Orientations of Graphs Avoiding Given Lists on Out-degrees

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Abstract

Let G be a graph and $F : V(G) \to 2^{\mathbb{N}}$ be a function. The graph G is said to be F-*avoiding* if there exists an orientation O of G such that $d^+_O(v) \notin F(v)$ for every $v \in V(G)$, where $d^+_O(v)$ denotes the out-degree of v in the directed graph G with respect to O. In this paper it is shown that if G is bipartite and $|F(v)| \le \frac{d_G(v)}{2}$ for every $v \in V(G)$, then G is F-avoiding. The bound $|F(v)| \le \frac{d_G(v)}{2}$ is best possible. For every graph G, we conjecture that if $|F(v)| \le \frac{1}{2}\big(d_G(v) - 1\big)$ for every $v \in V(G)$, then G is F-avoiding. We also argue this conjecture; the best possibility and some partial solutions, e.g. for the complete graphs.

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1 Introduction

In this paper, we denote by $\mathbb{N}$ the set of natural numbers (including 0). We only deal with finite simple graphs in this paper, except for those in Section 6.2.3. Let G be a graph. $d_G(v)$ denotes the degree of a vertex v in G. An *orientation* of G is an assignment of a direction to each edge of G. For an orientation O of G and a vertex v, we denote by $d^+_O(v)$ the out-degree of v in the directed graph G with respect to the orientation O. Sometimes for abbreviation, we use $d^+(v)$ instead of $d^+_O(v)$. Because of its fruitful applications, an orientation with specified properties has been extensively studied. See the books [4, Chapter 8] and [16, Chapter 61]. Frank and Gyárfás [6] proved that for a graph G and two functions $a, b : V(G) \to \mathbb{N}$ with $a(v) \le b(v)$ for every vertex v, G has an orientation such that

$$a(v) \;\le\; d^+(v) \;\le\; b(v) \tag{1}$$

for every vertex v, if and only if for any $U \subseteq V(G)$,

$$\sum_{v\in U} a(v) - d(U) \;\le\; \big|E\big(G[U]\big)\big| \;\le\; \sum_{v\in U} b(v),$$

where $d(U)$ is the number of edges connecting U and $V(G) \setminus U$, and $G[U]$ is the subgraph of G induced by U.

In this paper, we consider similar, but more general problems on orientations with some constraints on the out-degree of each vertex. Let $F : V(G) \to 2^{\mathbb{N}}$. A graph G is said to be F-*avoiding* if there exists an

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orientation O of G such that $d^+_O(v) \notin F(v)$ for every $v \in V(G)$. In this case, we say that O *avoids* F or O is an F-*avoiding* orientation. For $f : V(G) \to \mathbb{N}$ or for a non-negative integer c, we define the notion f-*avoiding* and c-*avoiding* by extending the definition of F-avoiding. Namely, G is said to be f-*avoiding* (resp. c-*avoiding*) if G is F-avoiding, where $F(v) = \{f(v)\}$ (resp. $F(v) = \{c\}$) for each $v \in V(G)$. Conversely, for $S \subseteq \mathbb{N}$, we call an orientation O of G an S-*orientation*, if $d^+_O(v) \in S$ for every $v \in V(G)$.

Remark. An orientation satisfying the Inequality (1) corresponds to an F-avoiding one, where

$$F(v) = \{i \in \mathbb{N} : i < a(v) \text{ or } i > b(v)\}.$$

Thus, an F-avoiding orientation is actually a generalization of the orientation given in the Inequality (1). Frank, Tardos and Sebő [8] also considered an orientation satisfying the Inequality (1) with the parity constraints at the out-degree. This is also a special case of an F-avoiding orientation.

Remark. The concept of an F-avoiding orientation is related to Tutte's 3-flow conjecture. In fact, the 3-flow conjecture is known to be equivalent with the statement that every 4-edge-connected 5-regular graph is F-avoiding, where $F = \{0, 2, 3, 5\}$, see [5, Exercises 21.3.2 b) and 21.3.6. a) on Pages 572–573]. Note that such an F-avoiding orientation is called a mod 3-orientation, see [10] and Section 6 in this paper.

The purpose of this article is to give several sufficient conditions for undirected graphs to be f-avoiding or F-avoiding. We divide the rest of paper into the following five sections;

- **(Section 2)** We consider f-avoiding and c-avoiding graphs for $f : V(G) \to \mathbb{N}$ and a non-negative integer c. In this section, we show that some families of graphs are f-avoiding or c-avoiding.
- **(Section 3)** We focus on bipartite graphs. We prove the following result: For a bipartite graph G and for $F : V(G) \to 2^{\mathbb{N}}$, if $|F(v)| \le \frac{d_G(v)}{2}$ for every vertex v, then G is F-avoiding (Theorem 9). We also show that the condition is best possible.
- **(Section 4)** We consider an algebraic method, using Combinatorial Nullstellensatz. We also give another proof to a lemma (Lemma 11), which plays a key role in the proof of Theorem 9 in Section 3.
- **(Section 5)** We propose a conjecture which states that: For a graph G and for $F : V(G) \to 2^{\mathbb{N}}$, if $|F(v)| \le \frac{1}{2}\big(d_G(v) - 1\big)$ for every vertex v, then G is F-avoiding (Conjecture 1). We also show several results related to it;
  - The best possibility of the condition.
  - A partial solution to it, using the theorem in Section 3 (Theorem 16).
  - The special case for complete graphs (Theorem 20).
- **(Section 6)** We give some partial solutions to Conjecture 1, using some results on a β-orientation.

Before proceeding the main arguments, we put several terminology that are used in this paper. Let G be a graph. By the *order* of G, we mean the cardinality of $V(G)$. In this paper, P_n, C_n and K_n denote the path, the cycle and the complete graph of order n, respectively. A *unicyclic* graph is a connected graph containing exactly one cycle. We denote the minimum degree of G by $\delta(G)$ and we use $\alpha(G)$ for the independence number of the graph G. A *block* of G is a maximal subgraph without a cut-vertex. A *leaf block* is a block that contains at most one cut-vertex of G. It is well-known that every graph has a unique block decomposition.

2 f-avoiding graphs

In this section, we prove that if G is a graph with $\delta(G) \geq 3$, then G is f-avoiding, for every function $f : V(G) \to \mathbb{N}$. Moreover, we determine all pairs (G, f) such that G is f-avoiding, where G is a 2-edge-connected graph. First, we prove the following lemma for unicyclic graphs.

Lemma 1. *Every unicyclic graph has a* $\{1\}$*-orientation.*

Proof. First, we claim that for a given tree T and a vertex $v \in V(T)$, there exists an orientation of T such that $d^+(v) = 0$ and $d^+(u) = 1$ for every $u \in V(G) \setminus \{v\}$. To prove the claim, consider T as a rooted tree with the root v. For every $e = xy \in E(T)$ such that y is the parent of x, let the orientation of e be from x to y. Clearly, this orientation has the desired property and the claim is proved.

Now, let G be a unicyclic graph and C be its cycle. Remove the edges of C from G and call the resulting graph by G'. Clearly, G' is a forest and for every component T of G', $|V(T) \cap V(C)| = 1$. Let $V(T) \cap V(C) = \{v_T\}$. By the claim, there exists an orientation of T such that $d^+(v_T) = 0$ and $d^+(u) = 1$, for each $u \in V(T) \setminus \{v_T\}$. Now, orient all edges of C clockwise, and then we obtain a $\{1\}$-orientation of G. □

In the following two theorems, we characterize all graphs which are c-avoiding, for $c \neq 1$.

Theorem 2. *A connected graph G is* 0*-avoiding if and only if G is not a tree.*

Proof. Assume that G is not a tree. Since G is connected, it has a spanning tree T. Let $e \in E(G \setminus T)$ and H be the subgraph with the edge set $E(T) \cup \{e\}$. Obviously, H is a unicyclic graph. By Lemma 1, H has a 1-orientation. Now, orient all edges of $G \setminus E(H)$ arbitrarily. So, G is 0-avoiding.

