5-connected toroidal graphs are Hamiltonian-connected

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ABSTRACT

The problem on the Hamiltonicity of graphs is well studied in discrete algorithm and graph theory, because of its relation to traveling salesman problem (TSP). Starting with Tutte's result, stating that every 4-connected planar graph is Hamiltonian, several researchers have studied the Hamiltonicity of graphs on surfaces. Extending Tutte's technique, Thomassen proved that every 4-connected planar graph is in fact Hamiltonian-connected, i.e., there is a Hamiltonian path connecting any two prescribed vertices. For graphs on the torus, Thomas and Yu showed that every 5-connected graph on the torus has a Hamiltonian cycle. In this paper, we prove the following result which generalizes Thomas and Yu's result.

Every 5-connected graph on the torus is Hamiltonian-connected.

Our result is best possible in the sense that we cannot lower the connectivity 5 (i.e., there is a 4-connected graph on the torus which is not Hamiltonian-connected). Moreover, our proof is constructive in a sense that it gives rise to a polynomial time (indeed \(O(n^2)\)-time) algorithm to construct a Hamiltonian path between any two specified vertices, if an input graph is a 5-connected graph on the torus.

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1 Introduction

Finding a Hamiltonian cycle is arguably one of the most popular subjects in graph theory. It is also one of the central problems in combinatorial optimization, since it is connected to the famous traveling salesman problem. A study on Hamiltonian cycles was started with the connection to the famous Four Color Problem (now Theorem). It had been conjectured since 1880’s that every 3-connected cubic planar graph has a Hamiltonian cycle, and if true, it would imply the Four Color Problem. However, Tutte \cite{17} in 1946 constructed a counterexample. Since then, finding a Hamiltonian cycle in planar graphs and graphs on surfaces is one of the most active topics in graph theory. In the last decade, a Hamiltonian cycle in planar graphs is also studied in graph algorithm (\cite{7}, for example), because it is connected to the traveling salesmen problem.

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It is now known that a 3-connected planar graph does not always have a Hamiltonian cycle, since there exist planar triangulations on \( n \) vertices whose longest cycle is of length \( O(n^\alpha) \), where \( \alpha = \log 2 / \log 3 \approx 0.63 \); cf. [9]. However, if one considers 4-connectivity for planar graphs, the situation dramatically changes. Whitney [18] was the first to give a positive result, showing that every planar triangulation without separating triangles has a Hamiltonian cycle. Tutte [16] proved that every 4-connected planar graph has a Hamiltonian cycle. Extending Tutte’s technique, Thomassen [15] proved that every 4-connected planar graph is in fact Hamiltonian-connected, i.e., there is a Hamiltonian path connecting any two prescribed vertices. (A small omission in [15] was corrected by Chiba and Nishizeki [1]). Chiba and Nishizeki [2] pointed out that Thomassen’s proof implies a polynomial time algorithm to find, given a 4-connected planar graph, a Hamiltonian path between any specified vertices.

With some additional (and complicated) techniques and new ideas, Thomas and Yu [12] managed to show that every edge in a 4-connected projective-planar graph is contained in a Hamiltonian cycle, and recently, the authors [6] improved it by showing that every 4-connected projective-planar graph is Hamiltonian-connected. Note that these two results establish a conjecture of Grünbaum [4] and a conjecture of Dean [3], respectively.

Grünbaum [4] and Nash-Williams [10] also conjectured that every 4-connected graph on the torus has a Hamiltonian cycle. Recently, it is shown in [14] that there is a Hamiltonian path in 4-connected graphs on the torus. Very recently, the authors [5] proved that every 4-connected triangulation on the torus has a Hamiltonian cycle. Let us point out that there are 4-connected graphs that do not have a Hamiltonian cycle in the double torus or in \( N_3 \), where \( N_3 \) is the surface obtained from the sphere by attaching three crosscaps. Thus this conjecture, if true, would be best possible. On the other hand, for 4-connected graphs on the torus, certain edges may not be contained in any Hamiltonian cycle, which means that there are some 4-connected graphs on the torus that is not Hamiltonian-connected.

The following example was provided by Thomassen [15]. Embed the product of two even cycles (of length at least 4) in the torus so that every face is bounded by a cycle of length 4, and add an edge joining two non-adjacent vertices in some facial cycle. So this graph is obtained from a toroidal grid with one edge added in a 4-cycle. Then this new edge is not contained in any Hamiltonian cycle of the new graph. (Note that the original graph is a balanced bipartite graph and the new edge connects two vertices in the same bipartition.)

While the conjecture due to Grünbaum and Nash-Williams remains open, it is shown in [13] that every 5-connected graph on the torus has a Hamiltonian cycle. In this paper, we show the following, which is an improvement of it.

**Theorem 1** Every 5-connected graph embedded on the torus is Hamiltonian-connected.

Our result is best possible in the sense that we cannot lower the connectivity 5. (As mentioned above, there are 4-connected graphs on the torus which is not Hamiltonian-connected.) Also, we cannot improve Theorem 1 to graphs on surfaces with higher genera, since it is known that there are infinitely many graphs on a surface with higher genus such that they are not Hamiltonian-connected.

Our proof is constructive in a sense that it gives rise to a polynomial time (indeed \( O(n^2) \)-time) algorithm to construct a Hamiltonian path between any two specified vertices, if an input graph is 5-connected graph on the torus.

