5-connected toroidal graphs are Hamiltonian-connected

Ken-ichi Kawarabayashi[1]

Kenta Ozeki[2]

National Institute of Informatics, 2-1-2
Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan
and
JST, ERATO, Kawarabayashi Large Graph Project, Japan

ABSTRACT

The problem on the Hamiltonicity of graphs is well studied in discrete algorithm and graph theory, because of its relation to traveling salesman problem (TSP). Starting with Tutte's result, stating that every 4-connected planar graph is Hamiltonian, several researchers have studied the Hamiltonicity of graphs on surfaces. Extending Tutte's technique, Thomassen proved that every 4-connected planar graph is in fact Hamiltonian-connected, i.e., there is a Hamiltonian path connecting any two prescribed vertices. For graphs on the torus, Thomas and Yu showed that every 5-connected graph on the torus has a Hamiltonian cycle. In this paper, we prove the following result which generalizes Thomas and Yu's result.

Every 5-connected graph on the torus is Hamiltonian-connected.

Our result is best possible in the sense that we cannot lower the connectivity 5 (i.e, there is a 4-connected graph on the torus which is not Hamiltonian-connected). Moreover, our proof is constructive in a sense that it gives rise to a polynomial time (indeed $O(n^2)$-time) algorithm to construct a Hamiltonian path between any two specified vertices, if an input graph is a 5-connected graph on the torus.

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1 Introduction

Finding a Hamiltonian cycle is arguably one of the most popular subjects in graph theory. It is also one of the central problems in combinatorial optimization, since it is connected to the famous traveling salesman problem. A study on Hamiltonian cycles was started with the connection to the famous Four Color Problem (now Theorem). It had been conjectured since 1880's that every 3-connected cubic planar graph has a Hamiltonian cycle, and if true, it would imply the Four Color Problem. However, Tutte [17] in 1946 constructed a counterexample. Since then, finding a Hamiltonian cycle in planar graphs and graphs on surfaces is one of the most active topics in graph theory. In the last decade, a Hamiltonian cycle in planar graphs is also studied in graph algorithm ([7], for example), because it is connected to the traveling salesmen problem.

[1]Email address: k_keniti@nii.ac.jp

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Email address: ozeki@nii.ac.jp

It is now known that a 3-connected planar graph does not always have a Hamiltonian cycle, since there exist planar triangulations on n vertices whose longest cycle is of length $O(n^{\alpha})$, where $\alpha = \log 2/\log 3 \approx 0.63$; cf. [9]. However, if one considers 4-connectivity for planar graphs, the situation dramatically changes. Whitney [18] was the first to give a positive result, showing that every planar triangulation without separating triangles has a Hamiltonian cycle. Tutte [16] proved that every 4-connected planar graph has a Hamiltonian cycle. Extending Tutte's technique, Thomassen [15] proved that every 4-connected planar graph is in fact Hamiltonian-connected, i.e., there is a Hamiltonian path connecting any two prescribed vertices. (A small omission in [15] was corrected by Chiba and Nishizeki [1]). Chiba and Nizhizeki [2] pointed out that Thomassen's proof implies a polynomial time algorithm to find, given a 4-connected planar graph, a Hamiltonian path between any specified vertices.

With some additional (and complicated) techniques and new ideas, Thomas and Yu [12] managed to show that every edge in a 4-connected projective-planar graph is contained in a Hamiltonian cycle, and recently, the authors [6] improved it by showing that every 4-connected projective-planar graph is Hamiltonian-connected. Note that these two results establish a conjecture of Grünbaum [4] and a conjecture of Dean [3], respectively.

Grünbaum [4] and Nash-Williams [10] also conjectured that every 4-connected graph on the torus has a Hamiltonian cycle. Recently, it is shown in [14] that there is a Hamiltonian path in 4-connected graphs on the torus. Very recently, the authors [5] proved that every 4-connected triangulation on the torus has a Hamiltonian cycle. Let us point out that there are 4-connected graphs that do not have a Hamiltonian cycle in the double torus or in N_3, where N_3 is the surface obtained from the sphere by attaching three crosscaps. Thus this conjecture, if true, would be best possible. On the other hand, for 4-connected graphs on the torus, certain edges may not be contained in any Hamiltonian cycle, which means that there are some 4-connected graphs on the torus that is not Hamiltonian-connected. The following example was provided by Thomassen [15]. Embed the product of two even cycles (of length at least 4) in the torus so that every face is bounded by a cycle of length 4, and add an edge joining two non-adjacent vertices in some facial cycle. So this graph is obtained from a toroidal grid with one edge added in a 4-cycle. Then this new edge is not contained in any Hamiltonian cycle of the new graph. (Note that the original graph is a balanced bipartite graph and the new edge connects two vertices in the same bipartition.)

While the conjecture due to Grünbaum and Nash-Williams remains open, it is shown in [13] that every 5-connected graph on the torus has a Hamiltonian cycle. In this paper, we show the following, which is an improvement of it.

Theorem 1 *Every 5-connected graph embedded on the torus is Hamiltonian-connected.*

Our result is best possible in the sense that we cannot lower the connectivity 5. (As mentioned above, there are 4-connected graphs on the torus which is not Hamiltonian-connected.) Also, we cannot improve Theorem 1 to graphs on surfaces with higher genera, since it is known that there are infinitely many graphs on a surface with higher genus such that they are not Hamiltonian-connected.

Our proof is constructive in a sense that it gives rise to a polynomial time (indeed $O(n^2)$-time) algorithm to construct a Hamiltonian path between any two specified vertices, if an input graph is 5-connected graph on the torus.

Technically, we shall prove Theorem 1 in Section 4 by adapting the notion a *Tutte subgraph*. In order to state our technical result (Theorem 2) which implies Theorem 1, we need some definitions that will be given in the next section, and some lemmas mentioned in Section 3. In Section 5, we give an outline of the proof of Theorem 2. All of the detailed proofs appear in Sections 6 and 7.

2 Technical statement

For a graph G, the order of G is denoted by $|G|$. Let H_1 and H_2 be two subgraphs of a graph G. Then $H_1 \cup H_2$ denotes the subgraph of G with $V\big(H_1 \cup H_2\big) = V(H_1) \cup V(H_2)$ and $E\big(H_1 \cup H_2\big) = E(H_1) \cup E(H_2)$, and $H_1 \cap H_2$ denotes the subgraph of G with $V\big(H_1 \cap H_2\big) = V(H_1) \cap V(H_2)$ and $E\big(H_1 \cap H_2\big) = E(H_1) \cap E(H_2)$. We use the similar notation also for a vertex subset U or an edge subset P of G; So, $H_1 \cup U$ is the subgraph of G with $V\big(H_1 \cup U\big) = V(H_1) \cup U$ and $E\big(H_1 \cup U\big) = E(H_1)$, and $H_1 \cup P$ is the subgraph of G with $V\big(H_1 \cup P\big) = V(H_1) \cup V(P)$ and $E\big(H_1 \cup P\big) = E(H_1) \cup P$, where $V(P)$ is the set of vertices that are end vertices of some edges in P.

A pair (H_1, H_2) of subgraphs of G is called a *separation of* G if $V(G) = V(H_1) \cup V(H_2)$ and each edge of G is contained in exactly one of H_1 and H_2. A separation (H_1, H_2) of G is a *k-separation* if $|H_1|, |H_2| \geq k+1$ and $\big|V(H_1 \cap H_2)\big| = k$. Note that G is k-connected if and only if G has no l-separation for each $l < k$.

Let G be a connected graph on a surface $\mathbb{F}^2$. A *facial walk* in G is the boundary walk of some face of G. Furthermore, if it is a cycle, then we call it a *facial cycle* in G. Now suppose that $\mathbb{F}^2$ is a non-spherical surface. The *representativity* of G is the minimum integer k such that every essential closed curve on $\mathbb{F}^2$ hits G at least k times. Let x be a vertex in G and let F be a face of G such that the boundary walk of F contains x. We define the *x-width* (resp. *(x, F)-width*) of G as the minimum integer k such that every essential closed curve on $\mathbb{F}^2$ passing through x (resp. x and F) hits G at least k times.

For a path P and two vertices $x, y \in V(P)$, $P[x, y]$ denotes the subpath of P between x and y. For a cycle C with fixed direction, and for two vertices $x, y \in V(C)$, $C[x, y]$ denotes the subpath of C between x and y along the direction. For a cycle C of a plane graph G, we usually give a direction to C in the clockwise order, unless C is the outer facial cycle of G.

Let T be a subgraph of a graph G. A *T-bridge* of G is either (i) an edge of $G - E(T)$ with both ends on T or (ii) a subgraph of G induced by the edges in a component of $G - V(T)$ and all edges from that component to T. A T-bridge satisfying (i) is said to be *trivial*; otherwise it is *non-trivial*. For a T-bridge B of G, the vertices in $B \cap T$ are the *attachments* of B (on T), and any vertex of B that is not an attachment is a *non-attachment*. We say that T is a *Tutte subgraph* (resp. an *extended Tutte subgraph*) of G if every T-bridge of G has at most three (resp. four) attachments on T. Note that if G is 4-connected (resp. 5-connected) and T is a Tutte subgraph (resp. an extended Tutte subgraph) of G with $|T| \geq 4$ (resp. $|T| \geq 5$), then T must contain all vertices in G; otherwise, there exists a T-bridge of G whose attachments form a cut set in G of order at most three (resp. four), contradicting that G is 4-connected (resp. 5-connected). For $C \subset G$, T is a *C-Tutte* subgraph (resp. an *extended C-Tutte subgraph*) of G if T is a Tutte subgraph (resp. extended Tutte subgraph) of G and every T-bridge of G containing an edge of C has at most two (resp. three) attachments on T. Note that a Tutte subgraph in G is also an extended C-Tutte subgraph for any $C \subset G$. When T is a path or a cycle, we call T a (extended) C-Tutte path or a (extended) C-Tutte cycle. When G is a plane graph, we usually regard the outer facial walk as the subgraph C.

Let C be a facial cycle in G or a subpath of it. We will define an *extended C-flap*, which is, in some sense, an extension of a C-flap, see [6, 12] for the definition of a C-flap. Notice that a C-flap was defined by Thomas and Yu [12], but at that time, they regarded the empty graph also as a C-flap. In the present paper, we exclude the empty graph from the definition of an extended C-flap, since we think that it is easier to understand.

An *extended C-flap* is an $\{a, b, c, d\}$-bridge H of G for some vertices a, b, c and d such that either

(H1) $a, b \in V(C) \cap V(H)$ and $a \neq b$,

(H2) H contains a subpath P of C from a to b with $c, d \notin V(P)$, and

(H3) H is embedded on the disk such that P, c and d appear in the outer facial walk C_H in this order,

or

(H1) $a, b \in V(C) \cap V(H)$ and $a \neq b$,

(H2) H contains a subpath P of C from a to b with $c \notin V(P)$, and

(H3) $b = d$, and there exists an essential closed curve on $\mathbb{F}^2$ that hits G only at a and b, and there exists a plane graph $\widehat{H}$ with the outer facial walk $\widehat{C_H}$ and four distinct vertices $a, \hat{b}, c, \hat{d}$ in $\widehat{C_H}$ such that $\widehat{C_H}$ contains P and H is obtained from $\widehat{H}$ by identifying $\hat{b}$ and $\hat{d}$ to b.[3]

We call an extended C-flap for the former *of Type I*, while the latter is of *Type II*. Note that $P = C_H[a, b]$ for an extended C-flap of Type I, while $P = \widehat{C_H}[a, \hat{b}]$ for the one of Type II. In each type, the subpath P of C in (H2) is the *base path* of H. See Figure 1.[4] Notice that if a graph G has an extended C-flap of Type II, then the (b, F)-width of G is exactly 2, since there exists an essential closed curve on $\mathbb{F}^2$ that passes through only a and b. Note that a, b, c and d are *attachments of H*.

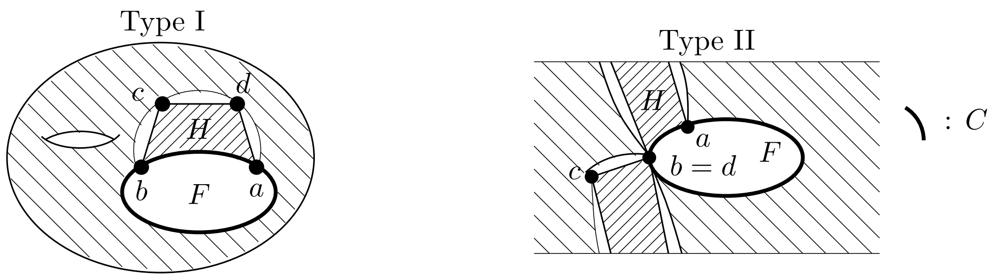

Figure 1: Two types of an extended C-flap H with attachments a, b, c, d.

The following is the technical theorem, which will be used for the proof of Theorem 1 in Section 4.

Theorem 2 *Let G be a 2-connected graph embedded on the torus with the representativity at least two, let F be a face of G, let C be the boundary cycle of F, let $x \in V(C)$, let $y \in V(G) - \{x\}$, and let f be an edge of C incident with x. Suppose that for every 2-separation (G_1, G_2) of G, we have $E(C) \cap E(G_1) \neq \emptyset$ and $E(C) \cap E(G_2) \neq \emptyset$. Then one of the following holds.*

(T1) *There exists an extended C-Tutte path T in G from x to y with $|T| \geq 4$.*

(T2) *There exists an extended C-flap H with attachments a, b, c, d and base path P and there exists an extended C-Tutte path T in $G - \big(V(H) - \{a, b, c, d\}\big)$ from b to y such that $|T| \geq 4$, $a, c, d \in V(T)$, P contains f, and a, f, x, b appear in P in this order.*

See Figure 2 for (T2) with an extended C-flap H of Type I. In Section 5, we will put the outline of the proof of Theorem 2, and in Section 6, we will show Theorem 2.

[3]In this definition, we regard all edges and all vertices in H other than b and d also as edges and vertices in $\widehat{H}$.

[4]In all figures of the present paper unless we mention explicitly, rising diagonals stroke from bottom left to top right represent an extended C-flap, while falling diagonals stroke from top left to bottom right represent other part of a given graph. White regions represent faces. Note that the right side of Figure 1 represents a part of a cylinder.

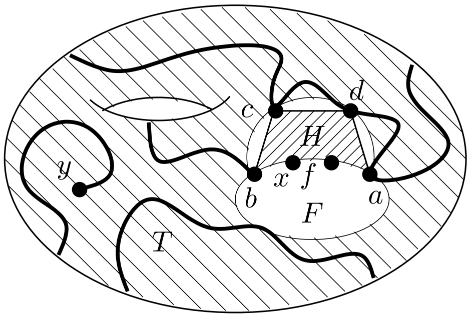

Figure 2: An extended C-Tutte path T desired in (T2) in Theorem 2.

3 Lemmas used in the proof of our theorems

Let us begin with some known results. The first lemma was proved by Sanders [11]. Note that he showed only the 2-connected case, but we can easily show the following, using a block decomposition. See also the paper [15] by Thomassen.

Theorem 3 *Let G be a connected plane graph, let C be a facial walk in G, let $x, y \in V(G)$ with $x \neq y$, and let $e \in E(C)$. Assume that G contains a path from x to y through e. Then G has a C-Tutte path from x to y through e.*

In this paper, we sometimes use Theorem 3 with the following form. Since we can easily prove it, we omit the proof here.

Theorem 4 *Let G be a 2-connected plane graph, let C be a facial cycle in G, let $x, y \in V(G)$ with $x \neq y$, and let $u \in V(C) - \{x, y\}$. Then G has a C-Tutte path T from x to y through u with $|T| \geq 4$, unless x and y are the only neighbors of u.*

On the other hand, as another directed corollary of Theorem 3, we obtain the following.

Theorem 5 *Let G be a connected plane graph, let C be a facial walk in G, let $x, u \in V(C)$ with $x \neq u$ and let $y \in V(G) - \{x, u\}$. Suppose that there exists a path in $G - u$ from x to y. Then G has a $C[x, u]$-Tutte subgraph consisting of u and a path T from x to y with $u \notin V(T)$.*

To show Theorem 5, consider the graph obtained from G by adding an edge connecting x and u, and use Theorem 3 to find a $\big(C[x, u] \cup \{xu\}\big)$-Tutte path from u to y through xu. Finally, deleting the added edge, we obtain the desired subgraph in Theorem 5.

