Characterizations of edge-colorings of complete graphs that forbid certain rainbow subgraphs

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Abstract

In this work, we characterize all edge-colorings of complete graphs without rainbow $K_{1,3}$ and those without rainbow P_4^+, where P_4^+ is the graph consisting of P_4 with one extra edge incident with an inner vertex. We also apply these classifications to other areas like highly connected monochromatic subgraphs with large size, Anti-Ramsey numbers, Gallai-Ramsey numbers and show some implications between forbidden rainbow subgraphs.

1 Introduction

The search for highly connected subgraphs is critical for many applications like the construction of electrical grids, computer processors and even network security. If, for example, there are several different types of security, this can be modeled by coloring the edges of the graphs, one color for each security type. Within an edge-colored complete graph, it is therefore natural to consider highly connected monochromatic subgraphs, and preferably large ones. If there are only two colors, at least one of the two monochromatic subgraphs will be connected and spanning but we cannot force higher connectivity without giving up on something, namely the "spanning" condition. When more colors are available, the situation is even worse. We must employ other strategies to force highly connected large monochromatic subgraphs. To this end, we forbid certain rainbow colored subgraphs from appearing.

We consider edge-colorings of (simple) complete graphs. Note that such a coloring is not necessarily proper. See [2] for any notation not defined here. The complete graph of order n is denoted by K_n. An *m-edge-coloring* is an

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edge-coloring with exactly m colors, where a set of colors is always $\{1, \ldots, m\}$ in this paper. We identify an edge-coloring of a graph with an edge-colored graph. A subgraph of an edge-coloring is said to be *rainbow* if no two edges in the subgraph have the same color. Let P_ℓ denote the path of order ℓ (so the length of P_ℓ is $\ell - 1$) and let P_4^+ denote the graph consisting of P_4 with one extra edge incident with an interior vertex (so it is the unique tree of order 5 with degree sequence $3, 2, 1, 1, 1$).

An edge-coloring of K_n without a rainbow copy of H for some certain graphs H has particular properties. The most famous one is due to Gallai [8] for the case $H \simeq K_3$. Indeed, the characterization gives a special partition of the vertices, which is called a *Gallai partition*.

Theorem 1 (Gallai [8]). *For $m \geq 2$ and $n \geq 3$, if G is an m-edge-coloring of K_n without rainbow K_3, then there exist two colors i and j and a partition of $V(G)$ into at least two parts such that for any two different parts, all edges between them are colored by the same color that is i or j.*

Similar to the case of K_3, Fujita and Magnant [4] gave a characterization for the case where a rainbow S_3^+ is forbidden, where S_3^+ is the graph consisting of a triangle with a pendant edge.

Another example of such a characterization is the case of paths. If an m-edge-coloring of K_n contains no rainbow P_3 then $m = 1$ must hold. Note that if $m < |E(H)|$ or $n < |V(H)|$, then an m-edge-coloring of K_n trivially contains no rainbow H. Let us call the case *trivial*. So, when $H \simeq P_3$, only the trivial cases can happen. In the following theorems, the condition (0) always corresponds to the trivial case.

For the case of P_4, it was shown that if an m-edge-coloring of K_n contains no rainbow P_4, then (0) $m \leq 2$ or $n \leq 3$ must hold, unless $n = 4$ and the edge-coloring is proper. (See [14, Theorem 1], or Theorem 11 in this paper). Therefore, for the case $n = 4$, only the trivial cases can happen, with one exception.

