Plane triangulations without a spanning Halin subgraph II

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Abstract

A Halin graph is a plane graph constructed from a planar drawing of a tree by connecting all leaves of the tree with a cycle which passes around the boundary of the graph. The tree must have four or more vertices and no vertices of degree two. Halin graphs have many nice properties such as being Hamiltonian and remain Hamiltonian after any single vertex deletion. In 1975, Lovász and Plummer conjectured that every 4-connected plane triangulation contains a spanning Halin subgraph. We recently gave a negative answer to this conjecture. In this paper, we construct an infinite class of 5-connected plane triangulations without a spanning Halin subgraph. Our smallest example contains 512 vertices.

Keywords: Halin graph, plane triangulation, k-connected graphs.
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1 Introduction

A *planar* graph is a graph that can be drawn on the plane in such a way that its edges intersect only at their endvertices. Any such drawing is called a *plane graph*. A *Halin graph*, denoted by $T \cup C$, is a plane graph constructed from a planar drawing of a tree T by connecting all leaves of the tree with a cycle C which passes around the boundary of the graph. The tree T must have four or more vertices and no vertices of degree two. Such a tree is called a *homeomorphically irreducible tree* (*HIT*). Halin graphs are named after Halin, who studied them in 1971 [8] as an example of a class of edge-minimal 3-connected graphs. His work was related to the characteristic properties of minimally k-connected graphs obtained by Dirac [7] and Tutte [12]. The family of Halin graphs is a natural generalization of the family of wheel graphs, where T is a star.

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Halin graphs have many interesting properties: Every Halin graph is Hamiltonian and remains Hamiltonian after any single vertex deletion [2]; every edge of a Halin graph belongs to a Hamilton cycle [5] and every edge is avoided by a Hamilton cycle [11]; every Halin graph is Hamilton-connected [1]; an n-vertex Halin graph has cycles of all lengths from 3 to n with the possible exception of a single even length [3, 10]. It is of interest to know nontrivial sufficient conditions for a graph to contain a spanning Halin subgraph. Lovász and Plummer [9] in 1975 conjectured that *every 4-connected plane triangulation has a spanning Halin subgraph*. In [4], we gave a negative answer for their conjecture by constructing 4-connected plane triangulations without a spanning Halin subgraph. These counterexamples are precisely 4-connected. In this paper, we show that, even if we assume 5-connectedness, the Lovász-Plummer conjecture is not true. Thus no stronger connectivity demand can be placed on a plane triangulation in order to guarantee the existence of a spanning Halin subgraph.

Theorem 1.1 *There exists an infinite family of 5-connected plane triangulations without a spanning Halin subgraph.*

The new construction of plane triangulations without a spanning Halin subgraph contains some ideas which are different from the ideas in [4]. In Section 2, we show some lemmas which will be used to prove that the constructed graphs do not have a spanning Halin subgraph. In Section 3, we construct an infinite family of 5-connected plane triangulations and prove that these graphs do not contain a spanning Halin subgraph.

2 Preliminaries

In this paper, we call a nonleaf vertex of a tree a *stem*. Thus in a HIT every stem has degree at least 3. By the handshaking lemma and the fact that every n-vertex tree has $n-1$ edges, we have the following proposition.

Proposition 2.1 *Every n-vertex HIT has at least $\lceil \frac{n}{2} \rceil + 1$ leaves.*

Let G be a plane graph containing a spanning Halin subgraph $H = T \cup C$. By Jordan's closed curve theorem, C separates the plane into two regions bounded by C. Throughout this paper we always call the region that contains $V(T)$ the *interior of* C and denote it by I_C and call the other region the *exterior of* C and denote it by O_C. A *chord* of C is an edge incident to two nonconsecutive vertices of C.

Lemma 2.2 *If a plane graph G contains a spanning Halin subgraph $H = T \cup C$, then the interior of C contains no chord of C.*

We will use the following well-known result in the proof of Claim 3.1, and give a proof for completeness.

