup_{z \in V(T)} \Big(\bigcup_{B \in \mathcal{B}_z^L} E_B^4 \cup \bigcup_{B \in \mathcal{B}_z^R} E_B^1 \Big), \qquad E_1^2 = \bigcup_{z \in V(T)} \Big(\bigcup_{B \in \mathcal{B}_z^L} E_B^5 \cup \bigcup_{B \in \mathcal{B}_z^R} E_B^3 \Big),$$

$$E_1^3 = \bigcup_{z \in V(T)} \Big(\bigcup_{B \in \mathcal{B}_z^L} E_B^3 \cup \bigcup_{B \in \mathcal{B}_z^R} E_B^4 \Big), \qquad E_1^4 = \bigcup_{z \in V(T)} \Big(\bigcup_{B \in \mathcal{B}_z^L} E_B^6 \cup \bigcup_{B \in \mathcal{B}_z^R} E_B^2 \Big),$$

$$E_1^5 = \bigcup_{z \in V(T)} \Big(\bigcup_{B \in \mathcal{B}_z^L} E_B^1 \cup \bigcup_{B \in \mathcal{B}_z^R} E_B^5 \Big), \quad \text{and} \quad E_1^6 = \bigcup_{z \in V(T)} \Big(\bigcup_{B \in \mathcal{B}_z^L} E_B^2 \cup \bigcup_{B \in \mathcal{B}_z^R} E_B^6 \Big).$$

See Figure 10. Since $\mathscr{E}_B = \{E_B^1, \ldots, E_B^6\}$ is a partition of $E(B)$ for any $B \in \mathcal{B}$, we see that $\{E_1^1, \ldots, E_1^6\}$ is a partition of $\bigcup_{B \in \mathcal{B}} E(B) = E(H) - E\big(H[V(T)]\big)$. For those sets $E_1^1, \ldots, E_1^6$, we show the following claim.

Claim 5 *For any integer i with $1 \le i \le 6$, any two edges in E_1^i do not cross at Q.*

Proof. Let $i \in \{1,2,3,4,5,6\}$ and let a_1a_2 and b_1b_2 be two edges in E_1^i. By the definition of E_1^i, the edge a_1a_2 is contained in some T-bridge B in $\mathcal{B}$. If $b_1b_2 \in E(B)$, then since the pair $(Q_B, \mathscr{E}_B)$ is a 6-page book embedding of B and $a_1a_2, b_1b_2 \in E_B^j \in \mathscr{E}_B$ for some $1 \le j \le 6$, we see that a_1a_2 and b_1b_2 do not cross at Q_B. Since the vertices in Q_B appear in Q in the same order, a_1a_2 and b_1b_2 do not cross at Q, neither. Thus, we may assume that $b_1b_2 \notin E(B)$, which implies that $b_1b_2 \in E(B')$ for some $B' \in \mathcal{B}$ with $B' \ne B$. In particular, since the vertices in $Q_B(u_B, v_B)$ appear in Q consecutively, we have $b_1, b_2 \in V(Q) - V\big(Q_B(u_B, v_B)\big)$. Thus, if $\{a_1, a_2\} \cap \{u_B, v_B, w_B\} = \emptyset$, then $a_1, a_2 \in V\big(Q_B(u_B, v_B)\big)$, and hence it follows from Lemma 3 that a_1a_2 and b_1b_2 do not cross at Q. Therefore, we may assume that $\{a_1, a_2\} \cap \{u_B, v_B, w_B\} \ne \emptyset$. By the symmetry between a_1a_2 and b_1b_2, we may also assume that $\{b_1, b_2\} \cap \{u_{B'}, v_{B'}, w_{B'}\} \ne \emptyset$.

Case 1. $a_1 = u_B$ or $a_2 = u_B$ or $b_1 = u_{B'}$ or $b_2 = u_{B'}$.

Say $a_1 = u_B$ by symmetry. It follows from the condition (Q3) for B that $a_1a_2 \in E_B^1$. Then by the definition of $E_1^1, \ldots, E_1^6$, we have $a_1a_2 \in E_1^1 \cup E_1^5$, and hence $i = 1, 5$ and $b_1b_2 \in E_1^1 \cup E_1^5$. Since $\{b_1, b_2\} \cap \{u_{B'}, v_{B'}, w_{B'}\} \ne \emptyset$, it follows from the conditions (Q3)–(Q5) for B' and the definitions of E_1^1 and E_1^5 that $b_1b_2 \in E_{B'}^1$, and either $b_1 = u_{B'}$ or $b_2 = u_{B'}$. By the symmetry, we may assume that $b_1 = u_{B'}$. If $u_B = u_{B'}$, then the edges a_1a_2 and b_1b_2 share the end vertex u_B, and hence they do not cross at Q. Therefore, we may assume that $u_B \neq u_{B'}$. Then it follows from the construction of Q that $b_1, b_2 \notin V\big(Q[u_B, u_B^+]\big)$, where u_B^+ is the successor of u_B at T. Since $a_1, a_2 \in V\big(Q[u_B, u_B^+]\big)$, it follows from Lemma 3 that a_1a_2 and b_1b_2 do not cross at Q. This completes the proof of Case 1.

Case 2. $a_1, a_2 \neq u_B$ and $b_1, b_2 \neq u_{B'}$.

By the condition of this case, we have $\{a_1, a_2\} \cap \{v_B, w_B\} \neq \emptyset$ and $\{b_1, b_2\} \cap \{v_{B'}, w_{B'}\} \neq \emptyset$. By the symmetry between a_1 and a_2, we may assume that $a_2 = v_B$ or $a_2 = w_B$. Then it follows from the conditions (Q3)–(Q5) for B that $a_1 \neq u_B$ and exactly one of the following holds;

(i) $i = 2$, $a_1a_2 \in E_B^3 \subseteq E_1^2$, $a_2 = w_B$ and $B \in \mathcal{B}_z^R$ for some $z \in V(T)$.

(ii) $i = 3$, $a_1a_2 \in E_B^3 \subseteq E_1^3$, $a_2 = w_B$ and $B \in \mathcal{B}_z^L$ for some $z \in V(T)$.

(iii) $i = 4$, $a_1a_2 \in E_B^2 \subseteq E_1^4$, $a_2 = v_B$ and $B \in \mathcal{B}_z^R$ for some $z \in V(T)$.

(iv) $i = 6$, $a_1a_2 \in E_B^2 \subseteq E_1^6$, $a_2 = v_B$ and $B \in \mathcal{B}_z^L$ for some $z \in V(T)$.

We here only prove the cases (i) and (ii), since the cases (iii) and (iv) can be shown in the same way as the cases (i) and (ii), respectively. (Indeed, the only difference of those proofs are the vertices v_B and w_B for a_2 and b_2. If we replace the vertex w_B with v_B and replace $w_{B'}$ with $v_{B'}$ in the following arguments, then we obtain the proofs of the cases (iii) and (iv), respectively.)

Case (i).

By the condition that $i = 2$ and the symmetry between a_1a_2 and b_1b_2, we may assume that $b_1 \neq u_{B'}$ and $b_2 = w_{B'}$ for some $B' \in \mathcal{B}_{z'}^R$ and some $z' \in V(T)$ with $B' \neq B$.

Suppose that $z \neq z'$. By the symmetry between a_1a_2 and b_1b_2, we may assume that z is closer to u in T than z'. (That is, the vertices u, z and z' appear in T in this order, possibly $u = z$.) Since both B and B' are placed on the right side of T, it follows from Lemma 2 that the edges zw_B and $z'w_{B'}$ do not cross at T. This implies that the vertices $z, w_B, z', w_{B'}$ appear in T in this order or in the order $z, z', w_{B'}, w_B$. Then it follows from the construction of Q that $z, Q_B(z, v_B), w_B, z', Q_{B'}(z', v_{B'}), w_{B'}$ appear in Q in this order or in the order $z, Q_B(z, v_B), z', Q_{B'}(z', v_{B'}), w_{B'}, w_B$. Note that $a_1 \in V\big(Q_B(z, v_B)\big)$, $b_1 \in V\big(Q_{B'}(z', v_{B'})\big)$, $a_2 = w_B$, and $b_2 = w_{B'}$. Therefore, the vertices a_1, a_2, b_1, b_2 appear in Q in this order or in the order a_1, b_1, b_2, a_2, respectively. In either case, the edges a_1a_2 and b_1b_2 do not cross at Q.

Therefore, the case $z = z'$ only remains. By the symmetry between a_1a_2 and b_1b_2, we may assume that $B = B_i$ and $B' = B_j$ for some $p + 1 \leq i < j \leq q$, where $\mathcal{B}_z^R = \big\{B_{p+1}, \ldots, B_q\big\}$. It follows from (2) that the vertices z, w_{B_i}, w_{B_j} appear in T in this order. This, together with the construction of Q, implies that $z, Q_{B_j}(z, v_{B_j}), Q_{B_i}(z, v_{B_i}), w_{B_i}, w_{B_j}$ appear in Q in this order. Note that $b_1 \in V\big(Q_{B_j}(z, v_{B_j})\big)$, $a_1 \in V\big(Q_{B_i}(z, v_{B_i})\big)$, $a_2 = w_{B_i}$, and $b_2 = w_{B_j}$. Therefore, the vertices b_1, a_1, a_2, b_2 appear in Q in this order, which implies that the edges a_1a_2 and b_1b_2 do not cross at Q. This completes the proof of the case (i). □

Case (ii).