Technically, we shall prove Theorem 1 in Section 4 by adapting the notion a Tutte subgraph. In order to state our technical result (Theorem 2) which implies Theorem 1, we need some definitions that will be given in the next section, and some lemmas mentioned in Section 3. In Section 5, we give an outline of the proof of Theorem 2. All of the detailed proofs appear in Sections 6 and 7.
2 Technical statement

For a graph $G$, the order of $G$ is denoted by $|G|$. Let $H_1$ and $H_2$ be two subgraphs of a graph $G$. Then $H_1 \cup H_2$ denotes the subgraph of $G$ with $V(H_1 \cup H_2) = V(H_1) \cup V(H_2)$ and $E(H_1 \cup H_2) = E(H_1) \cup E(H_2)$, and $H_1 \cap H_2$ denotes the subgraph of $G$ with $V(H_1 \cap H_2) = V(H_1) \cap V(H_2)$ and $E(H_1 \cap H_2) = E(H_1) \cap E(H_2)$. We use the similar notation also for a vertex subset $U$ or an edge subset $P$ of $G$: So, $H_1 \cup U$ is the subgraph of $G$ with $V(H_1 \cup U) = V(H_1) \cup U$ and $E(H_1 \cup U) = E(H_1)$, and $H_1 \cup P$ is the subgraph of $G$ with $V(H_1 \cup P) = V(H_1) \cup V(P)$ and $E(H_1 \cup P) = E(H_1) \cup P$, where $V(P)$ is the set of edges that are end vertices of some edges in $P$.

A pair $(H_1, H_2)$ of subgraphs of $G$ is called a separation of $G$ if $V(G) = V(H_1) \cup V(H_2)$ and each edge of $G$ is contained in exactly one of $H_1$ and $H_2$. A separation $(H_1, H_2)$ of $G$ is a $k$-separation if $|H_1|, |H_2| \geq k + 1$ and $|V(H_1 \cap H_2)| = k$. Note that $G$ is $k$-connected if and only if $G$ has no $l$-separation for each $l < k$.

Let $G$ be a connected graph on a surface $\mathbb{F}^2$. A facial walk in $G$ is the boundary walk of some face of $G$. Furthermore, if it is a cycle, then we call it a facial cycle in $G$. Now suppose that $\mathbb{F}^2$ is a non-spherical surface. The representativity of $G$ is the minimum integer $k$ such that every essential closed curve on $\mathbb{F}^2$ hits $G$ at least $k$ times. Let $x$ be a vertex in $G$ and let $F$ be a face of $G$ such that the boundary walk of $F$ contains $x$. We define the $x$-width (resp. $(x, F)$-width) of $G$ as the minimum integer $k$ such that every essential closed curve on $\mathbb{F}^2$ passing through $x$ (resp. $x$ and $F$) hits $G$ at least $k$ times.

For a path $P$ and two vertices $x, y \in V(P)$, $P[x, y]$ denotes the subpath of $P$ between $x$ and $y$. For a cycle $C$ with fixed direction, and for two vertices $x, y \in V(C)$, $C[x, y]$ denotes the subpath of $C$ between $x$ and $y$ along the direction. For a cycle $C$ of a plane graph $G$, we usually give a direction to $C$ in the clockwise order, unless $C$ is the outer facial cycle of $G$.

Let $T$ be a subgraph of a graph $G$. A $T$-bridge of $G$ is either (i) an edge of $G - E(T)$ with both ends on $T$ or (ii) a subgraph of $G$ induced by the edges in a component of $G - V(T)$ and all edges from that component to $T$. A $T$-bridge satisfying (i) is said to be trivial; otherwise it is non-trivial. For a $T$-bridge $B$ of $G$, the vertices in $B \cap T$ are the attachments of $B$ (on $T$), and any vertex of $B$ that is not an attachment is a non-attachment. We say that $T$ is a Tutte subgraph (resp. an extended Tutte subgraph) of $G$ if every $T$-bridge of $G$ has at most three (resp. four) attachments on $T$. Note that if $G$ is 4-connected (resp. 5-connected) and $T$ is a Tutte subgraph (resp. an extended Tutte subgraph) of $G$ with $|T| \geq 4$ (resp. $|T| \geq 5$), then $T$ must contain all vertices in $G$; otherwise, there exists a $T$-bridge of $G$ whose attachments form a cut set in $G$ of order at most three (resp. four), contradicting that $G$ is 4-connected (resp. 5-connected). For $C \subset G$, $T$ is a $C$-Tutte subgraph (resp. an extended $C$-Tutte subgraph) of $G$ if $T$ is a Tutte subgraph (resp. extended Tutte subgraph) of $G$ and every $T$-bridge of $G$ containing an edge of $C$ has at most two (resp. three) attachments on $T$. Note that a Tutte subgraph in $G$ is also an extended $C$-Tutte subgraph for any $C \subset G$. When $T$ is a path or a cycle, we call $T$ a (extended) $C$-Tutte path or a (extended) $C$-Tutte cycle. When $G$ is a plane graph, we usually regard the outer facial walk as the subgraph $C$.

Let $C$ be a facial cycle in $G$ or a subpath of it. We will define an extended $C$-flap, which is, in some sense, an extension of a $C$-flap, see [6, 12] for the definition of a $C$-flap. Notice that a $C$-flap was defined by Thomas and Yu [12], but at that time, they regarded the empty graph also as a $C$-flap. In the present paper, we exclude the empty graph from the definition of an extended $C$-flap, since we think that it is easier to understand.

