Every 5-connected planar triangulation is 4-ordered Hamiltonian

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Abstract

A graph $G$ is said to be $4$-ordered if for any ordered set of four distinct vertices of $G$, there exists a cycle in $G$ that contains all of the four vertices in the designated order. Furthermore, if we can find such a cycle as a Hamiltonian cycle, $G$ is said to be $4$-ordered Hamiltonian. It was shown that every 4-connected planar triangulation is (i) Hamiltonian (by Whitney) and (ii) 4-ordered (by Goddard). Therefore, it is natural to ask whether every 4-connected planar triangulation is 4-ordered Hamiltonian. In this paper, we give a partial solution to the problem, by showing that every 5-connected planar triangulation is 4-ordered Hamiltonian.

Keywords: 4-ordered, 4-ordered Hamiltonian, triangulations, plane graphs

1 Introduction

A graph $G$ is said to be $k$-ordered for an integer $3 \leq k \leq |V(G)|$, if for any ordered set of $k$ distinct vertices of $G$, there exists a cycle in $G$ that contains all the $k$ vertices in the designated order. Furthermore, if we can find such a cycle as a Hamiltonian cycle, $G$ is said to be $k$-ordered Hamiltonian. These topics have been extensively studied; see the survey [2].

In this paper, we focus on 4-connected planar triangulations. In fact, it is known that such graphs have good properties;

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Theorem 1 Let $G$ be a 4-connected planar triangulation. Then

(i) $G$ is Hamiltonian. (Whitney [12])

(ii) $G$ is 4-ordered. (Goddard [3])

Note that Theorem 1 (i) was improved to 4-connected planar graphs (by Tutte [11]) and 4-connected projective planar graphs (by Thomas and Yu [8]). However, we cannot lower the assumption on 4-connectedness to 3-connectedness, since there exist infinitely many 3-connected planar triangulations that are not Hamiltonian (see [4]). On the other hand, by using ideas of Goddard [3], we can construct infinitely many 3-connected planar triangulations that are not 4-ordered, and infinitely many 5-connected planar graphs that are not 4-ordered. Therefore, both of the assumptions “4-connected” and “triangulation” are needed for the property of being “4-ordered”.

Recall that 4-ordered Hamiltonian graphs are definitely Hamiltonian and 4-ordered. It follows from Theorems 1 (i) and (ii) that every 4-connected planar triangulation satisfies both properties, and hence it is natural to pose the following conjecture (Conjecture 2).

Conjecture 2 Every 4-connected planar triangulation is 4-ordered Hamiltonian.

Theorem 3 Every 5-connected planar triangulation is 4-ordered Hamiltonian.

This paper is organized as follows; in the next section, we will give terminologies and a known result, used in the proof of Theorem 3 in Section 3. In the last section, we will give a conclusion of this paper, together with some open problems.

2 Preliminaries

For a graph $G$, the order of $G$ is denoted by $|G|$. Let $H_1$ and $H_2$ be two subgraphs of a graph $G$. Then $H_1 \cup H_2$ denotes the subgraph of $G$ with $V(H_1 \cup H_2) = V(H_1) \cup V(H_2)$ and $E(H_1 \cup H_2) = E(H_1) \cup E(H_2)$, and $H_1 \cap H_2$ denotes the subgraph of $G$ with $V(H_1 \cap H_2) = V(H_1) \cap V(H_2)$ and $E(H_1 \cap H_2) = E(H_1) \cap E(H_2)$. We use a similar notation also for a vertex subset $U$ or an edge subset $P$ of $G$; so, $H_1 \cup U$ is the subgraph of $G$ with $V(H_1 \cup U) = V(H_1) \cup U$ and $E(H_1 \cup U) = E(H_1)$, and $H_1 \cup P$ is the subgraph of $G$ with $V(H_1 \cup P) = V(H_1) \cup V(P)$ and $E(H_1 \cup P) = E(H_1) \cup P$, where $V(P)$ is the set of vertices that are end vertices of some edges in $P$. A pair $(H_1, H_2)$ is a separation of $G$ if $H_1 \cup H_2 = G$ and $E(H_1) \cap E(H_2) = \emptyset$.