The next lemma was shown by Thomas and Yu [13].

Theorem 6 (Theorem (2.2) in [13]) *Let G be a connected plane graph, let C be a facial walk in G, let $x, y \in V(C)$ with $x \neq y$, and let $A \subset V(C)$. Suppose that $V(C[x, y]) \cap A = \emptyset$. Then G has a path T from x to y with $V(T) \cap A = \emptyset$ such that every $(T \cup A)$-bridge B of G has at most $\max\big\{|A| + 1, 3\big\}$ attachments and at most two attachments if B contains an edge of $C[x, y]$.*

We use Theorem 6 for A with $|A| \leq 3$. Notice that if $|A| = 1$, then Theorem 6 almost corresponds to Theorem 5. (However, Theorem 5 is stronger than Theorem 6 for $|A| = 1$, since we can specify even a vertex not contained in C as y.) When $|A| \leq 2$, the path obtained from Theorem 6 is a $C[x, y]$-Tutte path from x to y. Similarly, when $|A| \leq 3$, it is an extended $C[x, y]$-Tutte path from x to y.

The following lemma was shown in [12].

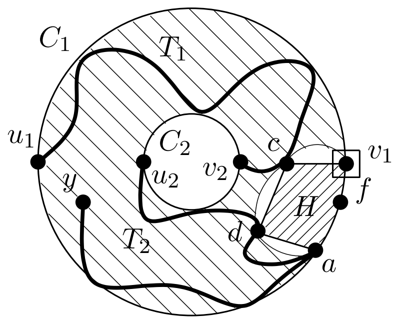

Figure 3: A $C_1[u_1, v_1]$-flap and an extended $\big(C_1[u_1, v_1] \cup C_2[v_2, u_2]\big)$-Tutte subgraph $T_1 \cup T_2$ desired in (T2) in Lemma 9.

Theorem 7 (Theorem (2.6) in [12]) *Let G be a 2-connected plane graph, let C be a facial cycle in G, let $x, y \in V(C)$, and let $e_1, e_2 \in E(C)$ such that $x \neq y$ and x, e_1, e_2, y appear on C in this clockwise order. Then there exists a $C[x, y]$-Tutte path T in G from x to y with $e_1, e_2 \in E(T)$.*

In addition to the above lemmas, we will use the following two new lemmas, which will be shown in Sections 7.1 and 7.2, respectively. We say that two paths T_1 and T_2 in G *connect* $\{u_1, u_2\}$ *and* $\{v_1, v_2\}$ if T_1 connects u_i and v_j and T_2 connects u_{3-i} and v_{3-j} for some $i, j \in \{1, 2\}$. Note that some vertices u_i and v_j might coincide each other; For example, if $u_1 = v_1$, then one of the paths T_1 and T_2 must consist of only the vertex u_1.

Lemma 8 *Let G be a 2-connected plane graph, let C_1, C_2 be facial cycles in G, let $u_1, v_1 \in V(C_1) - V(C_2)$ with $u_1 \neq v_1$, let $u_2 \in V(C_2) - V(C_1)$, and let $y \in V(G) - \{u_2, v_1\}$. Then G has two disjoint paths T_1 and T_2 such that T_1 and T_2 connect $\{u_1, u_2\}$ and $\{v_1, y\}$, $|T_1| + |T_2| \geq 5$, and $T_1 \cup T_2$ is a $C_1[u_1, v_1]$-Tutte subgraph in G.*

Lemma 9 *Let G be a 2-connected plane graph, let C_1, C_2 be facial cycles of G, let $u_1, v_1 \in V(C_1) - V(C_2)$ with $u_1 \neq v_1$, let $u_2, v_2 \in V(C_2) - V(C_1)$ with $u_2 \neq v_2$, let $y \in V(G) - \{u_1, v_1, v_2\}$, and let f be an edge of $C_1[u_1, v_1]$ incident with v_1. Suppose that $G - v_1$ is also 2-connected. Then one of the following holds.*

(T1′) *G has two disjoint paths T_1 and T_2 such that T_1 and T_2 connect $\{u_1, u_2\}$ and $\{v_2, y\}$, $v_1 \notin V(T_1 \cup T_2)$, $|T_1| + |T_2| \geq 5$, and $T_1 \cup T_2 \cup \{v_1\}$ is an extended $\big(C_1[u_1, v_1] \cup C_2[v_2, u_2]\big)$-Tutte subgraph in G.*

(T2′) *There exists an extended $C_1[u_1, v_1]$-flap H with attachments a, v_1, c, d and base path P, and there exist two disjoint paths T_1 and T_2 such that T_1 and T_2 connect $\{u_1, u_2\}$ and $\{v_2, y\}$, $v_1 \notin V(T_1 \cup T_2)$, $|T_1| + |T_2| \geq 5$, $a, c, d \in V(T_1 \cup T_2)$, $T_1 \cup T_2 \cup \{v_1\}$ is an extended $\big(C_1[u_1, v_1] \cup C_2[v_2, u_2]\big)$-Tutte subgraph in $G - \big(V(H) - \{a, v_1, c, d\}\big)$, P contains f, and a, f, v_1 appear in P in this order.*

In addition, we use the following important lemma (Lemma 10), which mentions how to cut a 2-connected graph on the torus with representativity exactly 2. Before that, we need some definitions.

A *block* in a graph G is a maximal subgraph that has no cut vertex in G. Note that any block is 2-connected unless it consists of only a vertex or an edge. A *chain of blocks* is a sequence $r_0, R_1, r_1, \ldots, r_{l-1}, R_l, r_l$ such that each R_i is a block in $\bigcup_{i=1}^{l} R_i$, each r_i is a vertex, in particular a cut vertex in $\bigcup_{i=1}^{l} R_i$ for $1 \leq i \leq l-1$, $V(R_i) \cap V(R_j) = \emptyset$ for each $0 \leq i < j \leq l$ with $j \neq i+1$, $\{r_i\} = V(R_i) \cap V(R_{i+1})$ for each $0 \leq i \leq l-1$, and if $|R_1| \geq 2$, then $r_0 \in V(R_0) - \{r_1\}$ and $r_l \in V(R_l) - \{r_{l-1}\}$. When a chain of blocks $r_0, R_1, r_1, \ldots, r_{l-1}, R_l, r_l$ satisfies that $|R_1| = 1$, then we call it *trivial*. (In this case, $l = 1$

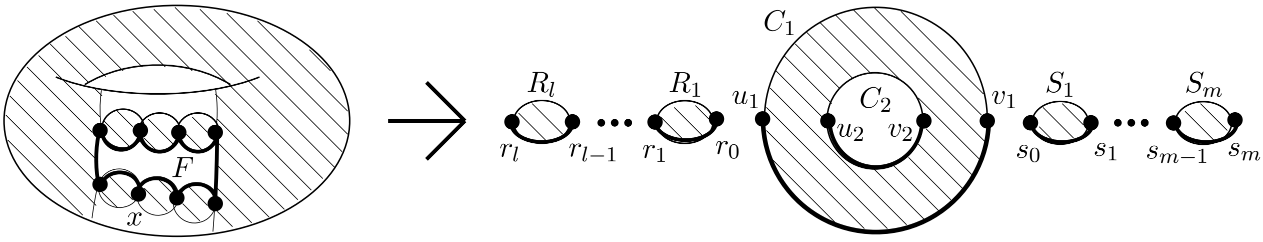

Figure 4: A 2-connected plane graph G_0 and chains of plane blocks $r_0, R_1, \dots, r_{l-1}, R_l, r_l$ and $s_0, S_1, \dots, s_{m-1}, S_m, s_m$ desired in Lemma 10.

and it consists of only one vertex $r_0 = r_1$.) Furthermore, if each block is a plane graph, then we call it a *chain of plane blocks*.

Lemma 10 *Let G be a 2-connected graph on the torus with the representativity exactly 2, let F be a face of G, let C be the boundary cycle of F, and let $x \in V(C)$. Suppose that the (x, F)-width of G is exactly 2. Then G can be decomposed into a 2-connected plane graph G_0 and two chains of plane blocks $r_0, R_1, r_1, R_2, \dots, r_{l-1}, R_l, r_l$ and $s_0, S_1, \dots, s_{m-1}, S_m, s_m$ satisfying the following properties (G1)–(G4). (Possibly, one of the chains or both may be trivial.) For each $1 \le i \le l$, let C_{R_i} be the unique facial walk in R_i that is not facial in G, and for each $1 \le j \le m$, let C_{S_j} be the unique facial walk in S_j that is not facial in G.*

(G1) *There exist two facial cycles C_1 and C_2 in G_0 and four distinct vertices u_1, u_2, v_1 and v_2 such that $u_1, v_1 \in V(C_1) - V(C_2)$ and $u_2, v_2 \in V(C_2) - V(C_1)$.*

(G2) *G is obtained from G_0 and the two chains of plane blocks $r_0, R_1, \dots, r_{l-1}, R_l, r_l$ and $s_0, S_1, \dots, s_{m-1}, S_m, s_m$ by identifying u_1 and r_0, u_2 and r_l, v_1 and s_0, and v_2 and s_m, respectively.*

(G3) $E(C) = E(C_1[u_1, v_1] \cup C_2[v_2, u_2]) \cup \bigcup_{i=1}^{l} E(C_{R_i}[r_i, r_{i-1}]) \cup \bigcup_{j=1}^{m} E(C_{S_j}[s_{j-1}, s_j])$.

(G4) $x = s_j$ *for some $0 \le j \le m$.*

4 Proof of Theorem 1, assuming Theorem 2

In this section, we give a proof of Theorem 1, assuming Theorem 2.

Let G be a 5-connected graph embedded on the torus, let $x \in V(G)$ and let $y \in V(G) - \{x\}$. It is enough to show that G has a Hamiltonian path from x to y.

Let $G^* = G - x$, let x^* be a neighbor of x in G, and let F^* be the unique face of G^* that is not a face in G. Note that the region F^* contains x and all edges incident with x in G. Let C^* be the boundary cycle of F^*. Since G is 5-connected, note that G^* is 4-connected and $|V(C^*)| \ge 5$.

Suppose first that the representativity of G^* is exactly 0. Then G^* is a planar graph. By Theorem 4, G^* has a C^*-Tutte path T^* from x^* to y with $|T^*| \ge 4$. (Note that the exceptional case in Theorem 4 does not occur if we take an appropriate vertex as u.) Let $T = \{xx^*\} \cup T^*$. Note that T is a path in G from x to y. Suppose that there exists a non-trivial T-bridge B in G. Since T^* is a C^*-Tutte path in G^*, $B \cap G^*$ has at most three attachments on T^*, and hence including x, B has at most four attachments on T. However, since $|T| \ge |T^*| + 1 \ge 5$, the attachments of B on T form a cut set in G of order at most four such that it separates some vertices in T from non-attachments of B, a contradiction. Therefore, there exists no non-trivial T-bridge of G, that is, T is a Hamiltonian path in G from x to y.

Suppose next that the representativity of G^* is exactly 1. Then there exists an essential curve γ on the torus hitting G^* at exactly one vertex, say z. Since γ hits G^* only at z,

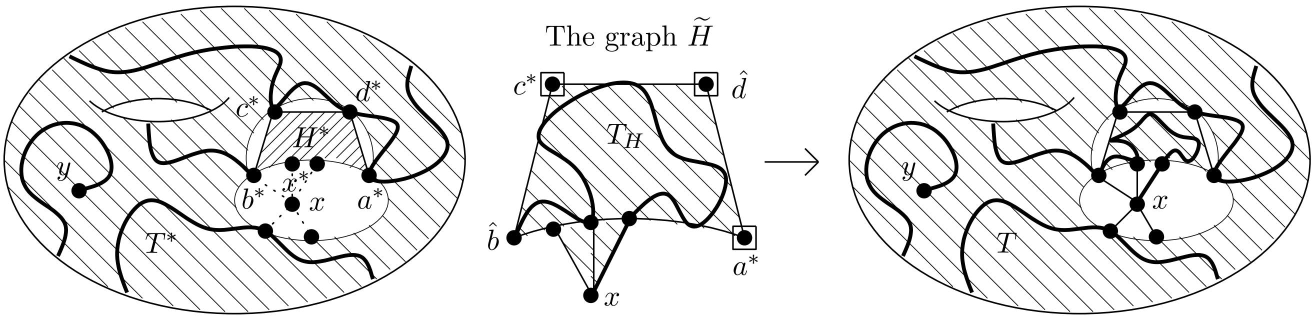

Figure 5: The case where the representativity of G^* is at least 2 and G^* satisfies (T2).

cutting the torus along γ, we obtain the plane graph G_0^* with two facial cycles C_1 and C_2 and two distinct vertices z_1 and z_2 with $z_1 \in V(C_1)$ and $z_2 \in V(C_2)$ such that G^* is obtained from G_0^* by identifying z_1 and z_2 into z. For $i = 1, 2$, let C_i^* be the unique facial cycle in $G_0^* - z_i$ that is not facial in G_0^*. Since $|C_i^*| \geq 3$ for $i = 1, 2$, there exists an edge e in C_1^* or in C_2^* such that e is incident with neither x^* nor y. By symmetry, we may assume that e is contained in C_1^*. By Theorem 3, $G_0^* - z_1$ has a C_1^*-Tutte path T^* from x^* to y through e. (Note that x^* or y might coincide z_2.) By the choice of e, we have $|T^*| \geq 4$. Let $T = \{xx^*\} \cup T^*$. Note that T is a path in G from x to y. Suppose that there exists a non-trivial T-bridge B in G. Suppose further that at least one of x and z_1 is not an attachment of B. Since T^* is a C_1^*-Tutte path in $G_0^* - z_1$, $B \cap (G_0^* - z_1)$ has at most three attachments on T^*, and hence B has at most four attachments on T. However, since $|T| \geq |T^*| + 1 \geq 5$, the attachments of B on T form a cut set in G of order at most four, a contradiction. Hence we may assume that both x and z_1 are attachments of B. In this case, $B \cap (G_0^* - z_1)$ is a T^*-bridge of $G_0^* - z_1$ containing an edge in C_1^*, and hence B has at most four attachments on T two of which are x and z_1. By the same argument above, this contradicts that G is 5-connected. Then T is a Hamiltonian path in G.

Therefore, we may assume that the representativity of G^* is at least 2. Let f be an edge of C^* incident with x^*. Since G^* is 4-connected, it trivially satisfies the assumptions in Theorem 2. Then by Theorem 2, G^* satisfies (T1) or (T2) with respect to G^*, C^*, x^*, y and f.

Suppose first that G^* satisfies (T1), and let T^* be an extended C^*-Tutte path satisfying the conditions in (T1) in Theorem 2. Let $T = \{xx^*\} \cup T^*$. Note that T is a path in G from x to y with $|T| \geq |T^*| + 1 \geq 5$. Suppose also that there exists a non-trivial T-bridge B of G. If x is not an attachment of B, then B is also a T^*-bridge of G^*, and hence B has at most four attachments on T. On the other hand, if x is an attachment of B, then $B - x$ is a T^*-bridge of G^* containing an edge in C^*, and hence B has at most four attachments one of which is x. In either case, since $|T| \geq 5$, those attachments separate some vertex in T^* from non-attachments of B, contradicting that G is 5-connected. Hence T is a Hamiltonian path in G.