Now, let G be a tree. By contradiction, assume that G is 0-avoiding. So the sum of the out-degrees of vertices in this orientation is at least $|V(G)|$, a contradiction. The proof is complete. □

Theorem 3. *Let $c > 1$ be a positive integer. Then every connected graph is c-avoiding.*

Proof. Let G be a graph, T be a spanning tree in G, and v be a leaf of T. Orient those edges of $E(G) \setminus E(T)$ which are incident with v inward to v. Also, orient all other edges of $E(G) \setminus E(T)$ arbitrarily. Now, consider T as a rooted tree with root v. For each leaf u, we have two choices for the orientation of the edge between u and its parent. Thus, there exists an orientation for this edge such that $d^+(u) \neq c$. We continue this procedure for all vertices of T, but v. Note that $d^+(v) \neq c$, since $d^+(v) \in \{0, 1\}$ and $c > 1$. So, we obtain the desired orientation, and the proof is complete. □

Now, we provide some sufficient conditions for a graph G to be f-avoiding. We first show a useful lemma.

Lemma 4. *Let G be a connected graph and let $f : V(G) \to \mathbb{N}$ be a function. Suppose that G has a vertex v with $d_G(v) \geq 3$ and $G \setminus \{v\}$ is connected. Then G is f-avoiding.*

Proof. Since $G \setminus \{v\}$ is connected, $G \setminus \{v\}$ has a spanning tree T'. Let $e = vw \in E(G)$ be an edge incident with v. So, the subgraph with the edge set $E(T') \cup \{e\}$ is a spanning tree T of G in which v is a leaf. Orient those edges of $E(G) \setminus E(T)$ that are incident with v as follows: If $f(v) \leq \frac{d_G(v)}{2}$, then orient these edges outward from v; Otherwise, orient them inward to v. Also orient all other edges of $E(G) \setminus E(T)$ arbitrarily.

Then consider T as a rooted tree with root v. For each leaf $u \neq v$, we have two choices for the orientation of the edge between u and its parent. Thus, there exists an orientation of this edge such that $d^+(u) \neq f(u)$. We continue this procedure for all vertices of T, but v. Note that $d^+(v) \in \{d_G(v), d_G(v)-1\}$ if $f(v) \leq \frac{d_G(v)}{2}$, and $d^+(v) \in \{0, 1\}$ otherwise. In either case, since $d_G(v) \geq 3$, we have $f(v) \neq d^+(v)$. The proof is complete. □

Theorem 5. *Let G be a graph and $\delta(G) \geq 3$. Then for every function $f : V(G) \to \mathbb{N}$, G is f-avoiding.*

Proof. Consider a spanning tree of G and a leaf v of it. Since $G \setminus \{v\}$ is connected and $\delta(G) \geq 3$, the vertex v satisfies the conditions in Lemma 4. Therefore, G is f-avoiding. □

In the following lemma, we characterize all functions f such that C_n is f-avoiding.

Lemma 6. *Let $n \geq 3$ be a positive integer and $f : V(C_n) \to \mathbb{N}$ be a function. Then C_n is f-avoiding if and only if either n is even, or n is odd and $f(v) \neq 1$ for some vertex v.*

Proof. One can easily prove that if n is odd and f is the constant function 1, then C_n has no orientation such that $d^+(v) \neq 1$, for every $v \in V(C_n)$. For the other direction, we apply induction on n. It can be easily checked that the assertion holds for $n = 3$. Thus, we may assume that $n \geq 4$.

First, assume that n is odd and $C_n = v_1, \ldots, v_n$. By assumption there exists some index i such that $f(v_i) \neq 1$. Without loss of generality suppose that $f(v_n) \neq 1$. Remove v_n and join v_1 to v_{n-1} to obtain C_{n-1}. Let f' be the restriction of f to $V(C_{n-1})$. By the induction hypothesis, C_{n-1} has an orientation such that $d^+(v_i) \neq f'(v_i) = f(v_i)$ for $i = 1, \ldots, n-1$. Without loss of generality, we may assume that the edge v_1v_{n-1} is oriented from v_{n-1} to v_1. Then remove the edge v_1v_{n-1} and orient the edges $v_{n-1}v_n$ and v_nv_1 from v_{n-1} to v_n, and from v_n to v_1, respectively, to obtain the desired orientation.

Next, assume that n is even. If f is the constant function 1, then orient two edges $v_{2i-1}v_{2i}$ and $v_{2i}v_{2i+1}$ from v_{2i} to v_{2i-1} and from v_{2i} to v_{2i+1}, respectively, for $i = 1, \ldots, \frac{n}{2}$ (taking the indices modulo n). Note that the resulting orientation avoids f. Thus, suppose that f is not the constant function 1 and $f(v_n) \neq 1$. Remove v_n and join v_1 to v_{n-1} to obtain C_{n-1}. Let f' be the restriction of f to $V(C_{n-1})$. If f' is not the constant function 1, then the proof is the same as for the case when n is odd. Now, suppose that f' is the constant function 1. Then in C_n, orient the edges $v_{2i-1}v_{2i}$ and $v_{2i}v_{2i+1}$ from v_{2i} to v_{2i-1} and from v_{2i} to v_{2i+1}, respectively, for $i = 1, \ldots, \frac{n}{2}$ (taking the indices modulo n). If $f(v_n) \neq 2 = d^+(v_n)$, then the resulting orientation of C_n avoids f; otherwise, reverse the orientation of all the edges to obtain the desired orientation. The proof is complete. □

Theorem 7. *Every 2-connected graph G is f-avoiding, for each $f : V(G) \to \mathbb{N}$, unless G is an odd cycle.*

Proof. First, suppose that there exists a vertex v in $V(G)$ such that $d_G(v) \geq 3$. Since G is 2-connected, $G \setminus \{v\}$ is connected. Then by Lemma 4, G is f-avoiding. Therefore, we may assume that $d_G(v) \leq 2$ for every $v \in V(G)$. By the assumption, G is an even cycle. Then it follows from Lemma 6 that G is also f-avoiding. So, the proof is complete. □

In the next theorem, we classify all 2-edge-connected graphs that are f-avoiding for any function f.

Theorem 8. *Let G be a 2-edge connected graph. Then for every $f : V(G) \to \mathbb{N}$, G is f-avoiding, unless every block of G is an odd cycle and for every $v \in V(G)$, $f(v)$ is the number of blocks containing v. Moreover, in the exceptional case, G is not f-avoiding.*

Proof. We prove the first statement by induction on $|E(G)|$. If G is 2-connected, then by Theorem 7, we are done. If there exists a block B that is not an odd cycle, then remove all edges of B and all resulting isolated vertices. Let G' be the obtained graph. Define $f' : V(G') \to \mathbb{N}$ such that $f'(v) = f(v)$ for $v \notin V(B)$, and $f'(v) = 0$ for $v \in V(B)$. Note that each component of G' has a vertex v in B, which means that $f'(v)$ is not the number of blocks containing v. Therefore, by the induction hypothesis, we see that every component of G' is f'-avoiding. So there exists an orientation of G', say O, which avoids f'. Now, define $g : V(B) \to \mathbb{N}$ such that $g(v) = f(v)$ for $v \notin V(G')$ and $g(v) = \max\{f(v) - d^+_O(v), 0\}$ for every $v \in V(B) \cap V(G')$. By Theorem 7, B is g-avoiding, and hence there exists an orientation O' for B which avoids g. It is not hard to see that O and O' together provide the desired orientation.