The case of P_5 was done by Thomason and Wagner [14, Theorem 2];

Theorem 2 (Thomason and Wagner [14]). *For positive integers m and n, if G is an m-edge-coloring of K_n without rainbow P_5, then after renumbering the colors, one of the following holds.*

(0) *$m \leq 3$ or $n \leq 4$.*

(1) *There exists a partition of $V(G)$ into $\{V_1, V_2, \ldots, V_m\}$ such that for each i, all edges connecting two vertices in V_i are colored by either 1 or i (so V_1 forms a monochromatic clique), and all edges between V_i and V_j with $i \neq j$ are colored by 1.*

(2) *$G - v$ is a monochromatic clique for some vertex v.*

(3) *There are three vertices v_1, v_2, and v_3 such that the edges v_1v_2, v_2v_3 and v_3v_1 have color $2, 3$ and 4, respectively, some edges incident with v_1 have color 3, and all other edges have color 1.*

(4) *There are four vertices* v_1, v_2, v_3 *and* v_4 *such that the edges* $v_1v_2, v_1v_3, v_1v_4, v_2v_3$ *and* v_2v_4 *have color* $2, 3, 4, 4$ *and* 3*, respectively, the edge* v_3v_4 *has color* 1 *or* 2*, and all other edges have color* 1*.*

(5) $n = 5$*,* $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$*, the edges* v_1v_4, v_1v_5 *and* v_2v_3 *have color* 1*, the edges* v_2v_4, v_2v_5 *and* v_1v_3 *have color* 2*, the edges* v_3v_4, v_3v_5 *and* v_1v_2 *have color* 3*, and the edge* v_4v_5 *has color* 4*.*

In this paper, we give some variants of those characterizations for the cases of $K_{1,3}$ and P_4^+. First, for the case of $K_{1,3}$, we define one specific edge-coloring.

For $n \geq 1$, let $G_1(n)$ be a 3-edge-coloring of K_n that satisfies the following conditions: The vertices of K_n are partitioned into three pairwise disjoint sets V_1, V_2 and V_3 such that for $1 \leq i \leq 3$ (with indices modulo 3), all edges between V_i and V_{i+1} have color i, and all edges connecting pairs of vertices within V_{i+1} have color i or $i+1$. Note that one of V_1, V_2 and V_3 is allowed to be empty, but at least two of them are non-empty. (Otherwise at most only two colors can appear.)

Theorem 3. *For positive integers* m *and* n*, if* G *is an* m*-edge-coloring of* K_n *without rainbow* $K_{1,3}$*, then after renumbering the colors, one of the following holds.*

(0) $m \leq 2$ *or* $n \leq 3$*.*

(1) $m = 3$ *and* $G \simeq G_1(n)$*.*

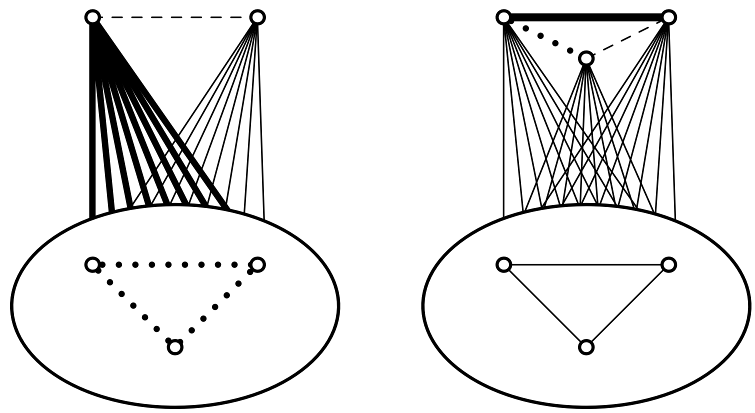
(2) $m \geq 4$ *and Item (1) in Theorem 2 holds.*

The proof of Theorem 3 appears in Section 4. Note that when $m = 3$, Item (1) in Theorem 2 corresponds to the case $G \simeq G_1(n)$ with $V_3 = \emptyset$.

Next we show the case of P_4^+, by proving a reduction to the case of $K_{1,3}$. Note that if an edge-coloring contains no rainbow $K_{1,3}$, then trivially it contains no rainbow P_4^+. We show that, somewhat surprisingly, the converse is also almost true. In order to state this result precisely, we need to describe two specific colorings of K_n.