Suppose next that G^* satisfies (T2). Let T^* be an extended C^*-Tutte path, and H^* be an extended C^*-flap with attachments a^*, b^*, c^*, d^* and base path P^* such that those satisfy the conditions in (T2) in Theorem 2. See the left side of Figure 5. By the same way as in the previous paragraph, we can show that there exists no non-trivial $\big(T^* \cup \{x\}\big)$-bridge of $G^* - \big(V(H^*) - \{a^*, b^*, c^*, d^*\}\big)$. Let $\widehat{H} = H^*$, $\hat{b} = b^*$ and $\hat{d} = d^*$ if H^* is of Type I; Otherwise let $\widehat{H}$ be the plane graph and $\hat{b}$ and $\hat{d}$ be the vertices as in (H3) of the definition of an extended C-flap of Type II. Note that $\widehat{H}$ is a plane graph. Let $\widetilde{H}$ be the graph obtained from $\widehat{H}$ by adding all edges in G between x and vertices in P^*. Since x^* is contained in P^* and x^* is a neighbor of x in G, $\widehat{H}$ is connected. It follows from Theorem 6 with $A = \{a^*, c^*, \hat{d}\}$ that $\widetilde{H}$ has a path T_H from x to $\hat{b}$ with $V(T_H) \cap A = \emptyset$ such that every $(T_H \cup A)$-bridge of $\widetilde{H}$ has at most four attachments. (We here ignore the last condition on T_H, since we do not need it.) See the middle of Figure 5. If there exists a non-trivial $(T_H \cup A)$-bridge of $\widetilde{H}$, then it has at most four attachments, but this contradicts that G is 5-connected. Thus, there exists no non-trivial $(T_H \cup A)$-bridge of $\widetilde{H}$. This implies that $T^* \cup T_H$ is a Hamiltonian path in G from x to y. See the right side of Figure 5. This completes the proof of Theorem 1, assuming Theorem 2. $\square$

5 Outline of the proof of Theorem 2

It remains to show Theorem 2. Here we briefly explain the outline of the proof of Theorem 2. Recall that the detailed proof appears in the next section. First, in Section 6.1, we show a claim (Claim 1) in order to avoid a 2-separation (G_1, G_2) of G with $x \in V(G_1 \cap G_2)$. In fact, if such a 2-separation exists, then we can replace one of G_1 and G_2, say G_2, by an edge and use the induction hypothesis to the obtained graph, say G_1^*. It follows from the assumption on a 2-separation in Theorem 2 that G_2 is a plane graph, and hence we can use some known results (Theorems 3, 5, and 6) to obtain a Tutte path, say T_2, inside G_2 with certain properties. Then we can expand the obtained extended Tutte path in G_1^* by combining T_2, and show that G satisfies (T1) or (T2).

Next, we divide the remaining proof into three cases depending on the (x, F)-width and x-width of G. The first two cases deal with graphs with low representativity; the (x, F)-width of G is exactly 2 (Section 6.2), and the (x, F)-width of G is at least 3 and the x-width is exactly 2 (Section 6.3). Note that these cases correspond to the base case of the induction. In these cases, since the (x, F)-width or x-width of G is small, it follows from Lemma 10 that we can cut the torus off to the cylinder and obtain a plane graph. Then depending on the places of x and y, we further divide the proofs into several subcases. In some subcases, we need new lemmas on Tutte paths (Lemmas 8 and 9); Otherwise we are done just using Theorems 3 and 5.

After dealing with the base case of the induction, we proceed to Section 6.4, which is the main part of the proof. Although we have to care the detail and divide the proof into some subcases, the main idea of this case is same and simple; By the assumption of this case, deleting the vertex x creates the new graph $G - x$, which satisfies all assumptions in Theorem 2. By the induction hypothesis, $G - x$ satisfies (T1) or (T2), and then expand the obtained extended Tutte path in $G - x$ to the one in G (Cases I, II-1, and III) using Theorem 6, or expand the obtained extended C-flap (Case II-2). $\square$

6 Proof of Theorem 2

We prove Theorem 2 by induction on $|G|$. Since G is 2-connected and has representativity at least two, it is easy to see that $|G| \geq 5$ and if $|G| = 5$, then G satisfies (T1). So, we may assume that $|G| \geq 6$.

We divide this proof into four steps: A claim dealing with a 2-separation with a certain condition (Section 6.1), the case where the (x, F)-width of G is exactly 2 (Section 6.2), the case where the (x, F)-width of G is at least 3 and the x-width of G is exactly 2 (Section 6.3), and the case where the x-width of G is at least 3 (Section 6.4).

6.1 A claim on a 2-separation

Claim 1 *If there exists a 2-separation (G_1, G_2) of G such that $x \in V(G_1 \cap G_2)$, then G satisfies (T1) or (T2).*

Proof. Suppose that G has a 2-separation (G_1, G_2) such that $x \in V(G_1 \cap G_2)$. Let $\{x, z\} = V(G_1 \cap G_2)$. Note that $z \in V(C)$ by the assumption of Theorem 2. If neither G_1 nor G_2 is bounded by a disk on the torus, then this contradicts that the representativity of G is at least 2. Hence by symmetry, we may assume that G_2 is bounded by a disk on the torus. See Figure 6.

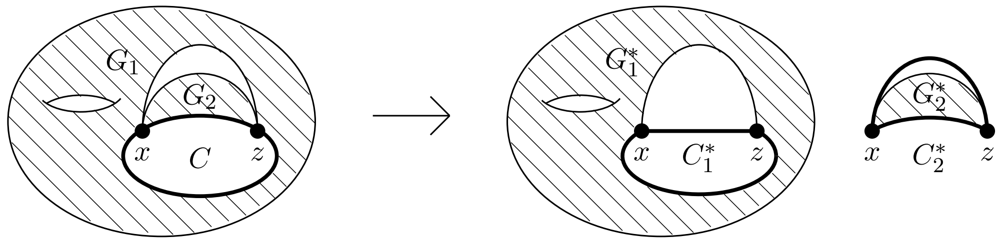

Figure 6: The 2-separation (G_1, G_2) of G and the graphs G_1^* and G_2^*.

For $i = 1, 2$, let G_i^* be the graph obtained from G by replacing G_{3-i} with an edge connecting x and z. (If G_i already has the edge connecting x and z, then we delete the original edge.) Notice that G_1^* is a 2-connected graph on the torus, and G_2^* is a 2-connected plane graph. We fix a direction on C and by symmetry, we may assume that $E(C[z, x]) = E(C) \cap E(G_1)$. Let $C_1^* = C[z, x] \cup \{xz\}$ and $C_2^* = C[x, z] \cup \{xz\}$. Note that for $i = 1, 2$, C_i^* is a facial cycle in G_i^*. Let $y^* = y$ if $y \in V(G_1^*)$; Otherwise let $y^* = z$. Let $f^* = f$ if $f \in E(G_1^*)$; Otherwise let $f^* = xz$.

Note that G_1^* is a 2-connected graph on the torus such that the representativity of G_1^* is at least 2 and for every 2-separation (G_1', G_2') of G_1^*, $E(C_1^*) \cap E(G_1') \neq \emptyset$ and $E(C_1^*) \cap E(G_2') \neq \emptyset$. Then by the induction hypothesis, G_1^* satisfies (T1) or (T2) with respect to G_1^*, C_1^*, x, y^* and f^*. Let T^* be an extended C_1^*-Tutte path, and H^* be an extended C_1^*-flap with attachments a^*, b^*, c^*, d^* and base path P^* satisfying the conditions in (T1) or (T2) in Theorem 2. (If G_1^* satisfies (T1), then ignore the definitions on H^*, a^*, and so on.)

Case I. $xz \in E(T^*)$.
If $y \in V(G_2^*)$, then $z = y^*$, and hence T^* consists of only xz, which contradicts that $|T^*| \geq 4$. Thus, we have $y \in V(G_1^*) - V(G_2^*)$.

By Theorem 3, there exists a C_2^*-Tutte path T_2 in G_2^* from x to z. In particular, we can take such a C_2^*-Tutte path T_2 so that $xz \notin E(T_2)$ by specifying an appropriate edge as e. Let $T = \big(T^* - \{xz\}\big) \cup T_2$. Note that $|T| \geq |T^*| + 1 \geq 5$.

Suppose first that G_1^* satisfies (T1). See Figure 7.[5] Note that every non-trivial T-bridge B of G is either (i) a T^*-bridge of G_1^* or (ii) a T_2-bridge of G_2^*. Then by the choice of T^* and T_2, B has at most four attachments and at most three attachments if B contains an edge of $C_1^* \cup C_2^*$. Since $E(C) \subset E(C_1^*) \cup E(C_2^*)$, T is an extended C-Tutte path in G. This implies that G satisfies (T1) in Theorem 2.

Suppose next that G_1^* satisfies (T2). See Figure 8. In this case, $f \in E(G_1)$. Let $H = H^*$, $a = a^*$, $b = b^*$, $c = c^*$, $d = d^*$, and $P = P^*$. Note that every non-trivial T-bridge B of $G - \big(V(H) - \{a, b, c, d\}\big)$ is either (i) a T^*-bridge of $G_1^* - \big(V(H^*) - \{a^*, b^*, c^*, d^*\}\big)$ or (ii) a T_2-bridge of G_2^*. By the same way as above, we can check that T is an extended C-Tutte path in $G - \big(V(H) - \{a, b, c, d\}\big)$, and G also satisfies (T2) in Theorem 2. This completes the proof of Case I.

Case II. $xz \notin E(T^*)$ and $y \in V(G_1^*)$.
Suppose first that G_1^* satisfies (T1). Let $T = T^*$. Note that the T-bridge of G containing G_2 is the unique T-bridge that is not a T^*-bridge of G_1^*, but the attachments of it are not changed. Thus, T is an extended C-Tutte path in G from x to y with $|T| \geq 4$, and hence G satisfies (T1).

[5]In Figure 7, the bottom curve represents a part of C. This situation is same for all figures in this subsection.

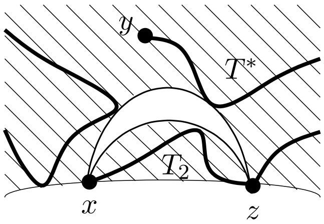

Figure 7: Case I with (T1) for G_1^*.

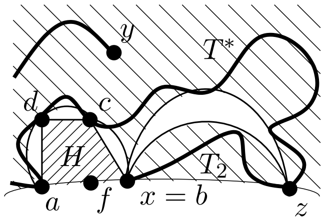

Figure 8: Case I with (T2) for G_1^*.

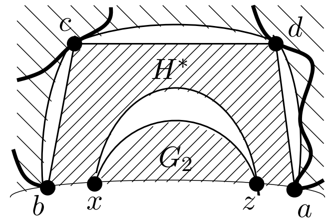

Figure 9: Case II with (T2) for G_1^* and $xz \in E(H^*)$.

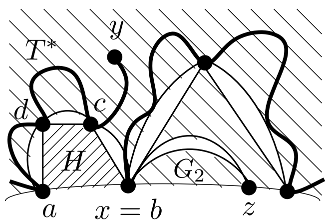

Figure 10: Case II with (T2) for G_1^* and $xz \notin E(H^*)$.

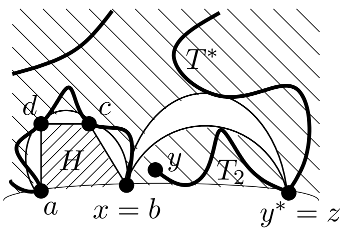

Figure 11: Case III with (T2) for G_1^* and $f^* \neq xz$.

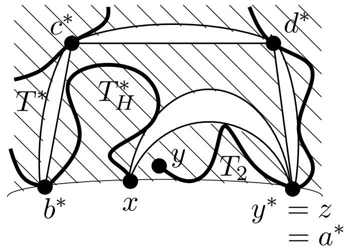

Figure 12: Case III with (T2) for G_1^* and $f^* = xz$.

Suppose next that G_1^* satisfies (T2). If $xz \in E(H^*)$, then let H be the subgraph of G induced by $\big(E(H^*) - \{xz\}\big) \cup E(G_2)$ and let $P = \big(P^* - \{xz\}\big) \cup C[x,z]$; otherwise let $H = H^*$ and $P = P^*$. See Figure 9 for the former, and Figure 10 for the latter. Let $a = a^*$, $b = b^*$, $c = c^*$, and $d = d^*$. In either case, H still has four attachments a, b, c, d, and hence H is an extended C-flap in G with attachments a, b, c, d and base path P. In particular, in the latter case, the bridge of G containing G_2 is the unique T-bridge that is not a T^*-bridge of G_1^*, but the attachments of it are not changed. These imply that G satisfies (T2) in Theorem 2.

Case III. $xz \notin E(T^*)$ and $y \in V(G_2^*) - V(G_1^*)$.
In this case, $y^* = z$ and $xy^* \in E(C_1^*)$. By Theorem 5, there exists a $C[x,z]$-Tutte subgraph in G_2^* consisting of x and a path T_2 from z to y with $x \notin V(T_2)$.

Suppose first that G_1^* satisfies (T1). Let $T = T^* \cup T_2$. By the same argument as in Case I, we can show that T is an extended C-Tutte path in G from x to y with $|T| \geq 4$, and hence G satisfies (T1) in Theorem 2.

Suppose next that G_1^* satisfies (T2) and $f^* \neq xz$. Since a^*, f^*, x, b^* appear in P^* in this order and $b^* \neq z = y^*$, we have that $x = b^*$. Let $T = T^* \cup T_2$, $H = H^*$, $a = a^*$, $b = b^* = x$, $c = c^*$, $d = d^*$ and $P = P^*$. See Figure 11. Note that every non-trivial T-bridge B of $G - \big(V(H) - \{a,b,c,d\}\big)$ is either (i) a T^*-bridge of $G_1^* - \big(V(H^*) - \{a^*, b^*, c^*, d^*\}\big)$, or (ii) a $\big(T_2 \cup \{x\}\big)$-bridge of G_2^*. By the same argument as in Case I, we can check that G satisfies (T2) in Theorem 2.

Suppose finally that G_1^* satisfies (T2) and $f^* = xz$. Since a^*, f^*, x, b^* appear in P^* in this order and $z = y^* \in V(T^*)$, we have that $a^* = y^* = z$. Let $\widehat{H^*} = H^*$, $\hat{b} = b^*$ and $\hat{d} = d^*$ if H^* is of Type I; otherwise let $\widehat{H^*}$ be the plane graph and $\hat{b}$ and $\hat{d}$ be the vertices as in (H3) in the definition of an extended C-flap of Type II. Note that $\widehat{H^*}$ is a plane graph. Let $\widehat{C_H}$ be the outer facial walk in $\widehat{H^*}$ containing $a^*, x, \hat{b}, c^*$ and $\hat{d}$ in this clockwise order. It follows from Theorem 6 with $A^* = \{a^*, c^*, \hat{d}\}$ that $\widehat{H^*}$ has a path T_H^* from x to $\hat{b}$ with $V(T_H^*) \cap A^* = \emptyset$ such that every $(T_H^* \cup A^*)$-bridge B of $\widehat{H^*}$ has at most four attachments and at most two attachments if B contains an edge of $\widehat{C_H}[x, \hat{b}]$. See Figure 12. Let $T = T_H^* \cup T^* \cup T_2$. Note that every non-trivial T-bridge B of G is either (i) a T^*-bridge of $G_1^* - \big(V(H^*) - \{a^*, b^*, c^*, d^*\}\big)$, or (ii) a $\big(T_2 \cup \{x\}\big)$-bridge of G_2^*, or (iii) a $(T_H^* \cup A^*)$-bridge of $\widehat{H^*}$. By the conditions of T^*, T_2 and T_H^* and the same way as in Case

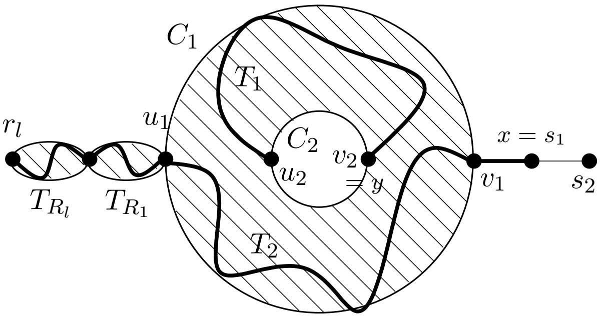

Figure 13: Case (d-1).

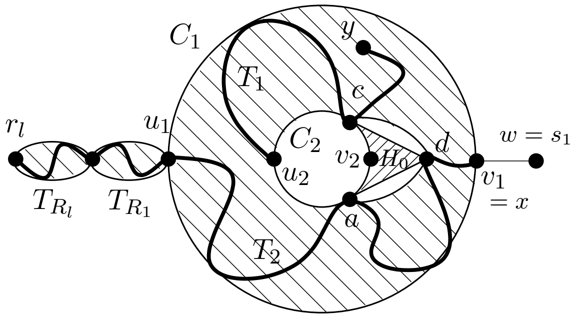

Figure 14: Case (c-2) with $(T_1 \cup T_2)$-bridge H_0.