Now, suppose that all blocks of G are odd cycles and there exists a vertex v such that $f(v)$ is not the number of blocks containing v. Let $B_1, \ldots, B_k$ be all blocks containing v. Note that $f(v) \neq k$. Let u_{2i-1} and u_{2i} be the two neighbors of v in B_i, for $i = 1, \ldots, k$. Now, join u_{2i} to u_{2i+1}, for every $i = 1, \ldots, k$ modulo $2k$. Then remove v and let H be the resulting graph. It is clear that any component of H has a block that is not an odd cycle, and $|E(H)| < |E(G)|$. So by the induction hypothesis, there exists an orientation O of H which avoids f. Now, for $i = 1, \ldots, k$, if the edge $u_{2i}u_{2i+1}$ has direction from u_{2i} to u_{2i+1}, then orient the edges $u_{2i}v$ and $u_{2i+1}v$ from u_{2i} to v and from v to u_{2i+1}, respectively. If the edge $u_{2i}u_{2i+1}$ has direction from u_{2i+1} to u_{2i}, then orient the edges $u_{2i}v$ and $u_{2i+1}v$ from v to u_{2i} and from u_{2i+1} to v, respectively. Let O' be this orientation. It is not hard to see that $d^+_{O'}(v) = d^-_{O'}(v) = k$, and $d^+_{O'}(u) = d^+_O(u)$ for every $u \in V(G)$ with $u \neq v$. Hence G is f-avoiding.

To complete the proof, assume that every block of G is an odd cycle and $f(v)$ is the number of blocks containing v, for each $v \in V(G)$. Now, we show that G is not f-avoiding by induction on the number of blocks. If G has only one block, then G is an odd cycle and f is the constant function 1. Therefore, the result follows from Lemma 6. Hence, we may assume that G has at least two blocks.

By contradiction, assume that there exists an orientation O of G which avoids f. Let B be a leaf block of G, that is, B contains exactly one cut vertex, say v. Since $f(x)$ is the number of blocks containing x, we see that $f(x) = 1$ for each $x \in V(B) \setminus \{v\}$. Let O_B be the orientation of B obtained by the restriction of O to B. Then, it is not hard to see that $d^+_{O_B}(v) = 1$; otherwise the sum of out-degrees on B is even, a contradiction. Now, remove all vertices of B except v from G and let H' be the resulting graph. Define $g : V(H') \to \mathbb{N}$ such that $g(u) = f(u)$, for every $u \in V(H') \setminus \{v\}$ and $g(v) = f(v) - 1$. It is straightforward to see that $g(u)$ is the number of blocks in H' containing u, for every $u \in V(H')$, and hence by the induction hypothesis, H' is not g-avoiding. But O induces an orientation of H' which avoids g, a contradiction. The proof is complete. □

3 F-avoiding bipartite graphs

In this section, we focus on F-avoiding bipartite graphs. The following is the main theorem in this section.

Theorem 9. *Let G be a bipartite graph and let $F : V(G) \to 2^{\mathbb{N}}$ be a function. If*

$$|F(v)| \leq \frac{d_G(v)}{2}$$

for every vertex v, then G is F-avoiding.

We first show that the condition $|F(v)| \leq \frac{d_G(v)}{2}$ in Theorem 9 is sharp. To see this, consider a bipartite graph G in which each vertex has even degree, and let $F(v) = \{0, 1, \ldots, \frac{d_G(v)-2}{2}\}$ for all $v \in V(G)$ except a vertex u, where $F(u) = \{0, 1, \ldots, \frac{d_G(u)}{2}\}$. Note that $|F(v)| = \frac{d_G(v)}{2}$ for any $v \neq u$ and $|F(u)| = \frac{d_G(u)}{2} + 1$.

Suppose that G is F-avoiding. Then G admits an orientation O such that $d_O^+(v) \notin F(v)$ for each $v \in V(G)$, which means that $d_O^+(v) \geq \frac{d_G(v)}{2}$ for all $v \in V(G)$ and $d_O^+(u) \geq \frac{d_G(u)}{2} + 1$. However, this implies that

$$\sum_{v \in V(G)} d_O^+(v) \;\geq\; \sum_{v \in V(G)} \frac{d_G(v)}{2} + 1 \;=\; |E(G)| + 1,$$

contradicting that $\sum_{v \in V(G)} d_O^+(v) = |E(G)|$. Therefore, G is not F-avoiding, and hence the condition $|F(v)| \leq \frac{d_G(v)}{2}$ is sharp.

3.1 Proof of Theorem 9

To prove Theorem 9, we use the following two lemmas. The former (Lemma 10) only plays an auxiliary role. It can be easily shown and appears in several books (for example, see [5, Exercises 3.4.13 on Page 93]), and hence we omit the proof of it. On the other hand, the latter (Lemma 11) is essential.

Lemma 10. *Every graph G admits an orientation D such that*

$$d_D^+(v) = \left\lceil \frac{d_G(v)}{2} \right\rceil \quad \text{or} \quad \left\lfloor \frac{d_G(v)}{2} \right\rfloor$$

for every $v \in V(G)$.

Lemma 11. *Let G be a bipartite graph and let $F : V(G) \to 2^{\mathbb{N}}$. If G admits an orientation D such that*

$$d_D^+(v) \geq |F(v)|$$

for every vertex v, then G is F-avoiding.

We give two different proofs to Lemma 11; An elementary one in Section 3.2 and an algebraic one in Section 4. Since they are based on different ideas and those might be useful to attack our conjecture (Conjecture 1) in Section 5, we decided to put both proofs in this paper.

Here, we prove Theorem 9 using the above two lemmas.

Proof of Theorem 9. By Lemma 10, G has an orientation D such that

$$d_D^+(v) \;\geq\; \left\lfloor \frac{d_G(v)}{2} \right\rfloor \;\geq\; |F(v)|$$

for every $v \in V(G)$. Then it follows from Lemma 11 that G is F-avoiding, and we are done. □

3.2 Proof of Lemma 11

Proof. We prove the lemma by induction on $|E(G)|$. Let A and B be the partite sets of G. Suppose that for any vertex v, $F(v)$ contains neither 0 nor $d_G(v)$. In this case, we orient all edges from A to B and let O be the resulting orientation. Note that $d_O^+(v) = d_G(v)$ for any $v \in A$ and $d_O^+(v) = 0$ for any $v \in B$. Recall that $F(v)$ contains neither 0 nor $d_G(v)$ for any vertex v, and hence we are done.

Therefore, we may assume that there exists a vertex u such that either 0 or $d_G(u)$ is contained in $F(u)$. First suppose that $F(u)$ contains 0. Then $d_D^+(u) \geq |F(u)| \geq 1$, and hence there exists an edge $e = uv$ in G

such that e is directed from u to v in D. Let $G' = G - e$ and D' be the orientation of G' obtained from D by the restriction to G'. For any vertex w, define $F'(w)$ so that

$$F'(w) = \begin{cases} F(w) & \text{for } w \neq u, \\ \{i - 1 : i \in F(u) - \{0\}\} & \text{for } w = u. \end{cases}$$

Note that $d^+_{D'}(w) = d^+_D(w) \geq |F(w)|$ for any vertex w with $w \neq u$ and $d^+_{D'}(u) = d^+_D(u) - 1 \geq |F(u)| - 1 = |F'(u)|$. Therefore, by the induction hypothesis, there exists an orientation O' for G' such that $d^+_{O'}(w) \notin F'(w)$ for every vertex w in G'. Let O be the orientation obtained from O' by adding the direction of e from u to v. Noting the direction of e, one can see that $d^+_O(u) \neq 0$. Thus, $d^+_O(w) = d^+_{O'}(w) \notin F'(w) = F(w)$ for any vertex w with $w \neq u$ and $d^+_O(u) = d^+_{O'}(u) + 1 \notin F(u)$, and hence we are done.

Next suppose that $F(u)$ contains $d_G(u)$. In this case, we can find a desired orientation in a similar way to the previous case. In fact, there exists an edge $e = uv$ in G such that e is directed from u to v in D. Then we define the graph G' and the orientation D' in the same way as before, and for any vertex w, define $F'(w)$ so that

$$F'(w) = \begin{cases} F(w) & \text{for } w \neq u, v, \\ F(u) - \{d_G(u)\} & \text{for } w = u, \\ \{i - 1 : i \in F(v) - \{0\}\} & \text{for } w = v. \end{cases}$$

Note that $d^+_{D'}(u) = d^+_D(u) - 1 \geq |F(u)| - 1 = |F'(u)|$ and $d^+_{D'}(v) = d^+_D(v) \geq |F(v)| \geq |F'(v)|$. Therefore, by the induction hypothesis, there exists an orientation O' for G' such that $d^+_{O'}(w) \notin F'(w)$ for every vertex w in G'. Let O be the orientation obtained from O' by adding the direction of e from v to u. Noting the direction of e, one can see that $d^+_O(u) \neq d_G(u)$, and $d^+_O(v) = d^+_{O'}(v) + 1 \notin F(v)$. Furthermore, we have $d^+_O(w) = d^+_{O'}(w) \notin F'(w) = F(w)$ for any vertex w with $w \neq u, v$. This completes the proof. □

3.3 Relation to spanning subgraphs with degree-constraints

Our arguments in this section were inspired by some works on (undirected) spanning subgraphs with degree-constraints. In particular, Theorem 9 and the following theorem on *list-avoiding* subgraphs have something in common.