Lemma 2.3 *In a plane triangulation, every minimal cut induces a cycle.*

Proof. Let G be a plane triangulation, X be a minimal cut of G and D be a component of $G - X$. Since every face of G is a triangle, $G[N(D) \cap X]$, the subgraph induced by $N(D) \cap X$, is a cycle C separating D from other components. Consequently, $V(C) \subseteq X$ is a cut of G. By the minimality of X, we have $V(C) = X$. We claim that C is an induced cycle of G. Otherwise, C has a chord c_1c_2 where $c_1, c_2 \in V(C)$ and $c_1c_2 \notin E(C)$. Let C_1 and C_2 be two paths on C from c_1 to c_2. Since C separates D and other components of $G - V(C)$, we can see that either $C_1 \cup \{c_1c_2\}$ or $C_2 \cup \{c_1c_2\}$ separates D and some components of $G - V(C)$. By symmetry, we may assume that $C_1 \cup \{c_1c_2\}$ is such a separating cycle. Then $V(C_1)$ is a cut of G and a proper subset of $V(C)$ which gives a contradiction to the assumption that $V(C) = X$ is a minimal cut of G. $\square$

3 Plane triangulations without spanning Halin subgraphs

In this section, we first construct an infinite family $\mathcal{G}_5$ of 5-connected plane triangulations. After that, we show that the triangulations have no spanning Halin subgraph. The following graphs play important roles in our constructions.

- **Dodecahedral graph:** An embedding of the dodecahedral graph is depicted in Figure 1. It is the unique cubic plane graph consisting of twenty vertices, thirty edges and twelve pentagonal faces, which will be called *Large Pentagons* (LP) in our construction.

- **Pentagonal Wheel (PW):** The PW is the plane graph depicted in Figure 2, where bold lines represent the exterior boundary cycle of it. It contains sixteen vertices, thirty edges, one pentagonal exterior face, five pentagonal interior faces named *Small Pentagons* (SP) and ten triangles sharing a common vertex (the center). Each SP shares one edge with the exterior face and each triangle shares one edge with an SP.

- **Pentagonal Chart of m sheaves (PC(m)):** The PC(m) is the plane graph depicted in Figure 3 with a vertex c (the center), m 5-cycles $v_iw_ix_iy_iz_iv_i$ for $i = 1, 2, \ldots, m$, five paths $cv_1v_2 \ldots v_m$, $cw_1w_2 \ldots w_m$, $cx_1x_2 \ldots x_m$, $cy_1y_2 \ldots y_m$, $cz_1z_2 \ldots z_m$ and $5(m-1)$ diagonal edges v_iw_{i+1}, w_ix_{i+1}, x_iy_{i+1}, y_iz_{i+1}, z_iv_{i+1} for each $i = 1, 2, \ldots, m-1$. Note that Figure 3 shows a PC(3), and bold lines (resp., dashed lines) denote the boundary cycle (resp., diagonal edges) of it. Clearly, a PC(m) has $5m+1$ vertices, $15m-5$ edges, an exterior pentagonal face and $10m-5$ interior triangular faces.

Construction of $\mathcal{G}_5$: We construct an infinite family of 5-connected plane triangulations as follows. We first place the dodecahedral graph on the plane with twelve LPs. We replace each of twelve LPs with a copy of the PW by identifying the 5-cycle of the LP with the boundary cycle of the PW. We then replace each five SPs of all PWs as a PC(m) for some $m \geq 2$. The resulted graph is denoted by G_m. In summary, in the graph G_m there are five PC(m)s embedded in each PW and there is a PW embedded in each of twelve LPs. A simple calculation shows that G_m has $300m - 88$ vertices. So, the smallest $G_m \in \mathcal{G}_5$ is obtained when $m = 2$, and has 512 vertices.

Let $G_m \in \mathcal{G}_5$ and P be an LP or an SP of G_m. If P is an LP, we call the region corresponding to the open pentagonal face of the dodecahedral graph the *interior region* of P. If P is an SP, we call the region corresponding to the pentagonal face of the PW the *interior region* of P. In either case, we denote by P^I the subgraph induced by the vertices of G_m in the interior region of P and by $\overline{P}$ the subgraph induced by $V(P \cup P^I)$. The vertices in P^I are called *inner vertices* of P.

Claim 3.1 *Every $G_m \in \mathcal{G}_5$ is 5-connected.*

Proof. Assume on the contrary that there exists a graph $G_m \in \mathcal{G}_5$ which is not 5-connected. By Lemma 2.3, G_m has a separating cycle D with at most four vertices.

By tedious check, PC(m) ($m \geq 2$) contains neither separating k-cycles ($k = 3, 4$), chords nor separating paths of length 2. This implies that G_m has no SP Q containing all vertices of D in $\overline{Q}$. Also, for all SP Q, $\overline{Q}$ contains neither chords nor separating paths of length 2.