An extended $C$-flap is an $\{a, b, c, d\}$-bridge $H$ of $G$ for some vertices $a, b, c$ and $d$ such that either

(H1) $a, b \in V(C) \cap V(H)$ and $a \neq b$,

(H2) $H$ contains a subpath $P$ of $C$ from $a$ to $b$ with $c, d \notin V(P)$, and
(H3) $H$ is embedded on the disk such that $P, c$ and $d$ appear in the outer facial walk $C_H$
 in this order,

or

(H1) $a, b \in V(C) \cap V(H)$ and $a \neq b$,

(H2) $H$ contains a subpath $P$ of $C$ from $a$ to $b$ with $c \notin V(P)$, and

(H3) $b = d$, and there exists an essential closed curve on $\mathbb{F}^2$ that hits $G$ only at $a$ and $b$, and

there exists a plane graph $\tilde{H}$ with the outer facial walk $C_{\tilde{H}}$ and four distinct vertices

$a, b, c, d$ in $C_H$ such that $C_{\tilde{H}}$ contains $P$ and $H$ is obtained from $\tilde{H}$ by identifying $b$
 and $d$ to $b$.

We call an extended $C$-flap for the former of Type I, while the latter is of Type II. Note
that $P = C_H[a, b]$ for an extended $C$-flap of Type I, while $P = C_{\tilde{H}}[a, b]$ for the one of Type
II. In each type, the subpath $P$ of $C$ in (H2) is the base path of $H$. See Figure 1. Notice
that if a graph $G$ has an extended $C$-flap of Type II, then the $(b, F)$-width of $G$ is exactly
2, since there exists an essential closed curve on $\mathbb{F}^2$ that passes through only $a$ and $b$. Note
that $a, b, c$ and $d$ are attachments of $H$.

![Figure 1: Two types of an extended C-flap H with attachments a, b, c, d.](image)

The following is the technical theorem, which will be used for the proof of Theorem 1
in Section 4.

**Theorem 2** Let $G$ be a 2-connected graph embedded on the torus with the representa-
tivity at least two, let $F$ be a face of $G$, let $C$ be the boundary cycle of $F$, let $x \in V(C)$,
let $y \in V(G) - \{x\}$, and let $f$ be an edge of $C$ incident with $x$. Suppose that for every
2-separation $(G_1, G_2)$ of $G$, we have $E(C) \cap E(G_1) \neq \emptyset$ and $E(C) \cap E(G_2) \neq \emptyset$.
Then one of the following holds.

(T1) There exists an extended $C$-Tutte path $T$ in $G$ from $x$ to $y$ with $|T| \geq 4$.

(T2) There exists an extended $C$-flap $H$ with attachments $a, b, c, d$ and base path $P$ and
there exists an extended $C$-Tutte path $T$ in $G - (V(H) - \{a, b, c, d\})$ from $b$ to $y$ such
that $|T| \geq 4$, $a, c, d \in V(T)$, $P$ contains $f$, and $a, f, x, b$ appear in $P$ in this order.

See Figure 2 for (T2) with an extended $C$-flap $H$ of Type I. In Section 5, we will put
the outline of the proof of Theorem 2, and in Section 6, we will show Theorem 2.

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3In this definition, we regard all edges and all vertices in $H$ other than $b$ and $d$ also as edges and vertices
in $\tilde{H}$.

4In all figures of the present paper unless we mention explicitly, rising diagonals stroke from bottom
left to top right represent an extended $C$-flap, while falling diagonals stroke from top left to bottom right
represent other part of a given graph. White regions represent faces. Note that the right side of Figure 1
represents a part of a cylinder.
3 Lemmas used in the proof of our theorems

Let us begin with some known results. The first lemma was proved by Sanders [11]. Note that he showed only the 2-connected case, but we can easily show the following, using a block decomposition. See also the paper [15] by Thomassen.

Theorem 3 Let $G$ be a connected plane graph, let $C$ be a facial walk in $G$, let $x, y \in V(G)$ with $x \neq y$, and let $e \in E(C)$. Assume that $G$ contains a path from $x$ to $y$ through $e$. Then $G$ has a $C$-Tutte path from $x$ to $y$ through $e$.

In this paper, we sometimes use Theorem 3 with the following form. Since we can easily prove it, we omit the proof here.

Theorem 4 Let $G$ be a 2-connected plane graph, let $C$ be a facial cycle in $G$, let $x, y \in V(G)$ with $x \neq y$, and let $u \in V(C) - \{x, y\}$. Then $G$ has a $C$-Tutte path $T$ from $x$ to $y$ through $u$ with $|T| \geq 4$, unless $x$ and $y$ are the only neighbors of $u$.

On the other hand, as another directed corollary of Theorem 3, we obtain the following.

Theorem 5 Let $G$ be a connected plane graph, let $C$ be a facial walk in $G$, let $x, u \in V(C)$ with $x \neq u$ and let $y \in V(G) - \{x, y\}$. Suppose that there exists a path in $G - u$ from $x$ to $y$. Then $G$ has a $C[x, u]$-Tutte subgraph consisting of $u$ and a path $T$ from $x$ to $y$ with $u \not\in V(T)$.

To show Theorem 5, consider the graph obtained from $G$ by adding an edge connecting $x$ and $u$, and use Theorem 3 to find a $(C[x, u] \cup \{xu\})$-Tutte path from $u$ to $y$ through $xu$. Finally, deleting the added edge, we obtain the desired subgraph in Theorem 5.

The next lemma was shown by Thomas and Yu [13].

Theorem 6 (Theorem (2.2) in [13]) Let $G$ be a connected plane graph, let $C$ be a facial walk in $G$, let $x, y \in V(C)$ with $x \neq y$, and let $A \subset V(C)$. Suppose that $V(C[x, y]) \cap A = \emptyset$. Then $G$ has a path $T$ from $x$ to $y$ with $V(T) \cap A = \emptyset$ such that every $(T \cup A)$-bridge $B$ of $G$ has at most $\max\{|A| + 1, 3\}$ attachments and at most two attachments if $B$ contains an edge of $C[x, y]$.

