

# On dominating even subgraphs in cubic graphs

Roman Čada\*    Shuya Chiba†    Kenta Ozeki‡    Kiyoshi Yoshimoto§

January 15, 2017

## Abstract

It is known that a 3-edge-connected graph has a spanning even subgraph in which every component contains at least five vertices, and the lower bound is best possible. A natural question arises whether we can improve the lower bound by changing the spanning property with the dominating property. In this paper, we show that a 3-edge-connected cubic graph has a dominating even subgraph in which every component contains at least six vertices.

## 1 Introduction

In this paper, we consider finite graphs without loops. An *even* graph is a graph in which every vertex has a positive even degree and a subgraph  $H$  of a graph  $G$  is said to be *dominating* if  $G - V(H)$  is edgeless. In this paper, a *cycle* is a connected 2-regular graph and a cycle with  $l$  vertices is called an  $l$ -*cycle*. A *2-factor* is a spanning 2-regular subgraph of a graph. An *edge-cut* is a minimal set of edges whose removal increases the number of components of the graph. We call an edge-cut with  $l$  edges

---

\*Department of Mathematics, University of West Bohemia; Centre of Excellence ITI – Institute for Theoretical Computer Science, Charles University; European Centre of Excellence NTIS – New Technologies for the Information Society; P.O. Box 314, 306 14 Pilsen, Czech Republic (E-mail address: [cadar@kma.zcu.cz](mailto:cadar@kma.zcu.cz)). Research partly supported by the project P202/12/G061 of the Czech Science Foundation and by the project LO1506 of the Czech Ministry of Education, Youth and Sports.

†Applied Mathematics, Faculty of Advanced Science and Technology, Kumamoto University, 2-39-1 Kurokami, Kumamoto 860-8555, Japan (E-mail address: [schiba@kumamoto-u.ac.jp](mailto:schiba@kumamoto-u.ac.jp)). This author's work was partly supported by JSPS KAKENHI Grant Numbers 26800083, 26400187.

‡National Institute of Informatics; JST, ERATO, Kawarabayashi Large Graph Project; 2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan (E-mail address: [ozeki@nii.ac.jp](mailto:ozeki@nii.ac.jp)). This author's work was partly supported by JSPS KAKENHI Grant Number 25871053 and by Grant for Basic Science Research Projects from The Sumitomo Foundation.

§Department of Mathematics, College of Science and Technology, Nihon University, Tokyo 101-8308, Japan (E-mail address: [yosimoto@math.cst.nihon-u.ac.jp](mailto:yosimoto@math.cst.nihon-u.ac.jp)). This author's work was partly supported by JSPS KAKENHI Grant Number 26400190.

an  $l$ -cut. An edge-cut is said to be *essential* if both of the two new components after deleting it have at least one edge.

For a vertex subset  $X \subset V(G)$ , the set of edges joining  $X$  and  $V(G) - X$  is denoted by  $\partial(X)$  or simply  $\partial X$ . If  $X$  consists of one vertex  $u$ , then we denote it simply by  $\partial(u)$ . For a subgraph  $H$  of  $G$ , we use  $\partial H$  instead of  $\partial(V(H))$ . For terminology and notation not defined in this paper, we refer the readers to [5].

In this paper we consider cubic graphs, i.e., 3-regular graphs. A classical result by Petersen [17] says that a bridgeless cubic graph has a 2-factor. This well-known result was generalized by Fleischner [10] as follows: a bridgeless graph with minimum degree at least three has a spanning even subgraph in which every component has at least three vertices. If we restrict ourselves to simple graphs, then the lower bound on the order of components is improved to four in [13]. Jackson and the fourth author considered 3-edge-connected graphs and showed the following.

**Theorem A** (Jackson and Yoshimoto [14]). *A 3-edge-connected graph with  $n$  vertices has a spanning even subgraph in which each component contains at least  $\min\{5, n\}$  vertices.*

They also gave an infinite family of 3-edge-connected cubic graphs in which every 2-factor contains 5-cycles. Thus the lower bound in the theorem is best possible in some sense. Kaiser and Škrekovski gave an interesting result, which also generalizes the Petersen's theorem.

**Theorem B** (Kaiser and Škrekovski [15]). *Every graph has an even subgraph which intersects all 3-cuts and 4-cuts.*

If a given graph is bridgeless and cubic, then for any  $u \in V(G)$ ,  $\partial(u)$  is always a 3-cut of the graph, and so the above theorem implies the following.

**Corollary C.** *A bridgeless cubic graph has a 2-factor which intersects all 3-cuts and 4-cuts.*

If a given cubic graph is 3-edge-connected, then for an  $l$ -cycle  $C$  where  $l \in \{3, 4\}$ ,  $\partial C$  is an  $l$ -cut of the graph, and so Corollary C implies that a 3-edge-connected cubic graph has a 2-factor in which every component contains at least five vertices.

Matthews and Sumner [16] conjectured that 4-connected claw-free graphs are hamiltonian and Ryjáček [18] showed the Matthews-Sumner conjecture is equivalent to the conjecture by Thomassen et al. [2, 4, 19] that 4-connected line graphs are

hamiltonian. Fleischner and Jackson [11] showed that the conjecture on line graphs, and so the Matthews-Sumner conjecture, is equivalent to the conjecture by Ash and Jackson [3] that an essentially 4-edge-connected cubic graph has a dominating cycle. Thus it is interesting and important to study the behavior of dominating subgraphs in cubic graphs. In this paper, we prove the following using Corollary C.

**Theorem 1.** *A 3-edge-connected cubic graph has a hamilton cycle or a dominating even subgraph  $F$  such that every component in  $F$  contains at least six vertices and  $F$  intersects all essential 3-cuts.*

In Section 2, we give several preparations for the proof of Theorem 1 and in Section 3, the proof will be given. Furthermore, we will give remarks on even subgraphs of 3-edge-connected cubic graphs and the Traveling salesman problem in Section 4.

We conjecture that Theorem 1 can be generalized as in Theorem A.

**Conjecture 1.** *A 3-edge-connected graph with  $n$  vertices has a dominating even subgraph in which each component has at least  $\min\{6, n\}$  vertices.*

Also it is a natural question to ask about the lower bound “6” in Theorem 1 and Conjecture 1.

**Problem 2.** *What is the maximum integer  $k$  such that any 3-edge-connected graph has a dominating even subgraph in which each component has at least  $\min\{k, n\}$  vertices?*

The following example implies that the upper bound must be at most nine.

**Fact 1.** *There is an infinite family of 3-edge-connected cubic graphs in which every dominating even subgraph has a cycle of order at most nine.*

*Proof.* We construct such a cubic graph. Let  $S$  be the graph as in Figure 1, where  $S$  has 34 vertices, 49 edges and four “half-edges” whose one end is in  $S$ . Later we define the other ends of the half-edges.

Let  $m$  and  $l$  be positive integers with  $3m = 4l$ . Let  $B$  be  $l$  copies of  $S$  and  $A$  be  $m$  mutually disjoint triangles with three half-edges incident to each vertex of the triangle. See Figure 2. Since  $A$  and  $B$  have  $3m$  and  $4l$  half-edges, respectively, and  $3m = 4l$ , we can pair up half-edges in  $A$  with half-edges in  $B$ . It is easy to pair up them so that the obtained graph  $G$  is 3-edge-connected.

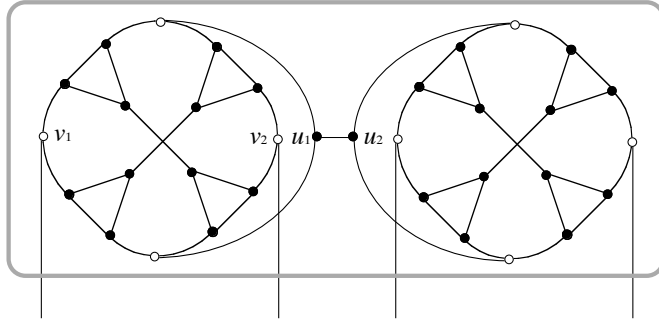


Figure 1:

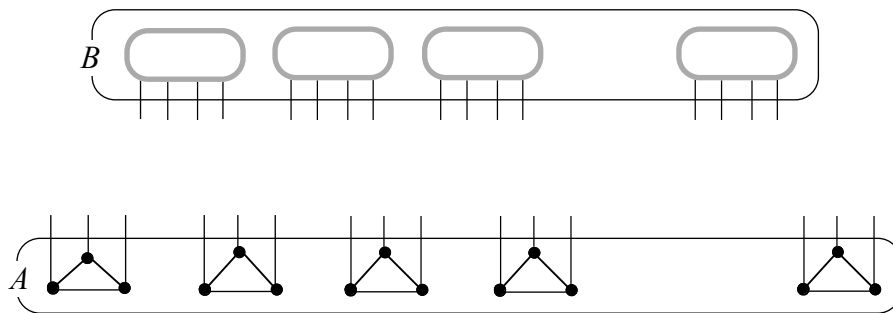


Figure 2:

We show that any dominating even subgraph  $F$  in  $G$  has a cycle of order at most nine. If there is a triangle  $T$  in  $A$  such that  $\partial T \cap F = \emptyset$ , then obviously  $F$  contains the 3-cycle  $T$  as a component. Suppose  $\partial T \cap F \neq \emptyset$  for all triangles  $T$  in  $A$ . Since  $F$  is an even subgraph,  $|\partial T \cap F| = 2$  for all triangles  $T$  in  $A$ , and so  $F$  contains  $2m$  edges joining  $A$  and  $B$ . Since  $2m = 8l/3$ , there is a component  $S$  in  $B$  such that  $|\partial S \cap F| = 4$ .