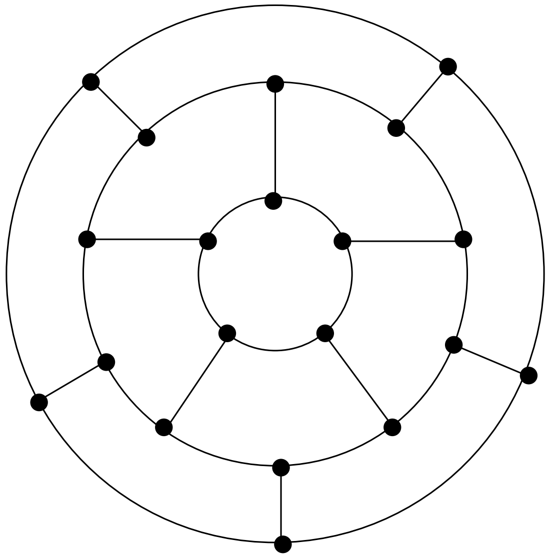

Figure 1: The dodecahedral graph.

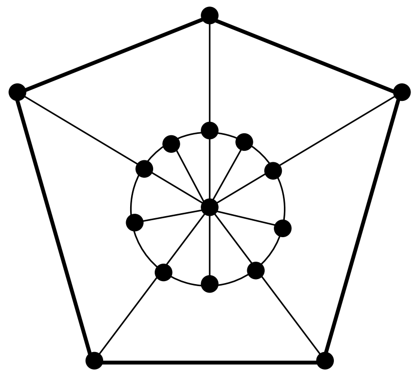

Figure 2: The pentagonal wheel.

Suppose that there exists an LP P such that $V(D) \subset V(\overline{P})$. Recall that $\overline{P}$ consists of a PW with five SPs and a center vertex c. If $c \notin V(D)$, then D is contained in two SPs Q_1 and Q_2, because there is no SP Q containing all vertices of D in $\overline{Q}$. However this implies that either $\overline{Q_1}$ or $\overline{Q_2}$ contains a chord or a separating path of length 2, a contradiction. So, we may assume $c \in V(D)$. Then D contains two neighbors of c and a vertex nonadjacent to c. If the two neighbors of c are adjacent, then there exists an SP Q such that $\overline{Q}$ contains a chord or a separating path of length 2. If the two neighbors of c are nonadjacent, then there exists an SP Q such that $\overline{Q}$ contains a separating path of length 2. In either case, we obtain a contradiction. Therefore, we can see that there is no LP P containing all vertices of D in $\overline{P}$.

So, there is an LP P such that $V(D)$ contains a vertex $x \in V(P^I)$, a vertex $y \in V(G_m) - V(\overline{P})$ and two vertices $u, v \in V(P)$. Since $x \in V(P^I)$, $N(x) \cap V(P)$ is either empty, a single vertex or two consecutive vertices of P. So, u and v are two consecutive vertices of P. Since $\overline{P}$ does not have a separating 3-cycle, G_m does not have any vertices in the interior region of the triangle $xuvx$. Similarly, there is no vertices in the interior region of the triangle $yuvy$. Hence D is not

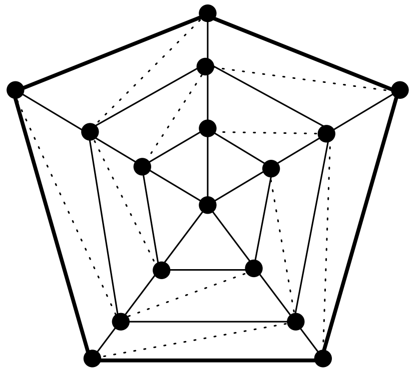

Figure 3: The pentagonal chart (PC(3)).

a separating cycle, giving a contradiction. □

Theorem 3.1 *No graph in $\mathcal{G}_5$ contains a spanning Halin subgraph.*

Proof. Suppose by way of contradiction, there is a graph $G := G_m \in \mathcal{G}_5$ containing a spanning Halin subgraph $H = T \cup C$. By our assumption, the interior region I_C contains the tree T and all chords of C are in the exterior region O_C.