For $n \geq 4$, let $G_2(n)$ be a 4-edge-coloring of K_n in which there is exactly one edge, say xy, having color 2. Every edge from x to all other vertices except y has color 3, and every edge from y to all other vertices except x have color 4. All edges not incident to vertices x, y have color 1. See Figure 1a. This graph contains no rainbow P_4^+ but contains a rainbow $K_{1,3}$ and (if $n \geq 5$) a rainbow P_5.

For $n \geq 4$, let $G_3(n)$ be a 4-edge-coloring of K_n in which there exists a rainbow K_3 having colors $1, 2$ and 3. Let every edge incident to at most one vertex in the rainbow K_3 have color 1. See Figure 1b. This graph contains no rainbow P_4^+ and no rainbow P_5, but contains a rainbow $K_{1,3}$.



(a) $G_2(n)$ (b) $G_3(n)$

Figure 1: The edge-colorings $G_2(n)$ and $G_3(n)$ of K_n.

Then we are ready to state our next theorem.

Theorem 4. *For positive integers m and n, if G is an m-edge-coloring of K_n without rainbow P_4^+, then after renumbering the colors, one of the following holds.*

(0) $m \leq 3$ *or* $n \leq 4$.

(1) $m = 4$ *and* $G \in \{G_2(n), G_3(n)\}$.

(2) $m \geq 4$ *and G contains no rainbow $K_{1,3}$. In particular, Item (1) in Theorem 2 holds.*

The proof of Theorem 4 appears in Section 5. The following implication is an important step for the proof of Theorem 4. Note that any edge-coloring of K_n without a rainbow $K_{1,3}$ does contain no rainbow P_5 (see Fact 1 in Section 3).

Lemma 5. *For positive integers m and n, if G is an m-edge-coloring of K_n without rainbow P_4^+, then after renumbering the colors, one of the following holds.*

(0) $m \leq 3$ *or* $n \leq 4$.

(1) $m = 4$ *and* $G \simeq G_2(n)$.

(2) $m \geq 4$ *and G contains no rainbow P_5.*

2 Applications of our characterizations

In this section, we give three applications of our characterizations.

2.1 Anti-Ramsey number

In recent years, there have been several attempts to extend Ramsey theory for edge-colorings with many colors. See the surveys [6, 7] for discussions of the topics in this and the following subsections.

Given a graph H, the *anti-Ramsey number*, denoted by $\mathrm{ar}(K_n, H)$, is defined to be the maximum number m of colors such that there exists an m-edge-coloring of K_n without rainbow H. Our characterizations directly imply the following. Note that Corollary 6 (I) is a special case of the results in [9, 13].

Corollary 6. (I) *For* $n \geq 4$, $\mathrm{ar}(K_n, K_{1,3}) = \lfloor \frac{n}{2} \rfloor + 1$.

(II) *For* $n \geq 6$, $\mathrm{ar}(K_n, P_4^+) = \lfloor \frac{n}{2} \rfloor + 1$.

Proof. We first show the upper bounds of both cases. Let $m = \lfloor \frac{n}{2} \rfloor + 2$, and consider an m-edge-coloring G of K_n. Suppose G contains no rainbow $K_{1,3}$ or no rainbow P_4^+, respectively. So we can apply Theorem 3 or 4 respectively. Note that $m \geq 4$ for (I) and $m \geq 5$ for (II). This implies that in either case, after renumbering the colors, there exists a partition of $V(G)$ into $\{V_1, V_2, \ldots, V_m\}$ satisfying the conditions in Item (1) in Theorem 2. Since any color i with $2 \leq i \leq m$ appears on at least one edge, we have $|V_i| \geq 2$. Therefore,

$$n = \sum_{i=1}^{m} |V_i| \geq 2(m-1) > n.$$

This contradiction shows that the upper bound.