For a path $P$ and two vertices $x, y \in V(P)$, $P[x, y]$ denotes the subpath of $P$ between $x$ and $y$. Furthermore, let $P(x, y) = P[x, y] - \{x\}$, $P[x, y) = P[x, y] - \{y\}$, and $P(x, y] = P[x, y] - \{x, y\}$.

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Let $G$ be a connected plane graph. A facial walk in $G$ is the boundary walk of some face of $G$. Furthermore, if it is a cycle, then we call it a facial cycle in $G$.

Let $T$ be a subgraph of a graph $G$. A $T$-bridge of $G$ is either (i) an edge of $G - E(T)$ with both ends on $T$ or (ii) a subgraph of $G$ induced by the edges in a component of $G - V(T)$ and all edges from that component to $T$. A $T$-bridge satisfying (i) is said to be trivial; otherwise it is non-trivial. For a $T$-bridge $B$ of $G$, the vertices in $B \cap T$ are the attachments of $B$ (on $T$), and any vertex of $B$ that is not an attachment is a non-attachment. We say that $T$ is a Tutte subgraph in $G$ if every $T$-bridge of $G$ has at most three attachments on $T$. For another subgraph $C$ of $G$, $T$ is a $C$-Tutte subgraph in $G$ if $T$ is a Tutte subgraph in $G$ and every $T$-bridge of $G$ containing an edge of $C$ has at most two attachments on $T$. When $T$ is a path or a cycle, we call $T$ a $C$-Tutte path or a $C$-Tutte cycle, respectively.

Note that if $G$ is 4-connected and $T$ is a Tutte subgraph in $G$ with $|T| \geq 4$, then $T$ must contain all vertices in $G$; otherwise, there exists a $T$-bridge in $G$ whose attachments form a cut set in $G$ of order at most three, contradicting that $G$ is 4-connected. Indeed, the concept of “Tutte subgraphs” was first introduced by Tutte [11] in order to prove his seminal result; every 4-connected planar graph is Hamiltonian. Since then it has been extended by several researchers, see [6, 7, 8, 9, 13]. The following theorem is a main tool to prove Theorem 3. See also the paper [9] by Thomassen.

**Theorem 4 (Sanders [7])** Let $G$ be a connected plane graph, let $C$ be a facial walk in $G$, let $x, y \in V(G)$ with $x \neq y$, and let $e \in E(C)$. Assume that $G$ contains a path from $x$ to $y$ through $e$. Then $G$ has a $C$-Tutte path from $x$ to $y$ through $e$.

Note that originally Sanders [7] showed only the 2-connected case, but we can easily show Theorem 4 using a block decomposition. Hence, we omit that proof of Theorem 4.

### 3 Proof of Theorem 3

Let $G$ be a 5-connected planar triangulation, and let $v_1, v_2, v_3$ and $v_4$ be four distinct vertices in $G$. We will show that $G$ has a Hamiltonian cycle passing through $v_1, v_2, v_3$ and $v_4$ in this order. It follows from Theorem 1 (ii) that $G$ has a cycle passing through those vertices in that order. This implies the following; $G - v_4$ has a path $P$ from $v_1$ to $v_3$ through $v_2$ such that

(P1) $G - V(P(v_1, v_3))$ contains a path from $v_3$ to $v_1$ through $v_4$.

In addition, by taking a path satisfying property (P1) as short as possible, we can also consider the following condition. Here a chord of $P$ is an edge $e$ not in $P$ such that both of end vertices of $e$ are contained in $P$. 

(P2) For any chord of \( P \), one end vertex of it is contained in \( P[v_1, v_2] \) and the other is contained in \( P(v_2, v_3) \).

Indeed, if there exists a chord \( xy \) of \( P \) such that both end vertices are contained in \( P[v_1, v_2] \) or in \( P[v_2, v_3] \), then we can detour \( P \) by \( xy \) instead of \( P[x, y] \). It is easy to see that the new path also satisfies condition (P1) and is shorter than \( P \). Therefore, a path that is as short as possible, subject to (P1), also satisfies condition (P2).