The proof of the case (ii) is similar to the proof of the case (i). By the symmetry between a_1a_2 and b_1b_2, we may also assume that $b_1 \neq u_{B'}$ and $b_2 = w_{B'}$ for some $B' \in \mathcal{B}_{z'}^L$ and some $z' \in V(T)$ with $B' \neq B$.

If $z \neq z'$, then by the same way as in the proof of the case (i), we see that the edges a_1a_2 and b_1b_2 do not cross at Q. Therefore, we may assume that $z = z'$. By the symmetry between a_1a_2 and b_1b_2, we may assume that $B = B_i$ and $B' = B_j$ for some $1 \leq i < j \leq p$, where $\mathcal{B}_z^L = \{B_1, \dots, B_p\}$. It follows from (1) that the vertices z, w_{B_j}, w_{B_i} appear in T in this order. This, together with the construction of Q, implies that $z, Q_{B_i}(z, v_{B_i}), Q_{B_j}(z, v_{B_j}), w_{B_j}, w_{B_i}$ appear in Q in this order. Note that $a_1 \in V\big(Q_{B_i}(z, v_{B_i})\big)$, $b_1 \in V\big(Q_{B_j}(z, v_{B_j})\big)$. $a_2 = w_{B_i}$, and $b_2 = w_{B_j}$. Therefore, the vertices a_1, b_1, b_2, a_2 appear in Q in this order, which implies that the edges a_1a_2 and b_1b_2 do not cross at Q. This completes the proof of the case (ii), and the proof of Claim 5. $\square$

Now we partition all edges in $H - \{uv, vw, wu\}$ into six sets $E^1, \dots, E^6$ such that no two edges in E^i cross at Q for any i with $1 \leq i \leq 6$ and they satisfy the conditions (Q3)–(Q5), which will complete the proof of Lemma 11.

$$\text{Let} \qquad E^i = \begin{cases} E_1^i \cup E_0^i, & \text{for } i = 1, 2, 3, 5, \\ E_1^i, & \text{for } i = 4, 6. \end{cases}$$

Note that $\{E_1^1, \dots, E_1^6\}$ are indeed a partition of $E(G)$. We first check that this partition satisfies the conditions (Q3)–(Q5).

Let e be an edge connecting u and $V(H)-\{u, v, w\}$. It follows from Claim 2 that e is contained in T or placed on the right side of T. If e is an edge in T or a chord of T, then $e \in E_0^1$; Otherwise, e is contained in some non-trivial T-bridge B in $\mathcal{B}_z^R$ for some $z \in V(T)$. It follows from Claim 3 (1) and the condition (Q3) for B that $e \in E_B^1$ and hence by the definition, we have $e \in E_1^1$. In either case, we have $e \in E^1$, and hence the condition (Q3) is satisfied.

Let e be an edge connecting v and $V(H)-\{u, v, w\}$. It follows from Claim 2 that e is contained in T or placed on the right side of T. If e is an edge in T or a chord of T, then $e \in E_0^2$; Otherwise, e is contained in some non-trivial T-bridge B in $\mathcal{B}_z^R$ for some $z \in V(T)$. It follows from Claim 3 (2) and the condition (Q5) for B that $e \in E_B^3$, and hence by the definition, we have $e \in E_1^2$. In either case, we have $e \in E^2$, and hence the condition (Q4) is satisfied.

Let e be an edge connecting w and $V(H) - \{u, v, w\}$. It follows from Claim 2 that e is placed on the left side of T. If e is a chord of T, then $e \in E_0^3$; Otherwise, e is contained in some non-trivial T-bridge B in $\mathcal{B}_z^L$ for some $z \in V(T)$. It follows from Claim 3 (3) and the condition (Q5) for B that $e \in E_B^3$, and hence by the definition, we have $e \in E_1^3$. In either case, we have $e \in E^3$, and hence the condition (Q5) is satisfied.

Therefore, it suffices to prove that no two edges in E^i cross at Q for any $1 \leq i \leq 6$. Let $i \in \{1, 2, 3, 4, 5, 6\}$ and let $a_1a_2, b_1b_2 \in E^i$. For $i = 4$ or 6, it follows from Claim 5 that the edges a_1a_2 and b_1b_2 do not cross at Q. So, we may assume that $i = 1, 2, 3$ or 5. By Claim 5, and the symmetry between a_1a_2 and b_1b_2, we may further assume that $a_1a_2 \in E_1^i$ and $b_1b_2 \in E_0^i$. In particular, a_1a_2 is contained in some non-trivial T-bridge B in $\mathcal{B}$. Since no vertices in T appear in $Q(u_B, u_B^+)$, where u_B^+ is the successor of u_B at T, we have $b_1, b_2 \notin V\big(Q(u_B, u_B^+)\big)$. So, if $\{a_1, a_2\} \cap \{v_B, w_B\} = \emptyset$, then $a_1, a_2 \in \{u_B\} \cup V\big(Q_B(u_B, v_B)\big) \subseteq V\big(Q[u_B, u_B^+)\big)$, and hence it follows from Lemma 3 that a_1a_2 and b_1b_2 do not cross at Q. Therefore, by the symmetry between a_1 and a_2, we may assume that $a_2 = v_B$ or $a_2 = w_B$. Since $i = 1, 2, 3, 5$, it follows from the conditions (Q3)–(Q5) for B and the definition of $E_1^1, \dots, E_1^6$ that $a_1 \neq u_B$ and exactly one of the following holds;

(i) $i = 2$, $a_1a_2 \in E_B^3 \subseteq E_1^2$, $a_2 = w_B$ and $B \in \mathcal{B}_z^R$ for some $z \in V(T)$.

(ii) $i = 3$, $a_1a_2 \in E_B^3 \subseteq E_1^3$, $a_2 = w_B$ and $B \in \mathcal{B}_z^L$ for some $z \in V(T)$.

Suppose first that the case (i) occurs. Then by the definition of E_0^2 and the symmetry between b_1 and b_2, we may assume that $b_2 = v$. Note that both of the edges zw_B and b_1v are contained in T or placed on the right side of T, and hence it follows from Lemma 2 that the edges zw_B and b_1v do not cross at T. Therefore, the vertices z, w_B, b_1, v appear in T in this order (possibly, $w_B = b_1$) or in the order b_1, z, w_B, v (possibly, $b_1 = z$ and/or $w_B = v$). Thus, it follows from the construction of Q that $z, Q_B(z, v_B), w_B, b_1, v$ appear in Q in this order or in the order $b_1, z, Q_B(z, v_B), w_B, v$, respectively. Since $a_1 \in V\big(Q_B(z, v_B)\big)$, $a_2 = w_B$ and $b_2 = v$, we see that the edges a_1a_2 and b_1b_2 do not cross at Q.

Suppose next that the case (ii) occurs. Then by the definition of E_0^3 and the symmetry between b_1 and b_2, we may assume that $b_2 = w$. Note that both of the edges zw_B and b_1w are contained in T or placed on the left side of T, and hence it follows from Lemma 2 that the edges zw_B and b_1w do not cross at T. Therefore, the vertices z, w_B, b_1, w appear in T in this order (possibly, $w_B = b_1$) or in the order b_1, z, w_B, w (possibly, $b_1 = z$ and/or $w_B = v$). Thus, it follows from the construction of Q that $z, Q_B(z, v_B), w_B, b_1, w$ appear in Q in this order or in the order $b_1, z, Q_B(z, v_B), w_B, w$, respectively. Since $a_1 \in V\big(Q_B(z, v_B)\big)$, $a_2 = w_B$ and $b_2 = w$, we see that the edges a_1a_2 and b_1b_2 do not cross at Q.

This completes the proof of Lemma 11. □

6.2 Proof of Theorem 1 (iii)

Let G be a graph on the projective plane. To find a book embedding of G, we may assume that G is a triangulation. By Lemma 9, G has a spanning disk triangulation H with outer cycle C satisfying the conditions (G1)–(G5).

By the condition (G2), H has a path from s to t that passes through x and then y. Then it follows from Theorem 5 that H has a C-Tutte path T from s to t that passes through x and then y. By the symmetry, we may assume that the edges in $C[s, x]$ are contained in T or placed on the left side of T. Thus, the edges in $\overleftarrow{C}[y, t]$ are contained in T or placed on the right side of T. Let E_L (resp., E_R) be the sets of edges e in $E(H[V(T)]) - E(T)$ such that e is placed on the left side (resp., the right side) of T. Note that $E(H[V(T)]) = E(T) \cup E_L \cup E_R$.

$$\begin{aligned} \text{Let} \quad \mathcal{B}_2 &= \{B : B \text{ is a non-trivial } T\text{-bridge of } H \text{ having exactly two attachments}\}, \\ \text{and} \quad \mathcal{B}_3 &= \{B : B \text{ is a non-trivial } T\text{-bridge of } H \text{ having exactly three attachments}\}. \end{aligned}$$

Since T is a C-Tutte path in H and H is 2-connected, every non-trivial T-bridge of H belongs to $\mathcal{B}_2$ or $\mathcal{B}_3$. Therefore, all vertices not contained in T are contained in B for some $B \in \mathcal{B}_2 \cup \mathcal{B}_3$, and any edge in H is either an edge connecting two vertices in T or contained in B for some $B \in \mathcal{B}_2 \cup \mathcal{B}_3$. Thus,

$$E(H) - E\big(H[V(T)]\big) = \bigcup_{B \in \mathcal{B}_2 \cup \mathcal{B}_3} E(B).$$