Theorem 12. *(**Shirazi and Verstraëte** [17]) Let G be a graph and $F : V(G) \to 2^{\mathbb{N}}$ be a function. If $|F(v)| \leq \frac{d_G(v)}{2}$ for every vertex v, then G has a spanning (undirected) subgraph H such that $d_H(v) \notin F(v)$ for every $v \in V(G)$.*

Though it was not explicitly mentioned, Shirazi and Verstraëte [17] essentially proved a statement similar to Lemma 11. Namely, for a graph G and a function $F : V(G) \to 2^{\mathbb{N}}$, if G admits an orientation D such that $d^+_D(v) \geq |F(v)|$ for every vertex v, then G has a spanning subgraph H such that $d_H(v) \notin F(v)$ for every $v \in V(G)$. This is exactly same as the statement of [7, Theorem 2]. By the same way as in the proof of Theorem 9 in Section 3.1, this statement and Lemma 10 implies Theorem 12.

Note that the proof in [17] was based on the Combinatorial Nullstellensatz, and later, Frank, Lau, and Szabó [7] gave an elementary proof to the statement. Indeed, the proof in Section 3.2 in this paper is based on the proof in [7], while the proof we will give in Section 4 is based on the one in [17].

4 An algebraic method to find an F-avoiding orientation

A subgraph H of a graph G with orientation D is called *Eulerian* if $d^-_H(v) = d^+_H(v)$ for all $v \in V(H)$. The subgraph H is called *even* if $|E(H)|$ is even and H is called *odd* otherwise. Let $EE(D)$ and $EO(D)$ denote the number of even and odd spanning Eulerian subgraphs of G with respect to D, respectively. For proving the next result, we need the Combinatorial Nullstellensatz Theorem due to Alon, see [1].

Theorem 13. *[Combinatorial Nullstellensatz] Let K be a field and $P \in K[x_1, \dots, x_m]$ be a polynomial of degree $p_1 + \cdots + p_m$, and let $S_1, \dots, S_m$ be subsets of K such that $|S_i| \geq p_i + 1$ for all i. If the coefficient of $x_1^{p_1} \dots x_m^{p_m}$ is non-zero, then there exists an element $s_i \in S_i$ for each $1 \leq i \leq m$ such that $P(s_1, \dots, s_m) \neq 0$.*

Now, we obtain a result based on Combinatorial Nullstellensatz. A similar method can be found in [2].

Theorem 14. *Let G be a graph and $F : V(G) \to 2^{\mathbb{N}}$ be a function. If D is an orientation of G such that $|F(v)| \leq d^+_D(v)$ for every vertex $v \in V(G)$ and $EE(D) \neq EO(D)$, then G is F-avoiding.*

Proof. It suffices to prove the theorem when $|F(v)| = d^+_D(v)$ for every $v \in V(G)$. First, we define a function $k_D : E(G) \times V(G) \longrightarrow \{-1, 0, 1\}$ as follows:

$$k_D(e, v) = \begin{cases} -1 & \text{if } e \in E^-_D(v), \\ 0 & \text{if } e \notin E_G(v), \\ 1 & \text{if } e \in E^+_D(v), \end{cases}$$

where $E_G(v)$, $E^+_D(v)$ and $E^-_D(v)$ denote the set of all edges, the set of all outgoing edges and the set of all incoming edges incident with v with respect to D, respectively. Define $L : V(G) \to 2^{\mathbb{N}}$ as follows: For each $v \in V(G)$,

$$L(v) = \Big\{\ell \in \mathbb{Z} : \frac{\ell + d_G(v)}{2} \in F(v)\Big\}.$$

Note that $|L(v)| = |F(v)|$ for each $v \in V(G)$. Now, let $E(G) = \{e_1, \dots, e_m\}$. Assign a variable x_i for each edge e_i and define $P \in \mathbb{Z}[x_1, \dots, x_m]$ as follows:

$$P(x_1, \dots, x_m) = \prod_{v \in V(G)} \prod_{\ell \in L(v)} \Big(\sum_{i=1}^{m} k_D(e_i, v) \cdot x_i - \ell\Big).$$

Note that $\deg(P) = \sum_{v \in V(G)} |L(v)| = \sum_{v \in V(G)} |F(v)| = \sum_{v \in V(G)} d^+_D(v) = |E(G)| = m$. Now, we show that if $P(t_1, \dots, t_m) \neq 0$ for some $t_1, \dots, t_m \in \{-1, 1\}$, then there is an orientation of G which avoids F. To see this, we construct this orientation by changing D. More precisely, for each i with $i = 1, \dots, m$, if $t_i = -1$, then reverse the direction of e_i in D, otherwise keep its direction. Denote this new orientation by O. It is not hard to see that if $k_D(e_i, v) \cdot t_i = 1$, then $e_i \in E^+_O(v)$ and if $k_D(e_i, v) \cdot t_i = -1$, then $e_i \in E^-_O(v)$. So for all $v \in V(G)$,

$$\sum_{i=1}^{m} k_D(e_i, v) \cdot t_i = d^+_O(v) - d^-_O(v).$$

Since $P(t_1, \dots, t_m) \neq 0$,

$$d^+_O(v) - d^-_O(v) = \sum_{i=1}^{m} k_D(e_i, v) \cdot t_i \neq \ell,$$

for every $\ell \in L(v)$. Since $d^+_O(v) + d^-_O(v) = d_G(v)$, we obtain $d^+_O(v) \neq \frac{\ell + d_G(v)}{2} \in F(v)$, i.e. O is F-avoiding.

To complete the proof, we show that $x_1 \dots x_m$ has non-zero coefficient in P. To do that, we consider orientations D' of G with $d^+_{D'}(v) = |F(v)|$ for all $v \in V(G)$. Call such an orientation *suitable*.

First, we consider the relation between the coefficient of $x_1 \dots x_m$ in P and suitable orientations. Every term $x_1 \dots x_m$ produced in the product of parenthesis corresponds to exactly one orientation which is defined as follows: for all $i = 1, \dots, m$, if x_i is selected from a parenthesis where its coefficient is $k_D(e_i, v) \neq 0$, orient e_i outward from v. Denote this orientation D'. Since each vertex v handles exactly $|L(v)| = |F(v)|$ incident edges, we see that D' is suitable. Furthermore, if we set $E^+_{D'}(v) = \{e_{i_1}, \dots, e_{i_{|F(v)|}}\}$, then the coefficient of $x_{i_1} \dots x_{i_{|F(v)|}}$ in

$$\prod_{\ell \in L(v)} \Big(\sum_{i=1}^{m} k_D(e_i, v) \cdot x_i - \ell \Big)$$

is $|F(v)|! \prod_{e \in E^+_{D'}(v)} k_D(e, v)$. Therefore, the orientation D' contributes $|F(v)|! \prod_{e \in E^+_{D'}(v)} k_D(e, v)$ to the coefficient at v. So in total, D' contributes to the term $x_1 \dots x_m$ by $r(D')$, where

$$r(D') = \prod_{v \in V(G)} \Big(|F(v)|! \prod_{e \in E^+_{D'}(v)} k_D(e, v) \Big).$$

This implies that the coefficient of $x_1 \dots x_m$ in P is $\sum r(D')$ where $\sum$ is taken over all suitable orientations of G.