Let \( G^1 = G - V(P(v_1, v_3)) \), and let \( C^1 \) be the unique facial walk of \( G^1 \) that is not facial in \( G \). Note that \( v_1, v_3 \in V(C^1) \). Now we consider a separation \((H_1, H_2)\) of \( G^1 \) such that \( |H_1 \cap H_2| \leq 2 \), \( v_1, v_3 \in V(H_1) \), and \( v_4 \in V(H_2) - V(H_1) \). When \( H'_1 \) consists of only the two vertices \( v_1 \) and \( v_3 \) and no edges and \( H'_2 = G^1 \), a pair \((H'_1, H'_2)\) is a separation of \( G^1 \) satisfying all of the above conditions. Therefore, such a separation \((H_1, H_2)\) of \( G^1 \) must exist. Take such a separation \((H_1, H_2)\) of \( G^1 \) so that \( |H_2| \) is as small as possible. If \( H_2 \) does not contain any edge in \( C^1 \), then \( H_1 \cap H_2 \) forms a cut set in \( G \) of order at most 2, contradicting that \( G \) is 5-connected. Hence \( H_2 \) contains an edge in \( C^1 \). Then it follows from condition (P1) that \( |H_1 \cap H_2| = 2 \) and there exists an edge \( e^1 \) in \( H_2 \cap C^1 \) such that \( G^1 \) has a path from \( v_3 \) to \( v_1 \) through \( e^1 \). (In fact, take an edge in \( H_2 \cap C^1 \) such that it is incident with a vertex in \( H_1 \cap H_2 \).)

It follows from Theorem 4 that \( G^1 \) has a \( C^1 \)-Tutte path \( T^1 \) from \( v_3 \) to \( v_1 \) through \( e^1 \). Note that by the choice of \( e^1 \), \( T^1 \) passes through both of the two vertices in \( H_1 \cap H_2 \). In addition, it satisfies the following property.

**Claim 1** \( T^1 \) contains \( v_4 \), but does not contain \( v_2 \).

**Proof.** Since \( v_2 \notin V(G^1) \), the second statement is trivial. So we only show the first one.

Suppose not, and let \( B \) be a \( T^1 \)-bridge of \( G^1 \) such that \( B \) contains \( v_4 \) as a non-attachment. Let \( S_B \) be the set of attachments of \( B \) on \( T^1 \). If \( B \) has no neighbors in \( P(v_1, v_3) \), then \( S_B \) would be a cut set of \( G \) such that \( S_B \) separates \( B - S_B \) from other vertices and \( |S_B| \leq 3 \), which contradicts that \( G \) is 5-connected. Therefore, \( B \) has neighbors in \( P(v_1, v_3) \). This implies that \( B \) contains an edge in \( C^1 \). Then since \( T^1 \) is a \( C^1 \)-Tutte path in \( G^1 \), we have \( |S_B| \leq 2 \). Let \( \overline{B} = G^1 - V(B - S_B) \). Then \((\overline{B}, B)\) is a separation of \( G^1 \) such that \( |\overline{B} \cap B| = |S_B| \leq 2 \), \( v_1, v_3 \in V(\overline{B}) \) and \( v_4 \in V(B) - V(\overline{B}) \). Furthermore, since \( T^1 \) passes through \( e^1 \) and \( B \) is a \( T^1 \)-bridge of \( G^1 \), we have \( e^1 \in E(\overline{B}) \), which implies that \( V(H_2) - V(B) \neq \emptyset \). Since \( T^1 \) passes through both of the two vertices in \( H_1 \cap H_2 \), we see that \( V(B) \subset V(H_2) \), which contradicts the choice of \((H_1, H_2)\). This completes the proof of Claim 1. \( \Box \)

Let \( G^2 = G - V(T^1(v_3, v_1)) \), and let \( C^2 \) be the unique facial walk of \( G^2 \) that is not facial in \( G \). Note that \( v_1, v_3 \in V(C^2) \). Then we consider a separation \((R_1, R_2)\) of \( G^2 \) such that \( |R_1 \cap R_2| \leq 2 \), \( v_1, v_3 \in V(R_1) \), and \( v_2 \in V(R_2) - V(R_1) \). When \( R'_1 \) consists of only
the two vertices $v_1$ and $v_3$ and no edges and $R_2' = G^2$, a pair $(R_1', R_2')$ is a separation of $G^2$ satisfying all of the above conditions. Therefore, such a separation $(R_1, R_2)$ of $G^2$ must exist. Take such a separation $(R_1, R_2)$ so that $|R_2|$ is as small as possible. Since $G$ is 5-connected, $R_2$ contains an edge in $C^2$. Note that $P$ is contained in $G^2$, and hence $G^2$ has a path from $v_1$ to $v_3$ through $v_2$. This implies $|R_1 \cap R_2| = 2$ and there exists an edge $e^2$ in $R_2 \cap C^2$ such that $G^2$ has a path from $v_1$ to $v_3$ through $e^2$.