For the lower bounds, it suffices to give an m-edge-coloring of K_n without rainbow $K_{1,3}$ or rainbow P_4^+, where $m = \lfloor \frac{n}{2} \rfloor + 1$. This can be done with a partition of $V(G)$ into $\{V_1, V_2, \ldots, V_m\}$ such that for $2 \leq i \leq m$, $|V_i| = 2$ and each edge inside V_i has color i, and all remaining edges have color 1. □

2.2 Gallai-Ramsey number

Given two graphs H and R, the *(m-edge-colored) Gallai-Ramsey number*, denoted by $\mathrm{gr}_m(H : R)$, is defined to be the minimum integer n such that $\binom{n}{2} \geq m$ and every edge-coloring of K_n using at most m colors contains either a rainbow H or a monochromatic R. Similar to Theorem 1 and the result by Fujita and Magnant [4], which have been used to determine the Gallai-Ramsey number of several graphs R with $H \simeq K_3$ and $H \simeq S_3^+$, we obtain the following corollaries.

Corollary 7. *For* $\binom{k}{2} \geq m \geq 4$,

$$\mathrm{gr}_m(K_{1,3}, K_{1,k}) = \mathrm{gr}_m(P_4^+, K_{1,k}) = \left\lfloor \frac{(m-1)k}{m-2} \right\rfloor.$$

Proof. Let $n = \lfloor \frac{(m-1)k}{m-2} \rfloor$ and consider an m-edge-coloring G of K_n. (Since $\binom{n}{2} \geq \binom{k}{2} \geq m$, there is an m-edge-coloring of K_n.) Suppose that G contains no rainbow $K_{1,3}$ or respectively no rainbow P_4^+. By Theorem 3 or 4 respectively and the condition $m \geq 4$, after renumbering the colors, Item (1) in Theorem 2 holds unless $m = 4$ and $G \in \{G_2(n), G_3(n)\}$ for the case of P_4^+. In the latter, we can easily find a monochromatic $K_{1,n-2}$ that contains a monochromatic $K_{1,k}$. Therefore, we may assume that there exists a partition of $V(G)$ into $\{V_1, V_2, \ldots, V_m\}$ satisfying the conditions in Item (1) in Theorem 2.

Let i be an integer with $2 \leq i \leq m$ such that $|V_i| = \min_{2 \leq j \leq m} |V_j|$. By this choice, we have $|V_i| \leq \frac{n}{m-1} \leq \frac{k}{m-2}$. Note that for a vertex v in V_i, all edges uv with $u \in V(G) \setminus V_i$ have color 1, and hence there exists a monochromatic star with $n - |V_i|$ leaves. Since

$$n - |V_i| \geq \left\lfloor \frac{(m-1)k}{m-2} \right\rfloor - \frac{k}{m-2} \geq k,$$

G contains a monochromatic $K_{1,k}$.

On the other hand, for $n = \left\lfloor \frac{(m-1)k}{m-2} \right\rfloor - 1$, we can construct an m-edge-coloring of K_n without rainbow $K_{1,3}$ (or rainbow P_4^+) nor monochromatic $K_{1,k}$; Consider the partition of K_n into $m-1$ parts $V_2, \ldots, V_m$ with almost the same size. Then put color i to all edges in V_i for $2 \leq i \leq m$, and color 1 to all edges between different parts. □

For the case of $m = 3$ in Corollary 7, we can similarly show $\mathrm{gr}_3(K_{1,3}, K_{1,k}) = 2k$. Note that a 3-edge-coloring $G \simeq G_1(2k-1)$ with V_1, V_2, V_3 of almost the same size attains the upper bound. Furthermore, we can calculate $\mathrm{gr}_m(K_{1,3}, R)$ and $\mathrm{gr}_m(P_4^+, R)$ for several graphs R, using our characterizations. But since we need more pages to show them and they are not a main purpose of this paper, we leave them for future works.