Claim 3.2 *Let P be either an LP or an SP of G. Then $V(C) \cap V(P^I) \neq \emptyset$.*

Proof. Suppose on the contrary $V(C) \cap V(P^I) = \emptyset$. Since T is a HIT, we have $d_T(v) \geq 3$ for all $v \in V(P^I)$. Let T' denote the subgraph of T induced by vertices in $\overline{P}$. For any $v \in V(T') \cap V(P^I)$, we have $d_{T'}(v) = d_T(v) \geq 3$. By applying Proposition 2.1 to each component of T', we get $|V(P)| \geq \lceil \frac{|V(P \cup P^I)|}{2} \rceil + 1$. Since $m \geq 2$, we have $|V(P^I)| \geq 6$. So $|V(P)| \geq |V(P^I)| + 2 \geq 8$, giving a contradiction to the assumption that P is a pentagon. □

Let $P := v_1v_2v_3v_4v_5v_1$ be an arbitrary LP or SP such that the vertices are listed in the clockwise order. Let C_P be the graph obtained from $C \cap \overline{P}$ by deleting components not containing vertices in P^I. Since $|V(P)| = 5$, C_P has at most two components.

Claim 3.3 *C_P is connected.*

Proof. Suppose on the contrary that C_P has two components W_1 and W_2. By the symmetry of these boundary vertices, we assume that v_2, v_3, v_4 and v_5 are four endvertices of C_P. Note that v_1 may be a vertex of C_P. Moreover, we assume that $v_3 \in V(W_1)$. Since W_1 and W_2 do not cross each other in P^I, we have $v_5 \notin V(W_1)$ and v_2 and v_4 are not on the same components. So there are two possibilities:

- **Case 1.** $v_2, v_3 \in V(W_1)$ and $v_4, v_5 \in V(W_2)$
- **Case 2.** $v_3, v_4 \in V(W_1)$ and $v_2, v_5 \in V(W_2)$

Case 1. Since W_1 connects v_2 and v_3 and contains an inner vertex of P^I, v_2v_3 is a chord of C. Since v_3 and v_4 are in two different components of C_P, $v_3v_4 \notin E(C)$, which in turn gives that v_3v_4 is a chord of C. Since, along cycle C, v_3 has one neighbor in P^I and the other one in the exterior region of P, the chords v_2v_3 and v_3v_4 lie in two different regions of the plane determined by C. So, one of them must be in I_C, giving a contradiction of Lemma 2.2.

Case 2. Since W_1 connects v_3 and v_4 and contains an inner vertex in P^I, v_3v_4 is a chord of C. Since v_2 and v_3 are in two different components of C_P, $v_2v_3 \notin E(C)$, which in turn gives v_2v_3 is a chord of C. Following the same argument as in Case 1, we have chords v_2v_3 and v_3v_4 lying in two different regions of the plane determined by C, which gives a contradiction of Lemma 2.2. □

Since $T - V(C)$, induced by the stems of T, is connected and C_P is a path, there are three possible scenarios for the locations of I_C and these are depicted in Figure 4:

- **Type (a):** Two endvertices of C_P are consecutive on the pentagon P, say v_4 and v_5. The subgraph P^I is embedded in the region bounded by C_P and the path $v_5v_1v_2v_3v_4$ (due to the fact that v_4v_5 is a chord of C).
- **Type (b):** Two endvertices of C_P are not consecutive on the pentagon P, say v_2 and v_5, and P^I is embedded in the region bounded by C_P and $v_5v_1v_2$.
- **Type (c):** Two endvertices of C_P are not consecutive on the pentagon P, say v_2 and v_5, and P^I is embedded in the region bounded by C_P and $v_2v_3v_4v_5$.

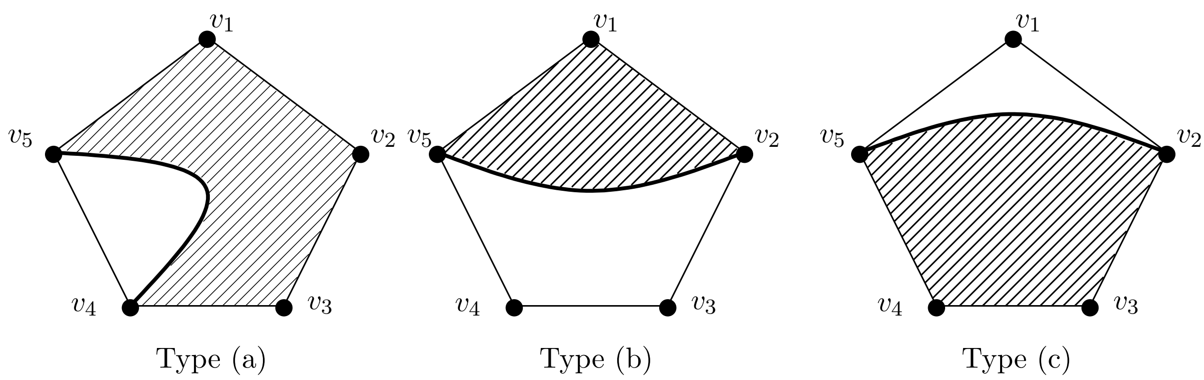

Figure 4: Type (a), type (b) and type (c) (P^I is embedded in the the shaded area).