Let  $u_1u_2$  be the edge in the middle of  $S$ , see Figure 1. Since  $F$  is a dominating subgraph in  $G$ , at least one of the vertices  $u_1$  and  $u_2$  is contained in  $F$ , say  $u_1$ . Let  $L_S$  be the left component of  $S - u_1u_2$ . Since  $F$  is an even subgraph, we have  $|\partial L_S \cap F| = 2$ , and hence  $u_1u_2 \notin F$  and  $\partial(u_1) - \{u_1u_2\} \subset F$ . Note that both  $v_1$  and  $v_2$  are contained in  $F$ . Then by the structure of  $L_S$ , it is an easy observation that  $v_1$  and  $v_2$  belong to the same component of  $F$  which is different from the one containing  $u_1$ , and furthermore the component containing  $u_1$  has at most nine vertices. This completes the proof of Fact 1.  $\square$

The following question is also natural.

**Problem 3.** *Does a 3-edge-connected cubic graph have a dominating even subgraph  $F$  such that every component in  $F$  contains at least six vertices and  $F$  intersects all essential 3-cuts and 4-cuts?*

## 2 Preparations

First, we give some additional notation. The set of all the neighbours of a vertex  $x \in V(G)$  is denoted by  $N_G(x)$  or simply  $N(x)$ , and its cardinality by  $d_G(x)$  or  $d(x)$ . For a subgraph  $H$  of  $G$ , we denote  $N_G(x) \cap V(H)$  by  $N_H(x)$  and its cardinality by  $d_H(x)$ . For simplicity, we denote  $|V(H)|$  by  $|H|$  and “ $u_i \in V(H)$ ” by “ $u_i \in H$ ”. Similarly  $G - V(H)$  is denoted by  $G - H$ .

Recall that an edge-cut is said to be *essential* if both of the two new components after deleting it have at least one edge. This definition directly implies the following fact, which will be implicitly used in our proofs.

**Fact 2.** *For a 2-edge-connected cubic graph  $G$  and for a  $k$ -cut  $T$ , both of the following two hold:*

- *If  $k = 2$ , then  $T$  is always an essential cut.*

- If  $k = 3$  and  $T = \partial S$  for some  $S \subseteq V(G)$  with  $|S| \geq 2$  and  $|V(G) - S| \geq 2$ , then  $T$  is an essential cut.

An  $i$ -cell is the union of two 5-cycles in a cubic graph which have  $i$  common edges. See Figure 3ab. We call a 5-cycle a  $0$ -cell. In the proof of Theorem 1, we

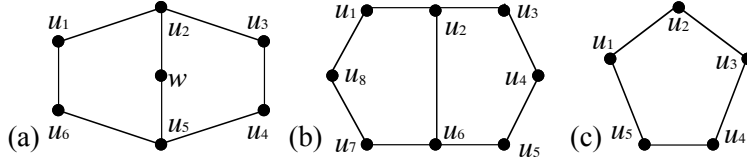


Figure 3:

will construct a dominating even subgraph from a 2-factor of a cubic graph which is obtained by reducing those cells. Hence we define reductions for those cells first.

Let  $D$  be a 2-cell in  $G$ , and  $u_1u_2 \cdots u_6u_1$  the 6-cycle and  $w$  the remaining vertex in  $D$ . See Figure 4. Let  $G'$  be the graph obtained from  $G$  by contracting all of

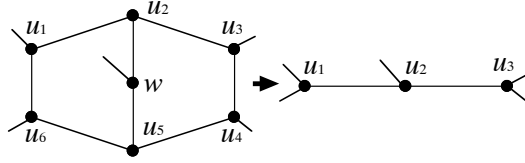


Figure 4: reduction of 2-cell

the paths  $u_1u_6$ ,  $u_2wu_5$ ,  $u_3u_4$  and removing the edges  $u_6u_5$  and  $u_5u_4$ . We denote this reduction by  $G' = G|D$ .

Let  $D$  be a 1-cell in  $G$  and  $u_1u_2 \cdots u_8u_1$  the 8-cycle of  $D$ . See Figure 5. Let  $G'$

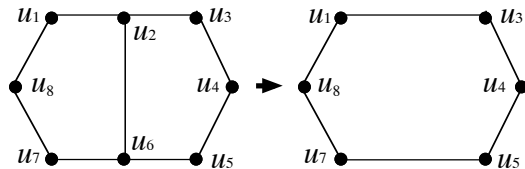


Figure 5: reduction of 1-cell

be the graph obtained from  $G$  by removing the edge  $u_2u_6$  and contracting both of the edges  $u_1u_2$  and  $u_7u_6$ . We denote by  $G' = G|D$  this reduction.

Let  $D = u_1 \cdots u_5 u_1$  be a 5-cycle without chord. Let  $u'_i \in V(G - C)$  which is adjacent to  $u_i$  for  $1 \leq i \leq 5$ . See Figure 6. Let  $G'$  be the graph obtained from  $G$  by

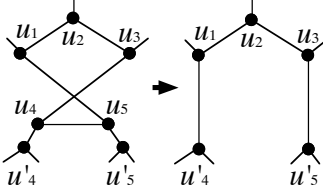


Figure 6: reduction of 5-cycle

removing the edges  $u_1 u_5, u_5 u_4, u_4 u_3$  and identifying  $u_1, u_4$  and  $u_3, u_5$ , respectively. We denote by  $G' = G|_{u_2} D$  this reduction.

We say that a 5-cycle  $C$  is *good* in  $G$  if there is an essential 3-cut  $T$  in  $G$  such that  $|T \cap \partial C| \geq 2$ . If  $C$  has exactly one chord and  $|G| \geq 8$ , then  $C$  is always good because  $\partial C$  is an essential 3-cut, If a 2- or 1-cell contains a good 5-cycle, then the cell is also called *good*. A cell which is not good is called *bad*. Notice that in a bad cell, every 5-cycle is bad.

We need the following fact in the proof of Theorem 1.

**Fact 3.** *Let  $i \in \{2, 1, 0\}$ . If a 3-edge-connected cubic graph  $G$  has a bad  $i$ -cell  $D$ , then  $G|D$  or  $G|_{u_2} D$  is 3-edge-connected.*

This fact is obtained from the following two lemmas.

**Lemma 4.** *Let  $D = u_1 u_2 u_3 u_4 u_5 u_1$  be a 5-cycle of a 3-edge-connected cubic graph. If there is an essential 3-cut  $T$  such that  $T \cap E(D) \neq \emptyset$ , then  $D$  is good.*

*Proof.* Suppose  $D$  is bad and there is an essential 3-cut  $T$  such that  $T \cap E(D) \neq \emptyset$ . Since  $D$  is bad and  $G$  is 3-edge-connected,  $D$  has no chord. Let  $u'_i \in N_{G-C}(u_i)$  for  $1 \leq i \leq 5$ . Since  $T$  is an essential edge-cut of a cubic graph and  $G$  is 3-edge-connected, no pair of edges in  $T$  is adjacent, and so  $T \cap E(D)$  contains two independent edges, say  $u_1 u_2, u_4 u_5$ . Then  $(T - \{u_1 u_2, u_4 u_5\}) \cup \{u_1 u'_1, u_5 u'_5\}$  is an essential 3-cut containing two edges in  $\partial D$ , a contradiction.  $\square$

**Lemma 5.** *For  $k \in \{2, 3\}$  and a  $k$ -edge-connected cubic graph  $G$ , the following hold.*

1. Let  $D$  be a 2-cell and  $u_1u_2 \cdots u_6u_1$  be the 6-cycle in  $D$ . See Figure 4. If  $G|D$  is not  $k$ -edge-connected, then  $G$  has an essential  $k$ -cut containing  $\{u_1u_2, u_5u_6\}$  or  $\{u_2u_3, u_4u_5\}$ .
2. Let  $D$  be a 1-cell and  $u_1u_2 \cdots u_8u_1$  be the 8-cycle in  $D$ . See Figure 5. If  $G|D$  is not  $k$ -edge-connected, then  $G$  has an essential  $k$ -cut containing  $\{u_1u_8, u_4u_5, u_2u_6\}$  or  $\{u_3u_4, u_7u_8, u_2u_6\}$ .
3. Let  $D = u_1u_2 \cdots u_5u_1$  be a 5-cycle and  $u'_j$  be the vertex in  $G - D$  which is adjacent to  $u_j$  for  $1 \leq j \leq 5$ . See Figure 6. If  $G|_{u_2}D$  is not  $k$ -edge-connected, then  $G$  has an essential  $k$ -cut containing  $\{u_1u'_1, u_4u'_4\}$  or  $\{u_3u'_3, u_5u'_5\}$ .

*Proof.* Let  $G' = G|D$  or  $G' = G|_{u_2}D$ , respectively,  $T$  be a minimum edge-cut of  $G'$ , and  $D'$  be the subgraph in  $G'$  corresponding to  $D$ . Let  $S \subset V(G')$  such that  $\partial S = T$  and  $u_1 \in S$ . Suppose  $|T| \leq k - 1$ . Since  $G$  is  $k$ -edge-connected,  $T$  is not an edge-cut of  $G$ , and so  $T = \partial S$  divides  $D'$ . For a vertex  $u \in V(D)$ , we denote a vertex in  $G - D$  adjacent to  $u$  by  $u'$  if it exists.

1. Since  $|T \cap D'| = 1$ , by symmetry, we may suppose  $T \cap D' = \{u_1u_2\}$ . Since  $T$  is a minimum cut, no pair of edges in  $T$  is adjacent, and so  $\{u'_1, u'_6\} \subset S$ . Thus  $\partial(S \cup \{u_6\})$  is an essential  $k$ -cut containing  $\{u_1u_2, u_5u_6\}$  of  $G$ .

2. Since  $T$  divides  $V(D')$ ,  $|T \cap D'| = 2$ . By symmetry, we have four cases.

If  $T \cap D' = \{u_1u_3, u_5u_7\}$ , then  $\partial S$  is also a  $(k - 1)$ -cut of  $G$ , a contradiction.

If  $T \cap D' = \{u_1u_3, u_4u_5\}$ , then  $\partial(S \cup \{u_2, u_6\})$  is a  $(k - 1)$ -cut of  $G$ , a contradiction.

If  $T \cap D' = \{u_1u_8, u_3u_4\}$ , then  $\partial(S - \{u_1, u_3\})$  is a  $(k - 1)$ -cut of  $G$ , a contradiction.