Now we appropriately insert $Q_B(u_B, v_B)$ for all $B \in \mathcal{B}_2 \cup \mathcal{B}_3$ into T so that we obtain a suitable sequence of $V(G)$. To do that, we define the type of each B in $\mathcal{B}_2 \cup \mathcal{B}_3$, depending on which side (left or right) where it is placed on and the place of its attachments.

$$\begin{aligned} \text{Let} \quad \mathcal{B}^L &= \{B \in \mathcal{B}_2 \cup \mathcal{B}_3 : B \text{ is placed on the left side of } T\}, \\ \text{and} \quad \mathcal{B}^R &= \{B \in \mathcal{B}_2 \cup \mathcal{B}_3 : B \text{ is placed on the right side of } T\}. \end{aligned}$$

Furthermore, let

$$\begin{aligned} \mathcal{B}^{L1} &= \{B \in \mathcal{B}^L : \text{all the attachments of } B \text{ are contained in } T[s,x]\}, \\ \mathcal{B}^{L2} &= \{B \in \mathcal{B}^L : B \text{ has at least two attachments on } T[x,y]\}, \\ \mathcal{B}^{L3} &= \{B \in \mathcal{B}^L : B \text{ has at least two attachments on } T(y,t]\}, \\ \mathcal{B}^{R1} &= \{B \in \mathcal{B}^R : \text{all the attachments of } B \text{ are contained in } T[y,t]\}, \\ \mathcal{B}^{R2} &= \{B \in \mathcal{B}^R : B \text{ has at least two attachments on } T[x,y]\}, \\ \text{and} \quad \mathcal{B}^{R3} &= \{B \in \mathcal{B}^R : B \text{ has an attachment on } T[s,x) \text{ and at most one attachment on } T[x,y]\}. \end{aligned}$$

See Figure 11. Note that for any $B \in \mathcal{B}_2$, B contains an edge in $C[s,x]$ if and only if $B \in \mathcal{B}^{L1}$, B contains an edge in $\overleftarrow{C}[s,y]$ if and only if $B \in \mathcal{B}^{R3}$, and B contains an edge in $\overleftarrow{C}[y,t]$ if and only if $B \in \mathcal{B}^{R1}$.

We claim that $\{\mathcal{B}^{L1}, \mathcal{B}^{L2}, \mathcal{B}^{L3}\}$ is a partition of $\mathcal{B}^L$. To see that, let $B \in \mathcal{B}^L$. If all the attachments of B are contained in $T[s,x]$, then $B \in \mathcal{B}^{L1}$ and $B \notin \mathcal{B}^{L2} \cup \mathcal{B}^{L3}$. Thus, we may assume that B has an attachment on $T(x,t]$. So, $B \notin \mathcal{B}^{L1}$. It follows from the planarity of H that B has no attachments on $T[s,x)$, and furthermore, $B \notin \mathcal{B}_2$, and hence $B \in \mathcal{B}_3$. Suppose that B has at least two attachments on $T[x,y]$. In this case, $B \in \mathcal{B}^{L2}$. Since B has at most three attachments on T, B has at most one attachment on $T(y,t]$, and hence $B \notin \mathcal{B}^{L3}$. Thus, we may further assume that B has at most one attachment on $T[x,y]$. In this case, $B \notin \mathcal{B}^{L2}$. Since B has exactly three attachments on T, this implies that B has at least two attachments on $T(y,t]$, and hence $B \in \mathcal{B}^{L3}$. Thus, the claim holds.

On the other hand, we also claim that $\{\mathcal{B}^{R1}, \mathcal{B}^{R2}, \mathcal{B}^{R3}\}$ is a partition of $\mathcal{B}^R$. Let $B \in \mathcal{B}^R$. If all the attachments of B are contained in $T[y,t]$, then $B \in \mathcal{B}^{R1}$ and $B \notin \mathcal{B}^{R2} \cup \mathcal{B}^{R3}$. Thus, we may assume that B has an attachment on $T[s,y)$. So, $B \notin \mathcal{B}^{R1}$. Suppose that B has at least two attachments on $T[x,y]$. In this case, clearly $B \in \mathcal{B}^{R2}$ and $B \notin \mathcal{B}^{R3}$. Thus, we may further assume that B has at most one attachment on $T[x,y]$. So, $B \notin \mathcal{B}^{R2}$. Since B has at least two attachments on T, this implies that B has at least one attachment on $T[s,x)$, and hence $B \in \mathcal{B}^{R3}$. Thus, the claim holds.

The claims above imply the following;

$$E(H) - E\big(H[V(T)]\big) = \bigcup_{B \in \mathcal{B}^{L1} \cup \mathcal{B}^{L2} \cup \mathcal{B}^{L3} \cup \mathcal{B}^{R1} \cup \mathcal{B}^{R2} \cup \mathcal{B}^{R3}} E(B). \tag{3}$$

Let $B \in \mathcal{B}_2$, and let u_B and v_B be the two attachments of B on T such that s, u_B, v_B appear in T in this order (possibly, $s = u_B$). Since H is a disk triangulation, $u_B v_B \in E(H)$ but $u_B v_B \notin E(B)$, and B has to contain an edge in the outer cycle C. In particular, it follows from the fact $s, x, y, t \in V(T)$ that B contains an edge in either $C[s,x]$, $\overleftarrow{C}[s,y]$ or $\overleftarrow{C}[y,t]$. (Note that since $xt \in E(C)$ and $x, t \in V(T)$, there exists no non-trivial T-bridge of H containing an edge in $C[x,t]$.) Furthermore, $B \cap C = C[u_B, v_B]$ if $B \in \mathcal{B}^{L1}$, and otherwise, $B \cap C = \overleftarrow{C}[u_B, v_B]$. Now we construct the graph B^+ that is obtained from B by adding the edge $u_B v_B$ and a new vertex, say w_B, and joining w_B to all vertices in $B \cap C$. Note that those added vertex and edges can

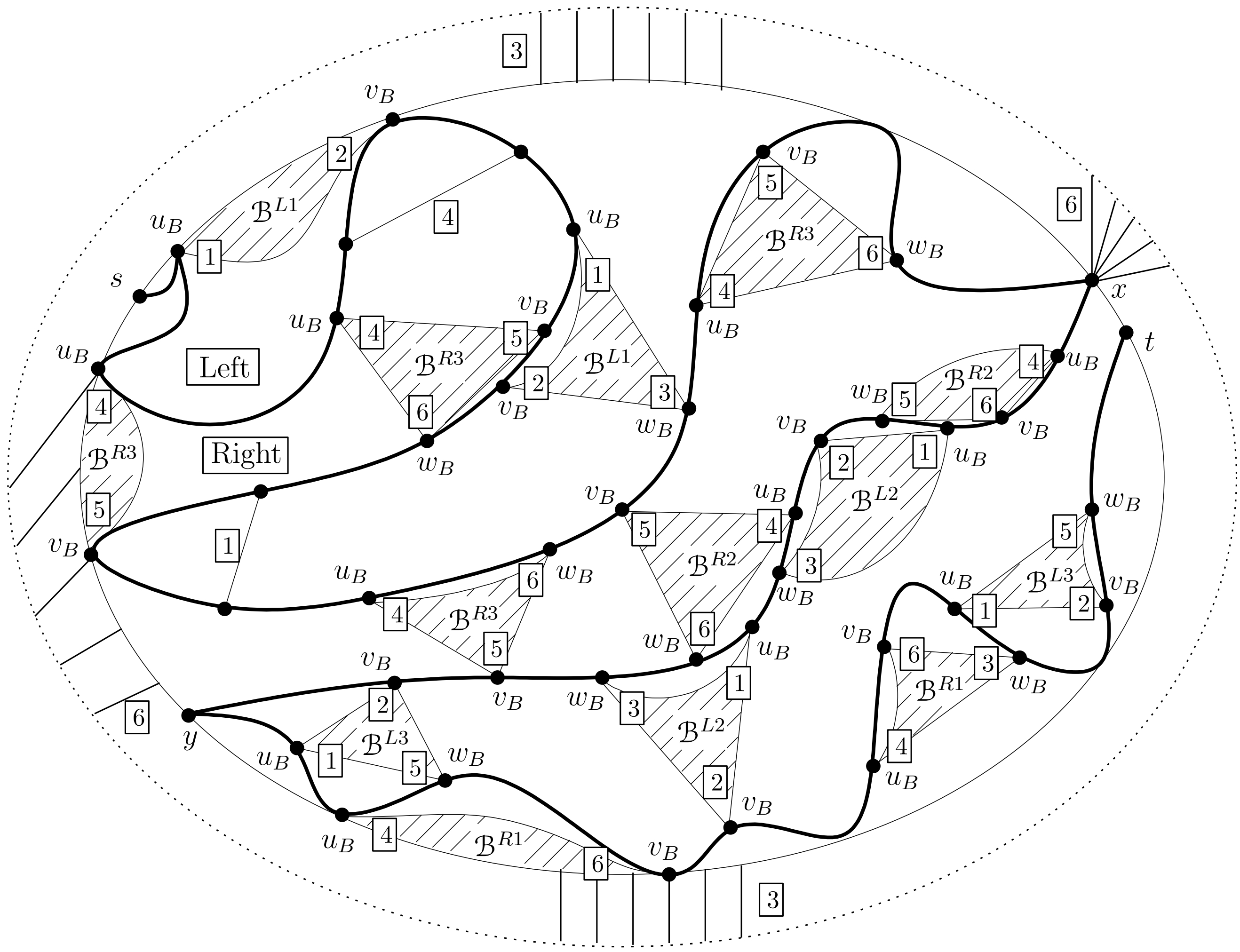

Figure 11: The C-Tutte path T in H and the T-bridges of H in $\mathcal{B}^{L1}, \mathcal{B}^{L2}, \mathcal{B}^{L3}, \mathcal{B}^{R1}, \mathcal{B}^{R2}$ and $\mathcal{B}^{R3}$.