I, we can check that T is an extended C-Tutte path in G. Hence G satisfies (T1).

This completes the proof of Claim 1. $\square$

In the remaining three sections (Sections 6.2–6.4), it follows from Claim 1 that we may assume that there exists no 2-separation (G_1, G_2) of G such that $x \in V(G_1 \cap G_2)$.

6.2 The case where the (x, F)-width of G is exactly 2

Let w be the vertex in C with $f = xw$. By Lemma 10, G can be decomposed into a 2-connected plane graph G_0 and two chains of plane blocks $r_0, R_1, r_1, R_2, \ldots, r_{l-1}, R_l, r_l$ and $s_0, S_1, \ldots, s_{m-1}, S_m, s_m$ satisfying (G1)–(G4). For each $1 \le i \le l$, let C_{R_i} be the unique facial walk in R_i that is not facial in G, and for each $1 \le j \le m$, let C_{S_j} be the unique facial walk in S_j that is not facial in G.

By (G4), $x = s_j$ for some $0 \le j \le m$. If $|S_{j'}| \ge 3$ for some $1 \le j' \le m$, then we can easily find a 2-separation (G_1, G_2) with $\{x, s_{j'-1}\} = V(G_1) \cap V(G_2)$ or $\{x, s_{j'}\} = V(G_1) \cap V(G_2)$, which was already done in Claim 1. Thus, for each $1 \le j' \le m$, $|S'_j| \le 2$. By the same argument and symmetry, we only have the following four possibilities.[6]

(a) $m = 1$, $|S_1| = 1$, $s_0 = s_1 = x$ and $w \in V(C_1[u_1, v_1])$.
(b) $m = 1$, $|S_1| = 2$, $s_0 = x \ne s_1$ and $w \in V(C_1[u_1, v_1])$.
(c) $m = 1$, $|S_1| = 2$, $s_0 = x \ne s_1$ and $w = s_1$.
(d) $m = 2$, $|S_1| = |S_2| = 2$, $s_1 = x$ and $w = s_2$.

(Note that Case (a) means that the chain $s_0, S_1, \ldots, s_{m-1}, S_m, s_m$ of plane blocks is trivial.) In either case, $y \in V(G_0) \cup \bigcup_{i=1}^{l} V(R_i)$. So we also have the following five possibilities;

(1) $y = v_1$ or $y = v_2$.
(2) $y \in V(G_0) - \{u_1, u_2, v_1, v_2\}$.
(3) $l = 1$, $|R_1| = 1$ and $y = r_0 = r_l$.
(4) $y = r_l \ne r_0$.
(5) $y \in \bigcup_{i=1}^{l} V(R_i)$ and $y \ne r_l$.

Case (1)–(4).

In these cases, we use the following paths; By Theorem 3, for each $1 \le i \le l$, R_i has a C_{R_i}-Tutte path T_{R_i} from r_{i-1} to r_i.

Case (1), (c-2) or (d-2).

[6] In the rest of Section 6.2, we deal with the vertices x and y also as vertices in G_0 and/or a vertex in $\bigcup_{i=1}^{l} R_i \cup \bigcup_{j=1}^{m} S_j$. This situation is same for other vertices, for example, $u_1, u_2, v_1, v_2, s_0, s_l$ and so on.

Since $y \neq x$, Case (a-1) does not occur. By the same reason (for Case (b-1) or (c-1)) or symmetry (for Case (d-1)), we may assume that $y = v_2$. (In Case (d-1), we do not use the vertex w and show that G satisfies (T1). Therefore we can indeed use the symmetry between s_0 and s_2.) If Case (c-2) or (d-2) occurs, then trivially $y \neq u_1, u_2, v_1, v_2$. In either case, we have $y \neq u_2, v_1$. Then by Lemma 8, G_0 has two disjoint paths T_1 and T_2 such that T_1 and T_2 connect $\{u_1, u_2\}$ and $\{v_1, y\}$, $|T_1| + |T_2| \geq 5$ and $T_1 \cup T_2$ is a $C_1[u_1, v_1]$-Tutte subgraph in G_0. Let

$$T = \begin{cases} T_1 \cup \bigcup_{i=1}^{l} T_{R_i} \cup T_2 & \text{if Case (b-1), (c-1) or (c-2) occurs,} \\ \{xs_0\} \cup T_1 \cup \bigcup_{i=1}^{l} T_{R_i} \cup T_2 & \text{if Case (d-1) or (d-2) occurs.} \end{cases}$$

Note that T is a path in G from x to y with $|T| \geq |T_1| + |T_2| - 1 \geq 4$.

Suppose that there exists no $(T_1 \cup T_2)$-bridge H_0 of G_0 such that H_0 contains v_2 as a non-attachment and H_0 has at least three attachments on $T_1 \cup T_2$. See Figure 13 for Case (d-1). (Note that in Case (1), we have $v_2 = y$, and hence such a $(T_1 \cup T_2)$-bridge H_0 cannot exist.) This condition implies that if $v_2 \notin V(T_1 \cup T_2)$, then the $(T_1 \cup T_2)$-bridge containing v_2 has at most two attachments. Note that for every non-trivial T-bridge B of G, either (i) B is a $(T_1 \cup T_2)$-bridge of G_0 and B does not contain v_2 as a non-attachment, or (ii) B contains v_2 as a non-attachment, or (iii) B is a T_{R_i}-bridge of R_i for some i with $1 \leq i \leq l$. Suppose first that B satisfies (i). Then since $T_1 \cup T_2$ is a $C_1[u_1, v_1]$-Tutte subgraph in G_0 and B does not contain v_2 as a non-attachment, $B = B \cap G_0$ has at most three attachments on $T_1 \cup T_2$. Suppose next that B satisfies (ii). Since B contains v_2 as a non-attachment, it follows from the assumption of this paragraph that B has at most three attachments one of which is x. If B satisfies (iii), then B has at most three attachments, since T_{R_i} is a C_{R_i}-Tutte path in R_i for $1 \leq i \leq l$. Thus, these imply that T is an extended C-Tutte path in G, and G satisfies (T1).

Suppose next that there exists a $(T_1 \cup T_2)$-bridge H_0 of G_0 such that H_0 contains v_2 as a non-attachment and H_0 has at least three attachments on $T_1 \cup T_2$. As mentioned in the second sentence of the previous paragraph, Case (c-2) or (d-2) must occur. See Figure 14 for Case (c-2). Hence we have $w = v_2$. Since $T_1 \cup T_2$ is a $C_1[u_1, v_1]$-Tutte subgraph in G_0, H_0 has exactly three attachments on $T_1 \cup T_2$. Moreover, one of them is on $C_2[v_2, u_2] - \{v_2\}$, say a, and another one is on $C_2[u_2, v_2] - \{v_2\}$, say c. Let d be the other attachment of H_0 on $T_1 \cup T_2$. Let H be the T-bridge of G containing H_0. In other words, H consists of H_0 and the edge $f = xw$. Let $b = x$, and let P be the subpath of C from a to b such that $E(P) \subset E(H)$. Note that H is an extended C-flap of G with attachments a, b, c, d and base path P such that a, f, x, b appear in P in this order. Furthermore, if $d = v_1$ and Case (c-2) occurs, then H is of Type II; otherwise it is of Type I. By the same argument as above, we can check that G satisfies (T2). This completes the proof of Cases (1), (c-2) and (d-2).

Case (a-2), (b-2), (a-3), (b-3), (a-4) or (b-4).

In this case, $w \in V(C_1[u_1, v_1])$, and hence $f \in E(C_1[u_1, v_1])$. Since v_1 corresponds to x in this case, it follows from Claim 1 that $G_0 - v_1$ is 2-connected. It follows from Lemma 9 with respect to $G_0, C_1, C_2, u_1, v_1, u_2, v_2, y$ and f that G_0 satisfies (T1$'$) or (T2$'$). Let T_1 and T_2 be two disjoint paths, and H be an extended $C_1[u_1, v_1]$-flap with attachments a, v_1, c, d and base path P satisfying the conditions in (T1$'$) or (T2$'$) in Lemma 9. (If G_0 satisfies (T1$'$), then ignore the definitions on H, a, and so on.) Let

$$T = \begin{cases} T_1 \cup \bigcup_{i=1}^{l} T_{R_i} \cup T_2 & \text{if Case (a-2), (a-3) or (a-4) occurs,} \\ \{xs_1\} \cup T_1 \cup \bigcup_{i=1}^{l} T_{R_i} \cup T_2 & \text{if Case (b-2), (b-3) or (b-4) occurs.} \end{cases}$$

Since $v_1 \notin V(T_1 \cup T_2)$, T is a path in G connecting x and y with $|T| \geq |T_1| + |T_2| - 1 \geq 4$.

Suppose first that G_0 satisfies (T1$'$). Notice that every non-trivial T-bridge B of G is either (i) a $(T_1 \cup T_2 \cup \{v_1\})$-bridge of G_0, or (ii) a T_{R_i}-bridge of R_i for some $1 \leq i \leq l$. Hence it follows from (T1$'$) and the choice of T_{R_i} that B has at most four attachments,

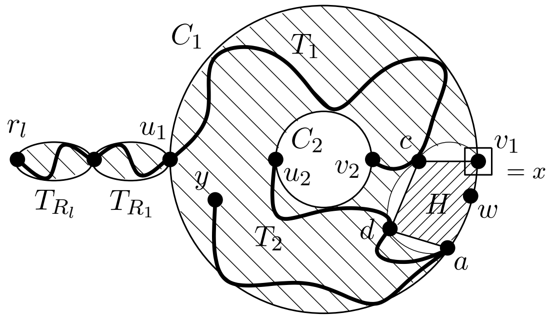

Figure 15: Case (a-2) with (T2′) for G_0.

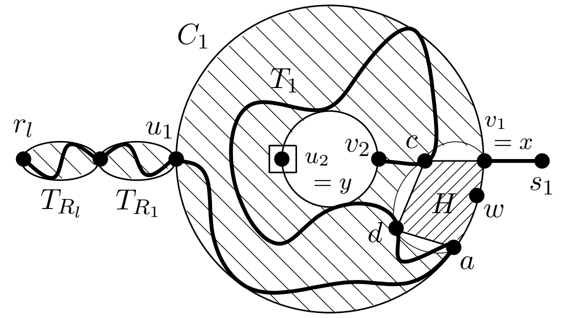

Figure 16: Case (b-4) with (T2′) for G_0.

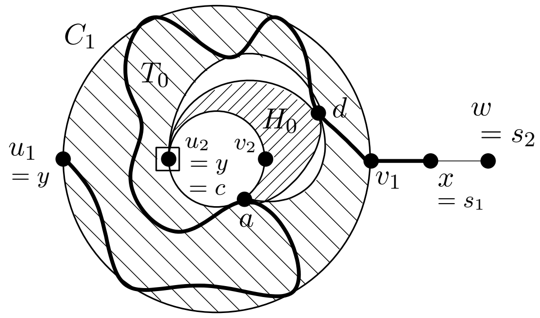

Figure 17: Case (d-3) with $(T_0 \cup \{u_2\})$-bridge H_0 with certain properties.

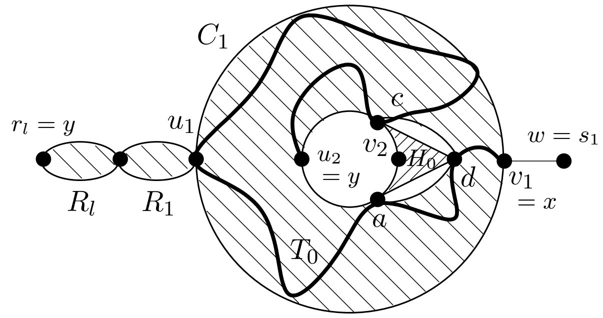

Figure 18: Case (c-4) with T_0-bridge H_0 with certain properties.

and at most three attachments if it contains an edge in $C_1[u_1, v_1] \cup C_2[v_2, u_2]$ or in C_{R_i}. Then by condition (G3), B has at most three attachments if it contains an edge in C. Hence T is an extended C-Tutte path, and G satisfies (T1).

Suppose next that G_0 satisfies (T2′). Note that H is an extended C-flap in G with attachments a, b, c, d and base path P, where $b = x$. See Figures 15 and 16 for Cases (a-2) and (b-4), respectively. Notice also that if Case (a-2), (a-3) or (a-4) occurs and $d = v_2$, then H is of Type II; otherwise it is of Type I. Moreover, a, f, x, b appear in P in this order. By the same argument as above, T is an extended C-Tutte path in $G - \big(V(H) - \{a, b, c, d\}\big)$. Hence G satisfies (T2), and this completes Cases (a-2), (b-2), (a-3), (b-3), (a-4) and (b-4).

Case (c-3) or (d-3).

In this case, note that $w = v_2$. Let $G_0' = G_0 - u_2$. Let C_2' be the unique facial walk in G_0' that is not facial in G_0. Then by applying Theorem 4 to the block of G_0' containing v_1 and u_1, G_0' has a C_2'-Tutte path T_0 from v_1 to u_1 with $|T_0| \geq 4$. (Note that the exceptional case in Theorem 4 does not occur if we take an appropriate vertex as u.) Let

$$T = \begin{cases} T_0 & \text{if Case (c-3) occurs,} \\ \{xs_0\} \cup T_0 & \text{if Case (d-3) occurs.} \end{cases}$$

Note that T is a path in G from x to y with $|T| \geq |T_0| \geq 4$.

Suppose that there exists no $(T_0 \cup \{u_2\})$-bridge H_0 of G_0 such that H_0 contains v_2 as a non-attachment and H_0 has at least three attachments on T_0. Each non-trivial T-bridge B of G satisfies that either (i) B is a $(T_0 \cup \{u_2\})$-bridge of G_0 that does not contain v_2 as a non-attachment, or (ii) B contains v_2 as a non-attachment and $B \cap G_0$ is a $(T_0 \cup \{u_2\})$-bridge of G_0 having at most two attachments on T_0. By the same argument as in Case (1), (c-2) or (d-2), we can show that T is an extended C-Tutte path in G. Hence G satisfies (T1).

Suppose next that there exists a $(T_0 \cup \{u_2\})$-bridge H_0 of G_0 such that H_0 contains v_2 as a non-attachment and H_0 has at least three attachments on $T_0 \cup \{u_2\}$. See Figure 17 for Case (d-3). Since H_0 contains v_2 as a non-attachment, $H_0 \cap G_0'$ contains an edge in C_2'. This implies that $H_0 \cap G_0'$ has at most two attachments on T_0, and hence H_0 has exactly three attachments on $T_0 \cup \{u_2\}$ one of which is u_2. Note that one of the attachments of $H_0 \cap G_0'$ on T_0 is on $C_2[v_2, u_2] - \{u_2, v_2\}$, say a. Let $b = x$, $c = u_2 = y$, and let d be the attachments of $H_0 \cap G_0'$ on T_0 with $d \neq a, c$. Let H be the T-bridge of G containing H_0, and let P be the subpath of C from a to b such that $E(P) \subset E(H)$. In other words, H consists of H_0 and f. In either case, H is an extended C-flap of G with attachments a, b, c, d and base path P such that a, f, x, b appear in P in this order. Notice also that when Case (c-3) occurs and $d = v_1$, then H is of Type II; otherwise it is of Type I. By the same argument as above, we can check that G satisfies (T2). This completes the proof of Cases (c-3) and (d-3).

Case (c-4) or (d-4).

In this case, note that $w = v_2$. Then it follows from Theorem 4 that G_0 has a C_1-Tutte path T_0 from v_1 to u_2 through u_1 with $|T_0| \geq 4$. (If the exceptional case in Theorem 4 occurs, then we can find an essential curve on the torus that hits G at only u_2, contradicting that the representativity of G is exactly 2.) Let

$$T = \begin{cases} T_0 & \text{if Case (c-4) occurs,} \\ \{xs_0\} \cup T_0 & \text{if Case (d-4) occurs.} \end{cases}$$

Note that T is a path in G from x to y with $|T| \geq |T_0| \geq 4$.