Now, for every suitable orientation D' of G, let $H(D')$ be the spanning subgraph of G consisting of all edges $e \in E(G)$ such that e has different directions in D' and in D. For every $e \in E(H(D'))$, there exist two distinct vertices u and v where $e \in E^+_{D'}(u)$ and $e \in E^+_D(v)$. So $k_D(e, v) = -k_D(e, u)$. Hence the following equality holds:

$$\prod_{v \in V(G)} \prod_{e \in E(H(D')) \cap E^+_{D'}(v)} k_D(e, v) = (-1)^{|E(H(D'))|} \prod_{u \in V(G)} \prod_{e \in E(H(D')) \cap E^+_{D}(u)} k_D(e, u).$$

Hence we obtain

$$\prod_{v \in V(G)} \prod_{e \in E^+_{D'}(v)} k_D(e, v) = (-1)^{|E(H(D'))|} \prod_{u \in V(G)} \prod_{e \in E^+_{D}(u)} k_D(e, u).$$

Thus $r(D') = (-1)^{|E(H(D'))|} r(D)$.

Since $d^+_{D'}(v) = |F(v)| = d^+_D(v)$, we have $d^+_{H(D')}(v) = d^-_{H(D')}(v)$ for all v, where we consider the graph $H(D')$ with the orientation D. So $H(D')$ is a spanning Eulerian subgraph of G with respect to D. On the other hand, for any spanning Eulerian subgraph H of G with respect to D, by revising the direction of all edges in H, we obtain a suitable orientation of G. These imply that we have a one-to-one correspondence between suitable orientations D' of G and spanning Eulerian subgraphs H of G with respect to D. Therefore, the coefficient of $x_1 \dots x_m$ is

$$\sum r(D') \; = \; \sum (-1)^{|E(H(D'))|} r(D) \; = \; \Big(EE(D) - EO(D) \Big) \cdot r(D),$$

where the sum is take over all suitable orientations of G. Since $EE(D) \neq EO(D)$, $x_1 \dots x_m$ has non-zero coefficient in P, as desired. So by Combinatorial Nullstellensatz Theorem, there are $t_1, \dots, t_m \in \{-1, 1\}$ such that $P(t_1, \dots, t_m) = 0$. The proof is complete. $\square$

For a bipartite graph G, since any spanning Eulerian subgraph is even, we have $EE(D) \neq 0$ and $EO(D) = 0$. Therefore, Theorem 14 immediately implies Lemma 11.

5 A conjecture for F-avoiding non-bipartite graphs

As we have seen in the previous two sections, we have a best possible result on F-avoiding orientations in bipartite graphs. In this section, we first pose the following conjecture for the general graphs, and then give several results related to it.

Conjecture 1. Let G be a graph and let $F : V(G) \to 2^{\mathbb{N}}$. If

$$|F(v)| \ \le \ \frac{1}{2}\big(d_G(v) - 1\big)$$

for every $v \in V(G)$, then G is F-avoiding.

5.1 Sharpness of the conditions

Now, we show that the upper bound $\frac{1}{2}\big(d_G(v) - 1\big)$ in Conjecture 1 is best possible, if it is true. This is a significant difference from the bipartite case, since as in Theorem 9, the condition $|F(v)| \le \frac{d_G(v)}{2}$ guarantees the existence of an F-avoiding orientation, if G is bipartite. We first prove the following lemma.

Lemma 15. *Let G be a $2k$-regular graph of order n and $F : V(G) \to 2^{\mathbb{N}}$ be a function with $F(v) = \{k, \ldots, 2k-1\}$ for any $v \in V(G)$. If*

$$\alpha(G) \ < \ \frac{n}{k+1}, \tag{2}$$

then G is not F-avoiding.

Proof. By contradiction, assume that there exists an F-avoiding orientation O of G. So $d^+_O(v) \in \{0, 1, \ldots, k-1, 2k\}$ for every $v \in V(G)$. Let

$$U_{2k} \ = \ \{v \in V(G) : d^+_O(v) = 2k\}.$$

Since G is $2k$-regular, it is easy to see that $uv \notin E(G)$ for any $u, v \in U_{2k}$. This implies that U_{2k} is an independent set in G, and hence $|U_{2k}| \le \alpha(G)$. Since $d^+_O(v) \le k-1$ for any $v \in V(G) \setminus U_{2k}$, it follows from the assumption (2) that

$$\begin{aligned}
\sum_{v \in V(G)} d^+_O(v) \ &\le \ 2k|U_{2k}| + (k-1)\big(n - |U_{2k}|\big) \\
&\le \ kn - n + (k+1)\alpha(G) \\
&< \ kn.
\end{aligned}$$

However, we have $\sum_{v \in V(G)} d^+_O(v) = |E(G)| = kn$, a contradiction. □

Note that $\big|\{k, \ldots, 2k-1\}\big| = k = \frac{d_G(v)}{2}$ and several $2k$-regular graphs satisfy the assumption (2) in Lemma 15. One example is an odd cycle for $k = 1$, which corresponds to the only if part of Lemma 6 for an odd integer n. Another example is the complete graph K_{2k+1}; we see that $\alpha(K_{2k+1}) = 1 < \frac{2k+1}{2k}$. More example can be constructed, such as the square of a cycle of certain length, and the line graph of a $(k+1)$-regular graph H such that H has no perfect matching, and so on. Like them, there are several graphs that show the best possibility of the bound $\frac{1}{2}\big(d_G(v) - 1\big)$ in Conjecture 1. This may be a reason why Conjecture 1 is difficult.

Recall that $\alpha(G) \ge \frac{n}{2}$ for any bipartite graph G, and hence Lemma 15 does not conflict with Theorem 9.

5.2 A partial solution to Conjecture 1

Using Theorem 9, we prove the following result, which gives a partial solution to Conjecture 1.

Theorem 16. *Let G be a graph and let $F : V(G) \to 2^{\mathbb{N}}$. If*

$$|F(v)| \leq \frac{d_G(v)}{4}$$

for every vertex v, then G is F-avoiding.

Proof. It is well known that G contains a spanning bipartite subgraph H such that $d_H(v) \geq d_G(v)/2$ for every $v \in V(G)$ (see [5, Theorem 2.4 on Page 49]). Orient all edges of $G \setminus E(H)$ arbitrarily, and let O' be this orientation. Then define $F'(v) = \{i - d^+_{O'}(v) : i \in F(v) \text{ and } i \geq d^+_{O'}(v)\}$. Since for every $v \in V(H)$,

$$|F'(v)| \ \leq \ |F(v)| \ \leq \ \frac{d_G(v)}{4} \ \leq \ \frac{d_H(v)}{2},$$

it follows from Theorem 9 that H is F'-avoiding. So it is easy to see that G is F-avoiding and the proof is complete. □

Then consider the following problem;

> Find the infimum real number c such that for every graph G, if $|F(v)| \leq c \cdot d_G(v)$ for each $v \in V(G)$, then G is F-avoiding.

By Theorem 16, we have $c \geq \frac{1}{4}$, and sharpness of Conjecture 1 says that $c \leq \frac{1}{2}$. Since there still exists a huge gap between them, we think that this is a challenging problem.

5.3 A possible idea similar to the bipartite case

With Lemma 11 in mind, the next conjecture might be a valuable step to attach Conjecture 1.

Conjecture 2. Let G be a graph and let $F : V(G) \to 2^{\mathbb{N}}$. If G has an orientation D such that

$$d^+_D(v) \geq |F(v)| + 1$$

for every vertex v, then G is F-avoiding.

We should notice that Conjecture 2 unfortunately cannot imply Conjecture 1. If we can change the assumption of Conjecture 2 with

$$d^+_D(v) \geq \begin{cases} |F(v)| + 1 & \text{if } d_G(v) \text{ is even,} \\ |F(v)| & \text{if } d_G(v) \text{ is odd,} \end{cases} \tag{3}$$

then a similar method used in the proof of Theorem 9 in Section 3.1 proves Conjecture 1. However, the assumption (3) is too weak. Consider K_4 with four vertices v_1, v_2, v_3 and v_4. Note that $d(v_i)$ is odd for any vertex v_i. Then let $F(v_1) = \{0, 1, 2\}$ and $F(v_i) = \{1\}$ for $i = 2, 3, 4$. We can easily find an orientation D satisfying the assumption (3), but the graph is not F-avoiding.