be naturally embedded on the outer region of B so that the three vertices u_B, v_B and w_B form the outer cycle. So, indeed B^+ is a plane triangulation with outer cycle $u_Bv_Bw_B$. Notice also that $B \cap C$ is contained in the outer cycle of $B^+ - w_B$. Then it follows from Lemma 11 that $B^+ - \{u_Bv_B, v_Bw_B, w_Bu_B\}$ has a 6-page book embedding $(Q_B^+, \mathscr{E}_B^+)$, where $\mathscr{E}_B^+ = \{E_B^{1+}, \ldots, E_B^{6+}\}$, such that (Q1) u_B is the first vertex in Q_B^+, v_B is the second last vertex, and w_B is the last vertex, (Q2) the vertices on $B \cap C$ appear in Q_B^+ in this order, (Q3) all edges connecting u_B and $V(B^+) - \{u_B, v_B, w_B\}$ belong to E_B^{1+}, (Q4) all edges connecting v_B and $V(B^+) - \{u_B, v_B, w_B\}$ belong to E_B^{2+}, and (Q5) all edges connecting w_B and $V(B^+) - \{u_B, v_B, w_B\}$ belong to E_B^{3+}. Let Q_B be the sequence obtained from Q_B^+ by deleting the last vertex w_B, and let $\mathscr{E}_B = \{E_B^1, \ldots, E_B^6\}$, where $E_B^i = E_B^{i+}$ for $1 \leq i \leq 6$ with $i \neq 3$, and $E_B^3 = E_B^{3+} \cap E(B)$. Note that the pair $(Q_B, \mathscr{E}_B)$ is a 6-page book embedding of B.

On the other hand, let $B \in \mathcal{B}_3$. Then we define the three attachments u_B, v_B and w_B of B on T as follows:

- If $B \in \mathcal{B}^{L1} \cup \mathcal{B}^{R1}$, then let u_B, v_B and w_B be the three attachments of B on T such that s, u_B, v_B, w_B and t appear in T in this order (possibly, $s = u_B$ or $w_B = t$).

- Suppose that $B \in \mathcal{B}^{L2}$. Note that B has at least two attachments on $T[x,y]$. If B has exactly three attachments on $T[x,y]$, then let u_B, v_B and w_B be the three attachments of B on T such that s, x, u_B, v_B, w_B, y and t appear in T in this order (possibly, $x = u_B$ and/or $w_B = y$); Otherwise, B has exactly one attachment on $T(y,t]$, say v_B, and exactly two attachments on $T[x,y]$, say u_B and w_B, where s, x, u_B, w_B, y, v_B and t appear in T in this order (possibly, $x = u_B$ and/or $w_B = y$ and/or $v_B = t$).

- Suppose that $B \in \mathcal{B}^{L3}$. Note that B has at least two attachments on $T(y,t]$. If B has exactly three attachments on $T(y,t]$, then let u_B, v_B and w_B be the three attachments of B on T such that s, x, y, u_B, v_B, w_B and t appear in T in this order (possibly, $w_B = t$); Otherwise, B has exactly one attachment on $T[x,y]$, say v_B, and B has exactly two attachments on $T(y,t]$, say u_B and w_B, where s, x, v_B, y, u_B, w_B and t appear in T in this order (possibly, $x = v_B$ or $v_B = y$ and/or $w_B = t$).

- Suppose that $B \in \mathcal{B}^{R2}$. Note that B has at least two attachments on $T[x,y]$. If B has exactly three attachments on $T[x,y]$, then let u_B, v_B and w_B be the three attachments of B on T such that s, x, u_B, v_B, w_B, y and t appear in T in this order (possibly $x = u_B$ and/or $w_B = y$); Otherwise, B has exactly one attachment on $T[s,x)$, say v_B, and B has exactly two attachments on $T[x,y]$, say u_B and w_B, where s, v_B, x, u_B, w_B, y and t appear in T in this order (possibly, $s = v_B$ and/or $x = u_B$ and/or $w_B = y$).

- Suppose that $B \in \mathcal{B}^{R3}$. Since B has exactly three attachments on T, B has at least two attachments on $T[s,x)$ and at most one attachment on $T[x,y]$. If B has exactly three attachments on $T[s,x)$, then let u_B, v_B and w_B be the three attachments of B on T such that s, u_B, v_B, w_B, x, y and t appear in T in this order (possibly $s = u_B$); Otherwise, B has exactly two attachments on $T[s,x)$, say u_B and w_B, and B has exactly one attachment on $T[x,y]$, say v_B, where s, u_B, w_B, x, v_B, y and t appear in T in this order (possibly, $s = u_B$ and/or $x = v_B$ or $v_B = y$).

Then see Table 1 for the places of the vertices u_B, v_B and w_B, depending on the types of B. Note that since H is a disk triangulation, we see that $u_Bv_B, v_Bw_B, w_Bu_B \in E(H)$ but $u_Bv_B, v_Bw_B, w_Bu_B \notin E(B)$.

Table 1: The possible places of the vertices u_B, v_B and w_B, depending on the types of B.

	$B \in \mathcal{B}^{L1}$	$B \in \mathcal{B}^{L2}$	$B \in \mathcal{B}^{L3}$	$B \in \mathcal{B}^{R1}$	$B \in \mathcal{B}^{R2}$	$B \in \mathcal{B}^{R3}$
u_B, w_B	$T[s,x]$	$T[x,y]$	$T(y,t]$	$T[y,t]$	$T[x,y]$	$T[s,x)$
v_B	$T[s,x]$	$T[x,y] \cup T(y,t]$	$T[x,y] \cup T(y,t]$	$T[y,t]$	$T[s,x) \cup T[x,y]$	$T[s,x) \cup T[x,y]$

Note that $B + \{u_Bv_B, v_Bw_B, w_Bu_B\}$ is a plane triangulation. Then it follows from Lemma 11 that B has a 6-page book embedding $(Q_B, \mathscr{E}_B)$, where $\mathscr{E}_B = \{E_B^1, \ldots, E_B^6\}$ such that (Q1) u_B is the first vertex in Q_B, v_B is the second last vertex, and w_B is the last vertex, (Q3) all edges connecting u_B and $V(B) - \{u_B, v_B, w_B\}$ belong to E_B^1, (Q4) all edges connecting v_B and $V(B) - \{u_B, v_B, w_B\}$ belong to E_B^2, and (Q5) all edges connecting w_B and $V(B) - \{u_B, v_B, w_B\}$ belong to E_B^3. (Since we do not use the condition (Q2) for $B \in \mathcal{B}_3$, we can ignore it.)

For each $z \in V(T)$, define $\mathcal{B}_z^L$ (resp., $\mathcal{B}_z^R$) as the set of T-bridges B in $\mathcal{B}_2 \cup \mathcal{B}_3$ such that $u_B = z$ and B is placed on the left (resp., right) side of T. Let $B_1, \dots, B_p$ be the elements in $\mathcal{B}_z^L$ along the clockwise order around z, where $p = |\mathcal{B}_z^L|$. Similarly, let $B_{p+1}, \dots, B_q$ be the elements in $\mathcal{B}_z^R$ along the clockwise order around z, where $q = p + |\mathcal{B}_z^R|$. Then we have the following claim.

Claim 6 *Let $z \in V(T)$, let $\mathcal{B}_z^L = \{B_1, \dots, B_p\}$, and let $\mathcal{B}_z^R = \{B_{p+1}, \dots, B_q\}$ as above. Then both of the following hold.*

(I) *The vertices $z, v_{B_1}, w_{B_1}, v_{B_2}, \dots, v_{B_p}, w_{B_p}, t$ appear in T in one of the following orders*[5]*:*

- (I-1) $z, v_{B_p}, w_{B_p}, v_{B_{p-1}}, \dots, w_{B_2}, v_{B_1}, (w_{B_1},)\ t$.
- (I-2) $z, v_{B_p}, w_{B_p}, v_{B_{p-1}}, \dots, w_{B_2}, w_{B_1}, y, v_{B_1}, t$.
- (I-3) $v_{B_1}, z, v_{B_p}, w_{B_p}, v_{B_{p-1}}, \dots, w_{B_2}, w_{B_1}, t$.

(II) *The vertices $z, v_{B_{p+1}}, w_{B_{p+1}}, v_{B_{p+2}}, \dots, v_{B_q}, w_{B_q}, t$ appear appear in T in one of the following orders*[6]*:*

- (II-1) $z, v_{B_{p+1}}, w_{B_{p+1}}, v_{B_{p+2}}, \dots, v_{B_q},\ (w_{B_q},)\ t$
- (II-2) $v_{B_q}, x, z, v_{B_{p+1}}, w_{B_{p+1}}, v_{B_{p+2}}, \dots, w_{B_{q-1}}, w_{B_q}, t$.
- (II-3) $z, v_{B_{p+1}}, w_{B_{p+1}}, v_{B_{p+2}}, \dots, w_{q-1}, w_{B_q}, x, v_{B_q}, t$.