It follows from Theorem 4 that $G^2$ has a $C^2$-Tutte path $T^2$ from $v_1$ to $v_3$ through $e^2$. Note that by the choice of $e^2$, $T^2$ passes through both of the two vertices in $R_1 \cap R_2$. Notice also that $T^1 \cup T^2$ is a cycle in $G$, and it satisfies the following, which is crucial in the proof of Theorem 3.

**Claim 2** There exist no non-trivial $(T^1 \cup T^2)$-bridges in $G$. In particular, $T^1 \cup T^2$ is a Hamiltonian cycle in $G$.

**Proof.** Suppose that there exists a non-trivial $(T^1 \cup T^2)$-bridge $D$ in $G$. Let $S_D$ be the set of attachments of $D$ on $T^1 \cup T^2$.

Suppose first that $S_D \cap V(T^1(v_3, v_1)) = \emptyset$. This condition implies that $D$ is a $T^2$-bridge of $G^2$. Since $T^2$ is a $C^2$-Tutte path in $G^2$, we have $|S_D| \leq 3$, which implies that $S_D$ is a cut set in $G$ of order at most three, contradicting that $G$ is 5-connected. Therefore, we may assume that $S_D \cap V(T^1(v_3, v_1)) \neq \emptyset$. By the same argument, we also see that $S_D \cap V(T^2(v_1, v_3)) \neq \emptyset$.

These conditions, together with the planarity, imply that $D$ contains an edge in $C^1$ and an edge in $C^2$. Then since $D - V(T^1(v_3, v_1))$ is a $T^2$-bridge of $G^2$ containing an edge in $C^2$, we have

$$|S_D \cap V(T^2)| \leq 2. \quad (1)$$

Suppose that $D$ contains no vertices in $P$ as non-attachments. See Figure 1. This condition implies that there exists a $(T^1 \cup P)$-bridge, say $B_D$, such that $D \subseteq B_D$. Note that $B_D - V(P(v_1, v_3))$ is connected and a $T^1$-bridge of $G^1$ containing an edge in $C^1$, and hence $B_D - V(P(v_1, v_3))$ has at most two attachments on $T^1$. Since any vertex in $S_D \cap V(T^1(v_3, v_1))$ is an attachment of $B_D - V(P(v_1, v_3))$ on $T^1$, we have

$$|S_D \cap V(T^1(v_3, v_1))| \leq \left| \left( B_D - V(P(v_1, v_3)) \right) \cap V(T^1) \right| \leq 2.$$

Then it follows from inequality (1) that $|S_D| \leq 4$, which contradicts that $G$ is 5-connected.

Therefore, we may assume that $D$ contains vertices in $P$ as non-attachments. See Figure 2. Since $P$ is a path in $G^2$ from $v_1$ to $v_3$ and $v_1, v_3 \in V(T^2)$, $D$ has at least two attachments on $P$. Then it follows from inequality (1) that $S_D \cap V(T^2) \subseteq V(P)$ and $|S_D \cap V(T^2)| = 2$. Let $\{x, y\} = S_D \cap V(T^2)$. Note that $P[x, y]$ is contained in $D$. Consider the region bounded by $P[x, y] \cup T^2[x, y]$. Since $S_D \cap V(T^2) = S_D \cap V(P)$ is
the set of attachments of $D$ on $T^2$, there are no edges between vertices in $P(x, y)$ and those in $T^2(x, y)$. Thus, since $G$ is a triangulation, there exists an edge in $G$ connecting $x$ and $y$. Note that $xy$ is a chord of $P$. It follows from condition (P2) and symmetry that we may assume that $x$ is contained in $P[v_1, v_2]$ and $y$ is contained in $P(v_2, v_3)$. Then $(\overline{D}, D)$ is a separation of $G^2$, where $\overline{D} = G^2 - V(D - S_D)$, such that $|\overline{D} \cap D| = |S_D| = 2$, $v_1, v_3 \in V(\overline{D})$, and $v_2 \in V(D)$. Furthermore, since $T^2$ passes through $e^2$ and $D$ is a $(T^1 \cup T^2)$-bridge of $G$, we have $e^2 \in E(\overline{D})$, which implies that $V(R_2) - V(D) \neq \emptyset$. Since $T^2$ passes through both of the two vertices in $R_1 \cap R_2$, we see that $V(D) \subset V(R_2)$, which contradicts the choice of $(R_1, R_2)$.