Though Conjecture 2 does not imply Conjecture 1, it can imply the following statement, which is slightly weaker than Conjecture 1: If $|F(v)| \leq \frac{1}{2}\big(d_G(v) - 2\big)$ for each $v \in V(G)$, then G is F-avoiding. Furthermore, this method might be useful to attack the problem mentioned in the previous subsection.

5.4 F-avoiding complete graphs

We prove Conjecture 1 for complete graphs with the same lists assigned to the vertices. Before we prove the main theorem of this section, we need some preparation including two new lemmas.

A *tournament* is a complete graph with an orientation. There is a classic theorem for the score sequence of tournaments due to Landau as follows.

Theorem 17. *(**Landau** [12]) A sequence of non-negative integers $s_1, s_2, \ldots, s_n$ with $s_1 \leq s_2 \leq \ldots \leq s_n$ is an out-degree sequence of a tournament if and only if for any i with $1 \leq i \leq n$, we have*

$$\sum_{j=1}^{i} s_j \geq \binom{i}{2}, \tag{4}$$

and the equality holds when $i = n$.

Lemma 18. *Let $0 \leq a \leq b \leq n-1$ be two non-negative integers. Then the complete graph K_n has an $\{a,b\}$-orientation if and only if the following condition holds: There exist two non-negative integers x and y with $x + y = n$ such that*

(i) $ax + by = \binom{n}{2}$, and

(ii) $a \geq \frac{x-1}{2}$.

Furthermore, x and y correspond to the number of vertices of out-degrees a and b, respectively.

Proof. Suppose first that K_n has an $\{a,b\}$-orientation. Then letting x and y be the number of vertices of out-degrees a and b, respectively, obviously, the equality (i) holds. Furthermore, by Theorem 17 with $i = x$, we have $xa \geq \binom{x}{2}$. This implies the inequality (ii) $a \geq \frac{x-1}{2}$.

For the other direction suppose that a and b satisfy the (in)equalities (i) and (ii) for some x and y with $x + y = n$. To see that K_n has an $\{a,b\}$-orientation, we will show that the sequence $a, \ldots, a, b, \ldots, b$, where the numbers of a's and b's are x and y, respectively, satisfies the inequality (4) for any i with $1 \leq i \leq n$. For every $i = 1, \ldots, x$, it follows from the inequality (ii) that

$$ia \geq \frac{i(x-1)}{2} \geq \frac{i(i-1)}{2} = \binom{i}{2}.$$

Therefore, the inequality (4) holds for $i = 1, \ldots, x$.

Now, we show the inequality (4) for $x + 1 \leq i \leq n$. Letting $t = i - x$, it suffices to prove $ax + bt \geq \binom{x+t}{2}$. By equality (i), we have $ax + b(n-x) = \binom{n}{2}$, and hence we have to show that $(n - x - t)b \leq \binom{n}{2} - \binom{x+t}{2} = (n - (x+t))\frac{n+x+t-1}{2}$, which is equivalent to show that $b \leq \frac{n+x+t-1}{2}$. Again by equality (i), $b = \frac{\binom{n}{2} - ax}{n-x}$. So, we should show that $\frac{\binom{n}{2} - ax}{n-x} \leq \frac{n+x+t-1}{2}$, which is equivalent to show that $-2ax - x \leq -x^2 + tn - tx$. By inequality (ii), $a \geq \frac{x-1}{2}$. So, $-2ax - x \leq -(x-1)x - x = -x^2$. Since $0 \leq tn - tx$, our inequality is proved and hence $ax + bt \geq \binom{x+t}{2}$.
Therefore, by (i) and (ii) the conditions in Theorem 17 are hence satisfied and the complete graph K_n has an $\{a,b\}$-orientation. Thus the proof is completed. □

Lemma 19. *For $i = 0, \ldots, k-1$, the complete graph K_{2k} has an $\{i, k+i\}$-orientation.*

Proof. By Lemma 18, the complete graph K_{2k} has an $\{i, k+i\}$-orientation if and only if the following condition holds: There exist two non-negative integers x and y with $x + y = 2k$ such that

(i) $ix + (2k - x)(k + i) = \binom{2k}{2}$, and

(ii) $i \geq \frac{x-1}{2}$.

It is easy to see that $x = 2i + 1$ and $y = 2k - 2i - 1$ satisfy the (in)equalities (i) and (ii), as desired. □

Now, we are ready to prove Conjecture 1 for complete graphs with the same lists assigned to the vertices. In fact, we show the following.

Theorem 20. *Let S be a set of non-negative integers of size $\lfloor \frac{1}{2}(n-2) \rfloor$. Then there exists an orientation for the complete graph K_n such that $d^+(v) \notin S$ for every $v \in V(K_n)$.*

Proof. First, suppose that $n = 2k$. Note that $|S| = k - 1$. Since the number of pairs $(i, i + k)$, where $0 \leq i \leq k-1$, is k, there exists an integer $i \in \{0, \ldots, k-1\}$ such that $i, k+i \notin S$. By Lemma 19, the desired orientation exists and the proof is completed.

Now, assume that $n = 2k+1$ and G is the complete graph of order n. Note that $|S| = k-1$. Let i be the smallest integer such that $i \notin S$ and let U be a subset of $V(G)$ such that $|U| = 2i+1$. By Lemma 10, $G[U]$ has an orientation such that both the out-degree and in-degree are equal to i. Now, for each $e = wz \in E(G)$ such that $w \in V(U)$ and $z \in V(G) \setminus V(U)$, orient e from z to w. Let

$$S' = \{\ell \in \mathbb{N} : \ell + 2i + 1 \in S\}.$$

Since $\{0, 1, \ldots, i-1\} \subseteq S$, it is clear that $|S'| \leq k - 1 - i$. Since $|V(G \setminus U)|$ is even, by the previous case there exists an orientation for $G \setminus U$ such that for every $v \in V(G \setminus U)$, $d^+(v) \notin S'$. Together with previously mentioned ones, we obtain an orientation with desired condition, and the proof is completed. □

6 Conjecture 1 and strongly $\mathbb{Z}_k$-connectivity

In this section, we give some results that are partial solutions to Conjecture 1, assuming Lai's conjecture on a β-orientation.

Let G be a graph and let k be an odd integer with $k \geq 3$. A mapping $\beta : V(G) \to \mathbb{Z}_k$ is called a $\mathbb{Z}_k$*-boundary* of G if $\sum_{v \in V(G)} \beta(v) \equiv 0 \pmod{k}$. Let β be a $\mathbb{Z}_k$-boundary of G. An orientation D of G is called a β*-orientation* if, for every vertex $v \in V(G)$, $d_D^+(v) - d_D^-(v) \equiv \beta(v) \pmod{k}$. When $\beta(v) = 0$ for any vertex v, a β-orientation is called a *mod k-orientation.* A graph G is said to be *strongly $\mathbb{Z}_k$-connected* if G has a β-orientation for every $\mathbb{Z}_k$-boundary β.

The following interesting conjecture was proposed in 2007 by Lai, modifying a famous Jaeger's conjecture [9] on a mod k-orientation.

Conjecture 3.[Lai [10]] Let G be a graph, and let k be an odd integer with $k \geq 3$. If G is $(2k-2)$-edge-connected, then G is strongly $\mathbb{Z}_k$-connected.

6.1 Partial solutions to Conjecture 1 assuming Lai's conjecture

We give two partial solutions to Conjecture 1. To do that, we first prepare the following lemma.

Lemma 21. *Let $p, n, s \geq 2$ be integers such that $p \leq n$, p is a prime and $s \in \mathbb{Z}_p$. For $i = 1, \ldots, n$, let a_i and b_i be two distinct elements in $\mathbb{Z}_p$. Then there exists $I \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in I} a_i + \sum_{i \in \{1,\ldots,n\} \setminus I} b_i \equiv s$* (mod p).