Suppose that there exists no T_0-bridge H_0 of G_0 such that H_0 contains v_2 as a non-attachment and H_0 has at least three attachments on T_0. Note that for each non-trivial T-bridge B of G, either (i) B is a T_0-bridge of G_0 that does not contain v_2 as a non-attachment, or (ii) B contains v_2 as a non-attachment and $B \cap G_0$ is a T_0-bridge of G_0 having at most two attachments on T_0, or (iii) $B = \bigcup_{i=1}^{l} R_i$ that is a $\{u_1, u_2\}$-bridge of G. By the same argument as in Case (1), (c-2) or (d-2), we can show that T is an extended C-Tutte path. Hence G satisfies (T1).

Suppose that there exists a T_0-bridge H_0 of G_0 such that H_0 contains v_2 as a non-attachment and H_0 has at least three attachments on T_0. See Figure 18 for Case (c-4). Since H_0 is a T_0-bridge of G_0 and T_0 is a C_1-Tutte subgraph in G_0, H_0 has exactly three attachments on T_0. Note that one of the attachments of H_0 on T_0 is on $C_2[v_2, u_2] - \{v_2\}$, say a, and another one is on $C_2[u_2, v_2] - \{v_2\}$, say c. Let d be the other attachment of H_0 on T_0 and let $b = x$. Let H be the T-bridge of G containing H_0, and let P be the subpath of C from a to b such that $E(P) \subset E(H)$. In other words, H consists of H_0 and f. In either case, H is an extended C-flap of G with attachments a, b, c, d and base path P such that a, f, x, b appear in P in this order. Notice also that when Case (c-4) occurs and $d = v_1$, then H is of Type II; otherwise it is of Type I. By the same argument as above, we can check that G satisfies (T2). This completes the proof of Cases (c-4) and (d-4).

Case (5).

If $y = r_0$, then let $p = 0$; Suppose otherwise, that is, suppose that $y \in \bigcup_{i=1}^{l} V(R_i) - \{r_0, r_l\}$. In this case, let p be the integer with $1 \leq p \leq l$ such that $y \in V(R_p) - \{r_{p-1}\}$. By Theorem 3, for each $1 \leq i \leq p-1$, R_i has a C_{R_i}-Tutte path T_{R_i} from r_{i-1} to r_i. If $p < l$, then let T_{R_p} be a C_{R_p}-Tutte path in R_p from r_{p-1} to y through r_p; Otherwise, let T_{R_p} be a path from r_{p-1} to y in R_p with $r_p \notin V(T_{R_p})$ such that $T_{R_p} \cup \{r_p\}$ is a $C_{R_p}[r_p, r_{p-1}]$-Tutte subgraph of R_p. Such paths exist by Theorems 3 and 5.

Case (a-5).

Let $G_0' = G_0 - v_2$. By Claim 1, G_0' is 2-connected. Let C_2' be the unique facial cycle in G_0' that is not facial in G_0. Then it follows from Theorem 4 that G_0' has a C_2'-Tutte path T_0 from v_1 to u_1 through u_2 with $|T_0| \geq 4$. (If the exceptional case in Theorem 4 occurs, then we can find an essential curve on the torus that hits G at only v_1, contradicting that

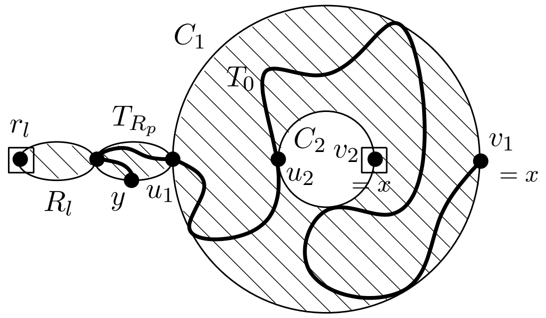

Figure 19: Case (a-5) for the case $p < l$.

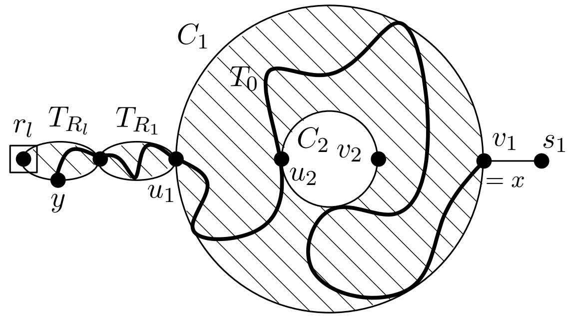

Figure 20: Case (b-5) for the case $p = l$.

the representativity of G is exactly 2.) Let

$$T = T_0 \cup \bigcup_{i=1}^{p} T_{R_i}.$$

See Figure 19 for the case $p < l$. Note that T is a path in G from x to y with $|T| \geq |T_0| \geq 4$. Notice also that for each non-trivial T-bridge B of G, either (i) B is a $\big(T_0 \cup \{v_2\}\big)$-bridge of G_0, or (ii) B is a T_{R_i}-bridge of R_i for some $1 \leq i \leq p-1$, or (iii) B is a $(T_{R_p} \cup \{r_p\})$-bridge of R_p, or (iv) $B = \bigcup_{i=p+1}^{l} R_i$ that is an $\{r_p, u_2\}$-bridge of G. (Note that (iv) occurs only when $p < l$.) In either case, by the choice of T_0 and T_{R_i}, B has at most three attachments. In fact, for (i), B clearly has at most three attachments if v_2 is not an attachment; Otherwise, $B \cap G_0'$ is a T_0-bridge of G_0' containing an edge of C_2', and hence B also has at most three attachments one of which is v_2. Similarly we can show that for other cases. Hence T is a Tutte path in G, in particular, an extended C-Tutte path in G. Therefore G satisfies (T1). This completes Case (a-5).

Case (b-5), (c-5) or (d-5).

It follows from Theorem 4 that G_0 has a C_2-Tutte path T_0 from v_1 to u_1 through u_2 with $|T_0| \geq 4$. (If the exceptional case in Theorem 4 occurs, then we can find an essential curve on the torus that hits G at only v_1, contradicting that the representativity of G is exactly 2.) Let

$$T = \begin{cases} T_0 \cup \bigcup_{i=1}^{p} T_{R_i} & \text{if Case (b-5) or (c-5) occurs,} \\ \{xs_0\} \cup T_0 \cup \bigcup_{i=1}^{p} T_{R_i} & \text{if Case (d-5) occurs.} \end{cases}$$

See Figure 20 for Case (b-5) with $p = l$. Note that T is a path in G from x to y with $|T| \geq |T_0| \geq 4$. Notice also that for each T-bridge B of G, either (i) B is a T_0-bridge of G_0 that does not contain v_2 as a non-attachment, or (ii) B contains v_2 as a non-attachment and $B \cap G_0$ is a T_0-bridge of G_0 containing an egde of C_2, or (iii) B is a T_{R_i}-bridge of R_i for some $1 \leq i \leq p-1$, or (iv) B is a $(T_{R_p} \cup \{r_p\})$-bridge of R_p, or (v) $B = \bigcup_{i=p+1}^{l} R_i$ that is an $\{r_p, u_2\}$-bridge of G. (Note that (v) occurs only when $p < l$.) Hence G satisfies (T1). This completes Cases (b-5), (c-5) and (d-5).

6.3 The case where the (x, F)-width is at least three and the x-width is exactly 2

In this case, we will show that G actually satisfies (T1), and hence we can ignore the edge f. Since the x-width is exactly 2, there exists a face F' of G such that x is contained in the boundary walk of F' and the (x, F')-width is exactly 2. Since the (x, F)-width is at least three, we have $F' \neq F$. Let C' be the boundary cycle of F'. By Lemma 10, G can be decomposed into a 2-connected plane graph G_0 and two chains of plane blocks

$r_0, R_1, r_1, R_2, \ldots, r_{l-1}, R_l, r_l$ and $s_0, S_1, \ldots, s_{m-1}, S_m, s_m$ satisfying (G1)–(G4). For each $1 \leq i \leq l$, let C_{R_i} be the unique facial walk in R_i that is not facial in G, and for each $1 \leq j \leq m$, let C_{S_j} be the unique facial walk in S_j that is not facial in G. Since the (x, F)-width is at least 3, F is also a face in G_0.

By the same argument as in Section 6.2, we see that only the four cases (a)–(d) in Section 6.2 could occur. In addition, Cases (b) and (c) are symmetric each other since we here ignore the edge f, and if Case (d) occurs, the degree of x is exactly 2 and the (x, F)-width of G would be at most two, a contradiction. Therefore, we only have the following two possibilities

(a) $m = 1$, $|S_1| = 1$, and $s_0 = s_1 = x$.
(b) $m = 1$, $|S_1| = 2$, and $s_0 = x \neq s_1$.

In either case, $x = v_1$. By symmetry for Case (a) and by the assumption that the (x, F)-width is at least 3, we may assume that F is incident with v_1 and F is not incident with a vertex in C_2. Note that $y \in V(G_0) \cup \bigcup_{i=1}^{l} V(R_i) - \{v_1\}$. So we also have the following five possibilities;

(1) $y = v_2$.
(2) $y \in V(G_0) - \{u_1, u_2, v_1, v_2\}$.
(3) $l = 1$, $|R_1| = 1$ and $y = r_0 = r_l$.
(4) $y = r_l \neq r_0$.
(5) $y \in \bigcup_{i=1}^{l} V(R_i)$ and $y \neq r_l$.

In either case, let

$$G'_0 \quad = \quad \begin{cases} G_0 - v_2 & \text{if Case (a) occurs,} \\ G_0 & \text{if Case (b) occurs.} \end{cases}$$

By Claim 1, we may assume that G'_0 is 2-connected. If Case (a) occurs, then let C'_2 be the unique facial cycle in G'_0 that is not facial in G_0; Otherwise, let $C'_2 = C_2$.

Case (1) or (2).

In either case, we use the following paths; By Theorem 3, for each $1 \leq i \leq l$, R_i has a C_{R_i}-Tutte path T_{R_i} from r_{i-1} to r_i. By Lemma 8, G'_0 has two disjoint paths T_1 and T_2 such that T_1 and T_2 connect $\{u_1, u_2\}$ and $\{v_1, y\}$, $|T_1| + |T_2| \geq 5$ and $T_1 \cup T_2$ is a $C_1[u_1, v_1]$-Tutte subgraph in G_0. Let

$$T = T_1 \cup \bigcup_{i=1}^{l} T_{R_i} \cup T_2.$$

See Figure 21 for Case (b-2). Note that T is a path in G from x to y with $|T| \geq |T_1| + |T_2| - 1 \geq 4$.

Note that for every non-trivial T-bridge B of G, either (i) B is also a $(T_1 \cup T_2)$-bridge of G_0, or (ii) B is not a $(T_1 \cup T_2)$-bridge of G_0, x is an attachment of B, and $B - x$ is a $(T_1 \cup T_2)$-bridge of G_0, or (iii) B is a T_{R_i}-bridge of R_i for some i with $1 \leq i \leq l$.

Suppose first that B satisfies (i). Then B has at most three attachments, since $T_1 \cup T_2$ is a $C_1[u_1, v_1]$-Tutte subgraph in G_0.

Suppose next that B satisfies (ii). In this case, B has at most four attachments on T one of which is x. Furthermore, suppose that B contains an edge in C. This assumption implies that there exist two attachments a_1 and a_2 of B on T such that both a_1 and a_2 are contained in C. If Case (a-2) occurs, then v_2 and one of a_1 and a_2, say a_1, appear in the same facial cycle in G'_0. However, this implies the existence of an essential curve on the torus passing through F and hitting G at a_1 and x. On the other hand, if Case (b-2) occurs, then $B - x$ has three attachments on $T_1 \cup T_2$ in G_0, and hence one of a_1 and a_2,

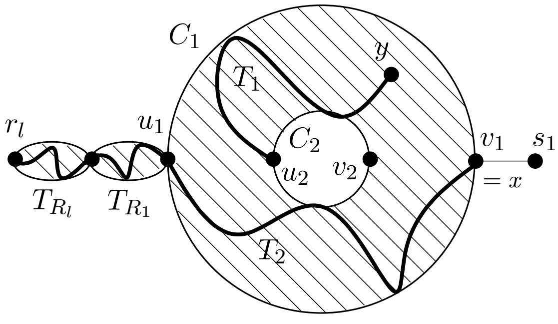

Figure 21: Case (b-2).

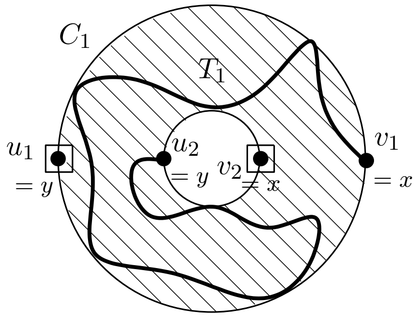

Figure 22: Case (a-3).

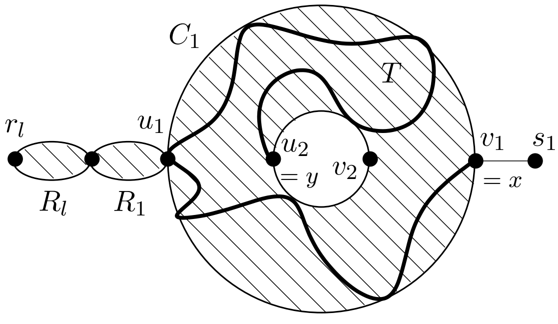

Figure 23: Case (b-4).

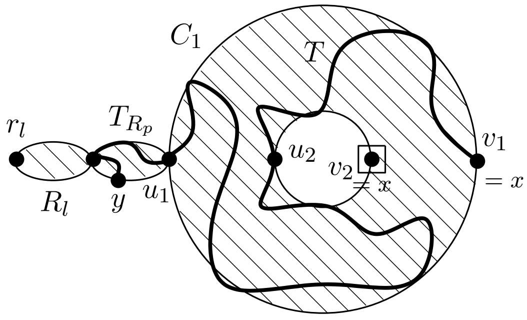

Figure 24: Case (a-5) with $p < l$.

say a_1, is also contained in C_2. This again implies the existence of an essential curve on the torus passing through F and hitting G at a_1 and x. In either case, this contradicts that (x, F)-width of G is at least 3. Hence B does not contain an edge in C.

Suppose finally that B satisfies (iii). Then B has at most three attachments since T_{R_i} is a C_{R_i}-Tutte path in R_i for $1 \le i \le l$.

Therefore, T is an extended C-Tutte path in G, and G satisfies (T1).

Case (3).

Let $G_0'' = G_0' - u_1$, and let C_1' be the unique facial walk in G_0'' that is not facial in G_0'. Then it follows from Theorem 4 that G_0'' has a C_1'-Tutte path T from v_1 to u_2 with $|T| \ge 4$. (Note that the exceptional case in Theorem 4 does not occur if we take an appropriate vertex as u.) See Figure 22 for Case (a-3). Note that T can be regarded also as a path in G from x to y.

Notice that for every non-trivial T-bridge B of G, either (i) neither v_2 nor u_1 is an attachment of B and B is also a T-bridge of G_0'', or (ii) B is not a T-bridge of G_0 and v_2 is not an attachment of B, (iii) B is not a T-bridge of G_0 and v_2 is an attachment of B.

If B satisfies (i), then B has at most three attachments on T, since T is a C_1'-Tutte path in G_0''.

Suppose next that B satisfies (ii). Since B is not a T-bridge of G_0 and v_2 is not an attachment of B, u_1 is an attachment of B and $B \cap G_0''$ is a T-bridge of G_0'' containing an edge of C_1'. Hence B has at most three attachments one of which is u_1.

Suppose finally that B satisfies (iii). In this case, it follows from the same arguments as above that $B - v_2$ has at most three attachments on T. Then by the same arguments as in (ii) in Case (1) or (2), we see that B has at most four attachments and contains no edge in C.

These imply that T is an extended C-Tutte path in G, and G satisfies (T1).

Case (4).