2.3 Monochromatic k-connected subgraphs

Given parameters k and m, a subgraph R of an m-edge-coloring of K_n is called *almost spanning* if $|R| \geq n - f(k, m)$ where f is a function that does not depend on n. The search for almost spanning monochromatic highly connected subgraphs was initiated, in this form, by Bollobás and Gyárfás.

Conjecture 1 (Bollobás and Gyárfás [1]). *If $n > 4(k-1)$, then every 2-edge-coloring of K_n contains a monochromatic k-connected subgraph of order at least $n - 2(k-1)$.*

After several partial solutions [10, 11, 3], recently Łuczak [12] published a proof of Conjecture 1 but a gap in the proof has been noted and, to date, not yet fixed.

We extend the research above to the case of more than two colors. In 2013, Fujita and Magnant classified those graphs H with the property that if G is an edge-coloring of K_n without rainbow H, then G contains an almost spanning monochromatic k-connected subgraph.

Theorem 8 (Fujita and Magnant [5]). *Let n, k, m be positive integers with $n \gg m \gg k$. A connected graph H has the property that "any m-edge-coloring of*

K_n without rainbow H contains an almost spanning monochromatic k-connected subgraph" if and only if $H \in \{K_3, P_4^+, P_6\}$ or a connected subgraph of them.

In an effort to encourage sharpening this result, they posed the following problem (adapted to the current work).

Problem 1 (Fujita and Magnant [5]). *Prove the best possible bounds on the order n of graphs and the order of the obtained almost spanning monochromatic k-connected subgraph in Theorem 8.*

In [5], they considered large n with respect to fixed k: For example, for the case of $K_{1,3}$ and P_4^+, they assume $n > 7.5(k-1)$ and obtain a monochromatic k-connected subgraph of order $n-3k+2$. Here we obtain a much better bound on n and a larger monochromatic k-connected subgraph, allowing slightly large m compared with k.

As mentioned in Section 1, if an m-edge-coloring of K_n contains no rainbow P_3 (resp. rainbow P_4), then $m = 1$ (resp. $m \leq 2$ or $n \leq 4$) must hold. So, these cases are not interesting. As a corollary of Theorem 2, we obtain the answer to the case of P_5. Namely, given positive integers $k \geq 1$, $m \geq \max\{\frac{k+4}{2}, 4\}$ and $n \geq \max\{k+2, 5\}$, an m-edge-coloring of K_n without rainbow P_5 contains a monochromatic k-connected subgraph of order at least $n-1$. (To be exact, we obtain this by the same way as the proof of Corollary 9 below.) The bound "at least $n-1$" is best possible, because of Item (2) in Theorem 2.

In this work, we solve Problem 1 for the cases when the forbidden rainbow subgraph is $K_{1,3}$ or P_4^+, using our characterizations in Section 1.

Corollary 9. *Given positive integers $k \geq 1$, $m \geq \max\{\frac{k+4}{2}, 4\}$ and $n \geq 5$, any m-edge-coloring of K_n without rainbow $K_{1,3}$ contains a monochromatic k-connected subgraph of order n (so, it is spanning).*

Proof. Let G be an m-edge-coloring of K_n without rainbow $K_{1,3}$. Since $m \geq 4$, it follows from Theorem 3 that, after renumbering the colors, there exists a partition of $V(G)$ into $\{V_1, V_2, \dots, V_m\}$ with the conditions as in Item (1) in Theorem 2. Note that $|V_i| \geq 2$ for any $2 \leq i \leq m$, and hence we have $\sum_{i=2}^{m} |V_i| \geq 2(m-1) \geq k+2$. Therefore, if we delete any $k-1$ vertices, there are two integers i and j such that at least one vertex in V_i and in V_j remains, respectively. Since any vertices are connected at least those remaining vertices in V_i or in V_j by edges with color 1, the graph consisting of edges of color 1 is still connected. Therefore, the subgraph of G with color 1 is spanning and k-connected. This completes the proof. $\square$