Proof. Define $Q \in \mathbb{Z}_p[x_1, \ldots, x_n]$ as follows:

$$Q(x_1, \ldots, x_n) = \sum_{i=1}^{n} (a_i - b_i) x_i.$$

Now, let $s' \equiv s - \sum_{i=1}^{n} b_i \pmod p$ with $s' \in \mathbb{Z}_p$ and let $P \in \mathbb{Z}_p[x_1, \ldots, x_n]$ be the polynomial defined as follows:

$$P(x_1, \ldots, x_n) = \prod_{j \in \mathbb{Z}_p \setminus \{s'\}} \Big(Q(x_1, \ldots, x_n) - j \Big).$$

Note that $\deg(P) = p-1$. Since $a_i - b_i \not\equiv 0 \pmod p$ and p is a prime number, the coefficient of $\prod_{i \in \mathbb{Z}_p \setminus \{s'\}} x_i$ in P is $(p-1)! \prod_{i \in \mathbb{Z}_p \setminus \{s'\}} (a_i - b_i)$, which is non-zero in $\mathbb{Z}_p$. So by the Combinatorial Nullstellensatz Theorem, there exists an integer $t_i \in \{0, 1\}$ for any i such that $P(t_1, \ldots, t_n) \not\equiv 0 \pmod p$. This directly implies $Q(t_1, \ldots, t_n) \equiv s' \pmod p$. Let $I = \{i : t_i = 1\}$. Then

$$\begin{aligned} s &\equiv s' + \sum_{i=1}^{n} b_i &&\equiv Q(t_1, \ldots, t_n) + \sum_{i=1}^{n} b_i \\ &= \sum_{i \in I} (a_i - b_i) + \sum_{i=1}^{n} b_i &&\equiv \sum_{i \in I} a_i + \sum_{i \in \{1, \ldots, n\} \setminus I} b_i. \end{aligned}$$

Then the proof is complete. □

Remark. The assumption that p is a prime number in Lemma 21 is necessary. To see that, for example, consider the case where $p = 4$, n is odd, $s = 0$ and $a_i = 1$ and $b_i = 3$ for any i.

Then we give two results assuming Conjecture 3. In these results, we consider d-regular graphs and assume the condition $|F(v)| \leq \frac{d}{2} - 1$ for any vertex v. If d is even, then it is equivalent with the one $|F(v)| \leq \frac{1}{2}\big(d - 1\big)$ in Conjecture 1, while it is stronger by one if d is odd.

Theorem 22. *Assume that Conjecture 3 is true. Let G be a d-regular d-edge-connected graph, and let $F : V(G) \to 2^{\mathbb{N}}$ be a function where $|F(v)| \leq \frac{d}{2} - 1$. If $\lfloor \frac{d}{2} \rfloor + 1$ is an odd prime number, then G is F-avoiding.*

Proof. Let $p = \lfloor \frac{d}{2} \rfloor + 1$. It is clear that for every $v \in V(G)$, there are two distinct elements $a_v, b_v \in \mathbb{Z}_p$ such that for any $i \in F(v)$ we have $i \not\equiv a_v, b_v \pmod p$. Since p is an odd prime number, there exists an element $s \in \mathbb{Z}_p$ with $2s \equiv nd \pmod p$, where n is the order of G. Then by Lemma 21, there exists $I \subseteq V(G)$ such that $\sum_{v \in I} a_v + \sum_{v \in V(G) \setminus I} b_v \equiv s \pmod p$. Now, define $\beta(v) = 2a_v - d$ if $v \in I$, and $\beta(v) = 2b_v - d$ otherwise. Then

$$\begin{aligned} \sum_{v \in V(G)} \beta(v) &= \sum_{v \in I} \big(2a_v - d\big) + \sum_{v \in V(G) \setminus I} \big(2b_v - d\big) \\ &\equiv 2s - nd \equiv 0 \pmod p. \end{aligned}$$

Now, since G is d-edge-connected, it is $(2p-2)$-edge-connected and so by Conjecture 3, G has a β-orientation O. Thus for a vertex v, $d_O^+(v) - d_O^-(v) \equiv \beta(v) \pmod p$. Since $d_O^+(v) + d_O^-(v) = d$, we obtain that $d_O^+(v) \equiv a_v$ or $b_v \pmod p$. In either case, $d_O^+(v) \notin F(v)$, and hence the orientation O is F-avoiding. □

Theorem 23. *Assume that Conjecture 3 is true. Let G be a d-regular d-edge-connected graph of order n, and let $F : V(G) \to 2^{\mathbb{N}}$ be a function, where $|F(v)| \leq \frac{d}{2} - 1$. Then for a given vertex u, there exists an orientation O such that $d_O^+(v) \notin F(v)$ for each $v \in V(G) \setminus \{u\}$.*

Proof. Let $k \in \{\lfloor \frac{d}{2} \rfloor, \lfloor \frac{d}{2} \rfloor + 1\}$ be an odd number. It is clear that for every $v \in V(G)$, there is an element $a_v \in \mathbb{Z}_p$ such that for any $i \in F(v)$ we have $i \not\equiv a_v \pmod{p}$. Define $\beta(v) = 2a_v - d$ for every $v \in V(G) \setminus \{u\}$ and $\beta(u) = -\sum_{v \in V(G) \setminus \{u\}} \beta(v)$. Since G is d-edge-connected, it is $(2k-2)$-edge connected and so by Conjecture 3, G has a β-orientation O. Then the same argument as in the proof of Theorem 22 completes the proof. □

By [14, Exercise 14 on Page 75], any Cayley graph is regular and the edge connectivity is equal to the degree. Therefore, we have the following corollary.

Corollary 24. *Assume that Conjecture 3 is true. Then the conclusion of both Theorems 22 and 23 holds for Cayley graphs, in particular, complete graphs.*

6.2 Further partial solutions using strongly $\mathbb{Z}_k$-connectivity

Note that some special cases of Conjecture 3 have been already proved. Using them, we show some relaxations of the theorems in the previous section.

6.2.1 K_4-minor-free graphs

The first special case is K_4-minor-free graphs, where we have the following.

Theorem 25.*[Lai, Liang, Liu, Meng, Miao, Shao and Zhang [11]] Every $(4s-1)$-edge-connected K_4-minor-free graph is strongly $\mathbb{Z}_{2s+1}$-connected.*

Then for K_4-minor-free graphs, using Theorem 25 in the proof of Theorem 23, instead of assuming Conjecture 3, we obtain the following corollary.

Corollary 26. *Let G be a d-regular d-edge-connected K_4-minor-free graph of order n, and let $F : V(G) \to 2^{\mathbb{N}}$ be a function, where $|F(v)| \leq \frac{d}{2} - 1$. Then for a given vertex u, there exists an orientation O such that $d^+_O(v) \notin F(v)$ for each $v \in V(G) \setminus \{u\}$.*

6.2.2 Complete graphs

Using the next theorem, we can give some partial solution to the case of complete graphs.

Theorem 27.*[[11, Theorem 1.6]] Let n be a positive integer and $k = 2\lfloor \frac{n-1}{4} \rfloor + 1$. Then K_n is strongly $\mathbb{Z}_k$-connected.*

Theorem 28. *Let n be a positive integer and $F : V(K_n) \to 2^{\mathbb{N}}$ be a function, where $|F(v)| \leq \frac{1}{2}(n-3)$ for all $v \in V(K_n)$. Then for a given vertex u, there exists an orientation O such that $d^+_O(v) \notin F(v)$ for each $v \in V(K_n) \setminus \{u\}$.*

Proof. Define $k = 2\lfloor \frac{n-1}{4} \rfloor + 1$. Since $k - 1 \geq \lfloor \frac{n-3}{2} \rfloor$, for every $v \in V(G)$, there is an element $a_v \in \mathbb{Z}_p$ such that for any $i \in F(v)$ we have $i \not\equiv a_v \pmod{p}$. Now, by similar arguments as in previous theorems and by Theorem 27, K_n has the desired orientation. □

6.2.3 Graphs with higher edge-connectivity

The next special case is the one for regular graphs with high edge-connectivity. Note that we allow multiple edges for graphs in this subsection. The following is a recent result on β-orientations. Originally the theorem is more technical (see [13, Theorem 3.1]), but we can easily modify the statement. (To be more precise, apply the statement with $V_0 = \emptyset$ and substitute $\tau(v)$ with $0 \leq |\tau(v)| \leq k$ and consider the parity of it.)