We use Theorem 6 for $A$ with $|A| \leq 3$. Notice that if $|A| = 1$, then Theorem 6 almost corresponds to Theorem 5. (However, Theorem 5 is stronger than Theorem 6 for $|A| = 1$, since we can specify even a vertex not contained in $C$ as $y$.) When $|A| \leq 2$, the path obtained from Theorem 6 is a $C[x, y]$-Tutte path from $x$ to $y$. Similarly, when $|A| \leq 3$, it is an extended $C[x, y]$-Tutte path from $x$ to $y$.

The following lemma was shown in [12].
Figure 3: A $C_1[u_1, v_1]$-flap and an extended $(C_1[u_1, v_1] \cup C_2[v_2, u_2])$-Tutte subgraph $T_1 \cup T_2$ desired in (T2) in Lemma 9.

**Theorem 7 (Theorem (2.6) in [12])** Let $G$ be a 2-connected plane graph, let $C$ be a facial cycle in $G$, let $x, y \in V(C)$, and let $e_1, e_2 \in E(C)$ such that $x \neq y$ and $x, e_1, e_2, y$ appear on $C$ in this clockwise order. Then there exists a $C[x, y]$-Tutte path $T$ in $G$ from $x$ to $y$ with $e_1, e_2 \in E(T)$.

In addition to the above lemmas, we will use the following two new lemmas, which will be shown in Sections 7.1 and 7.2, respectively. We say that two paths $T_1$ and $T_2$ in $G$ connect $\{u_1, u_2\}$ and $\{v_1, v_2\}$ if $T_1$ connects $u_i$ and $v_j$ and $T_2$ connects $v_{3-i}$ and $v_{3-j}$ for some $i, j \in \{1, 2\}$. Note that some vertices $u_i$ and $v_j$ might coincide each other; For example, if $u_1 = v_1$, then one of the paths $T_1$ and $T_2$ must consist of only the vertex $v_1$.

**Lemma 8** Let $G$ be a 2-connected plane graph, let $C_1, C_2$ be facial cycles in $G$, let $u_1, v_1 \in V(C_1) - V(C_2)$ with $u_1 \neq v_1$, let $u_2 \in V(C_2) - V(C_1)$, and let $y \in V(G) - \{u_2, v_2\}$, then $G$ has two disjoint paths $T_1$ and $T_2$ such that $T_1$ and $T_2$ connect $\{u_1, u_2\}$ and $\{v_1, y\}$, $|T_1| + |T_2| \geq 5$, and $T_1 \cup T_2$ is a $C_1[u_1, v_1]$-Tutte subgraph in $G$.

**Lemma 9** Let $G$ be a 2-connected plane graph, let $C_1, C_2$ be facial cycles of $G$, let $u_1, v_1 \in V(C_1) - V(C_2)$ with $u_1 \neq v_1$, let $u_2, v_2 \in V(C_2) - V(C_1)$ with $u_2 \neq v_2$, let $y \in V(G) - \{u_2, v_2\}$, and let $f$ be an edge of $C_1[u_1, v_1]$ incident with $v_1$. Suppose that $G - v_1$ is also 2-connected. Then one of the following holds.

- (T1') $G$ has two disjoint paths $T_1$ and $T_2$ such that $T_1$ and $T_2$ connect $\{u_1, u_2\}$ and $\{v_2, y\}$, $v_1 \notin V(T_1 \cup T_2)$, $|T_1| + |T_2| \geq 5$, and $T_1 \cup T_2 \cup \{v_1\}$ is an extended $(C_1[u_1, v_1] \cup C_2[v_2, u_2])$-Tutte subgraph in $G$.
- (T2') There exists an extended $C_1[u_1, v_1]$-flap $H$ with attachments $a, v_1, c, d$ and base path $P$, and there exist two disjoint paths $T_1$ and $T_2$ such that $T_1$ and $T_2$ connect $\{u_1, u_2\}$ and $\{v_2, y\}$, $v_1 \notin V(T_1 \cup T_2)$, $|T_1| + |T_2| \geq 5$, $a, c, d \in V(T_1 \cup T_2)$, $T_1 \cup T_2 \cup \{v_1\}$ is an extended $(C_1[u_1, v_1] \cup C_2[v_2, u_2])$-Tutte subgraph in $G - (V(H) - \{a, v_1, c, d\})$, $P$ contains $f$, and $a, f, v_1$ appear in $P$ in this order.

In addition, we use the following important lemma (Lemma 10), which mentions how to cut a 2-connected graph on the torus with representativity exactly 2. Before that, we need some definitions.

A **block** in a graph $G$ is a maximal subgraph that has no cut vertex in $G$. Note that any block is 2-connected unless it consists of only a vertex or an edge. A **chain of blocks** is a sequence $r_0, R_1, r_1, \ldots, r_{l-1}, R_l, r_l$ such that each $R_i$ is a block in $\bigcup_{i=1}^{l} R_i$, each $r_i$ is a vertex, in particular a cut vertex in $\bigcup_{i=1}^{l} R_i$ for $1 \leq i \leq l - 1$, $V(R_i) \cap V(R_j) = \emptyset$ for each $0 \leq i < j \leq l$ with $j \neq i + 1$, $\{r_i\} = V(R_i) \cap V(R_{i+1})$ for each $0 \leq i \leq l - 1$, and if $|R_1| \geq 2$, then $r_0 \in V(R_0) - \{r_1\}$ and $r_l \in V(R_l) - \{r_{l-1}\}$. When a chain of blocks $r_0, R_1, r_1, \ldots, r_{l-1}, R_l, r_l$ satisfies that $|R_1| = 1$, then we call it **trivial**. (In this case, $l = 1$)
Therefore, there exists no non-trivial four such that it separates some vertices in since
\(|T| = T\).