Claim 3.4 *Let P be an LP and suppose that C_P is as depicted in Figure 4 with the same labeling of vertices. Then the following vertices are stems of T (i.e., they are not contained in $V(C)$).*

- v_1, v_2, v_3 *if C_P is Type (a),*
- v_1 *if C_P is Type (b), and*
- v_3, v_4 *if C_P is Type (c).*

Proof. Suppose on the contrary that one of vertices given in Claim 3.4 is a leaf of T, so it is on C. Let v denote such a vertex. By the definition of C_P, v may not be in C_P.

Case 1. $v \notin V(C_P)$: If C_P is Type (b), then $v = v_1$, so both v_1v_2 and v_1v_5 are chords of C and in I_C. If C_P is Type (c), then $v = v_3$ or v_4, say $v = v_3$. In this case, v_2v_3 is a chord of C in I_C. In either case, we obtain a contradiction of Lemma 2.2. So, we may assume that C_P is Type (a). Since $v_1v_5, v_3v_4 \in E(G)$ and I_C does not contain a chord, following the same reasoning above, we get $v \neq v_1$ and $v \neq v_3$, so $v = v_2$.

For each i ($1 \le i \le 5$), let S_i be the SP in $\overline{P}$ that contains $v_i v_{i+1}$, where $v_6 = v_1$. Recall that C_{S_i} is the unique component of C in S_i which contains an inner vertex in S_i^I. Let w_i be the other common vertex of S_{i-1} and S_i, that is, $\{v_i, w_i\} = V(S_{i-1} \cap S_i)$ for each $1 \le i \le 5$, and let c be the center vertex of $\overline{P}$.

We claim that one of v_1, v_3 and w_2 is on the path C_P. Suppose not. Recall that we have assumed that $v = v_2$ is not on the path C_P. Since C does not pass through any of v_1, v_2 and w_2, C_{S_1} is Type (a) such that the endvertices are vertices on S_1 other than v_1, v_2 or w_2. For the same reason, C_{S_2} is also Type (a) such that the endvertices are vertices on S_2 other than v_2, v_3 or w_2. This implies that c is contained in C in order to connect C_{S_1} and C_{S_2}. Since C contains at least one inner vertex of S_4 by Claim 3.2, either w_4 or w_5 is an endvertex of C_{S_4}. So, either v_4w_4 or v_5w_5 is a chord of C in I_C, contrary to Lemma 2.2.

Since v_1, v_3, and w_2 are three neighbors of v_2, the above claim implies that one of the edges v_1v_2, v_3v_2, and w_2v_2 is a chord of C in I_C, contrary to Lemma 2.2.

Case 2. $v \in V(C_P)$. Let Q and R be the other two LPs containing v. Since $v \in V(C_P)$, $v \in \{v_1, v_2, v_3\}$ if C_P is Type (a), $v \in \{v_1\}$ if C_P is Type (b) and $v \in \{v_3, v_4\}$ if C_P is Type (c). By Jordan's closed curve theorem, if we apply the fact shown in Case 1 to Q, then we can see that $v \in V(C_Q)$ (otherwise $v \notin V(C_Q)$, and hence $v \notin V(C)$, giving a contradiction). Since both C_P and C_Q are paths and v is not an endvertex of C_P, $C_P \cap C_Q$ is the common edge of C_P and C_Q. Thus v is an endvertex of C_Q. By the same reasoning, v is an endvertex of C_R. However, this is a contradiction of the assumption that $v \in V(C_P)$. $\square$

Let L_i ($0 \le i \le 11$) be LPs depicted in Figure 5, and let w_j ($1 \le j \le 20$) be the original vertices of the dodecahedral graph, as depicted in Figure 5.