It follows from Theorem 4 that G_0' has a C_1-Tutte path T from v_1 to u_2 through u_1 with $|T| \ge 4$. (If the exceptional case in Theorem 4 occurs, then we can find an essential curve on the torus that hits G at only u_2, contradicting that the representativity of G is exactly 2.) See Figure 23 for Case (b-4). Note that T can be regarded also as a path in G from x to y.

Notice that for every non-trivial T-bridge B of G, either (i) B is also a T-bridge of G_0, or (ii) B is not a T-bridge of G_0, x is an attachment of B, and $B - x$ is a T-bridge of G_0, or (iii) $B = \bigcup_{i=1}^{l} R_i$ that is an $\{r_0, r_l\}$-bridge of G_0. By the same arguments as in Case (1) or (2), we see that if B satisfies (i), then B has at most three attachments, and if B satisfies (ii), then B has at most four attachments and contains no edge in C. If B satisfies (iii), then B trivially has at most two attachments. Therefore T is an extended C-Tutte path in G, and G satisfies (T1).

Case (5).

If $y = r_0$, then let $p = 0$; Suppose otherwise, that is, suppose that $y \in \bigcup_{i=1}^{l} V(R_i) - \{r_0, r_l\}$. In this case, let p be the integer with $1 \le p \le l$ such that $y \in V(R_p) - \{r_{p-1}\}$. By Theorem 3, for each $1 \le i \le p-1$, R_i has a C_{R_i}-Tutte path T_{R_i} from r_{i-1} to r_i. If $p < l$, then let T_{R_p} be a C_{R_p}-Tutte path from r_{p-1} to y through r_p in R_p; Otherwise, let T_{R_p} be a path from r_{p-1} to y in R_p with $r_p \notin V(T_{R_p})$ such that $T_{R_p} \cup \{r_p\}$ is a $C_{R_p}[r_p, r_{p-1}]$-Tutte subgraph of R_p. Such paths exist by Theorems 3 and 5.

It follows from Theorem 4 that G_0' has a C_2'-Tutte path T_0 from v_1 to u_1 through u_2 with $|T_0| \ge 4$. (If the exceptional case in Theorem 4 occurs, then we can find an essential curve on the torus that hits G at only v_1, contradicting that the representativity of G is exactly 2.) Let

$$T = T_0 \cup \bigcup_{i=1}^{p} T_{R_i}.$$

See Figure 24 for Case (a-5) with $p < l$. Note that T can be regarded also as a path in G from x to y.

Notice that for every non-trivial T-bridge B of G, either (i) B is a T_0-bridge of G_0, or (ii) B is not a T_0-bridge of G_0, x is an attachment of B, and $B - x$ is a T_0-bridge of G_0, or (iii) B is a T_{R_i}-bridge of R_i for some $1 \le i \le p-1$, or (iv) B is a $(T_{R_p} \cup \{r_p\})$-bridge of R_p, or (v) $B = \bigcup_{i=p+1}^{l} R_i$ that is an $\{r_p, u_2\}$-bridge of G. (Note that (v) occurs only when $p < l$.) By the same arguments as in Case (1) or (2) or in Case (a-5) in Section 6.2, we see that T is an extended C-Tutte path in G, and G satisfies (T1).

6.4 The case where the x-width of G is at least three

Now we are ready to proceed to the main part of the proof of Theorem 2. Let $G' = G - x$. Note that G' is also 2-connected by Claim 1 and the representativity of G' is at least two by the assumption of this case. Let F' be the face of G' containing F, and let C' be the boundary cycle of F'. Note that for every 2-separation (G_1', G_2') of G', $E(C') \cap E(G_1') \neq \emptyset$ and $E(C') \cap E(G_2') \neq \emptyset$.

If $xy \in E(C)$, then let $w = y$; Otherwise, that is, if $xy \notin E(C)$, then let w be the vertex in C such that $f = xw$. In either case, let x' be the neighbor of x in C with $x' \neq w$, and let f' be the edge of C' such that f' is incident with x' and $f' \notin E(C)$. Let $f' = x'w'$. We may assume that x', x, w appear in C in this order. This order implies that x', f', w', w appear in C' in this order and $E(C'[w, x']) = E(C) - \{x'x, xw\}$. See Figure 25.[7] By the induction hypothesis with respect to G', C', x', y and f', G' satisfies (T1) or (T2). Let T' be an extended C'-Tutte path, and H' be an extended C'-flap with attachments a', b', c', d' and base path P' satisfying the conditions in (T1) or (T2) in Theorem 2. (If G' satisfies

[7] As figures in Section 6.1, in Figure 25, the bottom curve represents a part of C, and the bold line represents the path $C'[x', w]$ that is edge-disjoint from C. This situation is same for all figures in this subsection.

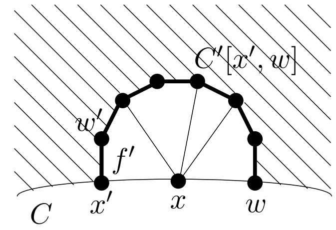

Figure 25: The vertices w and x and the edge f'.

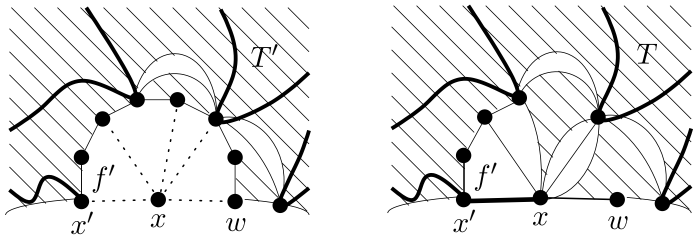

Figure 26: Case I without T-bridge with certain conditions.

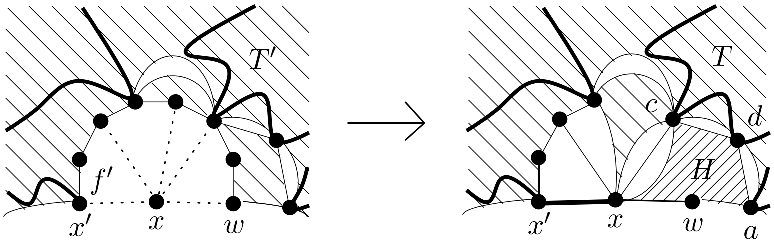

Figure 27: Case I with T-bridge H containing w and having at least four attachments.

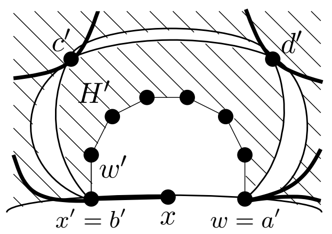

Figure 28: The T-bridge H' in G in Case II-1.

(T1), then ignore the definitions on H', a' and so on.) We divide this case into three subcases.

Case I. G' satisfies (T1).

Let $T = T' \cup \{x'x\}$.

Suppose that either $w \in V(T)$, or $w \notin V(T)$ and the T-bridge of G containing w has at most three attachments on T. See Figure 26. (Note that if $xy \in E(C)$, then this always occurs since $w = y \in V(T)$.) Let B be a non-trivial T-bridge of G. Note that B is a $\big(T' \cup \{x\}\big)$-bridge of G. If x is not an attachment of B, then B is also a T'-bridge of G'. Since T' is an extended C'-Tutte subgraph in G', B has at most four attachments, and at most three attachments if B contains an edge of C'. On the other hand, suppose that x is an attachment of B. Note that $B - x$ is a T'-bridge of G' and contains an edge in $E(C'[x', w])$, so $B - x$ has at most three attachments on T'. Hence B has at most four attachments on $T' \cup \{x\}$ one of which is x. Assume further that B contains an edge in C. If B does not contain w as a non-attachment, then B contains an edge in $C[w, x']$, but this contradicts that G' is 2-connected or the (x, F)-width is at least three. Therefore, B contains w as a non-attachment. Then it follows from the assumption of this paragraph that B has at most three attachments on T. These imply that T is an extended C-Tutte path in G. Therefore, G satisfies (T1).

Suppose next that $w \notin V(T)$ and the T-bridge H of G containing w has at least four attachments on T. See Figure 27. Note that x is an attachment of H since H contains w as a non-attachment and $x \in V(T)$. Note that $H - x$ is a T'-bridge of G' containing an edge of C', and hence $H - x$ has at most three attachments on T'. Notice also that one of the attachments of $H - x$ is on $C[w, x'] - \{w\}$, say a, and another one is on $C'[x', w] - \{w\}$, say c. Let d be the other attachment of $H - x$ on T', let $b = x$, and let P be the subpath of C between a and b with $E(P) \subset E(H)$. Hence $P = \{bw\} \cup C[w, a]$. This implies that H is an extended C-flap (of Type I) with attachments a, b, c, d and base path P. By the same argument as above, T is an extended C-Tutte path in $G - \big(V(H) - \{a, b, c, d\}\big)$. Hence G satisfies (T2), and completes the proof of Case I.

Case II. G' satisfies (T2), and a' is placed in $C[w, x'] - \{x'\}$.

Since a', f', x', b' appear in P' in this order, b' is contained in $C[a', x'] - \{a'\}$. Note that if $xy \in E(C)$, then $w = y \in V(T')$, and hence we have always $a' = w$. Let H be the subgraph of G induced by the edges in H' and all edges incident with x in G. Let $a = a'$,

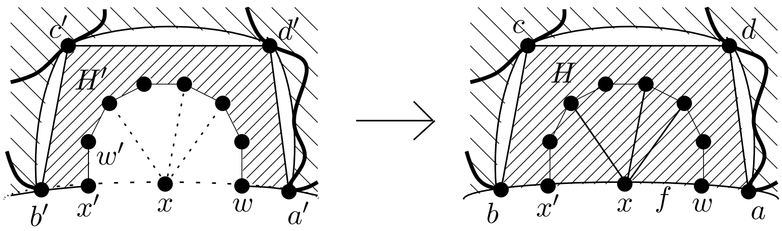

Figure 29: The extended C'-flap H' in G' and the extended C-flap H in G in Case II-2 with $f = xw$.

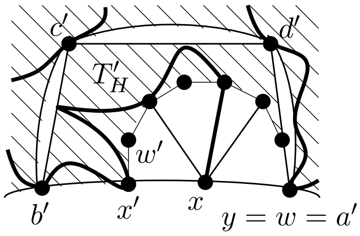

Figure 30: Case II-2 with $xy \in E(C)$.

$b = b'$, $c = c'$, $d = d'$ and P be the subpath of C from a to b with $E(P) \subset E(H)$, that is, $P = (P' - C'[x', w]) \cup \{x'x, xw\}$. Note that H satisfies the conditions of an extended C-flap of G with attachments a, b, c, d and base path P (Case II-2), unless $H - \{a, b, c, d\}$ is not connected (Case II-1). Indeed, in the exceptional case, H is not an $\{a, b, c, d\}$-bridge of G. Conversely, if $H - \{a, b, c, d\}$ is connected, then H is an extended C-flap satisfying the conditions in (T2). It is easy to see that the exceptional case happens only when $b' = x'$, $a' = w$ and x' and w are the only neighbors of x in G. Thus, we further divide this case into two subcases, to distinguish the two situations above.

Case II-1. $b' = x'$, $a' = w$ and x' and w are the only neighbors of x in G.

In this case, H is not an $\{a, b, c, d\}$-bridge of G since $H - \{a, b, c, d\}$ is not connected. See Figure 28. Then let $T = T' \cup \{x'x\}$. By the conditions, H' is a T-bridge of G having exactly four attachments. Since the x-width is at least 3, H' contains no edges in C. Moreover, every non-trivial T-bridge of G that is not H' is also a T'-bridge of G'. So, T is an extended C-Tutte path in G, and hence G satisfies (T1).

Case II-2. $b' \neq x'$ or $a' \neq w$ or x has a neighbor in G other than x' and w.

As mentioned above, H is an extended C-flap with attachments a, b, c, d and base path P. If $f = xw$ (that is, if $xy \notin E(C)$), then a, f, x, b appear in P in this order. Let $T = T'$, and we can check that G satisfies (T2) in Theorem 2. See Figure 29.

Therefore, we may assume that $xy \in E(C)$. Since $y \in V(T)$, we have $y = w = a'$. Let $\widehat{H'} = H'$, $\hat{b} = b'$ and $\hat{d} = d'$ if H' is of Type I; otherwise let $\widehat{H'}$ be the plane graph and $\hat{b}$ and $\hat{d}$ be the vertices as in (H3) in the definition of an extended C-flap of Type II. Note that $\widehat{H'}$ is a plane graph. Let $\widehat{C'_H}$ be the outer facial walk in $\widehat{H'}$ containing $a', x, x', \hat{b}, c'$ and $\hat{d}$ in this clockwise order. It follows from Theorem 6 with $A' = \{a', c', \hat{d}\}$ that $\widehat{H'}$ has a path T'_H from x to $\hat{b}$ with $V(T'_H) \cap A' = \emptyset$ such that every $(T'_H \cup A')$-bridge B of $\widehat{H'}$ has at most four attachments and at most two attachments if B contains an edge of $\widehat{C'_H}[x, \hat{b}]$. See Figure 30. Let $T = T'_H \cup T'$. Note that every non-trivial T-bridge B of G is either (i) a T'-bridge of $G' - \big(V(H') - \{a', b', c', d'\}\big)$, or (ii) a $(T'_H \cup A')$-bridge of $\widehat{H'}$. Therefore, by the conditions for T' and T'_H, we can check that T is an extended C-Tutte path in G. Hence G satisfies (T1).

Case III. G' satisfies (T2), and a' is placed in $C[x', w] - \{w\}$.

In this case, depending on the place of the vertex b', we further divide the proof into two subcases; b' is placed either (1) in $C'[w, x']$ or (2) in $C'[a', w] - \{a', w\}$. (If $xy \in E(C)$, then (1) must occur.) See the left side of Figures 31 and 32.

Let $\widehat{H'} = H'$, $\hat{b} = b'$ and $\hat{d} = d'$ if H' is of Type I; Otherwise let $\widehat{H'}$ be the plane graph and $\hat{b}$ and $\hat{d}$ be the vertices as in (H3) of the definition of an extended C-flap of Type II. Let $\widetilde{H}$ be the graph obtained from $\widehat{H'} \cup \{x\}$ by adding all edges of G connecting x and a vertex in $C'[x', a']$. Note that when (2) b' is placed in $C'[a', w] - \{a', w\}$, then we do not add any edges connecting x and a vertex in $C'[b', w]$. Notice also that $\widetilde{H}$ is a plane graph. Let $\widetilde{C_H}$ be the facial walk in $\widetilde{H}$ containing $x, x', \hat{b}, c', \hat{d}$ and a' in this order. By Theorem

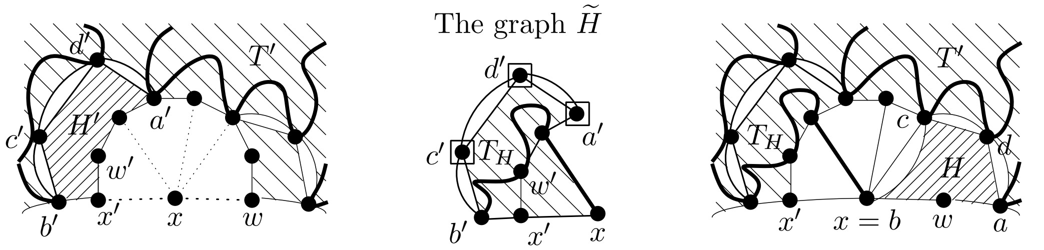

Figure 31: Case III-(1).

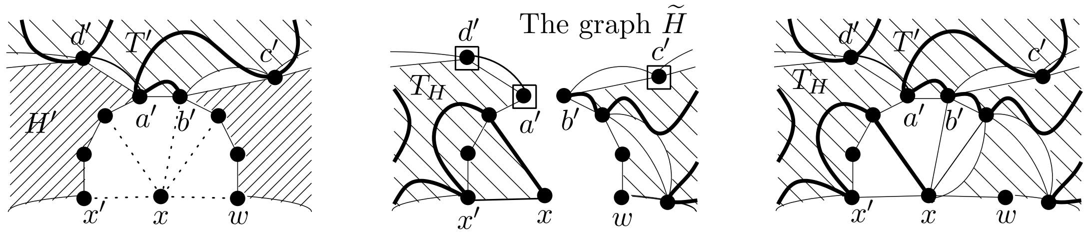

Figure 32: Case III-(2).