Proof. We prove (I) and (II) at the same time. Suppose first that $z \in V\big(T[s,x)\big)$. Then all non-trivial T-bridges B in $\mathcal{B}_z^L$ belong to $\mathcal{B}^{L1}$, and all non-trivial T-bridges B in $\mathcal{B}_z^R$ belong to $\mathcal{B}^{R3}$. (See Table 1.) It follows from the planarity that if $z = u_B$ for some $B \in \mathcal{B}_2$, then $B = B_1$ in the case when $B \in \mathcal{B}_z^L$, and $B = B_q$ in the case when $B \in \mathcal{B}_z^R$. Then the choice of u_B, v_B, w_B implies that the vertices $z, v_{B_1}, (w_{B_1},)\ v_{B_2}, \dots, v_{B_p}, w_{B_p}, t$ appear in T in the order as in (I-1). On the other hand, it follows from the planarity that if $z = u_B$ for some $B \in \mathcal{B}^{R3}$ such that B has an attachment on $T(x,y]$, then $B = B_q$ and $v_B \in V\big(T(x,y]\big)$. Therefore, the choice of u_B, v_B, w_B implies that the vertices $z, v_{B_{p+1}}, w_{B_{p+1}}, v_{B_{p+2}}, \dots, v_{B_q}, (w_{B_q},)\ t$ appear in T in the order as in (II-1) if B_q has no attachment on $T(x,y]$; Otherwise as in (II-3).

Suppose next that $z \in V\big(T[x,y)\big)$. Then all non-trivial T-bridges B in $\mathcal{B}_z^L$ belong to $\mathcal{B}^{L2}$, and all non-trivial T-bridges B in $\mathcal{B}_z^R$ belong to $\mathcal{B}^{R2}$. Note that no T-bridge B in $\mathcal{B}_2$ satisfies $z = u_B$. It follows from the planarity that if $z = u_B$ for some $B \in \mathcal{B}^{L2}$ such that B has an attachment on $T(y,t]$, then $B = B_1$ and $v_B \in V\big(T(y,t]\big)$. Then the choice of u_B, v_B, w_B implies that the vertices $z, v_{B_1}, w_{B_1}, v_{B_2}, \dots, v_{B_p}, w_{B_p}, t$ appear in T in the order as in (I-1) if B_1 has no attachment on $T(y,t]$; Otherwise as in (I-2). Similarly, if $z = u_B$ for some $B \in \mathcal{B}^{R2}$ such that B has an attachment on $T[s,x)$, then $B = B_q$, and the vertices $z, v_{B_{p+1}}, w_{B_{p+1}}, v_{B_{p+2}}, \dots, v_{B_q}, w_{B_q}, t$ appear in T in the order as in (II-1) or (II-2).

Suppose finally that $z \in V\big(T[y,t)\big)$. Then all non-trivial T-bridges B in $\mathcal{B}_z^L$ belong to $\mathcal{B}^{L3}$, and all non-trivial T-bridges B in $\mathcal{B}_z^R$ belong to $\mathcal{B}^{R1}$. If $z = u_B$ for some $B \in \mathcal{B}^{L3}$ such that B has an attachment on $T[x,y]$, then $B = B_1$ and $v_B \in V\big(T[x,y]\big)$. Thus, the choice of u_B, v_B, w_B

[5] Possibly, $w_{B_p} = v_{B_{p-1}}$ and/or $\cdots$, and/or $w_{B_1} = t$. In (I-1), w_{B_1} does not exist when $z \in V\big(T[s,x)\big)$ and $B_1 \in \mathcal{B}_2$. Furthermore, (I-2) (resp., (I-3)) occurs only when $z \in V\big(T[x,y)\big)$ (resp., only when $z \in V\big(T[y,t)\big)$).

[6] Possibly, $w_{B_{p+1}} = v_{B_{p+2}}$ and/or $\cdots$, and/or $w_{B_q} = t$. In the first case, w_{B_q} does not exist when $z \in V\big(T[s,x)\big) \cup V\big(T[y,t)\big)$ and $B_q \in \mathcal{B}_2$. Furthermore, (II-2) (resp., (II-3)) occurs only when $z \in V\big(T[x,y)\big)$ (resp., only when $z \in V\big(T[s,x)\big)$).

implies that the vertices $z, v_{B_1}, w_{B_1}, v_{B_2}, \dots, v_{B_p}, w_{B_p}, t$ appear in T in the order as in (I-1) or (I-3). On the other hand, it follows from the planarity that if $z = u_B$ for some $B \in \mathcal{B}_2$, then $B = B_q$ and $B \in \mathcal{B}_z^R$. Then the vertices $z, v_{B_{p+1}}, w_{B_{p+1}}, v_{B_{p+2}}, \dots, v_{B_q}, \ (w_{B_q},)\ t$ appear in T in the order as in (II-1).

This completes the proof of Claim 6. $\square$

Now, we insert the sequence $Q_{B_1}(z, v_{B_1})\, Q_{B_2}(z, v_{B_2}) \cdots\ Q_{B_p}(z, v_{B_p})\, Q_{B_q}(z, v_{B_q})\, Q_{B_{q-1}}(z, v_{B_{q-1}}) \cdots\ Q_{B_{p+1}}(z, v_{B_{p+1}})$ just after z in T. Therefore, we obtain the new sequence

$$T[s, z]\ Q_{B_1}(z, v_{B_1})\ \cdots\ Q_{B_p}(z, v_{B_p})\ Q_{B_q}(z, v_{B_q})\ \cdots\ Q_{B_{p+1}}(z, v_{B_{p+1}})\ T(z, t].$$

We do the above insertion for all $z \in V(T)$ independently, and let Q be the obtained sequence of the vertices in G. It follows from the construction of Q and the condition (Q2) for each $B \in \mathcal{B}_2$ that the following claim holds.

Claim 7 *The vertices in $C[s, x]\ \overleftarrow{C}[y, t]$ appear in Q in this order.*

Now we will partition all edges in G into the six sets so that no two edges in a same set cross at Q. To do that, we first partite all edges in $\bigcup_{B \in \mathcal{B}_2 \cup \mathcal{B}_3} B$ into six sets $E_1^1, \dots, E_1^6$.

$$\begin{aligned}
\text{Let}\qquad E_1^1 &= \bigcup_{B \in \mathcal{B}^{L1} \cup \mathcal{B}^{L2}} E_B^1 \cup \bigcup_{B \in \mathcal{B}^{L3}} E_B^1 \cup \bigcup_{B \in \mathcal{B}^{R1}} E_B^4 \cup \bigcup_{B \in \mathcal{B}^{R2} \cup \mathcal{B}^{R3}} E_B^4,\\
E_1^2 &= \bigcup_{B \in \mathcal{B}^{L1} \cup \mathcal{B}^{L2}} E_B^2 \cup \bigcup_{B \in \mathcal{B}^{L3}} E_B^2 \cup \bigcup_{B \in \mathcal{B}^{R1}} E_B^6 \cup \bigcup_{B \in \mathcal{B}^{R2} \cup \mathcal{B}^{R3}} E_B^6,\\
E_1^3 &= \bigcup_{B \in \mathcal{B}^{L1} \cup \mathcal{B}^{L2}} E_B^3 \cup \bigcup_{B \in \mathcal{B}^{L3}} E_B^5 \cup \bigcup_{B \in \mathcal{B}^{R1}} E_B^3 \cup \bigcup_{B \in \mathcal{B}^{R2} \cup \mathcal{B}^{R3}} E_B^5,\\
E_1^4 &= \bigcup_{B \in \mathcal{B}^{L1} \cup \mathcal{B}^{L2}} E_B^4 \cup \bigcup_{B \in \mathcal{B}^{L3}} E_B^4 \cup \bigcup_{B \in \mathcal{B}^{R1}} E_B^1 \cup \bigcup_{B \in \mathcal{B}^{R2} \cup \mathcal{B}^{R3}} E_B^1,\\
E_1^5 &= \bigcup_{B \in \mathcal{B}^{L1} \cup \mathcal{B}^{L2}} E_B^5 \cup \bigcup_{B \in \mathcal{B}^{L3}} E_B^3 \cup \bigcup_{B \in \mathcal{B}^{R1}} E_B^5 \cup \bigcup_{B \in \mathcal{B}^{R2} \cup \mathcal{B}^{R3}} E_B^2,\\
\text{and}\qquad E_1^6 &= \bigcup_{B \in \mathcal{B}^{L1} \cup \mathcal{B}^{L2}} E_B^6 \cup \bigcup_{B \in \mathcal{B}^{L3}} E_B^6 \cup \bigcup_{B \in \mathcal{B}^{R1}} E_B^2 \cup \bigcup_{B \in \mathcal{B}^{R2} \cup \mathcal{B}^{R3}} E_B^3.
\end{aligned}$$

Since $\mathscr{E}_B = \{E_B^1, \dots, E_B^6\}$ is a partition of $E(B)$ for any $B \in \mathcal{B}_2 \cup \mathcal{B}_3$, we see that $\{E_1^1, \dots, E_1^6\}$ is a partition of $\bigcup_{B \in \mathcal{B}_2 \cup \mathcal{B}_3} E(B) = E(H) - E\big(H[V(T)]\big)$. (See (3).) For those sets $E_1^1, \dots, E_1^6$, we show the following claim.