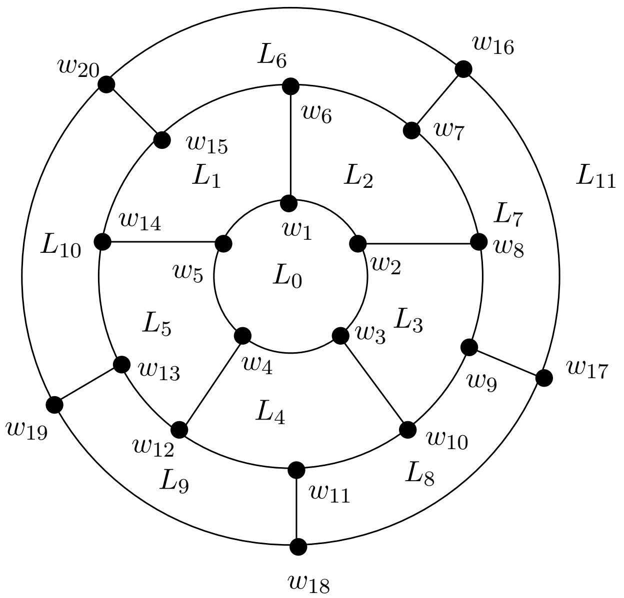

Figure 5: L_i ($0 \le i \le 11$) and w_j ($1 \le j \le 20$).

Claim 3.5 *For any $a, b \in \{0, \ldots, 11\}$ where $a \ne b$, C_{L_a} and C_{L_b} have at most one common vertex.*

Proof. Suppose on the contrary that there exist two LPs L_a and L_b such that C_{L_a} and C_{L_b} have two or more common vertices. Note that $C_{L_a} \cup C_{L_b}$ contains no cycle. This implies that

$C_{L_a} \cup C_{L_b}$ is a path, and hence C_{L_a} and C_{L_b} have a common edge e. Since $V(L_a^I) \cap V(\overline{L_b}) = \emptyset$, $e \in E(L_a)$. By the same reasoning, $e \in E(L_b)$. Let $e = w_x w_y$, that is, w_x and w_y are two endvertices of e. Since two pentagons in the dodecahedral graph share at most one edge, we may assume the other neighbor of w_x on C is not in L_a. Then w_x is an endvertex of C_{L_a}, which in turn shows that the other neighbor of w_y along C is in L_a. So, w_y is an endvertex of C_{L_b}. Along cycle C, one side of the edge $w_x w_y$ must be in I_C and the other side in O_C (see Figure 6).

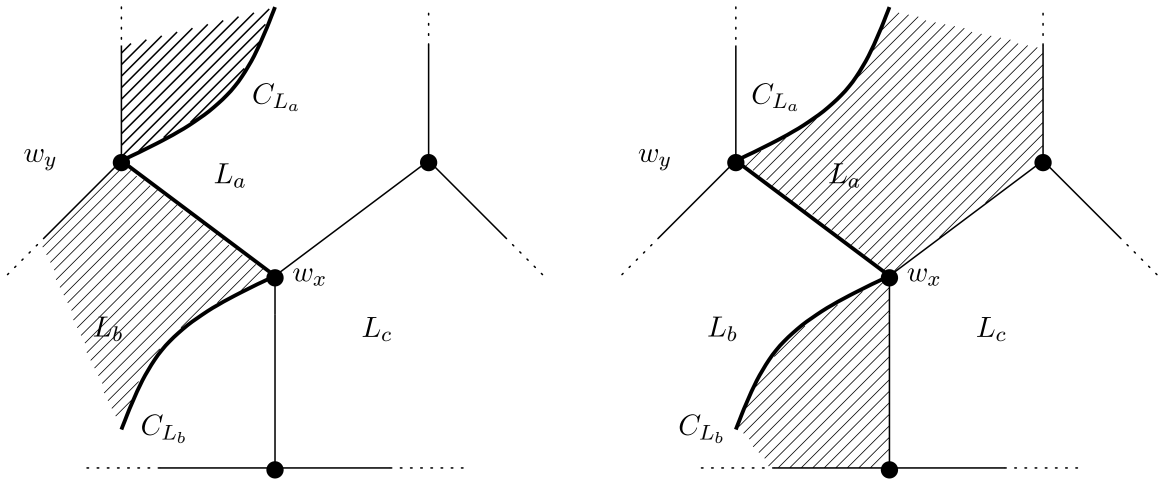

Figure 6: The edge $e = w_x w_y$ (the shaded area is the interior region of C).