If  $T \cap D' = \{u_1u_8, u_4u_5\}$ , then since  $\{u_1, u_3\} \subset S$ ,  $\partial(S \cup \{u_2\})$  is an essential  $k$ -cut containing  $\{u_1u_8, u_4u_5, u_2u_6\}$  of  $G$ .

3. Since  $T = \partial S$  divides  $D'$ ,  $|T \cap D'| = 1$ . By symmetry, we may suppose  $T \cap D' = \{u_1u_2\}$ . Since  $\{u'_1, u'_4\} \subset S$  and  $G$  is  $k$ -edge-connected,  $\partial(S - u_1)$  is an essential  $k$ -cut containing  $\{u_1u'_1, u_4u'_4\}$  of  $G$ .  $\square$

*Proof of Fact 3.* If  $D$  is a 5-cycle and  $G|_{u_2}D$  is not 3-edge-connected, then by Lemma 5,  $D$  is good. If  $D$  is a 2- or 1-cell and  $G|D$  is not 3-edge-connected, then there exist a 5-cycle  $C$  in  $D$  and an essential 3-cut  $T$  of  $G$  such that  $|D \cap T| \geq 2$  by Lemma 5. Thus by Lemma 4,  $D$  is good.  $\square$



### 3 Proof of Theorem 1

Let  $G$  be a 3-edge-connected cubic graph. We may assume that  $G$  is not hamiltonian; otherwise we are done. First we define a sequence of bad cells in  $G$  which will be reduced.

Let

$$\mathcal{D}_1 = \{D_1, D_2, \dots, D_p\}$$

be a maximal set of mutually disjoint 2-cells in  $G$  such that  $D_{i+1}$  is bad in  $G_i$  for each  $0 \leq i \leq p-1$ , where  $G_0 = G$  and  $G_i = G_{i-1} \setminus D_i$  for  $1 \leq i \leq p$ . If there is no bad 2-cell in  $G$ , then we define  $\mathcal{D}_1 = \emptyset$  and  $p = 0$ . We denote the subgraph in  $G_i$  corresponding to  $D_i$  by  $D'_i$ . See Figure 7. Notice that  $G - \bigcup_{l \leq i} D_l = G_i - \bigcup_{l \leq i} D'_l$

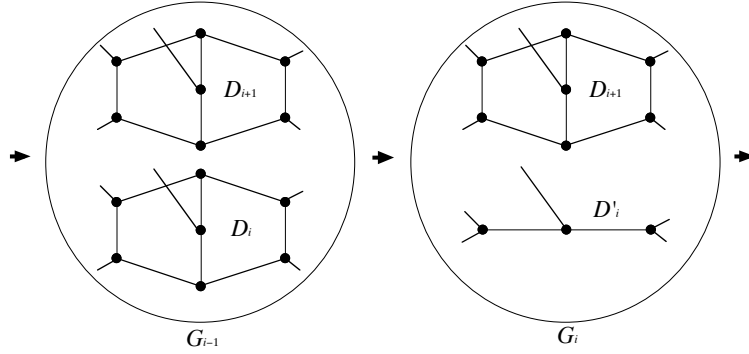


Figure 7:

and, by Fact 3, each  $G_i$  is 3-edge-connected for every  $0 \leq i \leq p$ . By the maximality of  $\mathcal{D}_1$ , obviously the following claim holds.

**Claim 1.** *There is no 2-cell in  $G - \bigcup_{l \leq p} D_l = G_p - \bigcup_{l \leq p} D'_l$  which is bad in  $G_p$ .*

Let

$$\mathcal{D}_2 = \{D_{p+1}, D_{p+2}, \dots, D_{p+q}\}$$

be a maximal set of mutually disjoint 1-cells in  $G - \bigcup_{1 \leq l \leq p} D_l$  such that  $D_{i+1}$  is bad in  $G_i$  for each  $p \leq i \leq p+q-1$ , where  $G_i = G_{i-1} \setminus D_i$  for  $p+1 \leq i \leq p+q$ . If there is no bad 1-cell in  $G - \bigcup_{1 \leq l \leq p} D_l$ , then we define  $\mathcal{D}_2 = \emptyset$  and  $q = 0$ . The subgraph in  $G_i$  corresponding to  $D_i$  is denoted by  $D'_i$ . In this case also,  $G - \bigcup_{l \leq i} D_l = G_i - \bigcup_{l \leq i} D'_l$  and, by Fact 3, each  $G_i$  is 3-edge-connected for any  $0 \leq i \leq p+q$ .

**Claim 2.** *There is no 1-cell in  $G - \bigcup_{l \leq p+q} D_l = G_{p+q} - \bigcup_{l \leq p+q} D'_l$  which is bad in  $G_{p+q}$  and there is no 2-cell  $C$  in  $G_{p+j} - \bigcup_{i \leq p+j} D'_i$  which is bad in  $G_{p+j}$  for any  $0 \leq j \leq q$ .*

*Proof.* By the maximality of  $\mathcal{D}_2$ , we have the first statement. If there is  $1 \leq j \leq q$  such that  $G_{p+j} - \bigcup_{i \leq p+j} D'_i$  contains a 2-cell  $C$  which is bad in  $G_{p+j}$ , then obviously  $C$  is bad in  $G_{p+j-1}$  also, and so  $C$  is bad in  $G_p$ . This contradicts Claim 1.  $\square$

Let  $\mathcal{D}_0$  be a maximal set of mutually disjoint bad 5-cycles in  $G - \bigcup_{i \leq p+q} D_i$ . For  $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_0$ , we define a vertex subset  $R^*$  of  $G$ , whose vertices may not be contained in a dominating even subgraph of  $G$  which is constructed later.

First, for each cell  $D_i$  in  $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_0$ , we define pairs of vertices in  $D_i$ .

1. Let  $D_i$  be a 2-cell in  $\mathcal{D}_1$  and  $u_1 u_2 u_3 u_4 u_5 u_6 u_1$  be the 6-cycle in  $D_i$ . See Figure 3a. The pairs of  $D_i$  are  $\{u_1, u_3\}$  and  $\{u_4, u_6\}$ .
2. Let  $D_i$  be a 1-cell in  $\mathcal{D}_2$  and  $u_1 u_2 \cdots u_8 u_1$  be the 8-cycle in  $D_i$ . See Figure 3b. We define the pair of  $D_i$  by  $\{u_8, u_4\}$ .
3. For a 5-cycle  $D_i = u_1 u_2 \cdots u_5 u_1$  in  $\mathcal{D}_0$ , the pair is defined by arbitrary two adjacent vertices in  $D_i$ , e.g.,  $\{u_1, u_2\}$ . See Figure 3c.

Let  $\mathcal{P}_0$  be the set of all the pairs for all  $D_l \in \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_0$ . For each pair  $\{u_i, u_j\} \in \mathcal{P}_0$ , let  $E_{u_i, u_j} = \partial(\{u_i, u_j\}) \cap \partial D_l$ , where  $\{u_i, u_j\} \subset D_l \in \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_0$ . Obviously  $0 \leq |E_{u_i, u_j}| \leq 2$ . Let

$$\mathcal{P} = \{\{u_i, u_j\} \in \mathcal{P}_0 : |E_{u_i, u_j}| = 2\} \text{ and } \mathcal{Q} = \bigcup_{\{u_i, u_j\} \in \mathcal{P}} E_{u_i, u_j}.$$

We define a bipartite graph  $H$  on the partite sets  $\mathcal{P}$  and  $\mathcal{Q}$  by defining the adjacency relation so that  $\{u_i, u_j\} \in \mathcal{P}$  and  $e \in \mathcal{Q}$  are adjacent if and only if  $e \in E_{u_i, u_j}$ . Since each element in  $\mathcal{Q}$  is adjacent to at most two pairs in  $\mathcal{P}$ , for any  $\mathcal{S} \subset \mathcal{P}$ ,

$$2|\mathcal{S}| = |E_H(\mathcal{S}, N(\mathcal{S}))| \leq |E_H(N(\mathcal{S}), \mathcal{P})| \leq 2|N(\mathcal{S})|.$$

Thus by Hall's theorem, there is a matching  $M$  in  $H$  covering  $\mathcal{P}$ . Let  $\varphi : \mathcal{P} \rightarrow \mathcal{Q}$  be the injection defined by  $M$ , i.e., for each  $\{u_i, u_j\} \in \mathcal{P}$ , the pair is adjacent to  $\varphi(\{u_i, u_j\}) \in \mathcal{Q}$  by  $M$ . Let

$$R^* = \{u_k : u_k \in \{u_i, u_j\} \text{ is the end of the edge } \varphi(\{u_i, u_j\}) \text{ for some } \{u_i, u_j\} \in \mathcal{P}\}.$$

Notice that there are no edges connecting two vertices in  $R^*$ , except for those connecting  $u_1$  and  $u_6$  or connecting  $u_3$  and  $u_4$  for some 2-cell in  $\mathcal{D}_1$ .

Let

$$\mathcal{D}_3 = \{D_{p+q+1}, D_{p+q+2}, \dots, D_{p+q+r}\} \subset \mathcal{D}_0$$

be a maximal subset of  $\mathcal{D}_0$  such that for  $p+q \leq i \leq p+q+r-1$ ,

1.  $D_{i+1}$  is bad in  $G_i$ , where  $u_i \in R^* \cap D_i$  and  $G_i = G_{i-1}|_{u_i} D_i$  and
2.  $D_{i+1}$  intersects neither 3-cycle nor 4-cycle in  $G_i$ .

If there is no such 5-cycle, then we define  $\mathcal{D}_3 = \emptyset$  and  $r = 0$ . We denote by  $D'_i$  the subgraph in  $G_i$  corresponding to  $D_i \in \mathcal{D}_3$ . Notice that each  $D_i$  has no chord since  $D_i$  is bad in  $G_{i-1}$  and each  $G_i$  is 3-edge-connected for any  $p+q \leq i \leq p+q+r$  by Fact 3.