Claim 8 *For any integer i with $1 \le i \le 6$, any two edges in E_1^i do not cross at Q.*

Proof. Let $i \in \{1, 2, 3, 4, 5, 6\}$ and let a_1a_2 and b_1b_2 be two edges in E_1^i. By the definition of E_1^i, the edge a_1a_2 is contained in some non-trivial T-bridge B in $\mathcal{B}_2 \cup \mathcal{B}_3$. If $b_1b_2 \in E(B)$, then since the pair $(Q_B, \mathscr{E}_B)$ is a 6-page book embedding of B and $a_1a_2, b_1b_2 \in E_B^j \in \mathscr{E}_B$ for some $1 \le j \le 6$, we see that a_1a_2 and b_1b_2 do not cross at Q_B, and hence at Q. Thus, we may assume that $b_1b_2 \notin E(B)$, which implies that $b_1b_2 \in E(B')$ for some $B' \in \mathcal{B}_2 \cup \mathcal{B}_3$ with $B' \ne B$. In particular, since the vertices in $Q_B(u_B, v_B)$ appear in Q consecutively, we have $b_1, b_2 \in V(Q) - V\big(Q_B(u_B, v_B)\big)$. So, if $\{a_1, a_2\} \cap \{u_B, v_B, w_B\} = \emptyset$, then $a_1, a_2 \in V\big(Q_B(u_B, v_B)\big)$, and hence it follows from Lemma 3 that a_1a_2 and b_1b_2 do not cross at Q. Therefore, we may assume that $\{a_1, a_2\} \cap \{u_B, v_B, w_B\} \neq \emptyset$. By the symmetry between a_1a_2 and b_1b_2, we may also assume that $\{b_1, b_2\} \cap \{u_{B'}, v_{B'}, w_{B'}\} \neq \emptyset$.

Case 1. $a_1 = u_B$ or $a_2 = u_B$ or $b_1 = u_{B'}$ or $b_2 = u_{B'}$.

Say $a_1 = u_B$ by symmetry. It follows from the condition (Q3) for B that $a_1a_2 \in E_B^1$, and it follows from the definitions of $E_1^1, \ldots, E_1^6$ that $a_1a_2 \in E_1^1 \cup E_1^4$. So, $i = 1$ or 4, and $b_1b_2 \in E_1^1 \cup E_1^4$. Since $\{b_1, b_2\} \cap \{u_{B'}, v_{B'}, w_{B'}\} \neq \emptyset$, it follows from the conditions (Q3)–(Q5) that $b_1 = u_{B'}$ or $b_2 = u_{B'}$, say $b_1 = u_{B'}$ by symmetry. If $u_B = u_{B'}$, then the edges a_1a_2 and b_1b_2 share u_B as the end vertex, and hence they do not cross. Thus, we may assume that $u_B \neq u_{B'}$. Then it follows from the construction of Q that $b_1, b_2 \notin V\big(Q[u_B, u_B^+]\big)$, where u_B^+ is the successor of u_B at T. Since $a_1, a_2 \in V\big(Q[u_B, u_B^+]\big)$, it follows from Lemma 3 that a_1a_2 and b_1b_2 do not cross at Q. This completes the proof of Case 1.

Case 2. $a_1, a_2 \neq u_B$ and $b_1, b_2 \neq u_{B'}$.

Since $\{a_1, a_2\} \cap \{u_B, v_B, w_B\} \neq \emptyset$, it follows from the symmetry between a_1 and a_2 that we may assume that $a_2 = v_B$ or $a_2 = w_B$. By the symmetry between a_1a_2 and b_1b_2, we may also assume that $b_2 = v_{B'}$ or $b_2 = w_{B'}$. It follows from the conditions (Q4) and (Q5) for B and B' that $a_1a_2 \in E_B^2 \cup E_B^3$ and $b_1b_2 \in E_{B'}^2 \cup E_{B'}^3$. In particular, we see that $i \neq 1, 4$. Note that $a_1 \in V\big(Q_B(u_B, v_B)\big)$ and $b_1 \in V\big(Q_{B'}(u_{B'}, v_{B'})\big)$.

We claim that B and B' are placed on the same side of T. For the contrary, suppose that B and B' are placed on the different side of T, which means either $B \in \mathcal{B}^L$ and $B' \in \mathcal{B}^R$, or $B \in \mathcal{B}^R$ and $B' \in \mathcal{B}^L$. By the symmetry between B and B', we may assume the former occurs. Since $a_1a_2 \in E_B^2 \cup E_B^3$ and $b_1b_2 \in E_{B'}^2 \cup E_{B'}^3$, it follows from the definition of $E_1^1, \ldots, E_1^6$ that one of the following hold.

(i) $i = 3$, $B \in \mathcal{B}^{L1}$, $a_1a_2 \in E_B^3$, $a_2 = w_B$, $B' \in \mathcal{B}^{R1}$, $b_1b_2 \in E_{B'}^3$, and $b_2 = w_{B'}$.

(ii) $i = 3$, $B \in \mathcal{B}^{L2}$, $a_1a_2 \in E_B^3$, $a_2 = w_B$, $B' \in \mathcal{B}^{R1}$, $b_1b_2 \in E_{B'}^3$, and $b_2 = w_{B'}$.

(iii) $i = 5$, $B \in \mathcal{B}^{L3}$, $a_1a_2 \in E_B^3$, $a_2 = w_B$, $B' \in \mathcal{B}^{R2} \cup \mathcal{B}^{R3}$, $b_1b_2 \in E_{B'}^2$, and $b_2 = v_{B'}$.

Suppose that the case (i) occurs. Then all the attachments of B are contained in $T[s, x]$. and all the attachments of B' are contained in $T[y, t]$. It follows from the construction of Q that $a_1, a_2 \in V\big(Q[s, x]\big)$ and $b_1, b_2 \in V\big(Q[y, t]\big)$, which imply that a_1a_2 and b_1b_2 do not cross at Q.

Suppose next that the case (ii) occurs. Since $B \in \mathcal{B}^{L2}$, we have $u_B, w_B \in V\big(T[x, y]\big)$. (See Table 1.) Therefore, it follows from the construction of Q that all vertices in $Q_B[u_B, v_B) \cup \{w_B\}$ appear in $Q[x, y]$. Since $a_1a_2 \in E_B^3$, it follows from the condition (Q4) for B that $a_1, a_2 \neq v_B$, and hence $a_1, a_2 \in V\big(Q_B[u_B, v_B)\big) \cup \{w_B\}$. Therefore, we have $a_1, a_2 \in V\big(Q[x, y]\big)$. On the other hand, since $B' \in \mathcal{B}^{R1}$, it is easy to see that $b_1, b_2 \in V\big(Q[y, t]\big)$. These imply that a_1a_2 and b_1b_2 do not cross at Q.

Suppose finally that the case (iii) occurs. Since $B \in \mathcal{B}^{L3}$, we have $u_B, w_B \in V\big(T[y, t]\big)$. (See Table 1.) Therefore, it follows from the construction of Q that all vertices in $Q_B[u_B, v_B) \cup \{w_B\}$ appear in $Q[y, t]$. Since $a_1a_2 \in E_B^3$, it follows from the condition (Q4) for B that $a_1, a_2 \neq v_B$, and hence $a_1, a_2 \in V\big(Q_B[u_B, v_B)\big) \cup \{w_B\}$. Therefore, we have $a_1, a_2 \in V\big(Q[y, t]\big)$. On the other hand, since $B' \in \mathcal{B}^{R2} \cup \mathcal{B}^{R3}$, it is easy to see that $b_1, b_2 \in V\big(Q[s, y]\big)$. These imply that a_1a_2 and b_1b_2 do not cross at Q, and hence the claim holds.

Therefore, we may assume that B and B' are placed on the same side of T, that is, either $B, B' \in \mathcal{B}^L$ or $B, B' \in \mathcal{B}^R$. Suppose that $u_B \neq u_{B'}$. Since the edges $u_B a_2$ and $u_{B'} b_2$ are placed on the same side of T (both on the left side or the right side), it follows from Lemma 2 that $u_B a_2$ and $u_{B'} b_2$ do not cross at T. Recall that $a_1 \in V\big(Q_B(u_B, v_B)\big)$ and hence $a_1 \in V\big(Q(u_B, u_B^+)\big)$, where u_B^+ is the successor of u_B at T. Similarly, $b_1 \in V\big(Q(u_{B'}, u_{B'}^+)\big)$, where $u_{B'}^+$ is the successor of $u_{B'}$ at T. Furthermore, we have $a_2, b_1, b_2 \notin V\big(Q(u_B, u_B^+)\big)$ and $a_1, a_2, b_2 \notin V\big(Q(u_{B'}, u_{B'}^+)\big)$. These imply that the edges $a_1 a_2$ and $b_1 b_2$ do not cross at Q, neither.