The exceptional case in Theorem 4 does not occur if we take an appropriate vertex as

\((G, \{x, y\})\) be a face of

and it consists of only one vertex \(r_0 = r_1\). Furthermore, if each block is a plane graph, then we call it a chain of plane blocks.

**Lemma 10** Let \(G\) be a 2-connected graph on the torus with the representativity exactly 2, let \(F\) be a face of \(G\), let \(C\) be the boundary cycle of \(F\), and let \(x \in V(C)\). Suppose that the \((x, F)\)-width of \(G\) is exactly 2. Then \(G\) can be decomposed into a 2-connected plane graph \(G_0\) and two chains of plane blocks \(r_0, R_1, r_1, R_2, \ldots, r_{l-1}, R_l, r_l\) and \(s_0, S_1, \ldots, s_{m-1}, S_m, s_m\) satisfying the following properties (G1)-(G4). (Possibly, one of the chains or both may be trivial.) For each \(1 \leq i \leq l\), let \(C_{R_i}\) be the unique facial walk in \(R_i\) that is not facial in \(G\), and for each \(1 \leq j \leq m\), let \(C_{S_j}\) be the unique facial walk in \(S_j\) that is not facial in \(G\).

(G1) There exist two facial cycles \(C_1\) and \(C_2\) in \(G_0\) and four distinct vertices \(u_1, u_2, v_1\) and \(v_2\) such that \(u_1, v_1 \in V(C_1) - V(C_2)\) and \(u_2, v_2 \in V(C_2) - V(C_1)\).

(G2) \(G\) is obtained from \(G_0\) and the two chains of plane blocks \(r_0, R_1, \ldots, r_{l-1}, R_l, r_l\) and \(s_0, S_1, \ldots, s_{m-1}, S_m, s_m\) by identifying \(u_1\) and \(r_0, u_2\) and \(r_l, v_1\) and \(s_0, v_2\) and \(s_m\), respectively.

(G3) \(E(C) = E(C_1[u_1, v_1] \cup C_2[v_2, u_2]) \cup \bigcup_{i=1}^{l} E(C_{R_i}[r_i, r_{i-1}]) \cup \bigcup_{j=1}^{m} E(C_{S_j}[s_{j-1}, s_j])\).

(G4) \(x = s_j\) for some \(0 \leq j \leq m\).

### 4 Proof of Theorem 1, assuming Theorem 2

In this section, we give a proof of Theorem 1, assuming Theorem 2.

Let \(G\) be a 5-connected graph embedded on the torus, let \(x \in V(G)\) and let \(y \in V(G) - \{x\}\). It is enough to show that \(G\) has a Hamiltonian path from \(x\) to \(y\).

Let \(G^* = G - x\), let \(x^*\) be a neighbor of \(x\) in \(G\), and let \(F^*\) be the unique face of \(G^*\) that is not a face in \(G\). Note that the region \(F^*\) contains \(x\) and all edges incident with \(x\) in \(G\). Let \(C^*\) be the boundary cycle of \(F^*\). Since \(G\) is 5-connected, note that \(G^*\) is 4-connected and \(|V(C^*)| \geq 5\).

Suppose first that the representativity of \(G^*\) is exactly 0. Then \(G^*\) is a planar graph. By Theorem 4, \(G^*\) has a \(C^*\)-Tutte path \(T^*\) from \(x^*\) to \(y\) with \(|T^*| \geq 4\). (Note that the exceptional case in Theorem 4 does not occur if we take an appropriate vertex as \(u\).) Let \(T = \{x^*\} \cup T^*\). Note that \(T\) is a path in \(G\) from \(x\) to \(y\). Suppose that there exists a non-trivial \(T\)-bridge \(B\) in \(G\). Since \(T^*\) is a \(C^*\)-Tutte path in \(G^*\), \(B \cap G^*\) has at most three attachments on \(T^*\), and hence including \(x\), \(B\) has at most four attachments on \(T\). However, since \(|T| \geq |T^*| + 1 \geq 5\), the attachments of \(B\) on \(T\) form a cut set in \(G\) of order at most four such that it separates some vertices in \(T\) from non-attachments of \(B\), a contradiction. Therefore, there exists no non-trivial \(T\)-bridge of \(G\), that is, \(T\) is a Hamiltonian path in \(G\) from \(x\) to \(y\).

Suppose next that the representativity of \(G^*\) is exactly 1. Then there exists an essential curve \(\gamma\) on the torus hitting \(G^*\) at exactly one vertex, say \(z\). Since \(\gamma\) hits \(G^*\) only at \(z\),
The graph $\tilde{H}$

Figure 5: The case where the representativity of $G^*$ is at least 2 and $G^*$ satisfies (T2).