**Claim 3.** *If there exist  $0 \leq j \leq r-1$  and  $i \in \{2, 1, 0\}$  such that  $G_{p+q+j} - \bigcup_{l \leq p+q+j} D'_l - D_{p+q+j+1}$  contains an  $i$ -cell  $C$  which is good in  $G_{p+q+j}$ , then  $C$  is good in  $G_{p+q+j+1}$  also.*

*Proof.* Since  $C$  is good in  $G_{p+q+j}$ , there exist an essential 3-cut  $T$  of  $G_{p+q+j}$  and a 5-cycle  $C_1$  in  $C$  such that  $|T \cap \partial C_1| \geq 2$ . Let  $H_1$  and  $H_2$  be the two components in  $G_{p+q+j} - T$ . Since  $C \subset G_{p+q+j} - \bigcup_{l \leq p+q+j} D'_l - D_{p+q+j+1}$ , the 5-cycle  $D_{p+q+j+1}$  is contained in  $H_1 - C$  or  $H_2 - C$ , and so both of  $T$  and  $C$  exist in  $G_{p+q+r+1}$  also. Thus  $C$  is good in  $G_{p+q+r+1}$ .  $\square$

**Claim 4.** *1. If  $G_{p+q+j} - \bigcup_{i \leq p+q+j} D'_i$  has a bad 5-cycle, then the 5-cycle is bad in  $G_{p+q}$  also.*

2. *There is neither 2- nor 1-cell  $C$  in  $G_{p+q+j} - \bigcup_{i \leq p+q+j} D'_i$  which is bad in  $G_{p+q+j}$  for any  $0 \leq j \leq r$ .*

*Proof.* Claim 3 implies the first statement immediately. By Claim 3, if there is  $0 \leq j \leq r$  such that  $G_{p+q+j} - \bigcup_{i \leq p+q+j} D'_i$  contains a 2- or 1-cell  $C$  which is bad in  $G_{p+q+j}$ , then  $C$  is bad in  $G_{p+q+j-1}$ , and so  $C$  is bad in  $G_{p+q}$ . This contradicts Claim 2.  $\square$

Let

$$S_0 = \emptyset \text{ and } S_i = \bigcup_{1 \leq l \leq i} D'_l$$

for  $1 \leq i \leq p + q + r$ . We call a vertex in  $S_i$  a *yellow* vertex.

We extend  $R^*$ . Let  $D_i \in \mathcal{D}_1$  and  $u_1u_2u_3u_4u_5u_6u_1$  be the 6-cycle. See Figure 3a. We define  $R(D_i) = \{u_2, u_5\}$ . Let  $D_i \in \mathcal{D}_2$  and  $u_1u_2u_3u_4u_5u_6u_7u_8u_1$  be the 8-cycle. See Figure 3b. We define  $R(D_i) = \{u_2, u_6\}$ . Let

$$R_0 = R^* \cup \bigcup_{1 \leq l \leq p+q} R(D_l) \ (\subset V(G))$$

and

$$R_i = R_0 - \bigcup_{1 \leq l \leq i} V(D_l) \ (\subset V(G_i))$$

for  $1 \leq i \leq p + q + r$ . We call a vertex in  $R_i$  a *red* vertex. By the definition of  $S_i$  and  $R_i$ , there is no vertex in  $G_i$  which is both red and yellow. Notice that

$$R_i = \begin{cases} R_{i+1} \cup (R^* \cap D_{i+1}) \cup R(D_{i+1}) & \text{for } 0 \leq i \leq p + q - 1 \\ R_{i+1} \cup (R^* \cap D_{i+1}) & \text{for } p + q \leq i \leq p + q + r - 1 \end{cases} \quad (1)$$

For Theorem 1, it is enough to show that for all  $0 \leq i \leq p + q + r$ ,  $G_i$  has an even subgraph  $F_i$  such that

1.  $F_i$  intersects all essential 3-cuts in  $G_i$ .
2. Every component of  $F_i$  contains at least five vertices. Especially if a 5-cycle  $C$  is a component of  $F_i$ , then  $C$  contains a yellow vertex, i.e.,  $C \cap S_i \neq \emptyset$ .
3. Every vertex in  $G_i - F_i$  is red, i.e.,  $G_i - F_i \subset R_i$ , and  $|D_l - F_i| \leq 1$  for  $i + 1 \leq l \leq p + q + r$ .

Indeed by the second condition, any component in  $F_0$  contains at least six vertices as  $S_0 = \emptyset$ . Although  $(R^* \cap V(D_i)) \cup R(D_i) \subset R_0$  ( $1 \leq i \leq p$ ) and  $R(D_i) \subset R_0$  ( $p+1 \leq i \leq p+q$ ) may not be independent, since  $|D_l - F_0| \leq 1$  for all  $1 \leq l \leq p+q+r$ , we have  $G_0 - F_0 (\subset R_0)$  is independent, i.e.,  $F_0$  is dominating  $G = G_0$ . Therefore  $F_0$  is a desired even subgraph.

We construct  $F_i$  inductively. First we show the existence of  $F_{p+q+r}$ . Notice that  $G_i$  is 3-edge-connected for every  $0 \leq i \leq p + q + r$  by Fact 3 as we reduced bad cells.

**Claim 5.** *There is a 2-factor  $F_{p+q+r}$  in  $G_{p+q+r}$  such that*

1.  $F_{p+q+r}$  intersects all 3-cuts and 4-cuts in  $G_{p+q+r}$  and
2. each 5-cycle  $C$  in  $F_{p+q+r}$  contains a yellow vertex, i.e.,  $C \cap S_{p+q+r} \neq \emptyset$ .

*Proof.* Since  $G_{p+q+r}$  is a 3-edge-connected cubic graph, by Corollary C,  $G_{p+q+r}$  has a 2-factor  $F_{p+q+r}$  which intersects all 3-cuts and 4-cuts. We choose  $F_{p+q+r}$  such that the number of components is as small as possible.

Suppose  $F_{p+q+r}$  contains a 5-cycle  $C$  without a yellow vertex. If  $C$  is good in  $G_{p+q+r}$ , then there is an essential 3-cut  $T$  such that  $|T \cap \partial C| \geq 2$ , and  $F_{p+q+r}$  does not intersect the 3-cut  $T$ , a contradiction. Therefore  $C$  is bad in  $G_{p+q+r}$ . Since  $C$  has no yellow vertex,  $C \subset G_{p+q+r} - S_{p+q+r}$ , and so the 5-cycle  $C$  exists in  $G_{p+q}$  and, by Claim 4,  $C$  is bad in  $G_{p+q}$  also.

Suppose  $C \notin \mathcal{D}_0$ . By the maximality of  $\mathcal{D}_0$ , there is a bad 5-cycle  $D \in \mathcal{D}_0$  intersecting  $C$ . If  $|E(C \cap D)| \leq 2$ , then  $C \cup D$  is a 2- or 1-cell in  $G_{p+q}$ . Since both of  $C$  and  $D$  are bad in  $G_{p+q}$ ,  $C \cup D$  is bad in  $G_{p+q}$ . This contradicts Claim 2.

If  $|E(C \cap D)| = 3$ , then  $D$  is a 5-cycle in  $G_{p+q+r}$  also. However,  $F_{p+q+r}$  does not contain the vertex in  $D - C$  as  $C$  is a component of  $F_{p+q+r}$ . This is a contradiction.

Therefore  $C \in \mathcal{D}_0$ . Since  $C$  is bad in  $G_{p+q+r}$  and  $C \notin \mathcal{D}_3$ ,  $C$  intersects a 3- or a 4-cycle  $C_1$  in  $G_{p+q+r}$ . Since  $C$  is bad,  $C$  has no chord, and so  $C_1 - C \neq \emptyset$ . If  $C_1 - C$  is a vertex  $w$ , then  $w$  is not contained in  $F_{p+q+r}$  as  $C$  is a component of the 2-factor  $F_{p+q+r}$ . This is a contradiction.

If  $C_1 - C$  contains an edge  $ww'$ , then there is a component  $C_2$  in  $F_{p+q+r}$  containing the edge  $ww'$ . Since the symmetric difference  $\tilde{C} = C \triangle C_1 \triangle C_2$  is a cycle, the subgraph  $(F_{p+q+r} - C \cup C_2) \cup \tilde{C}$  is a 2-factor of  $G_{p+q+r}$  in which the number of components is less than  $F_{p+q+r}$ . This contradicts the choice of  $F_{p+q+r}$ . Thus  $C$  contains a yellow vertex.  $\square$

Suppose  $G_{i+1}$  has a desired even subgraph  $F_{i+1}$  for  $1 \leq i+1 \leq p+q+r$ . Since  $G_{i+1}$  has no vertex which is both yellow and red and  $F_{i+1}$  contains every vertex which is not red in  $G_{i+1}$ , we have  $\bigcup_{l \leq i+1} D'_l \subset F_{i+1}$ .

**Claim 6.** *If  $F_i$  is an even subgraph of  $G_i$  obtained from  $F_{i+1}$  by replacing edges in  $D'_{i+1}$  with edges in  $D_{i+1}$ , i.e.,*

$$E(F_{i+1}) - E(D'_{i+1}) = E(F_i) - E(D_{i+1}),$$

then the following holds.

1.  $F_i$  intersects all essential 3-cuts in  $G_i$ ,
2. Every component  $C$  of  $F_i$  intersecting no edge of  $D_{i+1}$  contains at least five vertices. Especially if  $C$  is a 5-cycle, then  $C$  contains a yellow vertex, i.e.,  $C \cap S_i \neq \emptyset$ .
3. Every vertex in  $G_i - F_i - D_{i+1}$  is red, i.e.,  $G_i - F_i - D_{i+1} \subset R_i$ , and  $|D_j - F_i| \leq 1$  for  $j \geq i + 2$ .

*Proof.* 1. Let  $T$  be any essential 3-cut of  $G_i$ . Since  $D_{i+1}$  is bad in  $G_i$ ,  $T \cap E(D_{i+1}) = \emptyset$  by Lemma 4. This implies  $T$  is an essential 3-cut of  $G_{i+1}$  and  $T \cap E(D'_{i+1}) = \emptyset$ . Thus  $F_{i+1} - E(D'_{i+1}) = F_i - E(D_{i+1})$  intersects  $T$ .