6 with $A' = \{a', c', \hat{d}\}$, there exists a path T_H in $\widetilde{H}$ from x to $\hat{b}$ with $V(T_H) \cap A' = \emptyset$ such that every $(T_H \cup A')$-bridge B has at most four attachments and at most two attachments if B contains an edge of $\widetilde{C_H}[x, \hat{b}]$. See the middle of Figures 31 and 32. Note that we can regard T_H as a path also in G, and let $T = T' \cup T_H$.

Suppose first that either $w \in V(T)$, or $w \notin V(T)$ and the T-bridge of G containing w has at most three attachments on T. See the right side of Figure 32. By the same argument as in Case I, T is an extended C-Tutte path in G, and hence G satisfies (T1).

Suppose next that $w \notin V(T)$ and the T-bridge of G containing w has at least four attachments on T. Let H be the T-bridge of G containing w. Note that x is an attachment of H since H contains w as a non-attachment and $x \in V(T)$. If (2) b' is placed in $C'[a', w] - \{a', w\}$, then $H - x$ is a $(T_H \cup A')$-bridge of $\widetilde{H}$ containing an edge of $\widetilde{C_H}[x, \hat{b}]$. This implies that $H - x$ has at most two attachments, and hence H has at most three attachments one of which is x, a contradiction. Thus, we have that (1) b' is placed in $C'[w, x']$. Then H is a $(T' \cup \{x\})$-bridge of $G - (V(H') - \{a', b', c', d'\})$. This implies that $H - x$ is a T'-bridge of $G' - (V(H') - \{a', b', c', d'\})$ containing an edge of C', and hence $H - x$ has exactly three attachments in $G' - (V(H') - \{a', b', c', d'\})$. Note that one of the attachments of $H - x$ is on $C[w, x'] - \{w\}$, say a, and another one is on $C'[a', w] - \{w\}$, say c. Let d be the other attachment of $H - x$ on T', let $b = x$, and let P be the subpath of C from a to b such that $E(P) \subset E(H)$. Hence $P = \{bw\} \cup C[w, a]$. This implies that H is an extended C-flap (of Type I) with attachments a, b, c, d and base path P. Moreover, a, f, x, b appear in P in this order. See the right side of Figure 31.

Now we will show that T is an extended C-Tutte path in $G - (V(H) - \{a, b, c, d\})$ satisfying the desired conditions. Let B be a non-trivial T-bridge of $G - (V(H) - \{a, b, c, d\})$. Note that B is either (i) a $(T' \cup \{x\})$-bridge of $G - (V(H') - \{a', b', c', d'\})$, or (ii) a $(T_H \cup A')$-bridge of $\widetilde{H}$. If B satisfies (i), then by the same argument as in Case I, we can show that B has at most four attachments and at most three attachments if B contains an edge of C. Next suppose that B satisfies (ii). Recall that B has at most four attachments on $T_H \cup A'$ and exactly two attachments if B contains an edge of $\widetilde{C_H}[x, \hat{b}]$. Since the x-width of G is at least three, we have that $E(C) \cap E(\widetilde{H}) = E(C[b', x]) = E(\widetilde{C_H}[x, \hat{b}])$. So, if B contains an edge of C, then B contains an edges of $\widetilde{C_H}[x, \hat{b}]$, and hence B has exactly two attachments. These imply that T is an extended C-Tutte path in $G - (V(H) - \{a, b, c, d\})$.

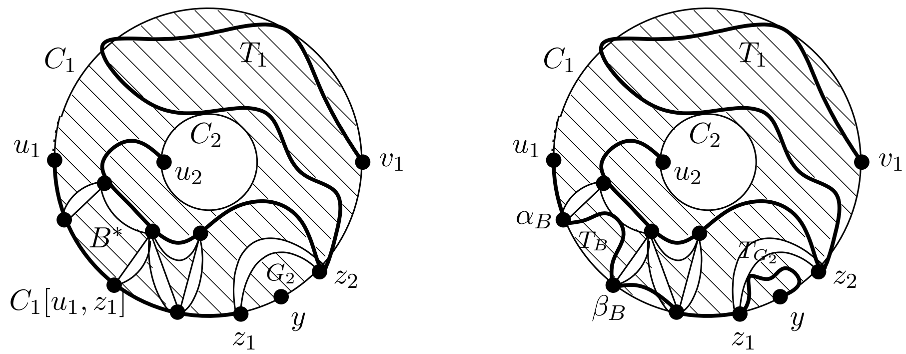

Figure 33: The paths T_1 and $C_1[u_1, z_1]$ (the left side) and the Tutte subgraph $T_1 \cup T_2$ (the right side) in Case I in the proof of Lemma 8.

Hence G satisfies (T2) and this completes the proof of Theorem 2. $\square$

7 Proofs of Lemmas

7.1 Proof of Lemma 8

Let G be a 2-connected plane graph, let C_1, C_2 be facial cycles in G, let $u_1, v_1 \in V(C_1) - V(C_2)$ with $u_1 \neq v_1$, let $u_2 \in V(C_2) - V(C_1)$, and let $y \in V(G) - \{u_2, v_1\}$. We divide the proof into two cases.

Case I. There exists no path in $G - V(C_1[u_1, v_1])$ from u_2 to y.

In this case, either there exists a 2-separation (G_1, G_2) of G with $V(G_1) \cap V(G_2) \subseteq V(C_1[u_1, v_1])$ such that $G_1 - V(G_2)$ contains u_2 and $G_2 - V(G_1)$ contains y, or $y \in V(C_1[u_1, v_1])$. Suppose first that the former occurs. Then there exists a 2-separation (G_1, G_2) of G, say $\{z_1, z_2\} = V(G_1) \cap V(G_2)$, such that $z_1, z_2 \in V(C_1[u_1, v_1])$, z_1 is closer to u_1 on $C_1[u_1, v_1]$ than z_2, $u_2 \in V(G_1) - V(G_2)$ and $y \in V(G_2) - \{z_2\}$. In this case, we take such a 2-separation (G_1, G_2) so that $|G_2|$ is as large as possible. On the other hand, suppose otherwise. In this case, let $G_1 = G$, $z_1 = y$, z_2 be the neighbor of y in $C_1[y, v_1]$, and G_2 be the graph consisting of only yz_2.

Let $G' = G - V(C_1[u_1, z_1])$ and let C'_1 be the unique facial walk in G' that is not facial in G. Since G is a 2-connected plane graph and $u_2 \neq v_1$, it follows from the maximality of $|G_2|$ that u_2, v_1 and z_2 are contained in the same block of G'. Then by Theorem 4, G' has a C'_1-Tutte path T_1 from u_2 to v_1 through z_2 with $|T_1| \geq 4$. (To be exact, if $z_2 = v_1$, then such a path T_1 exists by taking an appropriate vertex as u in Theorem 4. If $z_2 \neq v_1$ and the exceptional case in Theorem 4 occurs, then v_1 is contained in both C_1 and C_2, a contradiction.)

By Theorem 5 or trivially, G_2 has a $C_1[z_1, z_2]$-Tutte subgraph consisting of z_2 and a path T_{G_2} from z_1 to y with $z_2 \notin V(T_{G_2})$.

Let $\mathcal{B}$ be the set of $\big(T_1 \cup C_1[u_1, z_1]\big)$-bridges B of G such that B has at least two attachments on $C_1[u_1, z_1]$. For each $B \in \mathcal{B}$, let α_B and β_B be the attachments of B on $C_1[u_1, z_1]$ such that α_B is as close to u_1 on $C_1[u_1, z_1]$ as possible and β_B is as close to z_1 on $C_1[u_1, z_1]$ as possible. Since B has at least two attachments on $C_1[u_1, z_1]$, we have $\alpha_B \neq \beta_B$ for any $B \in \mathcal{B}$. For $B, B' \in \mathcal{B}$, we write $B' \preceq B$ if either (i) $B = B'$ or (ii) B' is contained in the disk bounded by $P \cup C_1[\alpha_B, \beta_B]$, where P is a path in B connecting α_B and β_B. Since G is a plane graph, $\preceq$ is a partial order on $\mathcal{B}$. Let $\widetilde{\mathcal{B}}$ be the set of maximal elements of $\mathcal{B}$ with respect to the partial order $\preceq$. By the planarity, for any $B, B' \in \widetilde{\mathcal{B}}$, $C_1[\alpha_B, \beta_B]$ and $C_1[\alpha_{B'}, \beta_{B'}]$ are disjoint.

Let $B \in \widetilde{\mathcal{B}}$, and let $A_B = V(B) \cap V(T_2)$. Since B has at least two attachments on

$C_1[u_1, z_1]$, $B \cap G'$ is a T_1-bridge of G' containing an edge of C_1', and hence $|A_B| \le 2$. Let B^* be the subgraph of G induced by the union of all elements $B' \in \mathcal{B}$ such that $B' \preceq B$, together with $C_1[\alpha_B, \beta_B]$. Note that $C_1[\alpha_B, \beta_B]$ is a subpath of the unique facial walk in B^* that is not facial in G, and $V\big(C_1[\alpha_B, \beta_B]\big) \cap A_B = \emptyset$. Then by Theorem 6, B^* has a path T_B from α_B to β_B such that $V(T_B) \cap A_B = \emptyset$ and $T_B \cup A_B$ is a $C_1[\alpha_B, \beta_B]$-Tutte subgraph. Let

$$T_2 = \Big(C_1[u_1, z_1] - \bigcup_{B \in \widetilde{\mathcal{B}}} C_1[\alpha_B, \beta_B]\Big) \cup \bigcup_{B \in \widetilde{\mathcal{B}}} T_B \cup T_{G_2}.$$

Note that T_2 is a path in G from u_1 to y such that T_2 is disjoint from T_1. Note that $|T_1| + |T_2| \ge 5$. See Figure 33. We will show that $T_1 \cup T_2$ is a $C_1[u_1, v_1]$-Tutte subgraph in G.

Let D be a non-trivial $\big(T_1 \cup T_2\big)$-bridge of G. By the choice of $\widetilde{\mathcal{B}}$, D is either (i) a $\big(T_1 \cup C_1[u_1, z_1]\big)$-bridge of G having at most one attachment on $C_1[u_1, z_1]$, or (ii) a $\big(T_{G_2} \cup \{z_2\}\big)$-bridge of G_2, or (iii) a $\big(T_B \cup A_B\big)$-bridge of B^* for some $B \in \widetilde{\mathcal{B}}$.

Suppose first that D satisfies (i). If D has no attachment on $C_1[u_1, z_1]$, then D is also a T_1-bridge of G', and hence D has at most three attachments and at most two attachments if D contains an edge in C_1'. Note that $C_1[z_2, v_1]$ is a subpath of C_1' and $G' \cap C_1[u_1, v_1] = C_1[z_2, v_1]$, and hence if D contains an edge in $C_1[u_1, v_1]$, then D has at most two attachments. So we may assume that D has an attachment on $C_1[u_1, z_1]$. This implies that $D \cap G'$ is a T_1-bridge of G' containing an edge of C_1', and hence $D \cap G'$ has at most two attachments on T_1. Hence D has at most three attachments one of which is on $C_1[u_1, z_1]$. Furthermore, since $v_1, z_2 \in V(T_1)$, D cannot contain an edge in $C_1[z_2, v_1]$.

Suppose that D satisfies (ii). Recall that $T_{G_2} \cup \{z_2\}$ is a $C_1[z_1, z_2]$-Tutte subgraph in G_2. Thus, D has at most three attachments on $T_{G_2} \cup \{z_2\}$ and at most two attachments if D contains an edge in $C_1[z_1, z_2]$. Note that $C_1[u_1, v_1] \cap G_2 = C_1[z_1, z_2]$.

Suppose that D satisfies (iii). Recall that $T_B \cup A_B$ is a $C_1[\alpha_B, \beta_B]$-Tutte subgraph of B^*. Thus, D has at most three attachments on $T_B \cup A_B$ and at most two attachments if D contains an edge in $C_1[\alpha_B, \beta_B]$. Note that $C_1[u_1, v_1] \cap B^* = C_1[\alpha_B, \beta_B]$.

Therefore, $T_1 \cup T_2$ is a $C_1[u_1, v_1]$-Tutte subgraph in G, and this completes the proof of Case I.

Case II. There exists a path in $G - V(C_1[u_1, v_1])$ from u_2 to y.

Let $G' = G - V(C_1[u_1, v_1])$, and let C_1' be the unique facial walk in G' that is not facial in G. Note that $C_1[v_1, u_1] - \{u_1, v_1\}$ is contained in C_1'. Let w be a neighbor of u_1 in C_1 with $w \in V\big(C_1[v_1, u_1]\big)$, except for the case when $C_1[v_1, u_1]$ consists of only an edge v_1u_1. In the exceptional case, such a neighbor w does not exist, and then let w be any neighbor of u_1 in C_1'. Let e be an edge of C_1' such that G' has a path from u_2 to y through e, and we take such an edge e so that it is as close to w on C_1' as possible. Moreover, we can take such an edge e so that $e \ne u_2y$. It follows from Theorem 3 that G' has a C_1'-Tutte path T_1 from u_2 to y through e. It is easy to see that $|T_1| \ge 3$.

Now we deal with $\big(T_1 \cup C_1[u_1, v_1]\big)$-bridges of G as in Case I, but this case is slightly more complicated because of the existence of a *special* bridge, defined below. Let $\mathcal{B}$ be the set of $\big(T_1 \cup C_1[u_1, v_1]\big)$-bridges B of G such that B has at least two attachments on $C_1[u_1, v_1]$. An element H in $\mathcal{B}$ is said to be *special* if both u_1 and v_1 are attachments of H and $C_1[u_1, v_1] \cup P$ bounds a region containing C_2, where P is a path in H from u_1 to v_1; Otherwise it is *non-special*.

For each $H \in \mathcal{B}$ that is special, let α_H and β_H be the attachments of H on $C_1[u_1, v_1]$ such that all attachments of H on $C_1[u_1, v_1]$ appear in $C_1[u_1, \alpha_H] \cup C_1[\beta_H, v_1]$ and $C_1[\alpha_H, \beta_H] \cup P$ bounds a region containing C_2, where P is a path in H from α_H to β_H. Possibly $\alpha_H = u_1$ and/or $\beta_H = v_1$. For each $B \in \mathcal{B}$ that is not special, let α_B and β_B be the attachments of B on $C_1[u_1, v_1]$ such that α_B is as close to u_1 on $C_1[u_1, v_1]$ as possible and β_B is as close to v_1 on $C_1[u_1, v_1]$ as possible. Note that $\alpha_B \ne \beta_B$ for any $B \in \mathcal{B}$.

For $B, B' \in \mathcal{B}$, we write $B' \preceq B$ if either (i) $B = B'$, or (ii) B is special and B' is contained in a disk bounded by $P \cup C_1[\beta_B, \alpha_B]$, or (iii) B is non-special and B' is contained

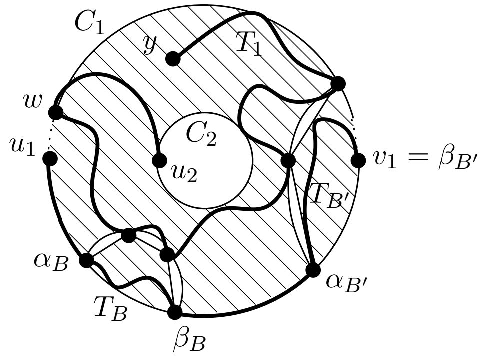

Figure 34: The Tutte subgraph $T_1 \cup T_2$ in Case II-1 in the proof of Lemma 8.