Theorem 29._[Lovász, Thomassen, Wu, and Zhang [13]] Let k be a positive odd integer, let G be a graph with $\mathbb{Z}_k$-boundary β, and let $z_0 \in V(G)$. Suppose that the following hold;_

- _$d_G(z_0) \leq 2k - 1$, and_
- _for any vertex set A with $z_0 \notin A$ and $|V(G) \setminus A| \geq 2$, we have $d(A) \geq 3k - 3$, where $d(A)$ denotes the number of edges between A and $V(G) \setminus A$._

Then for any orientation on the edges incident with z_0, if it satisfies $d^+(z_0) - d^-(z_0) \equiv \beta(z_0) \pmod{k}$, then G admits a β-orientation that is an extension of the prescribed orientation.

For a graph G and two vertices u and v, let $\lambda_G(u, v)$ be the maximum number of edge-disjoint paths between u and v. By Menger's theorem, we see that $\lambda_G(u, v) \geq k$ if and only if $d(A) \geq k$ for any vertex set A with $u \in A$ and $v \notin A$.

We also use the operation so-called _lifting_. Let z be a vertex of a graph G, and let xz and yz be two edges incident with z such that x and y are distinct vertices. Then the deletion of xz and yz and the addition of the new edge xy is called _lifting_ of xz and yz. The following is a well-known theorem by Mader [15], see also a textbook [3, Theorem 4.10];

Theorem 30. _Let G be a graph and let z be a vertex of G. Suppose that $d_G(z) \geq 4$. Then there exists a pair $\{e_1, e_2\}$ of edges incident with z such that the lifting of e_1 and e_2 obtains the graph G' satisfying that $\lambda_{G'}(u, v) = \lambda_G(u, v)$ for any $u, v \in V(G) \setminus \{z\}$._

Now we are ready to prove our next theorem.

Theorem 31. _Let G be a d-regular d-edge-connected graph, and let $F : V(G) \to 2^{\mathbb{N}}$. If $|F(v)| \leq \frac{d-5}{3}$ for all $v \in V(G)$, then G is F-avoiding._

Proof. Let $k \in \left\{\lfloor \frac{d}{3} \rfloor, \lfloor \frac{d}{3} \rfloor + 1\right\}$ be an odd number. Note that $3k - 3 \leq d \leq 3k + 2$. Let

$$t = \begin{cases} \frac{k-1}{2} & \text{if } d = 3k-3 \text{ or } 3k-2, \\ \frac{k+1}{2} & \text{if } d = 3k-1 \text{ or } 3k, \\ \frac{k+3}{2} & \text{if } d = 3k+1 \text{ or } 3k+2. \end{cases}$$

Then t is an integer in either case. For each vertex v, we let

$$L(v) = \{\ell \in \mathbb{Z}_k : \ell \not\equiv i \pmod{k} \text{ for any } i \in F(v)\}.$$

Since $|F(v)| \leq \frac{d-5}{3} \leq k - 1$, we have $L(v) \neq \emptyset$. We take a vertex z_0 such that $|L(z_0)|$ is as small as possible. We first claim the following;

(∗) There exists an element $\ell_v \in L(v)$ for each vertex v with $v \neq z_0$ and $i_0 \in \{t, t+1, \dots, d-t\} \setminus F(z_0)$ such that

$$\sum_{v \in V(G)\setminus\{z_0\}} \Big(2\ell_v - d\Big) + \big(2i_0 - d\big) \equiv 0 \pmod{k}.$$

To see this, suppose that it does not hold. Let u be a vertex of G with $u \neq z_0$. By the choice of z_0, we have $|L(z_0)| \leq |L(u)|$. Fix $\ell_v \in L(v)$ for each vertex v with $v \neq z_0, u$. Since we assume the claim does not hold, for any $\ell_u \in L(u)$, if we let $i(\ell_u)$ be an integer such that $t \leq i(\ell_u) \leq d - t$ and

$$\sum_{v \in V(G)\setminus\{z_0\}} \big(2\ell_v - d\big) + \big(2i(\ell_u) - d\big) \equiv 0 \pmod{k},$$

then $i(\ell_u) \in F(z_0)$. By the choice of t, we have

$$\Big|\{t, t+1, \dots, d-t\}\Big| = \big(d - t\big) - t + 1 = d - 2t + 1 = 2k - 1 \text{ or } 2k,$$

and the former occurs if and only if $d = 3k - 3, 3k - 1$ or $3k + 1$. This implies that for all $\ell_u \in L(u)$, except for at most one when $d = 3k - 3, 3k - 1$ or $3k + 1$, we have at least two choices for $i(\ell_u)$. Since those two choices have the same value on $\mathbb{Z}_k$, if $d = 3k - 2, 3k$ or $3k + 2$, then

$$|L(z_0)| \geq |\mathbb{Z}_k| - \Big(|F(z_0)| - |L(u)|\Big) \geq k - (k - 1) + |L(u)| > |L(u)|,$$

contradicting the choice of z_0. On the other hand, if $d = 3k - 3, 3k - 1$ or $3k + 1$, then $|F(z_0)| \leq \frac{d-5}{3} \leq \frac{3k-4}{3}$, and the integrality implies $|F(z_0)| \leq k - 2$. Therefore,

$$|L(z_0)| \geq |\mathbb{Z}_k| - \Big(|F(z_0)| - \big(|L(u)| - 1\big)\Big) \geq k - (k - 2) + |L(u)| - 1 > |L(u)|,$$

a contradiction again. This shows the claim (∗).

We take ℓ_u and i_0 as in the claim (∗). By applying Theorem 30 t times with z_0 playing the role of z, we obtain the graph G' satisfying that $d_{G'}(z_0) = d - 2t = 2k - 2$ or $2k - 1$, and $\lambda_{G'}(u, v) = \lambda_G(u, v)$ for any $u, v \in V(G) \setminus \{z_0\}$. For each vertex v, let $\beta(v) = 2\ell_v - d$ if $v \neq z_0$; Otherwise let $\beta(z_0) = 2i_0 - d$. By the claim (∗), β is a $\mathbb{Z}_k$-boundary of G'. We can check that G' satisfies the assumptions of Theorem 29. In fact, if we let A be a vertex set with $z_0 \notin A$ and $|V(G') \setminus A| \geq 2$, then there exist two vertices $u \in A$ and $v \in V(G') \setminus \big(A \cup \{z_0\}\big)$, and we have $d(A) \geq \lambda_{G'}(u, v) = \lambda_G(u, v) \geq d \geq 3k - 3$.

Then first we orient all the edges incident with z_0 so that exactly $i_0 - t$ edges are out-going from z_0 and other $d_{G'}(z_0) - (i_0 - t)$ edges are in-coming to z_0. Since $t \leq i_0 \leq d - t$ and $d_{G'}(z_0) = d - 2t$, such an orientation indeed exists. By Theorem 29, G' admits a β-orientation O' that is an extension of the prescribed orientation. We take O' to the orientation O of G by the natural way; If we obtain the edge xy by the lifting of xz_0 and yz_0 and xy is oriented from x to y, then we give a direction to each of xz_0 and yz_0 so that they are oriented from x to z_0, and from z_0 to y, respectively. Since this does not change the out-degree of each vertex v with $v \neq z_0$, we have $d^+_O(v) = d^+_{O'}(v) \equiv \ell_v \pmod{k}$, and hence $d^+_O(v) \notin F(v)$. On the other hand, $d^+_O(z_0) = d^+_{O'}(z_0) + t = i_0 \notin F(z_0)$. Therefore, the orientation O avoids F, and the proof is completed. □

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