2. Obviously  $C$  is a component of  $F_{i+1}$  also. Thus  $|C| \geq 5$  and  $C$  contains a yellow vertex in  $S_{i+1} - V(D'_{i+1}) = S_i$  if  $|C| = 5$ .

3. Since  $G_i - D_{i+1} = G_{i+1} - D'_{i+1}$ , we have  $G_i - D_{i+1} - F_i = G_{i+1} - D'_{i+1} - F_{i+1} \subset R_{i+1} \subset R_i$  by (1). For  $j \geq i + 2$ ,

$$D_j - F_i = D_j - F_{i+1},$$

and so we have  $|D_j - F_i| = |D_j - F_{i+1}| \leq 1$ . □

In the remaining part of this paper, we will construct a desired even subgraph  $F_i$  of  $G_i$  from  $F_{i+1}$  by replacing edges in  $D'_{i+1}$  with edges in  $D_{i+1}$ . From the above claim, it is enough to show that  $F_i$  satisfies the following:

A1. Every component  $C$  containing an edge of  $D_{i+1}$  in  $F_i$  contains at least five vertices. Especially if  $C$  is a 5-cycle, then  $C$  contains a yellow vertex, i.e.,  $C \cap S_i \neq \emptyset$ .

A2. A vertex in  $D_{i+1} - F_i$  is red, i.e.,  $D_{i+1} - F_i \subset R_i$ , and  $|D_{i+1} - F_i| \leq 1$ .

We divide our argument into the following three cases.

1.  $0 \leq i \leq p - 1$ .
2.  $p + q \leq i \leq p + q + r - 1$ .

3.  $p \leq i \leq p + q - 1$ .

The first case is easier than the other cases. If there is a vertex in  $G_i - D_{i+1}$  which is adjacent to  $u \in D_{i+1}$ , then we denote the vertex by  $u'$ .

1.  $0 \leq i \leq p - 1$ , i.e.,  $D_{i+1} \in \mathcal{D}_1$ .

Since  $V(D'_{i+1}) \subset F_{i+1}$  and  $F_{i+1}$  is an even subgraph,  $|F_{i+1} \cap \partial D'_{i+1}|$  is 2 or 4. If the subgraph induced by  $V(D_{i+1})$  contains an edge that is not in  $E(D_{i+1})$ , then  $|\partial D_{i+1}| = 3$ . This implies  $D_{i+1}$  contains a good 5-cycle, i.e.,  $D_{i+1}$  is good in  $G_i$ . This contradicts our assumption. Therefore, both of  $\{u_1, u_3\}$  and  $\{u_4, u_6\}$  contain a red vertex.

*Case 1.*  $|F_{i+1} \cap \partial D'_{i+1}| = 4$ .

Since  $D'_{i+1} \subset F_{i+1}$ , by symmetry, we may suppose

$$F_{i+1} \cap \partial D'_{i+1} = \{u_1u'_6, u_1u'_1, u_2w', u_3u'_3\},$$

and then  $F_{i+1}$  contains the edge  $u_2u_3$ . See Figure 8. Let  $F_i$  be the even subgraph

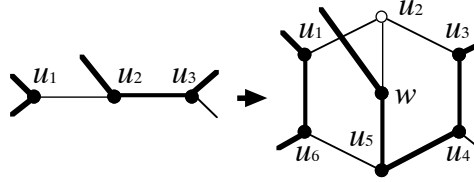


Figure 8:

obtained from  $F_{i+1}$  by replacing

$$u'_1u_1u'_6 \text{ and } w'u_2u_3u'_3 \text{ by } u'_1u_1u_6u'_6 \text{ and } w'wu_5u_4u_3u'_3.$$

Since every component  $C$  containing an edge of  $D_{i+1}$  in  $F_i$  contains at least six vertices, A1 holds. Since  $u_2 \in R(D_{i+1}) \subset R_i$ , A2 holds.

*Case 2.*  $|F_{i+1} \cap \partial D'_{i+1}| = 2$ .

Since  $D'_{i+1} \subset F_{i+1}$ ,  $F_{i+1} \cap \partial D'_{i+1}$  does not contain the edge  $u_2w'$ . Thus by symmetry, we may suppose

$$F_{i+1} \cap \partial D'_{i+1} \text{ is } \{u_1u'_6, u_3u'_3\} \text{ or } \{u_1u'_6, u_3u'_4\}.$$

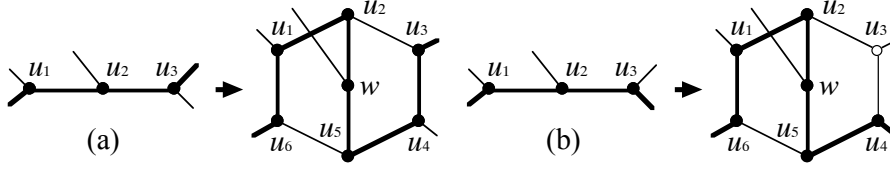


Figure 9:

See Figure 9ab. If the intersection is  $\{u_1u'_6, u_3u'_3\}$ , then the even subgraph  $F_i$  obtained from  $F_{i+1}$  by replacing

$$u_1u_2u_3 \text{ by } u_6u_1u_2wu_5u_4u_3$$

is a desired even subgraph because both of A1 and A2 hold as  $V(D_{i+1}) \subset F_i$ .

Suppose  $F_{i+1} \cap \partial D'_{i+1} = \{u_1u'_6, u_3u'_4\}$ . For the pair  $\{u_1, u_3\}$ , if  $u_3$  is red, i.e.,  $u_3 \in R_i$ , then the even subgraph  $F_i$  obtained from  $F_{i+1}$  by replacing

$$u'_6u_1u_2u_3u'_3 \text{ by } u'_6u_6u_1u_2wu_5u_4u'_4$$

is a desired even subgraph because both of A1 and A2 hold. Similarly if  $u_1 \in R_i$ , then the even subgraph  $F_i$  obtained from  $F_{i+1}$  by replacing

$$u'_6u_1u_2u_3u'_4 \text{ by } u'_6u_6u_5wu_2u_3u_4u'_4$$

is a desired even subgraph.

**2.**  $p + q \leq i \leq p + q + r - 1$ , i.e.,  $D_{i+1} \in \mathcal{D}_3$ .

In this case,  $D_{i+1} = u_1u_2 \cdots u_5u_1$  is a 5-cycle. By symmetry, we may suppose  $u_2$  is red, i.e.,  $u_2 \in R_i$ . Since  $V(D'_{i+1}) \subset F_{i+1}$ ,  $|F_{i+1} \cap \partial D'_{i+1}|$  is 2 or 4.

*Case 1.*  $|F_{i+1} \cap \partial D'_{i+1}| = 2$ .

Notice that  $F_{i+1} \cap \partial D'_{i+1}$  does not contain  $u_2u'_2$  because  $V(D'_{i+1}) \subset F_{i+1}$ . Hence by symmetry, we have the following three cases:

$$F_{i+1} \cap \partial D'_{i+1} \text{ is } \{u_1u'_4, u_3u'_5\}, \{u_1u'_4, u_3u'_3\} \text{ or } \{u_1u'_1, u_3u'_3\}.$$

(i) Suppose  $F_{i+1} \cap \partial D'_{i+1} = \{u_1u'_4, u_3u'_5\}$ . See Figure 10a. As  $V(D'_{i+1}) \subset F_{i+1}$ ,  $F_{i+1}$



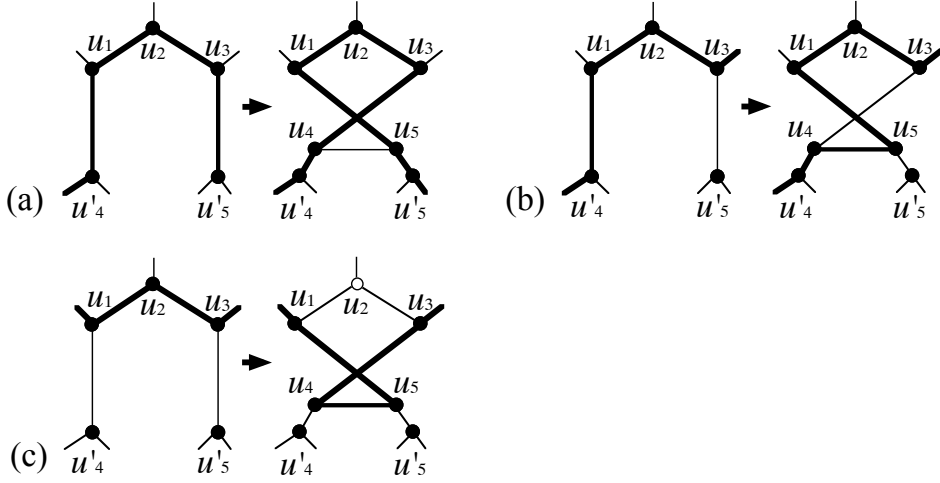


Figure 10:

contains the path  $u'_4u_1u_2u_3u'_5$ . Let  $F_i$  be the even subgraph in  $G_i$  which is obtained from  $F_{i+1}$  by replacing

$$u'_4u_1u_2u_3u'_5 \text{ by } u'_4u_4u_3u_2u_1u_5u'_5.$$

Obviously A1 holds. Since  $V(D_{i+1}) \subset F_i$ , A2 holds.

(ii) Suppose  $F_{i+1} \cap \partial D'_{i+1} = \{u_1u'_4, u_3u'_3\}$ . See Figure 10b. Then  $F_{i+1}$  contains the path  $u'_4u_1u_2u_3u'_3$  as  $V(D'_{i+1}) \subset F_{i+1}$ . Let  $F_i$  be the even subgraph in  $G_i$  which is obtained from  $F_{i+1}$  by replacing

$$u'_4u_1u_2u_3u'_3 \text{ by } u'_4u_4u_5u_1u_2u_3u'_3.$$

Obviously both of A1 and A2 hold.