Thus, the case $u_B = u_{B'}$ only remains. Suppose that $B, B' \in \mathcal{B}^L$. By the symmetry between $a_1 a_2$ and $b_1 b_2$, we may assume that $B = B_i$ and $B' = B_j$ with $1 \leq i < j \leq p$, where $\mathcal{B}_z^L = \big\{B_1, \ldots, B_p\big\}$. It follows from Claim 6 (I) that the vertices z, a_2, b_2 appear in T in the order z, b_2, a_2 unless the case (I-3) occurs and $v_{B_1} = a_2$. Note that in the exceptional case, they appear in T in the order a_2, z, b_2. This, together with the construction of Q, implies that $z, Q_{B_i}(z, v_{B_i}), Q_{B_j}(z, v_{B_j}), b_2, a_2$ appear in Q in this order, or in the order $a_2, z, Q_{B_i}(z, v_{B_i}), Q_{B_j}(z, v_{B_j}), b_2$. Note that $a_1 \in V\big(Q_{B_i}(z, v_{B_i})\big)$ and $b_1 \in V\big(Q_{B_j}(z, v_{B_j})\big)$. Therefore, the edges $a_1 a_2$ and $b_1 b_2$ do not cross at Q.

Suppose next that $B, B' \in \mathcal{B}^R$. By the symmetry between $a_1 a_2$ and $b_1 b_2$, we may assume that $B = B_i$ and $B' = B_j$ with $p+1 \leq i < j \leq q$, where $\mathcal{B}_z^R = \big\{B_{p+1}, \ldots, B_q\big\}$. It follows from Claim 6 (II) that the vertices z, a_2, b_2 appear in T in this order, unless the case (II-2) occurs and $v_{B_q} = b_2$. Note that in the exceptional case, they appear in T in the order b_2, z, a_2. This, together with the construction of Q, implies that $z, Q_{B_j}(z, v_{B_j}), Q_{B_i}(z, v_{B_i}), a_2, b_2$ appear in Q in this order or in the order $b_2, z, Q_{B_j}(z, v_{B_j}), Q_{B_i}(z, v_{B_i}), a_2$. Note that $b_1 \in V\big(Q_{B_j}(z, v_{B_j})\big)$ and $a_1 \in V\big(Q_{B_i}(z, v_{B_i})\big)$. Therefore, the edges $a_1 a_2$ and $b_1 b_2$ do not cross at Q.

This completes the proof of Claim 8. $\square$

Now we partition all edges in G into six sets $E^1, \ldots, E^6$ as follows;

$$\begin{array}{lrcl}
\text{Let} & E^1 & = & E_1^1 \cup E(T) \cup E_L, \\
 & E^2 & = & E_1^2, \\
 & E^3 & = & E_1^3 \cup \big(E(G) - E(H) - E_x\big), \\
 & E^4 & = & E_1^4 \cup E_R, \\
 & E^5 & = & E_1^5, \\
\text{and} & E^6 & = & E_1^6 \cup E_x.
\end{array}$$

Recall that $E(H[V(T)]) = E(T) \cup E_L \cup E_R$. Then it follows from the definition of $E_1^1, \ldots, E_1^6$ and equality (3) that $\{E^1, \ldots, E^6\}$ is indeed a partition of $E(G)$. We will show that no two edges in E^i cross at Q for any $1 \leq i \leq 6$, which will complete the proof of Theorem 1 (iii). Let $a_1 a_2, b_1 b_2 \in E^i$. For $i = 2$ or 5, it follows from Claim 8 that the edges $a_1 a_2$ and $b_1 b_2$ do not cross at Q. So, we may assume that $i = 1, 3, 4$ or 6. Recall that all edges in E_x share x as an end vertex. Thus, by the condition (G3), Lemma 2, Claims 7 and 8, and the symmetry between $a_1 a_2$ and $b_1 b_2$, we may further assume that $a_1 a_2 \in E_1^i$ and $b_1 b_2 \in E^i - E_1^i$. In particular, $a_1 a_2$ is contained in some T-bridge B in $\mathcal{B}_2 \cup \mathcal{B}_3$.

Case 1. $i = 1$.

In this case, $a_1 a_2 \in E_1^1$ and $b_1 b_2 \in E(T) \cup E_L$. It follows from the definition of E_1^1 that $a_1 a_2 \in E_B^1 \cup E_B^4$. By the conditions (Q3)–(Q5) for B, we see that $a_1, a_2 \in V\big(Q_B[u_B, v_B)\big)$. Note that $Q_B[u_B, v_B)$ is contained in $Q[u_B, u_B^+)$, where u_B^+ is the successor of u_B at T, and hence $a_1, a_2 \in V\big(Q[u_B, u_B^+)\big)$. On the other hand, since $b_1, b_2 \in V(T)$, it follows from the construction of Q that $b_1, b_2 \notin V\big(Q(u_B, u_B^+)\big)$. Then it follows from Lemma 3 that a_1a_2 and b_1b_2 do not cross at Q, and complete the proof of Case 1.

Case 2. $i = 3$.

In this case, $a_1a_2 \in E_1^3$ and $b_1b_2 \in E(G) - E(H) - E_x$. It follows from the condition (G3) and the symmetry between b_1 and b_2 that we may assume that $b_1 \in V\big(C[s, x]\big)$ and $b_2 \in V\big(\overleftarrow{C}[y, t]\big)$.

Case 2.1. $B \in \mathcal{B}^{L1} \cup \mathcal{B}^{R1}$.

By the definition of E_1^3, we have $a_1a_2 \in E_B^3$. Then, it follows from the conditions (Q3) and (Q4) that $\{a_1, a_2\} \cap \{u_B, v_B\} = \emptyset$. Now we only prove the case where $B \in \mathcal{B}^{L1}$, but the case where $B \in \mathcal{B}^{R1}$ can be shown by the same way. (In fact, exchanging the role of b_1 and b_2 and the role of $C[s, x]$ and $\overleftarrow{C}[y, t]$, we obtain a proof of the case where $B \in \mathcal{B}^{R1}$.)

Let u and v be two vertices in $T[s, x] \cap C[s, x]$ such that u_B is contained in $T[u, v)$. (Note that s and x satisfy the conditions of u and v, respectively, and hence those vertices indeed exist.) Taking such vertices u and v so that $C[u, v)$ is as short as possible, we may assume that no vertices in $C[s, x]$ are contained in $T(u, v)$. By this choice and the planarity, we see that $u, u_B, v_B, (w_B,)\ v, x$ appear in T in this order (possibly, w_B does not exist, and/or $u = u_B$, and/or $w_B = v$, and/or $v = x$). If $b_1 \notin V\big(C(u, v)\big)$, then we have $a_1, a_2 \in V\big(Q[u, v]\big)$ and $b_1, b_2 \notin V\big(Q(u, v)\big)$. Then it follows from Lemma 3 that the edges a_1a_2 and b_1b_2 do not cross at Q. Thus, we may assume that $b_1 \in V\big(C(u, v)\big)$, and hence there exists a non-trivial T-bridge B' in $\mathcal{B}_2 \cap \mathcal{B}_u^L$ with $b_1 \in V(B') - \{u_{B'}, v_{B'}\}$. It follows from the construction of Q that $B' = B_1$ and

$$Q[u, b_1] \;=\; Q_{B'}[u, b_1],$$

where $\mathcal{B}_u^L = \{B_1, \ldots, B_p\}$. In particular, all vertices in B' appear in $Q[u, u^+)$, where u^+ is the successor of u at T. So, if $u \neq u_B$, then we have $a_1, a_2 \in V\big(Q[u^+, v]\big)$ and $b_1, b_2 \notin V\big(Q(u^+, v)\big)$. Then again, it follows from Lemma 3 that the edges a_1a_2 and b_1b_2 do not cross at Q.

Therefore, we may assume that $u = u_B$, and hence $B \in \mathcal{B}_u^L$. Let $B = B_j$ for some j with $1 \leq j \leq p$. If $j \neq 1$, then $a_1, a_2 \notin V\big(Q[u, b_1]\big)$, and hence it follows from Lemma 3 that the edges a_1a_2 and b_1b_2 do not cross at Q. Therefore, we may assume that $j = 1$, which means that $a_1, a_2, b_1 \in V(B)$. Recall that $a_1a_2 \in E_B^3$. Then it follows from the conditions (Q3) and (Q4) that $a_1, a_2, b_1 \neq u_B, v_B$, $\{a_1, a_2, b_1\} \cap \{u_B, v_B\} \neq \emptyset$, and hence $a_1, a_2, b_1 \in V\big(Q_B(u_B, v_B)\big)$. Since $(Q_B^+, \mathscr{E}_B^+)$ is a 6-page book embedding of B^+, we see that the edges a_1a_2 and b_1w_B do not cross at Q_B^+. Therefore, it follows from the symmetry between a_1 and a_2 that a_1, a_2, b_1, w_B appear in Q_B^+ in this order or in the order b_1, a_1, a_2, w_B. Since $Q_B(u, v_B)$ is a subsequence of Q, we see that a_1, a_2, b_1, b_2 appear in Q in this order or in the order b_1, a_1, a_2, b_2, respectively. Therefore, the edges a_1a_2 and b_1b_2 do not cross at Q, and this completes the proof of Case 2.1. □

Case 2.2. $B \in \mathcal{B}^{L2} \cup \mathcal{B}^{L3} \cup \mathcal{B}^{R2} \cup \mathcal{B}^{R3}$.