By symmetry, we may assume that the L_a-side of $w_x w_y$ is I_C (i.e., the illustration shown on the right in Figure 6). Applying Claim 3.4 to L_b, we have that w_x is a stem of T, that is, w_x is not on C, giving a contradiction. $\square$

Claim 3.6 *$E(C) = \bigcup_{i=0}^{11} E(C_{L_i})$, that is, every edge on C is contained in C_{L_i} for some $i = 0, 1, \ldots, 11$.*

Proof. Suppose that there exists an edge e on C which is not contained in C_{L_i} for any i. By the definition of C_P for a pentagon P, if at least one vertex of e is in P^I, then $e \in E(C_P)$. So, we may assume $e \in E(L_c) \cap E(L_d)$, and we can then denote the edge e as $w_x w_y$. Then neither w_x nor w_y is an endvertex of C_{L_c} (resp., C_{L_d}), otherwise $w_x w_y$ is contained in C_{L_c} (resp., C_{L_d}). Moreover, by Claim 3.4 and the fact that $w_x w_y$ on C (i.e., neither w_x nor w_y is a stem of T), we can see that both C_{L_c} and C_{L_d} are Type (b), and $w_x w_y$ corresponds to $v_3 v_4$ in the center of Figure 4. However, by Jordan's closed curve theorem, one side of $w_x w_y$ is contained in I_C, giving a contradiction. $\square$

Claim 3.7 *For each endvertex v of C_{L_i}, there is some $j \neq i$ such that v is an endvertex of C_{L_j}.*

Proof. Let v be an endvertex of C_{L_i} for some $i = 0, 1, \ldots, 11$. Let w be the neighbor of v in C which is not contained in L_i. By Claim 3.6, vw is in C_{L_j} for some j. Clearly, $L_j \neq L_i$. By Claim 3.5, C_{L_i} and C_{L_j} share at most one vertex. So, v is an endvertex C_{L_j}. $\square$

Along cycle C, by Claim 3.7, we obtain an alternating cyclic order of LPs and endvertices of C_{LP}. We call such a cyclic order the *LP-order*, and denote it $L_a w_x L_b w_y \cdots L_a$.

Claim 3.8 *Let L_a, L_b and L_c be three LPs that share a vertex w_d. If the LP-order contains $\cdots L_a w_d \cdots$ and the area indicated by the arrow in Figure 7 is the interior region of C, then the LP-order contains $\cdots L_a w_d L_b \cdots$.*

Proof. Suppose that the LP-order contains $\cdots L_a w_d L_c \cdots$. By Jordan's closed curve theorem and Claim 3.4 for L_b, w_d is a stem of T, a contradiction. $\square$

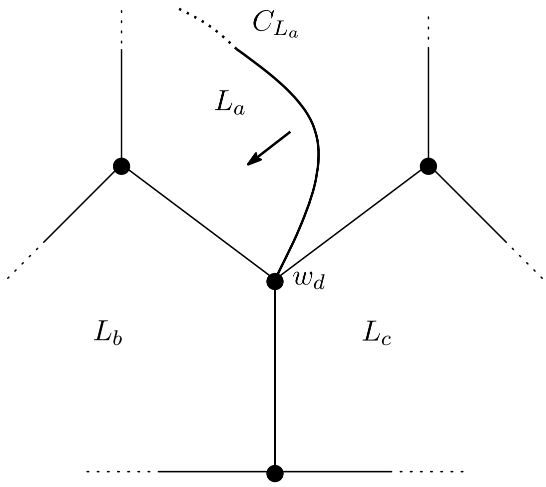

Figure 7: L_a, L_b, L_c and w_d.

By Claim 3.2, the following statement holds and will be used to complete our proof.

C goes through all LPs, that is, the LP-order contains L_i for all i $(0 \leq i \leq 11)$. $(*)$

Claim 3.9 *For any $i \in \{0, 1, \ldots, 11\}$, C_{L_i} is not Type (a) .*

Proof. Suppose on the contrary that C_{L_0} is Type (a). We may assume that w_4 and w_5 are the two endvertices of C_{L_0}. By Claim 3.4, we can see that all of w_1, w_2 and w_3 are interior vertices. Applying Claim 3.8 twice, we may assume the segment of the LP-order around L_0 is

$$\cdots L_4 w_4 L_0 w_5 L_1 \cdots .$$

Hence we can see that C_{L_5} is Type (b) such that w_{12} and w_{14} are both endvertices of C_{L_5}. Then, w_{13} is a stem of T by Claim 3.4. By Jordan's closed curve theorem and Claim 3.8, the LP-order is

$$\cdots L_4 w_4 L_0 w_5 L_1 \cdots L_{10} w_{14} L_5 w_{12} L_9 \cdots .$$

By symmetry, we may assume further that the LP-order is

$$\cdots L_4 w_4 L_0 w_5 L_1 \cdots L_{10} w_{14} L_5 w_{12} L_9 \cdots L_{11} \cdots .$$