(iii) Suppose  $F_{i+1} \cap \partial D'_{i+1} = \{u_1u'_1, u_3u'_3\}$ . See Figure 10c. Let  $F_i$  be the even subgraph in  $G_i$  which is obtained from  $F_{i+1}$  by replacing

$$u'_1u_1u_2u_3u'_3 \text{ by } u'_1u_1u_5u_4u_3u'_3.$$

Since the component in  $F_i$  containing an edge in  $D_{i+1}$  contains at least six vertices, A1 holds. As  $D_{i+1} - F_i = \{u_2\} \subset R_i$ , A2 holds.

*Case 2.*  $|F_{i+1} \cap \partial D'_{i+1}| = 4$ .

As  $V(D'_{i+1}) \subset F_{i+1}$ ,  $F_{i+1} \cap \partial D'_{i+1}$  is not  $\{u_1u'_4, u_1u'_1, u_3u'_3, u_3u'_5\}$ . Thus by symmetry, we have two cases.

(i) Suppose

$$F_{i+1} \cap \partial D'_{i+1} = \{u_1 u'_4, u_1 u'_1, u_2 u'_2, u_3 u'_3\},$$

and then  $u_2 u_3 \in F_{i+1}$ . See Figure 11a. Let  $F_i$  be the even subgraph in  $G_i$  which is

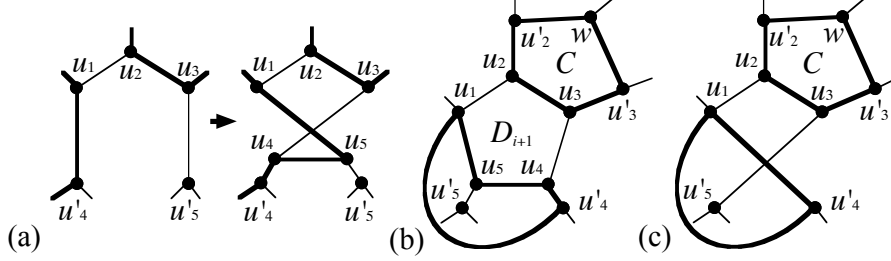


Figure 11:

obtained from  $F_{i+1}$  by replacing

$$u'_4 u_1 u'_1 \text{ by } u'_4 u_4 u_5 u_1 u'_1.$$

Since  $V(D_{i+1}) \subset F_i$ , A2 holds.

The component continuing  $u'_4 u_4 u_5 u_1 u'_1$  of  $F_i$  contains at least six vertices. Suppose  $C = u_2 u_3 u'_3 w u'_2 u_2$  is a 5-cycle and  $C \cap S_i = \emptyset$ . See Figure 11b. Then  $\tilde{C} = C \cup D_{i+1}$  is a 1-cell in  $G_i - \bigcup_{j \leq i} D'_j$ . Since  $D_{i+1} \in \mathcal{D}_3$ ,  $D_{i+1}$  is bad in  $G_i$ . Suppose that  $C$  is good and let  $T$  be an essential 3-cut such that  $|T \cap \partial C| \geq 2$ . Since  $D_{i+1}$  is bad,  $T \cap \partial C \subset \partial C - \{u_2 u_1, u_3 u_4\}$  by Lemma 4. Since  $T$  is an essential 3-cut of  $G_{i+1}$  also and  $F_{i+1}$  contains  $C$  as a component,  $F_{i+1}$  does not intersect  $T$ , a contradiction. See Figure 11c. Thus both of  $D_{i+1}$  and  $C$  are bad, and so  $\tilde{C}$  is a bad 1-cell in  $G_i$ . This contradicts Claim 4.

(ii) Suppose

$$F_{i+1} \cap \partial D'_{i+1} = \{u_1 u'_4, u_1 u'_1, u_2 u'_2, u_3 u'_5\},$$

and then  $u_2 u_3 \in F_{i+1}$ . See Figure 12a. Let  $F_i$  be the even subgraph in  $G_i$  which is obtained from  $F_{i+1}$  by replacing

$$u'_4 u_1 u'_1 \text{ and } u'_2 u_2 u_3 u'_5 \text{ by } u'_4 u_4 u_3 u_2 u'_2 \text{ and } u'_1 u_1 u_5 u'_5.$$

Since  $V(D_{i+1}) \subset F_i$ , A2 holds.

Let  $C_1$  and  $C_2$  be the components in  $F_i$  containing  $u_1 u_5$  and  $u_2 u_3 u_4$ , respectively. Suppose  $C_1$  or  $C_2$  contains at most five vertices, and then  $C_1 \neq C_2$ . Since  $D_{i+1}$  intersects neither 3-cycle nor 4-cycle,  $C_1$  or  $C_2$  is a 5-cycle.

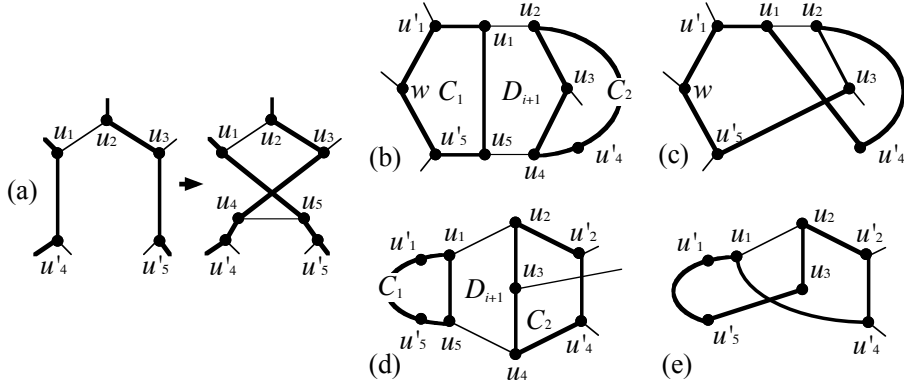


Figure 12:

Suppose  $C_1 = u_1 u_5 u'_5 w u'_1 u_1$  is a 5-cycle and  $C_1 \cap S_i = \emptyset$ . See Figure 12b. Then  $\widetilde{C}_1 = C_1 \cup D_{i+1}$  is a 1-cell in  $G_i$ . By Claim 4,  $\widetilde{C}_1$  is good. Since  $D_{i+1}$  is bad,  $C_1$  is good, and so there is an essential 3-cut  $T$  in  $G_i$  such that  $|T \cap \partial C_1| \geq 2$ , and by Lemma 4  $T \cap \partial C_1 \subset \partial C_1 - \{u_1 u_2, u_5 u_4\}$ . Thus  $T$  is an essential 3-cut of  $G_{i+1}$ . Since  $F_{i+1}$  contains the path  $u_1 u'_1 w u'_5 u_3$ ,  $F_{i+1}$  does not intersect  $T$ , a contradiction. See Figure 12c.

Suppose  $C_2 = u_2 u_3 u_4 u'_4 u'_2 u_2$  is a 5-cycle and  $C_2 \cap S_i = \emptyset$ . See Figure 12d. Then  $\widetilde{C}_2 = C_2 \cup D_{i+1}$  is a 2-cell in  $G_i$ . By Claim 4,  $\widetilde{C}_2$  is good, and so, as in the above case,  $G_i$  has an essential 3-cut  $T$  such that  $|T \cap \partial C_2| \geq 2$ , and  $T \cap \partial C_2 \subset \partial C_2 - \{u_1 u_2, u_4 u_5\}$ . Hence  $T$  is an essential 3-cut of  $G_{i+1}$ . Since  $F_{i+1}$  contains the path  $u'_5 u_3 u_2 u'_2 u'_4 u_1$ ,  $F_{i+1}$  does not intersect  $T$ , a contradiction. See Figure 12e.

**3.**  $p \leq i \leq p + q - 1$ , i.e.,  $D_{i+1} \in \mathcal{D}_2$ .

Since  $V(D'_{i+1}) \subset F_{i+1}$ ,  $|F_{i+1} \cap \partial D'_{i+1}|$  is 0, 2, 4 or 6.

*Case 1.*  $|F_{i+1} \cap \partial D'_{i+1}| = 0$ .

In this case, the 6-cycle  $u_1 u_3 u_4 u_5 u_7 u_8$  is contained in  $F_{i+1}$ , and replacing it with the 8-cycle  $u_1 u_2 u_3 u_4 u_5 u_6 u_7 u_8$ , we obtain the even subgraph  $F_i$  in  $G_i$ . Obviously, both A1 and A2 hold.

*Case 2.*  $|F_{i+1} \cap \partial D'_{i+1}| = 2$ .

Since  $V(D'_{i+1}) \subset F_{i+1}$  and  $F_{i+1}$  is an even subgraph, by symmetry we have two cases. If  $F_{i+1} \cap \partial D'_{i+1} = \{u_7 u'_7, u_8 u'_8\}$ , then  $F_{i+1}$  contains the path  $u_8 u_1 u_3 u_4 u_5 u_7$ . See Figure 13a. Then the even subgraph  $F_i$  obtained from  $F_{i+1}$  by replacing

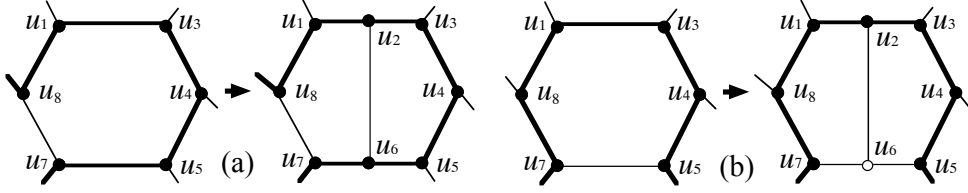


Figure 13:

$$u'_8 u_8 u_1 u_3 u_4 u_5 u_7 u'_7 \text{ by } u'_8 u_8 u_1 u_2 u_3 u_4 u_5 u_6 u_7 u'_7$$

is a desired even subgraph because both of A1 and A2 hold. Similarly, we can show the case of  $F_{i+1} \cap \partial D'_{i+1} = \{u_7 u'_7, u_5 u'_5\}$  since  $u_6 \in R(D_{i+1}) \subset R_i$ . See Figure 13b.