Corollary 10. *Given positive integers $k \geq 1$, $m \geq \max\{\frac{k+4}{2}, 4\}$ and $n \geq 5$, any m-edge-coloring of K_n without rainbow P_4^+ contains a spanning monochromatic k-connected subgraph, except one example when $m = 4$ containing a monochromatic k-connected subgraph of order $n-2$.*

Proof. Let G be an m-edge-coloring of K_n without rainbow P_4^+. Since $m \geq 4$, it follows from Theorem 4 that, after renumbering the colors, either (1) $G \in$

$\{G_2(n), G_3(n)\}$ or (2) G contains no rainbow $K_{1,3}$. In the former case, we can easily verify the conclusion. Note that $G_2(n) - \{x, y\}$ is a monochromatic clique by the color 1, and $G_3(n)$ has a spanning monochromatic subgraph by the color 1. In the latter case, it follows from Corollary 9 that G contains a spanning monochromatic k-connected subgraph. Note that the exceptional case occurs if and only if $G \simeq G_2(n)$. This completes the proof. □

As we see from the characterizations (Theorems 3 and 4), the bounds of the order of obtained almost spanning k-connected subgraphs in Corollaries 9 and 10 are both best possible. Therefore, the remaining cases of Problem 1 are only for K_3 and P_6.

3 Preliminaries

In our proofs, we make use of the following result.

Theorem 11 (Thomason and Wagner [14]). *For $n \geq 5$, every coloring of K_n using at least 3 colors contains a rainbow P_4.*

We also give an easy implication between forbidden rainbow subgraphs.

Fact 1. *If G is an edge-coloring of K_n without rainbow $K_{1,3}$, then G contains no rainbow P_5.*

Proof. Suppose that there exists a rainbow P_5 in G. Let this rainbow P_5 be $v_1v_2v_3v_4v_5$ and let the colors of its edges be 1, 2, 3 and 4 respectively. Consider the edge v_2v_4. By the symmetry between the colors 1 and 4 and between the colors 2 and 3, we may assume that the edge v_2v_4 has neither color 1 nor color 2. Then v_1, v_2, v_3 and v_4 forms a rainbow claw with center v_2. This completes the proof. □

4 Proof of Theorem 3

Proof. Let m and n be positive integers. Suppose that $m \geq 3$, $n \geq 4$ and G is an m-edge-coloring of K_n without a rainbow copy of $K_{1,3}$,

Suppose first that $m \geq 4$. If $n = 4$, then we can easily check that G contains a rainbow $K_{1,3}$. (In fact, it is known that $\mathrm{ar}(K_4, K_{1,3}) = 3$.) Thus, we may assume that $n \geq 5$. It follows from Fact 1 that G contains no rainbow P_5, and hence we can use the characterization in Theorem 2. After renumbering the colors, if Item (2) in Theorem 2 holds, then since $m \geq 4$, there is a rainbow $K_{1,3}$ with center v, a contradiction. If any one of Items (3)–(5) holds, then we can easily find a rainbow $K_{1,3}$, a contradiction again. Therefore, we may assume Item (1) in Theorem 2 must hold.

We may therefore assume that $m = 3$. By Theorem 11, G contains a rainbow P_4, and let this rainbow P_4 be $v_1v_2v_3v_4$ with color $1, 2$ and 3, respectively. Since

G contains no rainbow $K_{1,3}$, no edge incident with v_2 has color 3 and no edge incident with v_3 has color 1. Let

$$\begin{aligned} X_1 &= \{x \in V(G) : \text{the edges } xv_2 \text{ and } xv_3 \text{ have color 1 and 3, respectively}\}, \\ X_2 &= \{x \in V(G) \setminus X_1 : x \text{ is incident with an edge of color 1}\}, \quad \text{and} \\ X_3 &= V(G) \setminus \big(X_1 \cup X_2\big). \end{aligned}$$

Note that $v_1 \in X_1 \cup X_2$ and $v_2 \in X_2$. Since no edge incident with v_3 has color 1, we have $v_3 \in X_3$. Therefore, $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$, while X_1 may possibly be empty.