cutting the torus along $\gamma$, we obtain the plane graph $G^*_0$ with two facial cycles $C_1$ and $C_2$ and two distinct vertices $z_1$ and $z_2$ with $z_1 \in V(C_1)$ and $z_2 \in V(C_2)$ such that $G^*$ is obtained from $G^*_0$ by identifying $z_1$ and $z_2$ into $z$. For $i = 1, 2$, let $C^*_i$ be the unique facial cycle in $G^*_0 - z_i$ that is not facial in $G^*_0$. Since $|C^*_i| \geq 3$ for $i = 1, 2$, there exists an edge $e$ in $C^*_1$ or in $C^*_2$ such that $e$ is incident with neither $x^*$ nor $y$. By symmetry, we may assume that $e$ is contained in $C^*_1$. By Theorem 3, $G^*_0 - z_1$ has a $C^*_1$-Tutte path $T^*$ from $x^*$ to $y$ through $e$. (Note that $x^*$ or $y$ might coincide $z_2$.) By the choice of $e$, we have $|T^*| \geq 4$. Let $T = \{xx^*\} \cup T^*$. Note that $T$ is a path in $G$ from $x$ to $y$. Suppose that there exists a non-trivial $T$-bridge $B$ in $G$. Suppose further that at least one of $x$ and $z_1$ is not an attachment of $B$. Since $T^*$ is a $C^*_1$-Tutte path in $G^*_0 - z_1$, $B \cap (G^*_0 - z_1)$ has at most three attachments on $T^*$, and hence $B$ has at most four attachments on $T$. However, since $|T| \geq |T^*| + 1 \geq 5$, the attachments of $B$ on $T$ form a cut set in $G$ of order at most four, a contradiction. Hence we may assume that both $x$ and $z_1$ are attachments of $B$. In this case, $B \cap (G^*_0 - z_1)$ is a $T^*$-bridge of $G^*_0 - z_1$ containing an edge in $C^*_1$, and hence $B$ has at most four attachments on $T$ two of which are $x$ and $z_1$. By the same argument above, this contradicts that $G$ is 5-connected. Then $T$ is a Hamiltonian path in $G$.

Therefore, we may assume that the representativity of $G^*$ is at least 2. Let $f$ be an edge of $C^*$ incident with $x^*$. Since $G^*$ is 4-connected, it trivially satisfies the assumptions in Theorem 2. Then by Theorem 2, $G^*$ satisfies (T1) or (T2) with respect to $G^*, C^*, x^*, y$ and $f$.

Suppose first that $G^*$ satisfies (T1), and let $T^*$ be an extended $C^*$-Tutte path satisfying the conditions in (T1) in Theorem 2. Let $T = \{xx^*\} \cup T^*$. Note that $T$ is a path in $G$ from $x$ to $y$ with $|T| \geq |T^*| + 1 \geq 5$. Suppose also that there exists a non-trivial $T$-bridge $B$ of $G$. If $x$ is not an attachment of $B$, then $B$ is also a $T^*$-bridge of $G^*$, and hence $B$ has at most four attachments on $T$. On the other hand, if $x$ is an attachment of $B$, then $B - x$ is a $T^*$-bridge of $G^*$ containing an edge in $C^*$, and hence $B$ has at most four attachments one of which is $x$. In either case, since $|T| \geq 5$, those attachments separate some vertex in $T^*$ from non-attachments of $B$, contradicting that $G$ is 5-connected. Hence $T$ is a Hamiltonian path in $G$.

Suppose next that $G^*$ satisfies (T2). Let $T^*$ be an extended $C^*$-Tutte path, and $H^*$ be an extended $C^*$-flap with attachments $a^*, b^*, c^*$ and $d^*$ and base path $P^*$ such that those satisfy the conditions in (T2) in Theorem 2. See the left side of Figure 5. By the same way as in the previous paragraph, we can show that there exists no non-trivial $(T^* \cup \{x\})$-bridge of $G^* - (V(H^*) - \{a^*, b^*, c^*, d^*\})$. Let $\hat{H} = H^*$, $b = b^*$ and $\hat{d} = d^*$ if $H^*$ is of Type I; Otherwise let $\hat{H}$ be the plane graph and $b$ and $\hat{d}$ be the vertices as in (H3) of the definition of an extended $C$-flap of Type II. Note that $\hat{H}$ is a plane graph. Let $\tilde{H}$ be the graph obtained from $\hat{H}$ by adding all edges in $G$ between $x$ and vertices in $P^*$. Since $x^*$ is contained in $P^*$ and $x^*$ is a neighbor of $x$ in $G$, $\tilde{H}$ is connected. It follows from Theorem 6 with $A = \{a^*, c^*, \hat{d}\}$ that $\tilde{H}$ has a path $T_{\tilde{H}}$ from $x$ to $\hat{b}$ with $V(T_{\tilde{H}}) \cap A = \emptyset$ such that every $(T_{\tilde{H}} \cup A)$-bridge of $\tilde{H}$ has at most four attachments. (We here ignore the last condition
5 Outline of the proof of Theorem 2

It remains to show Theorem 2. Here we briefly explain the outline of the proof of Theorem 2. Recall that the detailed proof appears in the next section. First, in Section 6.1, we show a claim (Claim 1) in order to avoid a 2-separation \((T_H \cup A)\)-bridge of \(H\), then it has at most four attachments, but this contradicts that \(G\) is 5-connected. Thus, there exists no non-trivial \((T_H \cup A)\)-bridge of \(H\). This implies that \(T^* \cup T_H\) is a Hamiltonian path in \(G\) from \(x\) to \(y\). See the right side of Figure 5. This completes the proof of Theorem 1, assuming Theorem 2. □

6 Proof of Theorem 2

We prove Theorem 2 by induction on \(|G|\). Since \(G\) is 2-connected and has representativity at least two, it is easy to see that \(|G| \geq 5\) and if \(|G| = 5\), then \(G\) satisfies (T1). So, we may assume that \(|G| \geq 6\).

We divide this proof into four steps: A claim dealing with a 2-separation with a certain condition (Section 6.1), the case where the \((x, F)\)-width of \(G\) is exactly 2 (Section 6.2), the case where the \((x, F)\)-width of \(G\) is at least 3 and the \(x\)-width of \(G\) is small, it follows from Lemma 10 that we can cut the torus off to the cylinder and obtain a plane graph. Then depending on the places of \(x\) and \(y\), we further divide the proofs into several subcases. In some subcases, we need new lemmas on Tutte paths (Lemmas 8 and 9); Otherwise we are done just using Theorems 3 and 5.