*Case 3.*  $|F_{i+1} \cap \partial D'_{i+1}| = 4$ .

By symmetry, we have four cases.

(i)  $F_{i+1} \cap \partial D'_{i+1} = \{u_7 u'_7, u_8 u'_8, u_1 u'_1, u_3 u'_3\}$ .

Since  $V(D'_{i+1}) \subset F_{i+1}$ ,  $F_{i+1}$  contains the paths  $u_8 u_1$  and  $u_3 u_4 u_5 u_7$ . See Figure 14a.

Let  $P_1$  and  $P_2$  be the two paths obtained from the cycles in  $F_{i+1}$  intersecting  $D'_{i+1}$

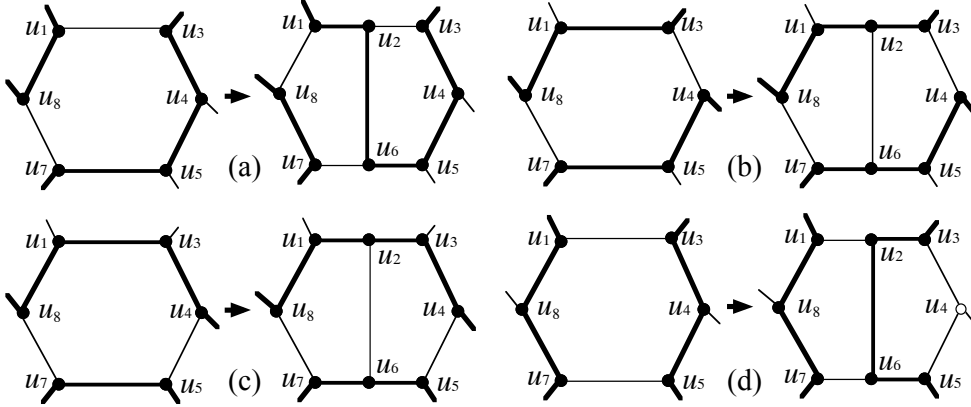


Figure 14:

by removing edges in  $E(D'_{i+1})$  and isolated vertices. By symmetry, we may suppose  $u_3 \in P_1$ . If  $u_7 \in P_1$  or  $u_8 \in P_1$ , then the even subgraph  $F_i$  obtained from  $F_{i+1}$  by replacing

$$u'_1 u_1 u_8 u'_8 \text{ and } u'_3 u_3 u_4 u_5 u_7 u'_7 \text{ by } u'_7 u_7 u_8 u'_8 \text{ and } u'_3 u_3 u_4 u_5 u_6 u_2 u_1 u'_1$$

is a desired even subgraph because both of A1 and A2 hold. See Figure 14a.

In the case of  $u_1 \in P_1$ , let  $F_i$  be the even subgraph obtained from  $F_{i+1}$  by replacing

$$u'_3u_3u_4u_5u_7u'_7 \text{ by } u'_3u_3u_4u_5u_6u_7u'_7.$$

Obviously A1 holds. Since  $u_2 \in R(D_{i+1}) \subset R_i$ , A2 holds.

(ii)  $F_{i+1} \cap \partial D'_{i+1} = \{u_7u'_7, u_8u'_8, u_3u'_3, u_4u'_4\}$ .

Then  $F_{i+1}$  contains the paths  $u_8u_1u_3$  and  $u_4u_5u_7$ . See Figure 14b. Then the even subgraph  $F_i$  obtained from  $F_{i+1}$  by replacing

$$u'_8u_8u_1u_3u'_3 \text{ and } u'_4u_4u_5u_7u'_7 \text{ by } u'_8u_8u_1u_2u_3u'_3 \text{ and } u'_4u_4u_5u_6u_7u'_7$$

is a desired even subgraph because both of A1 and A2 hold.

(iii) Similarly we can show the case that  $F_{i+1} \cap \partial D'_{i+1} = \{u_7u'_7, u_8u'_8, u_4u'_4, u_5u'_5\}$ .

See Figure 14c.

(iv)  $F_{i+1} \cap \partial D'_{i+1} = \{u_7u'_7, u_1u'_1, u_3u'_3, u_5u'_5\}$ .

Suppose first that  $u_4u_8 \in E(G)$ . Then,  $F_{i+1}$  contains either the paths  $u_7u_8u_1$  and  $u_3u_4u_5$ , or the paths  $u_1u_8u_4u_3$  and  $u_5u_7$ , or  $u_1u_3$  and  $u_5u_4u_8u_7$ . See Figure 15 for the case of the paths  $u_7u_8u_1$  and  $u_3u_4u_5$ . Let  $P_1$  and  $P_2$  be the two paths obtained

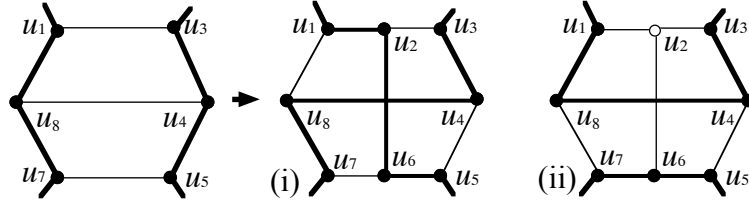


Figure 15:

from the cycles in  $F_{i+1}$  intersecting  $D'_{i+1}$  by removing edges in  $E(D'_{i+1})$  and isolated vertices. By symmetry, we may suppose  $u_3 \in P_1$ . If  $u_1 \in P_1$  or  $u_5 \in P_1$ , then the even subgraph  $F_i$  obtained from  $F_{i+1}$  by replacing the paths inside  $D'_{i+1}$

$$\text{by } u_7u_8u_4u_3 \text{ and } u_1u_2u_6u_5$$

is a desired even subgraph because both of A1 and A2 hold. See Figure 15i.

In the case of  $u_7 \in P_1$ , let  $F_i$  be the even subgraph obtained from  $F_{i+1}$  by replacing the paths inside  $D'_{i+1}$

$$\text{by } u_1u_8u_4u_3 \text{ and } u_7u_6u_5.$$

See Figure 15ii. Obviously A1 holds. Since  $u_2 \in R(D_{i+1}) \subset R_i$ , A2 holds.

Therefore, we may assume that  $u_4u_8 \notin E(G)$ . Since there are components in  $F_{i+1}$  containing  $u_7u_8u_1$  and  $u_3u_4u_5$ , both of  $u'_4$  and  $u'_8$  exist. Thus  $|E_{u_4, u_8}| = 2$ , and so one of  $u_4$  and  $u_8$  is in  $R_i$ . By symmetry, we may suppose  $u_4 \in R_i$ . Let  $F_i$  be the even subgraph obtained from  $F_{i+1}$  by replacing

$$u'_3u_3u_4u_5u'_5 \text{ by } u'_3u_3u_2u_6u_5u'_5.$$

See Figure 14d. Obviously A2 holds.

Suppose the component  $C_1$  of  $F_i$  containing  $u_7u_8u_1$  is a 5-cycle and  $C_1 \cap S_i = \emptyset$ . Let  $C_1 = u_1u_8u_7u'_7u'_1u_1$  and  $C_2 = u_1u_2u_6u_7u_8u_1$ . See Figure 16a. Then  $C = C_1 \cup C_2$

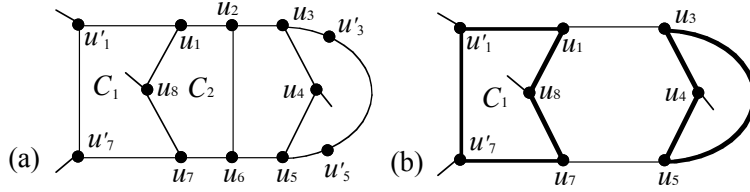


Figure 16:

is a 2-cell in  $G_i$ . By Claim 2,  $C$  is good in  $G_i$ . Since  $D_{i+1}$  is bad,  $C_2$  is bad, and so  $C_1$  is good. Thus there is an essential 3-cut  $T$  such that  $|T \cap \partial C_1| \geq 2$ . Since  $C_2$  is bad,  $T \cap \partial C_1 \subset \partial C_1 - \{u_1u_2, u_7u_6\}$ . Hence  $T$  is an essential 3-cut of  $G_{i+1}$ . Since  $F_{i+1}$  contains  $C_1$  as a component,  $F_{i+1}$  does not intersect  $T$ , a contradiction. See Figure 16b.

*Case 4.*  $|F_{i+1} \cap \partial D'_{i+1}| = 6$ .

In this case,  $F_{i+1}$  contains all the edges in  $\partial D'_{i+1}$ . Let  $P_1, P_2$  and  $P_3$  be the three paths obtained from the cycles in  $F_{i+1}$  intersecting  $D'_{i+1}$  by removing edges in  $E(D'_{i+1})$ . By symmetry, we may suppose  $u_7 \in P_1$ . It is easy to confirm that for all the following cases, both of A1 and A2 hold.

i). *The ends of  $P_1$  are  $u_7$  and  $u_8$ .*

a). If the ends of  $P_2$  are  $u_1$  and  $u_3$ , then let  $F_i$  be the even subgraph obtained from  $F_{i+1}$  by replacing the cycles in  $F_{i+1}$  intersecting  $D'_{i+1}$  with the cycle

$$u_7P_1u_8u_1P_2u_3u_4P_3u_5u_6u_7.$$

See Figure 17a.