We now show that $G \simeq G_1(n)$ with $V_i = X_i$ for $i \in \{1, 2, 3\}$. To do that, we first see the following.

a) Every vertex in X_1 is incident to an edge of color 1 and an edge of color 3.

b) Every vertex in X_2 is incident to an edge of color 1 and an edge of color 2.

c) No vertex in X_3 is incident to an edge of color 1.

From the definitions, a) and c) trivially hold. Let $x \in X_2$. By the definition of X_2, x is incident with an edge of color 1. Then it suffices to show that the edge xv_3 has color 2. If $x = v_2$, then it is trivially true. Thus, we may assume that $x \neq v_2$. Since no edge incident with v_3 has color 1, the edge xv_3 has color either 2 or 3. Suppose that the latter occurs, and then consider the edge xv_2. If it has color 2, then there is a rainbow $K_{1,3}$ with center x, a contradiction. Therefore, since no edge incident with v_2 has color 3, the edge xv_2 must have color 1. However, this implies that $x \in X_1$, a contradiction. This shows b).

Since G contains no rainbow $K_{1,3}$, these directly imply all of the following.

- By a), any edge connecting two vertices in X_1 has color 1 or 3.
- By b), any edge connecting two vertices in X_2 has color 1 or 2.
- By c), any edge connecting two vertices in X_3 has color 2 or 3.
- By a) and b), any edge between X_1 and X_2 has color 1.
- By b) and c), any edge between X_2 and X_3 has color 2.
- By a) and c), any edge between X_3 and X_1 has color 3.

Therefore, $G \simeq G_1(n)$, and this completes the proof. □

5 Proof of Theorem 4

In this section, we denote P_4^+ with vertices u_1, u_2, u_3, u_4 and u_5 by $[u_1u_2u_3u_4; u_5]$, where $u_1u_2u_3u_4$ forms a path P_4 and u_5 is adjacent to u_3.

We first prove Lemma 5, and then show that it implies Theorem 4.

Proof of Lemma 5. Let m and n be positive integers. Suppose that $m \geq 4$, $n \geq 5$ and G is an m-edge-coloring of K_n without rainbow P_4^+. Suppose further that there exists a rainbow P_5 in G. Let $v_1v_2v_3v_4v_5$ be this rainbow P_5 and let its edges be colored $1, 2, 3$ and 4 respectively.

We first consider the edge v_2v_4. If the edge v_2v_4 has color 2 or i with $i \geq 5$, then $[v_1v_2v_4v_5; v_3]$ is a rainbow P_4^+, a contradiction. The case when the edge v_2v_4 has color 3 is symmetric. Therefore, we may assume that the edge v_2v_4 has color 1 or 4, and without loss of generality, we assume it is 4.

We will show $G \simeq G_2(n)$ with v_3 and v_4 playing the role of x and y, respectively, and by exchanging the colors 2 and 3. Let $U = V(G) \setminus \{v_1, v_2, v_3, v_4, v_5\}$.

Let $u \in U$ and consider the edge v_4u.

$$\text{If } v_4u \text{ has} \begin{cases} \text{color 1, then } [v_2v_3v_4v_5; u] \text{ is a rainbow } P_4^+. \\ \text{color 2, then } [v_1v_2v_4v_3; u] \text{ is a rainbow } P_4^+. \\ \text{color 3 or } i \text{ with } i \geq 5, \text{ then } [uv_4v_2v_1; v_3] \text{ is a rainbow } P_4^+. \end{cases}$$

Therefore, the edge v_4u has color 4 for any $u \in U$. This means that u can also play the role of v_5, and hence we can use the symmetry for all the vertices in $U \cup \{v_5\}$.