By this assumption, Claim 3.8 and the fact $(*)$, it follows that $C_{L_{10}}$ contains neither w_{19} nor w_{20}. This implies that the other endvertex of $C_{L_{10}}$ is w_{15}. Thus the LP-order is

$$\cdots L_4 w_4 L_0 w_5 L_1 \cdots w_{15} L_{10} w_{14} L_5 w_{12} L_9 \cdots L_{11} \cdots .$$

By Claim 3.8 and the fact that w_1 is a stem of T, the LP-order is

$$\cdots L_4 w_4 L_0 w_5 L_1 w_6 L_2 \cdots L_6 w_{15} L_{10} w_{14} L_5 w_{12} L_9 \cdots L_{11} \cdots .$$

By the same arguments in L_6 (i.e., w_{16} is not an endvertex of C_{L_6}), w_7 is an endvertex of C_{L_6}. Also, by Claim 3.8 and the fact that w_2 is a stem of T, w_8 is an endvertex of $C(L_2)$. Thus, the LP-order is

$$\cdots L_4 w_4 L_0 w_5 L_1 w_6 L_2 w_8 L_3 \cdots L_7 w_7 L_6 w_{15} L_{10} w_{14} L_5 w_{12} L_9 \cdots L_{11} \cdots .$$

By the same arguments in L_7 (i.e., w_{17} is not an endvertex of C_{L_7}), w_9 is an endvertex of C_{L_7}. Also, by Claim 3.8 and the fact that w_3 is a stem of T, w_{10} is an endvertex of C_{L_3}. Thus, the LP-order is

$$\cdots L_4 w_4 L_0 w_5 L_1 w_6 L_2 w_8 L_3 w_{10} \cdots L_8 w_9 L_7 w_7 L_6 w_{15} L_{10} w_{14} L_5 w_{12} L_9 \cdots L_{11} \cdots .$$

However, by Claim 3.8 the LP-order contains

$$L_4 w_4 L_0 w_5 L_1 w_6 L_2 w_8 L_3 w_{10} L_4,$$

giving a contradiction of the fact $(*)$. □

Claim 3.10 *For any $i \in \{0, 1, \ldots, 11\}$, C_{L_i} is not Type (b).*

Proof. Suppose on the contrary that C_{L_0} is Type (b). We may assume that w_2 and w_5 are the two endvertices of C_{L_0}. Thus w_1 is a stem of T by Claim 3.4. Applying Claim 3.8 twice, we may also assume the segment of the LP-order around L_0 is

$$\cdots L_1 w_5 L_0 w_2 L_2 \cdots .$$

By Claim 3.9, C_{L_1} is either Type (b) or Type (c), that is, the other endvertex of C_{L_1} is either w_6 or w_{15}. If w_6 is an endvertex of C_{L_1}, then, by Claim 3.8, the LP-order is

$$L_2 w_6 L_1 w_5 L_0 w_2 L_2,$$

giving a contradiction.

Thus we may assume that w_{15} is the other endvertex of C_{L_1}. By the symmetrical arguments in L_2, w_7 is the other endvertex of C_{L_2}. By Claim 3.8, the LP-order is

$$L_6 w_{15} L_1 w_5 L_0 w_2 L_2 w_7 L_6,$$

giving a contradiction of the fact $(*)$. □

By Claims 3.9 and 3.10, C_{L_i} is Type (c) for all i. We may assume that w_5 and w_2 are the two endvertices of C_{L_0}. Applying Claim 3.8 twice, we may also assume the segment of the LP-order around L_0 is

$$\cdots L_5 w_5 L_0 w_2 L_3 \cdots .$$

By Claim 3.8 and the fact that C_P is Type (c) for all LPs P, we know that the LP-order is

$$\cdots L_9 w_{13} L_5 w_5 L_0 w_2 L_3 w_9 L_8 \cdots .$$

By the same arguments in L_8 and L_9, the LP-order is

$$L_9 w_{13} L_5 w_5 L_0 w_2 L_3 w_9 L_8 w_{18} L_9,$$

giving a contradiction of the fact $(*)$.

This completes the proof of Theorem 3.1. □

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