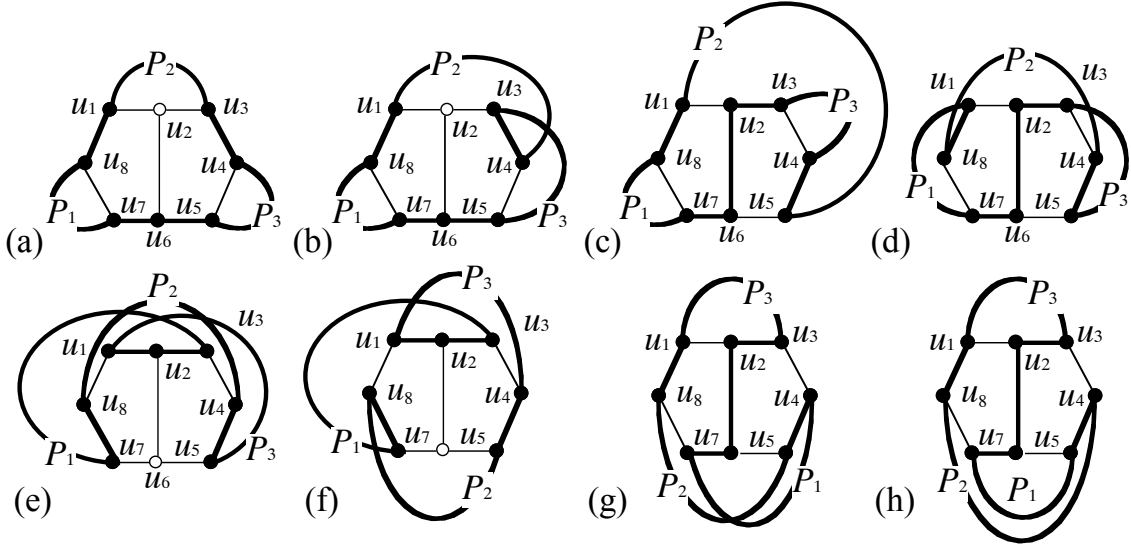


Figure 17:

- b). If the ends of  $P_2$  are  $u_1$  and  $u_4$ , then let  $F_i$  be the even subgraph obtained from  $F_{i+1}$  by replacing the cycles in  $F_{i+1}$  intersecting  $D'_{i+1}$  with the cycle

$$u_7P_1u_8u_1P_2u_4u_3P_3u_5u_6u_7.$$

See Figure 17b.

- c). If the ends of  $P_2$  are  $u_1$  and  $u_5$ , then let  $F_i$  be the even subgraph obtained from  $F_{i+1}$  by replacing the cycles in  $F_{i+1}$  intersecting  $D'_{i+1}$  with the cycle

$$u_7P_1u_8u_1P_2u_5u_4P_3u_3u_2u_6u_7.$$

See Figure 17c.

Notice that by symmetry, we finished showing all the cases where there is a path joining  $u_i$  and  $u_{i+1}$  for any  $i$  by the case i).

- ii). *The ends of  $P_1$  are  $u_7$  and  $u_1$ .*

In this case, the ends of  $P_2$  are  $u_8$  and  $u_4$ ; otherwise there is a path joining  $u_i$  and  $u_{i+1}$  for some  $i$ . Let  $F_i$  be the even subgraph obtained from  $F_{i+1}$  by replacing the cycles in  $F_{i+1}$  intersecting  $D'_{i+1}$  with the cycle

$$u_7P_1u_1u_8P_2u_4u_5P_3u_3u_2u_6u_7.$$

See Figure 17d.

iii). *The ends of  $P_1$  are  $u_7$  and  $u_3$ .*

If the ends of  $P_2$  are  $u_8$  and  $u_4$ , then let  $F_i$  be the even subgraph obtained from  $F_{i+1}$  by replacing the cycles in  $F_{i+1}$  intersecting  $D'_{i+1}$  with the cycle

$$u_7P_1u_3u_2u_1P_3u_5u_4P_2u_8u_7.$$

See Figure 17e.

If the ends of  $P_2$  are  $u_8$  and  $u_5$ , then let  $F_i$  be the even subgraph obtained from  $F_{i+1}$  by replacing the cycles in  $F_{i+1}$  intersecting  $D'_{i+1}$  with the cycle

$$u_7P_1u_3u_2u_1P_3u_4u_5P_2u_8u_7.$$

See Figure 17f.

iv). *The ends of  $P_1$  are  $u_7$  and  $u_4$ .*

If the ends of  $P_2$  are  $u_8$  and  $u_5$ , then let  $F_i$  be the even subgraph obtained from  $F_{i+1}$  by replacing the cycles in  $F_{i+1}$  intersecting  $D'_{i+1}$  with the cycle

$$u_7P_1u_4u_5P_2u_8u_1P_3u_3u_2u_6u_7.$$

See Figure 17g. The case that the ends of  $P_2$  are  $u_8$  and  $u_3$  is same to the case iii). See Figure 17f.

v). *The ends of  $P_1$  are  $u_7$  and  $u_5$ .*

If the ends of  $P_2$  are  $u_8$  and  $u_4$ , then let  $F_i$  be the even subgraph obtained from  $F_{i+1}$  by replacing the cycles in  $F_{i+1}$  intersecting  $D'_{i+1}$  with the cycle

$$u_7P_1u_5u_4P_2u_8u_1P_3u_3u_2u_6u_7.$$

See Figure 17h. The case that the ends of  $P_2$  are  $u_8$  and  $u_3$  is same to the case iv). See Figure 17g.

Now we completed the proof. ■

## 4 Closing Remarks

The *Travelling Salesman Problem (TSP)* is the one to find a spanning closed walk of short length in a given graph. The typical method for TSP on 3-edge-connected



cubic graphs is as follows; First, we find a 2-factor  $F$  in a given 3-edge-connected cubic graph  $G$ , and take a certain connected graph  $T$  (e.g. a spanning tree) in the graph  $G/F$  obtained from  $G$  by contracting all components in  $F$ , and then we obtain a connected subgraph  $F \cup T$  of  $G$ . By modifying it suitably, we can get a spanning closed walk of length some function on  $|E(T)|$ . Since  $T$  must be a connected subgraph of  $G/F$ ,  $|E(T)|$  is at least the number of components of  $F$  minus one, and so the fewer number of components in  $F$  gives the better bounds.

Aggarwal, Garg and Gupta [1] used Theorem A to begin with a 2-factor having at most  $n/5$  components, and showed the existence of a spanning closed walk of length at most  $4n/3$  in a 3-edge-connected cubic graph of order  $n$ . This result was further improved to 2-edge connected or connected cubic graphs, graphs of maximum degree at most 3, or better bounds than  $4n/3$ , see [6, 8, 9].

Because of the above reasons, several researchers have been interested in a 2-factor in cubic graphs such that the number of 5-cycles are small, see [7]. Instead of using a 2-factor, we can use an even subgraph satisfying certain conditions on the order of each component. In fact, such structures have appeared in [8, 9] as intermediate products, which is called an *R-factor* in [8]. For those intermediate products, it is not necessarily dominating, but the dominating property may help us to obtain good bounds, i.e, we expect that Theorem 1 has a potential application to problems on TSP.

## Acknowledgments

The authors would like to thank the referees for valuable suggestions and comments.

## References

- [1] N. Aggarwal, N. Garg and S. Gupta, *A  $4/3$ -approximation for TSP on cubic 3-edge-connected graphs*, <http://arxiv.org/pdf/1101.5586v1.pdf> (2011).
- [2] B.R. Alspach and C.D. Godsil (eds), *Cycles in Graphs*, Annals of Discrete Mathematics, vol. 27. Unsolved problems 2.6, 463. North-Holland, Amsterdam (1985)
- [3] P. Ash and B. Jackson, *Dominating cycles in bipartite graphs*, in J. In: Bondy, A., Murty, U.S.R. (eds.) Progress in Graph Theory, pp. 81-87. Academic Press, New York (1984)
- [4] J.C. Bermond and C. Thomassen, *Cycles in digraphs—a survey*, J. Graph Theory 5 (1981) 1–43

- [5] J.A. Bondy and U.S.R. Murty, *Graph theory*, Graduate Texts in Mathematics, 244. Springer, New York, (2008)
- [6] S. Boyd, R. Sitters, S. van der Ster, and L. Stougie, *The traveling salesman problem on cubic and subcubic graphs*, Mathematical Programming 144 (2014) 227–245.
- [7] B. Candráková and R. Luko’ka, *Avoiding 5-circuits in 2-factors of cubic graphs*, SIAM J. Discrete Math. 29 (2015) 1387–1405.
- [8] B. Candráková and R. Luko’ka, *Cubic TSP - a 1.3-approximation*, <http://arxiv.org/pdf/1506.06369v1.pdf> (2015).
- [9] J. Correa, O. Larré, and A. Soto, *TSP tours in cubic graphs: beyond 4/3*, SIAM J. Discrete Math. 29 (2015) 915–939.
- [10] H. Fleischner, *Spanning Eulerian subgraphs, the splitting lemma, and Petersen’s theorem*, Discrete Math. 101 (1992) 33–37.
- [11] H. Fleischner and B. Jackson, *A note concerning some conjectures on cyclically 4-edge-connected 3-regular graphs*, In: Andersen, L.D., Jakobsen, I.T., Thomassen, C., Toft, B., Vestergaard, P.D.(eds.) Graph Theory in Memory of G. A. Dirac. Annals of Discrete Mathematics, vol. 41, pp. 171–177. North-Holland, Amsterdam (1989)
- [12] D. Gamarnik, M. Lewenstein, and M. Sviridenko, *An improved upper bound for the TSP in cubic 3-edge-connected graphs*, Operations Research Letters 33 (2005) 467–474.
- [13] B. Jackson and K. Yoshimoto, *Even subgraphs of bridgeless graphs and 2-factors of line graphs*, Discrete Math. 307 (2007) 2775–2785.
- [14] B. Jackson and K. Yoshimoto, *Spanning Even Subgraphs of 3-edge-connected Graphs*, J. Graph Theory 62 (2009) 37–47.
- [15] T. Kaiser and R. Škrekovski, *Cycles intersecting edge-cuts of prescribed sizes*, SIAM J. Discrete Math. 22 (2008) 861–874.
- [16] M.M. Matthews and D.P. Sumner, *Hamiltonian results in  $K_{1,3}$ -free graphs*, J. Graph Theory 8 (1984) 139–146.
- [17] J. Petersen, *Die Theorie der regulären Graphs*, Acta Math. 15 (1891) 193–220.
- [18] Z. Ryjáček, *On a closure concept in claw-free graphs*, J Combin Theory Ser B 70 (1997) 217–224.
- [19] C. Thomassen, *Reflections on graph theory*, J. Graph Theory 10 (1986) 309–324.