After dealing with the base case of the induction, we proceed to Section 6.4, which is the main part of the proof. Although we have to care the detail and divide the proof into some subcases, the main idea of this case is same and simple: By the assumption of this case, deleting the vertex \(x\) creates the new graph \(G - x\), which satisfies all assumptions in Theorem 2. By the induction hypothesis, \(G - x\) satisfies (T1) or (T2), and then expand the obtained extended Tutte path in \(G - x\) to the one in \(G\) (Cases I, II-1, and III) using Theorem 6, or expand the obtained extended \(C\)-flap (Case II-2). □
of $G$ is at least 2. Hence by symmetry, we may assume that $G_2$ is bounded by a disk on the torus. See Figure 6.

For $i = 1, 2$, let $G_i^*$ be the graph obtained from $G$ by replacing $G_{3-i}$ with an edge connecting $x$ and $z$. (If $G_i$ already has the edge connecting $x$ and $z$, then we delete the original edge.) Notice that $G_i^*$ is a 2-connected graph on the torus, and $G_2^*$ is a 2-connected plane graph. We fix a direction on $C$ and by symmetry, we may assume that $E(C[z, x]) = E(C) \cap E(G_1)$. Let $C^*_1 = C[z, x] \cup \{xz\}$ and $C^*_2 = C[z, x] \cup \{xz\}$. Note that for $i = 1, 2$, $C^*_i$ is a facial cycle in $G_i^*$. Let $y^*_i = y$ if $y \in V(G_1^*)$; Otherwise let $y^*_i = z$. Let $f^* = f$ if $f \in E(G_1^*)$; Otherwise let $f^* = xz$.

Note that $G_1^*$ is a 2-connected graph on the torus such that the representativity of $G_1^*$ is at least 2 and for every 2-separation $(G_1^*, G_2^*)$ of $G_1^*$, $E(C^*_1) \cap E(G_2^*) \neq \emptyset$ and $E(C^*_1) \cap E(G_2^*) \neq \emptyset$. Then by the induction hypothesis, $G_1^*$ satisfies (T1) or (T2) with respect to $G_1^*, C^*_1, x, y^*_1, f^*_1$. Let $T^*$ be an extended $C^*_1$-Tutte path, and $H^*$ be an extended $C^*_1$-flap with attachments $a^*, b^*, c^*, d^*$ and base path $P^*$ satisfying the conditions in (T1) or (T2) in Theorem 2. (If $G_1^*$ satisfies (T1), then ignore the definitions on $H^*, a^*, \text{and so on}.)

**Case I.** $xz \in E(T^*)$.

If $y \in V(G_2^*)$, then $z = y^*_1$, and hence $T^*$ consists of only $xz$, which contradicts that $|T^*| \geq 4$. Thus, we have $y \notin V(G_1^*) - V(G_2^*)$.

By Theorem 3, there exists a $C_2^*$-Tutte path $T_2$ in $G_2^*$ from $x$ to $z$. In particular, we can take such a $C_2^*$-Tutte path $T_2$ so that $xz \notin E(T_2)$ by specifying an appropriate edge as $e$. Let $T = (T^* - \{xz\}) \cup T_2$. Note that $|T| \geq |T^*| + 1 \geq 5$.

Suppose first that $G_1^*$ satisfies (T1). See Figure 7. Note that every non-trivial $T$-bridge $B$ of $G$ is either (i) a $T^*$-bridge of $G_1^*$ or (ii) a $T_2$-bridge of $G_2^*$. Then by the choice of $T^*$ and $T_2$, $B$ has at most four attachments and at most three attachments if $B$ contains an edge of $C^*_1 \cup C^*_2$. Since $E(C) \subseteq E(C^*_1) \cup E(C^*_2)$, $T$ is an extended $C$-Tutte path in $G$. This implies that $G$ satisfies (T1) in Theorem 2.

Suppose next that $G_1^*$ satisfies (T2). See Figure 8. In this case, $f \in E(G_1)$. Let $H = H^*, a = a^*, b = b^*, c = c^*, d = d^*$, and $P = P^*$. Note that every non-trivial $T$-bridge $B$ of $G - (V(H) - \{a, b, c, d\})$ is either (i) a $T^*$-bridge of $G_1^* - (V(H^*) - \{a^*, b^*, c^*, d^*\})$ or (ii) a $T_2$-bridge of $G_2^*$. By the same way as above, we can check that $T$ is an extended $C$-Tutte path in $G - (V(H) - \{a, b, c, d\})$, and $G$ also satisfies (T2) in Theorem 2. This completes the proof of Case I.

**Case II.** $xz \notin E(T^*)$ and $y \in V(G_1^*)$.

Suppose first that $G_1^*$ satisfies (T1). Let $T = T^*$. Note that the $T$-bridge of $G$ containing $G_2$ is the unique $T$-bridge that is not a $T^*$-bridge of $G_1^*$, but the attachments of it are not changed. Thus, $T$ is an extended $C$-Tutte path in $G$ from $x$ to $y$ with $|T| \geq 4$, and hence $G$ satisfies (T1).

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5In Figure 7, the bottom curve represents a part of $C$. This situation is same for all figures in this subsection.
In order. It follows from Theorem 6 with \( c \) in Case I, we can show that \( y \) is not a 4-
particular, in the latter case, the bridge of \( G \) induced by \( a = a^*, b = b^*, c = c^*, d = d^* \). In either case, \( H \) still has four attachments \( a, b, c, d \), and hence \( H \) is an extended \( C \)-flap in \( G \) with attachments \( a, b, c, d \) and base path \( P \). In particular, in the latter case, the bridge of \( G \) containing \( G_2 \) is the unique \( T \)-bridge that is not a \( T^*\)-bridge of \( G_1^* \), but the attachments of it are not changed. These imply that \( G \) satisfies (T2) in Theorem 2.