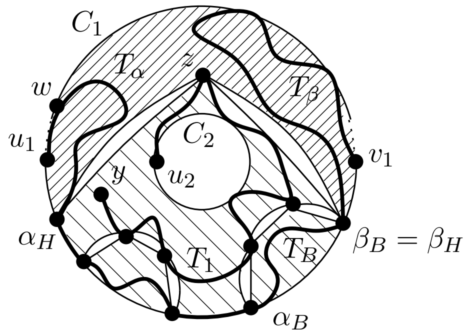

Figure 35: The Tutte subgraph $T_1 \cup T_2$ in Case II-2 in the proof of Lemma 8.

in a disk bounded by $P \cup C_1[\alpha_B, \beta_B]$, where P is a path in B connecting α_B and β_B. Since G is a plane graph, $\preceq$ is a partial order on $\mathcal{B}$. Let $\widetilde{\mathcal{B}}$ be the set of maximal elements B of $\mathcal{B}$ with respect to the partial order $\preceq$. Note that at most one element in $\widetilde{\mathcal{B}}$ is special. For any $B, B' \in \widetilde{\mathcal{B}}$ such that both B and B' are non-special, $C_1[\alpha_B, \beta_B]$ and $C_1[\alpha_{B'}, \beta_{B'}]$ are disjoint.

Let B be a non-special element in $\widetilde{\mathcal{B}}$, and let $A_B = V(B) \cap V(T_1)$. Since B has at least two attachments on $C_1[u_1, v_1]$, $B \cap G'$ is a T_1-bridge of G' containing an edge of C_1', and hence $|A_B| \leq 2$. Let B^* be the subgraph of G induced by the union of all elements $B' \in \mathcal{B}$ such that $B' \preceq B$, together with $C_1[\alpha_B, \beta_B]$. Note that $C_1[\alpha_B, \beta_B]$ is a subpath of the unique facial walk in B^* that is not facial in G. By Theorem 6, B^* has a path T_B from α_B to β_B such that $V(T_B) \cap A_B = \emptyset$ and $T_B \cup A_B$ is a $C_1[\alpha_B, \beta_B]$-Tutte subgraph.

We divide the rest of the proof into two cases, depending on whether G has a special $\big(T_1 \cup C_1[u_1, v_1]\big)$-bridge.

Case II-1. There exists no special $\big(T_1 \cup C_1[u_1, v_1]\big)$-bridge of G.

Let

$$T_2 = \Big(C_1[u_1, v_1] - \bigcup_{B \in \widetilde{\mathcal{B}}} C_1[\alpha_B, \beta_B]\Big) \cup \bigcup_{B \in \widetilde{\mathcal{B}}} T_B.$$

Note that T_2 is a path of G from u_1 to v_1 that is disjoint from T_1. See Figure 34. We will show that $T_1 \cup T_2$ is a $C_1[u_1, v_1]$-Tutte subgraph in G.

Let D be a non-trivial $\big(T_1 \cup T_2\big)$-bridge of G. By the choice of T_1 and the assumption of Case II-1, D is either (i) a non-special $\big(T_1 \cup C_1[u_1, v_1]\big)$-bridge of G having at most one attachment on $C_1[u_1, v_1]$ or (ii) a $\big(T_B \cup A_B\big)$-bridge of B^* for some $B \in \widetilde{\mathcal{B}}$. Suppose first that D satisfies (i). If D has no attachment on $C_1[u_1, v_1]$, then D is also a T_1-bridge of G', and hence D has at most three attachments on T_1 and no attachment on T_2. So we may assume that D has an attachment on $C_1[u_1, v_1]$. This implies that $D \cap G'$ is a T_1-bridge of G' containing an edge of C_1', and hence $D \cap G'$ has at most two attachments on T_1. Since D has only one attachment on $C_1[u_1, v_1]$, D has no edge in $C_1[u_1, v_1]$. If D satisfies (ii), then D has at most three attachments since $T_B \cup A_B$ is a $C_1[\alpha_B, \beta_B]$-Tutte subgraph of B^*. In either case, D has at most three attachments, and hence $T_1 \cup T_2$ is a $C_1[u_1, v_1]$-Tutte subgraph in G. This completes Case II-1.

Case II-2. There exists a special $\big(T_1 \cup C_1[u_1, v_1]\big)$-bridge of G.

In this case, there exists the unique element, say H, in $\widetilde{\mathcal{B}}$ such that H is special. See Figure 35.[8] Let H^* be the subgraph of G induced by the union of all elements $B' \in \mathcal{B}$ such that $B' \preceq H$, together with $C_1[u_1, \alpha_H] \cup C_1[\beta_H, v_1]$. Note that $w \in V(H^*) \cap V(C_1')$, and $e \notin E(H^*)$ by the choice of e. Let A_H be the set of attachments of H^* on T_1. If $|A_H| \geq 2$,

[8] In this figure, the special $\big(T_1 \cup C_1[u_1, v_1]\big)$-bridge H is also represented by rising diagonals stroke from bottom left to top right.

then $H^* \cap C_1'$ has an edge e' such that G' has a path from u_2 to y through e'. (Consider the path obtained by the concatenation of the subpath of T_1 from u_2 to a vertex in A_H, the path in H^* connecting two vertices in A_H through e', and the subpath of T_1 from the other vertex in A_H to y.) Since $w \in V(H) \cap V(C_1')$, e' is closer to w than e, contradicting the choice of e. Therefore, we have $|A_H| \leq 1$.

If $A_H \neq \emptyset$ then let $\{z\} = A_H$; otherwise let z be a new vertex, so $z \notin V(H^*)$. Let H^{**} be the graph obtained from $H^* \cup \{z\}$ by adding an edge f_1 connecting α_H and z and an edge f_2 connecting z and β_H so that $C_1[u_1, \alpha_H] \cup \{f_1, f_2\} \cup C_1[\beta_H, v_1] \cup C_1[v_1, u_1]$ forms the outer facial walk in H^{**}. By Theorem 7, H^{**} has a $\big(C_1[u_1, \alpha_H] \cup \{f_1, f_2\} \cup C_1[\beta_H, v_1]\big)$-Tutte path T_H from u_1 to v_1 with $f_1, f_2 \in E(T_H)$. Deleting the vertex z together with the two edges f_1 and f_2 from T_H, we obtain the two disjoint paths T_α and T_β such that T_α connects u_1 and α_H, and T_β connects β_H and v_1.

Let $\widetilde{\mathcal{B}}' = \widetilde{\mathcal{B}} - \{H\}$. Note that for any $B \in \widetilde{\mathcal{B}}'$, we have $C_1[\alpha_B, \beta_B] \subset C_1[\alpha_H, \beta_H]$. Let

$$T_2 = T_\alpha \cup \Big(C_1[\alpha_H, \beta_H] - \bigcup_{B \in \widetilde{\mathcal{B}}'} C_1[\alpha_B, \beta_B]\Big) \cup \bigcup_{B \in \widetilde{\mathcal{B}}'} T_B \cup T_\beta.$$

Note that T_2 is a path of G from u_1 to v_1 that is disjoint from T_1.

Let D be a $\big(T_1 \cup T_2\big)$-bridge of G. By the choice of T_2, D is either (i) a non-special $\big(T_1 \cup C_1[u_1, v_1]\big)$-bridge of G having at most one attachment on $C_1[u_1, v_1]$, or (ii) a $\big(T_B \cup A_B\big)$-bridge of B^* for some $B \in \widetilde{B}$, or (iii) a $\big(T_H \cup A_H\big)$-bridge of H^{**}.

By the same argument as in Case II-1 or by the fact that T_H is a $\big(C_1[u_1, \alpha_H] \cup \{f_1, f_2\} \cup C_1[\beta_H, v_1]\big)$-Tutte path in H^{**}, in either case, D has at most three attachments, and hence $T_1 \cup T_2$ is a $C_1[u_1, v_1]$-Tutte subgraph in G. This completes the proof of Lemma 8. $\square$

7.2 Proof of Lemma 9

Let G be a 2-connected plane graph, let C_1, C_2 be facial cycles in G, let $u_1, v_1 \in V(C_1) - V(C_2)$ with $u_1 \neq v_1$, let $u_2, v_2 \in V(C_2) - V(C_1)$ with $u_2 \neq v_2$, let $y \in V(G) - \{u_1, v_1, v_2\}$, and let f be an edge of $C_1[u_1, v_1]$ incident with v_1. Suppose that $G - v_1$ is also 2-connected.

Let $G' = G - v_1$, and let C_1' be the unique facial walk in G' that is not facial in G. Then it follows from applying Lemma 8 with changing the roles of the index i for C_i, u_i and v_i that G' has two disjoint paths T_1 and T_2 such that T_1 and T_2 connect $\{u_1, u_2\}$ and $\{v_2, y\}$, and $T_1 \cup T_2$ is a $C_2[v_2, u_2]$-Tutte subgraph in G'. Note that T_1 and T_2 are also paths in G with $v_1 \notin V(T_1 \cup T_2)$.

Suppose first that there exists no $\big(T_1 \cup T_2 \cup \{v_1\}\big)$-bridge of G containing an edge of $C_1[u_1, v_1]$ and having at least four attachments on $T_1 \cup T_2 \cup \{v_1\}$. By the choice of T_1 and T_2, each $\big(T_1 \cup T_2 \cup \{v_1\}\big)$-bridge B of G has at most four attachments. Suppose that B contains an edge of $C_1[u_1, v_1] \cup C_2[v_2, u_2]$. If B contains an edge of $C_1[u_1, v_1]$, then it follows from the assumption of this paragraph that B has at most three attachments. Then we may assume that B contains an edge of $C_2[v_2, u_2]$. Since $T_1 \cup T_2$ is a $C_2[v_2, u_2]$-Tutte subgraph in G', $B \cap G'$ has at most two attachments, and hence B has at most three attachments. Therefore, $T_1 \cup T_2 \cup \{v_1\}$ is an extended $\big(C_1[u_1, v_1] \cup C_2[v_2, u_2]\big)$-Tutte subgraph in G. Thus, G satisfies (T1$'$).

Suppose next that there exists a $\big(T_1 \cup T_2 \cup \{v_1\}\big)$-bridge H of G containing an edge of $C_1[u_1, v_1]$ and having at least four attachments on $T_1 \cup T_2 \cup \{v_1\}$. Since $H \cap G'$ is a $\big(T_1 \cup T_2\big)$-bridge of G', $H \cap G'$ has at most three attachments on $T_1 \cup T_2$, and hence H has exactly four attachments one of which is v_1. Let a be the attachment of $H - v_1$ on $C_1[u_1, v_1] - \{v_1\}$ and let c and d be the attachments of $H - v_1$ on $T_1 \cup T_2$ such that a, v_1, c, d appear in a facial walk of H in this order. Let P be the subpath of $C_1[u_1, v_1]$ from a to v_1. Then H is an extended $C_1[u_1, v_1]$-flap with attachments a, v_1, c, d and base path P such that $a, c, d \in V(T_1 \cup T_2)$. By the same argument as the previous case, $T_1 \cup T_2 \cup \{v_1\}$ is an extended $\big(C_1[u_1, v_1] \cup C_2[v_2, u_2]\big)$-Tutte subgraph in $G - \big(V(H) - \{a, v_1, c, d\}\big)$. Moreover, we can check that G satisfies (T2$'$).

This completes the proof of Lemma 9. $\square$

7.3 Proof of Lemma 10

Let G be a 2-connected graph on the torus with the representativity exactly 2. Let F be a face of G, let C be the boundary cycle of F and let $x \in V(C)$. Suppose that the (x, F)-width of G is exactly 2.

Since the (x, F)-width of G is exactly 2, there exists an essential closed curve γ on the torus that passes through F and hits G at exactly two vertices, which are vertices in C, say u, and x. Cutting open the torus along γ, we obtain a connected plane graph G' with two facial walks C'_1 and C'_2 and four distinct vertices u'_1, u'_2, x'_1 and x'_2 with $u'_1, x'_1 \in V(C'_1)$, $u'_2, x'_2 \in V(C'_2)$ and the following property; G is obtained from G' by identifying u'_1 and u'_2 into u, and x'_1 and x'_2 into x. Moreover, we may assume that $C'_1[u'_1, x'_1] \cup C'_2[x'_2, u'_2]$ becomes C after these identifications.

Suppose that G' has a 1-separation (K_1, K_2). If $|V(K_i) \cap \{u'_1, x'_1, u'_2, x'_2\}| \geq 2$ for each $i = 1, 2$, then there exists an essential curve on the torus which passes through G only at $V(K_1 \cap K_2)$, contradicting that the representativity of G is at least 2. Thus, by symmetry, we may assume that $|V(K_1) \cap \{u'_1, x'_1, u'_2, x'_2\}| \leq 1$. On the other hand, if $V(K_1 - K_2) \cap \{u'_1, x'_1, u'_2, x'_2\} = \emptyset$, then $(K_1, \overline{K_1})$ is a 1-separation of G, a contradiction, where $\overline{K_1} = G - V(K_1 - K_2)$. Therefore, $|V(K_1 - K_2) \cap \{u'_1, x'_1, u'_2, x'_2\}| = 1$. Then we may assume that for each 1-separation (K_1, K_2) of G', $|V(K_1 - K_2) \cap \{u'_1, x'_1, u'_2, x'_2\}| = 1$. If G' has at least one 1-separation, then let

$$G_0 = \bigcap_{(K_1,K_2):\text{1-separation of } G'} K_2;$$

otherwise, that is, if G' does not have any 1-separations, then let $G_0 = G'$. Note that G_0 is 2-connected and has both edges in C'_1 and C'_2. For $i = 1, 2$, let C_i be the facial cycle in G_0 with $E(C_i) \subset E(C'_i)$. If there exists a 1-separation (K_1, K_2) of G' with $\{u'_1\} = V(K_1 - K_2) \cap \{u'_1, x'_1, u'_2, x'_2\}$, then let $u_1 \in V(G_0)$ such that u_1 separates u'_1 from G_0 in G'; otherwise let $u_1 = u'_1$. If there exists a 1-separation (K_1, K_2) of G' with $\{x'_1\} = V(K_1 - K_2) \cap \{u'_1, x'_1, u'_2, x'_2\}$, then let $v_1 \in V(G_0)$ such that v_1 separates x'_1 from G_0 in G'; otherwise let $v_1 = x'_1$. Similarly, we define u_2 and v_2, respectively. Note that $u_1, v_1 \in V(C_1)$ and $u_2, v_2 \in V(C_2)$. If $u_1 \in V(C_2)$, then there exists an essential curve on the torus that hits G only at u_1, and hence the representativity of G is exactly one, a contradiction. Thus, we have that $u_1 \notin V(C_2)$. Similarly, $v_1 \notin V(C_2)$ and $u_2, v_2 \notin V(C_1)$. Hence condition (G1) holds.

Note that $G' - \big(V(G_0) - \{u_1, v_1, u_2, v_2\}\big)$ has exactly four components each of which contains exactly one vertex in $\{u'_1, x'_1, u'_2, x'_2\}$. Hence each of the components is a chain of plane blocks. Then identifying u'_1 and u'_2, and x'_1 and x'_2, respectively, we obtain two chains of plane blocks, say $r_0, R_1, r_1, R_2, \dots, r_{l-1}, R_l, r_l$ and $s_0, S_1, \dots, s_{m-1}, S_m, s_m$, which form $G - \big(V(G_0) - \{u_1, v_1, u_2, v_2\}\big)$. By symmetry, we may assume that G is obtained from G_0 and the two chains of plane blocks by identifying u_1 and r_0, u_2 and r_l, v_1 and s_0, and v_2 and s_m, respectively. Then condition (G2) also holds.

For each $1 \leq i \leq l$ and each $1 \leq j \leq m$, let C_{R_i} (resp. C_{S_j}) be the facial walk in R_i (resp. S_j) that is not facial in G. Since C'_1 and C'_2 become C, by symmetry, we may also assume that $E(C) = E(C_1[u_1, v_1] \cup C_2[v_2, u_2]) \cup \bigcup_{i=1}^{l} E(C_{R_i}[r_i, r_{i-1}]) \cup \bigcup_{j=1}^{m} E(C_{S_j}[s_{j-1}, s_j])$. Hence condition (G3) holds. Since G is obtained from G' by identifying u'_1 and u'_2 to u and x'_1 and x'_2 to x, by symmetry, we may assume that $x = s_j$ for some $0 \leq j \leq m$, which implies condition (G4).

This completes the proof of Lemma 10. $\square$

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