Therefore, there exist no non-trivial $(T^1 \cup T^2)$-bridges in $G$, which easily implies that $T^1 \cup T^2$ is a Hamiltonian cycle in $G$. This completes the proof of Claim 2. □

By Claim 1, $v_2$ appears in $T^2$ and $v_4$ appears in $T^1$, which implies that $T^1 \cup T^2$ contains $v_1, v_2, v_3$ and $v_4$ in this order. By Claim 2, $T^1 \cup T^2$ is a Hamiltonian cycle in $G$. These complete the proof of Theorem 3. □

4 Conclusion

In this paper, we have focused on the property of being 4-ordered Hamiltonian. In fact, considering known results (Theorem 1) on 4-connected planar triangulations, it is natural to pose Conjecture 2. We gave a partial solution to it, by showing that every 5-connected planar triangulation is 4-ordered Hamiltonian.

In the rest, we would like to put some problems related to $k$-ordered Hamiltonian. The first one is Conjecture 2, which already appeared in Section 1.

The second problem is the property of being 4-ordered Hamiltonian of graphs on non-spherical surfaces. In fact, there are some results that are the counter parts of Theorem 1. Recall that for a graph $G$ on a non-spherical surface $F^2$, the edge-width of $G$ is the length of a shortest non-contractible cycle in $G$.

Theorem 5 (Mukae and Ozeki [5]) Let $G$ be a 4-connected triangulation on a surface. Then $G$ is 4-ordered.
Theorem 6 (Yu [13]) For any surface $F^2$, there exists an integer $N = N(F^2)$ satisfying the following; for any 5-connected triangulation $G$ of $F^2$, if the edge-width of $G$ is at least $N$, then $G$ is Hamiltonian.

Note that the assumptions on 5-connectedness and edge-width in Theorem 6 are both best possible, in some sense. In fact, Theorem 6 cannot be improved to 4-connected graphs (see [10]) and to the statement without the edge-width assumption (see [1]).

Considering these two theorems, the following seems also a natural conjecture. Because of the facts mentioned above, the assumptions on the edge-width and 5-connectedness are best possible, if the conjecture is true. We leave it to readers as an open problem.

Problem 7 For any surface $F^2$, there exists an integer $N = N(F^2)$ satisfying the following; for any 5-connected triangulation $G$ of $F^2$, if the edge-width of $G$ is at least $N$, then $G$ is 4-ordered Hamiltonian.

Goddard [3] also mentioned about the property of being 5-ordered; no planar graph can be 5-ordered. However, his idea cannot work for graphs on non-spherical surfaces, and hence the following might also hold. Those are the last problems in this paper.

Problem 8 Any 5-connected triangulation of a non-spherical surface $F^2$ is 5-ordered.

Problem 9 For any surface $F^2$, there exists an integer $N = N(F^2)$ satisfying the following; for any 5-connected triangulation $G$ of $F^2$, if the edge-width of $G$ is at least $N$, then $G$ is 5-ordered Hamiltonian.

Note that if a 4-connected triangulation $G$ of a surface has two adjacent vertices of degree 4, then $G$ cannot be 5-ordered. In fact, if $v_1, v_3$ and $v_4$ are specified as in Figure 3 and $v_2$ and $v_5$ are specified as vertices outside of the structure, then the graph cannot have a cycle containing $v_1, v_2, v_3, v_4$ and $v_5$ in this order. Hence the assumption on 5-connectedness in Problems 8 and 9 are best possible, in a sense.
References