In this case, we see that $B - \{u_B, v_B, w_B\}$ contains neither b_1 nor b_2. This implies that $b_1, b_2 \notin V\big(Q_B(u_B, v_B)\big)$. So, if $\{a_1, a_2\} \cap \{u_B, v_B, w_B\} = \emptyset$, then $a_1, a_2 \in V\big(Q_B(u_B, v_B)\big)$, and hence it follows from Lemma 3 that the edges a_1a_2 and b_1b_2 do not cross at Q. (Recall that the vertices in $Q_B(u_B, v_B)$ are contained in Q consecutively.) Therefore, we may assume that $\{a_1, a_2\} \cap \{u_B, v_B, w_B\} \neq \emptyset$. By the symmetry between a_1 and a_2, we may also assume that $a_2 = u_B$

or $a_2 = v_B$ or $a_2 = w_B$. It follows from the conditions (Q3)–(Q5) that $a_1a_2 \in E_B^1 \cup E_B^2 \cup E_B^3$. Thus, by the definition of E_1^3, we see that $B \in \mathcal{B}^{L2}$, $a_1a_2 \in E_B^3$, and $a_2 = w_B$. By the definition of u_B and w_B for $B \in \mathcal{B}^{L2}$, we have $u_B, w_B \in V\big(T[x,y]\big)$. (See Table 1.) Since $b_1 \in V\big(C[s,x]\big)$ and $b_2 \in V\big(\overleftarrow{C}[y,t]\big)$, it follows from Claim 7 that b_1, u_B, w_B, b_2 appear in Q in this order. Then it follows from the construction of Q that $b_1, u_B, Q_B(u_B, v_B), w_B, b_2$ appear in Q in this order. Since $a_1 \in V\big(Q_B(u_B, v_B)\big)$ and $a_2 = w_B$, we see that the edges a_1a_2 and b_1b_2 do not cross at Q. This completes the proof of Case 2. $\square$

Case 3. $i = 4$.

We can show this case by the same way as Case 1. In fact, we have $a_1a_2 \in E_1^4$ and $b_1b_2 \in E_R$. It follows from the definition of E_1^4 that $a_1a_2 \in E_B^1 \cup E_B^4$. By the conditions (Q3)–(Q5) for B, we see that $a_1, a_2 \in V\big(Q_B[u_B, v_B)\big)$. Note that $Q_B[u_B, v_B)$ is contained in $Q[u_B, u_B^+)$, where u_B^+ is the successor of u_B at T, and hence $a_1, a_2 \in V\big(Q[u_B, u_B^+)\big)$. On the other hand, since $b_1, b_2 \in V(T)$, it follows from the construction of Q that $b_1, b_2 \notin V\big(Q(u_B, u_B^+)\big)$. Then it follows from Lemma 3 that a_1a_2 and b_1b_2 do not cross at Q, and this completes the proof of Case 3. $\square$

Case 4. $i = 6$.

In this case, $a_1a_2 \in E_1^6$ and $b_1b_2 \in E_x$. It follows from the condition (G3) and the symmetry between b_1 and b_2 that we may assume that $b_1 \in V\big(\overleftarrow{C}[s,y)\big)$ and $b_2 = x$. Note that the vertices s, b_1, x and y appear in Q in this order.

Case 4.1. $B \in \mathcal{B}^{L1} \cup \mathcal{B}^{L2} \cup \mathcal{B}^{L3} \cup \mathcal{B}^{R1} \cup \mathcal{B}^{R2}$.

In this case, we see that $B - \{u_B, v_B, w_B\}$ contains neither b_1 nor b_2. This implies that $b_1, b_2 \notin V\big(Q_B(u_B, v_B)\big)$. So, if $\{a_1, a_2\} \cap \{u_B, v_B, w_B\} = \emptyset$, then $a_1, a_2 \in V\big(Q_B(u_B, v_B)\big)$, and hence it follows from Lemma 3 that the edges a_1a_2 and b_1b_2 do not cross at Q. (Recall that the vertices in $Q_B(u_B, v_B)$ are contained in Q consecutively.) Therefore, we may assume that $\{a_1, a_2\} \cap \{u_B, v_B, w_B\} \neq \emptyset$. By the symmetry between a_1 and a_2, we may also assume that $a_2 = u_B$ or $a_2 = v_B$ or $a_2 = w_B$. It follows from the conditions (Q3)–(Q5) that $a_1a_2 \in E_B^1 \cup E_B^2 \cup E_B^3$. By the definition of E_1^6, we see that either (I) $B \in \mathcal{B}^{R1}$, $a_1a_2 \in E_B^2$, and $a_2 = v_B$, or (II) $B \in \mathcal{B}^{R2}$, $a_1a_2 \in E_B^3$, and $a_2 = w_B$.

Suppose first that the case (I) occurs. Then by the definition of u_B and v_B for $B \in \mathcal{B}^{R1}$, we have $u_B, v_B \in V\big(T[y,t]\big)$. (See Table 1.) Since $b_1 \in V\big(\overleftarrow{C}[s,y)\big)$ and $b_2 = x$, it follows from the planarity that b_1, b_2, u_B, v_B appear in Q in this order. Then it follows from the construction of Q that $b_1, b_2, u_B, Q_B(u_B, v_B), v_B$ appear in Q in this order. Since $a_1 \in V\big(Q_B(u_B, v_B)\big)$ and $a_2 = v_B$, the edges a_1a_2 and b_1b_2 do not cross at Q.

Suppose next that the case (II) occurs. Then by the definition of u_B and w_B for $B \in \mathcal{B}^{R2}$, we have $u_B, w_B \in V\big(T[x,y]\big)$. (See Table 1.) Since $b_1 \in V\big(\overleftarrow{C}[s,y)\big)$ and $b_2 = x$, it follows from the planarity that b_1, b_2, u_B and w_B appear in Q in this order (possibly $b_2 = x = u_B$). Then it follows from the construction of Q that $b_1, b_2, u_B, Q_B(u_B, v_B)$ and w_B appear in Q in this order. Since $a_1 \in V\big(Q_B(u_B, v_B)\big)$, we see that the edges a_1a_2 and b_1b_2 do not cross at Q. This completes the proof of Case 4.1. $\square$

Case 4.2. $B \in \mathcal{B}^{R3}$.

By the definition of E_1^6, we have $a_1a_2 \in E_B^3$. Then, it follows from the conditions (Q3) and (Q4) that $\{a_1, a_2\} \cap \{u_B, v_B\} = \emptyset$. Therefore, $a_1, a_2 \in V\big(T[s,x)\big)$. (See Table 1.)

Let u and v be two vertices in $T[s,y] \cap \overleftarrow{C}[s,y]$ such that u_B is contained in $T[u,v)$. (Note that s and y satisfy the conditions of u and v, respectively, and hence those vertices indeed exist.) Taking such vertices u and v so that $\overleftarrow{C}[u,v)$ is as short as possible, we may assume that no vertices in $\overleftarrow{C}[s,y]$ are contained in $T(u,v)$. By this choice and the planarity, we see that $u, u_B, (w_B,)\ x$ appear in T in this order (possibly, w_B does not exist, and/or $u = u_B$, and/or $v = x$). If $b_1 \notin V\big(\overleftarrow{C}(u,v)\big)$, then we have $a_1, a_2 \in V\big(Q[u,v]\big)$ and $b_1, b_2 \notin V\big(Q(u,v)\big)$. Then it follows from Lemma 3 that the edges a_1a_2 and b_1b_2 do not cross at Q. Thus, we may assume that $b_1 \in V\big(C(u,v)\big)$, and hence there exists a non-trivial T-bridge B' in $\mathcal{B}_2 \cap \mathcal{B}_u^R$ with $b_1 \in V(B') - \{u_{B'}, v_{B'}\}$. It follows from the construction of Q that $B' = B_q$ and

$$Q[u,b_1] \;=\; Q_{B_1}[u,v_{B_1})\; Q_{B_2}(u,v_{B_2}) \ldots\; Q_{B_p}(u,v_{B_p})\; Q_{B_q}(u,b_1],$$

where $\mathcal{B}_u^L = \{B_1, \ldots, B_p\}$ and $\mathcal{B}_u^R = \{B_{p+1}, \ldots, B_q\}$. In particular, all vertices in B' appear in $Q[u,u^+)$, where u^+ is the successor of u at T. So, if $u \neq u_B$, then we have $a_1, a_2 \in V\big(Q[u^+,v]\big)$ and $b_1, b_2 \notin V\big(Q(u^+,v)\big)$. Then it follows from Lemma 3 that the edges a_1a_2 and b_1b_2 do not cross at Q.

Therefore, we may assume that $u = u_B$. Let $B = B_j$ for some j with $p+1 \leq j \leq q$. If $j \neq q$, then we see that $a_1, a_2 \notin V\big(Q[u,b_1]\big)$, and hence the edges a_1a_2 and b_1b_2 do not cross at Q. Therefore, we may assume that $j = q$, which means that $a_1, a_2, b_1 \in V(B)$. Recall that $a_1a_2 \in E_B^3$. Then it follows from the conditions (Q3) and (Q4) that $a_1, a_2, b_1 \neq u_B, v_B$, and hence $a_1, a_2, b_1 \in V\big(Q_B(u_B, v_B)\big)$. Since $(Q_B^+, \mathscr{E}_B^+)$ is a 6-page book embedding of B^+, we see that the edges a_1a_2 and b_1w_B do not cross at Q_B^+. Therefore, it follows from the symmetry between a_1 and a_2 that a_1, a_2, b_1, w_B appear in Q_B^+ in this order or in the order b_1, a_1, a_2, w_B. Since $Q_B(u, v_B)$ is a subsequence of Q, we see that the edges a_1a_2 and b_1b_2 do not cross at Q, This completes the proof of Case 4.2, and the proof of Theorem 1 (iii). $\square$

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