Next, let $u \in U \cup \{v_5\}$, and consider the edge v_2u.

$$\text{If } v_2u \text{ has} \begin{cases} \text{color 2, then } [v_3v_4v_2v_1; u] \text{ is a rainbow } P_4^+. \\ \text{color 3, then } [v_4uv_2v_1; v_3] \text{ is a rainbow } P_4^+. \\ \text{color } i \text{ with } i \geq 4, \text{ then } [v_4v_3v_2v_1; u] \text{ is a rainbow } P_4^+. \end{cases}$$

Therefore, the edge v_2u has color 1 for any $u \in U \cup \{v_5\}$.

Then consider the edge v_1v_4.

$$\text{If } v_1v_4 \text{ has} \begin{cases} \text{color 1, then } [v_2v_3v_4v_1; v_5] \text{ is a rainbow } P_4^+. \\ \text{color 2, then } [v_2v_1v_4v_3; v_5] \text{ is a rainbow } P_4^+. \\ \text{color 3 or } i \text{ with } i \geq 5, \text{ then } [v_1v_4v_2v_3; v_5] \text{ is a rainbow } P_4^+. \end{cases}$$

Therefore, the edge v_1v_4 has color 4. This implies that v_1 and v_5 have the same adjacency to $\{v_2, v_4\}$ in the sense of colors, and hence we can now use the symmetry for the vertices in $U^+ := U \cup \{v_1, v_5\}$.

Next, let $u \in U^+$ and consider the edge v_3u. Let $u' \in U^+ \setminus \{u\}$.

$$\text{If } v_3u \text{ has} \begin{cases} \text{color 1, then } [u'v_4v_3v_2; u] \text{ is a rainbow } P_4^+. \\ \text{color 3, then } [uv_3v_2v_4; u'] \text{ is a rainbow } P_4^+. \\ \text{color } i \text{ with } i \geq 4, \text{ then } [u'v_2v_3v_4; u] \text{ is a rainbow } P_4^+. \end{cases}$$

Therefore, the edge v_3u has color 2 for any $u \in U^+$.

Finally, let $u, u' \in U^+$, and consider the edge uu'.

$$\text{If } uu' \text{ has} \begin{cases} \text{color 2, then } [v_3v_4uv_2; u'] \text{ is a rainbow } P_4^+. \\ \text{color 3, then } [uu'v_2v_3; v_4] \text{ is a rainbow } P_4^+. \\ \text{color } i \text{ with } i \geq 4, \text{ then } [v_4v_3uv_2; u'] \text{ is a rainbow } P_4^+. \end{cases}$$

Therefore, the edge uu' has color 1 for any $u, u' \in U^+$.

These arguments imply that $G \simeq G_2(n)$, and complete the proof. $\square$

Proof of Theorem 4. Let m and n be positive integers. Suppose that $m \geq 4$, $n \geq 5$ and G is an m-edge-coloring of K_n without rainbow P_4^+. It follows from Lemma 5 that, after renumbering the colors, either (1) $G \simeq G_2(n)$ or (2) G contains no rainbow P_5. In the former case, there is nothing to prove. Therefore, we may assume that G contains no rainbow P_5, and hence we can use the characterization in Theorem 2.

If Item (1) in Theorem 2 holds, then it is easy to see that G contains no rainbow $K_{1,3}$. If Item (2) holds, then since $m \geq 4$, there is a rainbow P_4^+ with v being of degree 3, a contradiction. If one of Item (4) or Item (5) holds, then we can easily find a rainbow P_4^+, a contradiction again. Therefore we may assume that Item (3) holds. If there is an edge uv_1 incident with v_1 having color 3 with $u \neq v_1, v_2, v_3$, then $[u'uv_1v_2; v_3]$ is a rainbow P_4^+ for $u' \in V(G) \setminus \{v_1, v_2, v_3, u\}$, a contradiction. Therefore, all edges incident with v_1 have color 1. This means $G \simeq G_3(n)$ with $v_1v_2v_3$ being a rainbow K_3, and this completes the proof. $\square$

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