**Case III.** \( xz \notin E(T^*) \) and \( y \in V(G_2^*) - V(G_1^*) \).

In this case, \( y^* = z \) and \( xy^* \in E(C_1^*) \). By Theorem 5, there exists a \( C[x, z] \)-Tutte subgraph in \( G_2^* \) consisting of \( x \) and a path \( T_2 \) from \( z \) to \( y \) with \( x \notin V(T_2) \).

Suppose first that \( G_1^* \) satisfies (T1). Let \( T = T^* \cup T_2 \). By the same argument as in Case I, we can show that \( T \) is an extended \( C \)-Tutte path in \( G \) from \( x \) to \( y \) with \( |T| \geq 4 \), and hence \( G \) satisfies (T1) in Theorem 2.

Suppose next that \( G_1^* \) satisfies (T2) and \( f^* \neq xz \). Since \( a^*, f^*, x, b^* \) appear in \( P^* \) in this order and \( b^* \neq z = y^* \), we have that \( x = b^* \). Let \( T = T^* \cup T_2, H = H^*, a = a^*, b = b^* = x, c = c^*, d = d^* \) and \( P = P^* \). See Figure 11. Note that every non-trivial \( T \)-bridge \( B \) of \( G - \{V(H) - \{a, b, c, d\}\} \) is either (i) a \( T^* \)-bridge of \( G_1^* - \{V(H^*) - \{a^*, b^*, c^*, d^*\}\} \), or (ii) a \( (T_2 \cup \{x\}) \)-bridge of \( G_2^* \). By the same argument as in Case I, we can check that \( G \) satisfies (T2) in Theorem 2.

Suppose finally that \( G_1^* \) satisfies (T2) and \( f^* = xz \). Since \( a^*, f^*, x, b^* \) appear in \( P^* \) in this order and \( z = y^* \in V(T^*) \), we have that \( a^* = y^* = z \). Let \( \tilde{H}^* = H^*, \tilde{b} = b^* \) and \( \tilde{d} = d^* \) if \( H^* \) is of Type I; otherwise let \( \tilde{H}^* \) be the plane graph and \( \tilde{b} \) and \( \tilde{d} \) be the vertices as in (H3) in the definition of an extended \( C \)-flap of Type II. Note that \( \tilde{H}^* \) is a plane graph. Let \( \tilde{C}_{\tilde{H}} \) be the outer facial walk in \( \tilde{H}^* \) containing \( a^*, \tilde{b}, c^*, \tilde{d} \) and \( \tilde{d} \) in this clockwise order. It follows from Theorem 6 with \( A^* = \{a^*, c^*, \tilde{d}\} \) that \( \tilde{H}^* \) has a path \( T_{\tilde{H}}^* \) from \( x \) to \( \tilde{b} \) with \( V(T_{\tilde{H}}^*) \cap A^* = \emptyset \) such that every \( (T_{\tilde{H}}^* \cup A^*) \)-bridge of \( \tilde{H}^* \) has at most four attachments and at most two attachments if \( B \) contains an edge of \( \tilde{C}_{\tilde{H}}[x, \tilde{b}] \). See Figure 12. Let \( T = T_{\tilde{H}}^* \cup T^* \cup T_2 \). Note that every non-trivial \( T \)-bridge \( B \) of \( G \) is either (i) a \( T^* \)-bridge of \( G_1^* - \{V(H^*) - \{a^*, b^*, c^*, d^*\}\} \), or (ii) a \( (T_2 \cup \{x\}) \)-bridge of \( G_2^* \), or (iii) a \( (T_{\tilde{H}}^* \cup A^*) \)-bridge of \( \tilde{H}^* \). By the conditions of \( T^* \), \( T_2 \) and \( T_{\tilde{H}}^* \) and the same way as in Case
6.2 The case where the \((x, F)\)-width of \(G\) is exactly 2

Let \(w\) be the vertex in \(C\) with \(f = xw\). By Lemma 10, \(G\) can be decomposed into a 2-connected plane graph \(G_0\) and two chains of plane blocks \(r_0, R_1, R_2, \ldots, r_{i-1}, R_i, r_i\) and \(s_0, S_1, \ldots, s_m, S_m, s_m\) satisfying (G1)–(G4). For each 1 \(\leq i \leq l\), let \(C_{R_i}\) be the unique facial walk in \(R_i\) that is not facial in \(G\), and for each 1 \(\leq j \leq m\), let \(C_{S_j}\) be the unique facial walk in \(S_j\) that is not facial in \(G\).

By (G4), \(x = s_j\) for some 0 \(\leq j \leq m\). If \(|S_j'| \geq 3\) for some 1 \(\leq j' \leq m\), then we can easily find a 2-separation \((G_1, G_2)\) with \(\{x, s_{j'-1}\} = V(G_1) \cap V(G_2)\) or \(\{x, s_{j'}\} = V(G_1) \cap V(G_2)\), which was already done in Claim 1. Thus, for each 1 \(\leq j' \leq m\), \(|S_{j'}| \leq 2\).

By the same argument and symmetry, we only have the following four possibilities.\(^6\)

(a) \(m = 1\), \(|S_1| = 1\), \(s_0 = s_1 = x\) and \(w \in V(C_1[u_1, v_1])\).
(b) \(m = 1\), \(|S_1| = 2\), \(s_0 = x \neq s_1\) and \(w \in V(C_1[u_1, v_1])\).
(c) \(m = 1\), \(|S_1| = 2\), \(s_0 = x \neq s_1\) and \(w = s_1\).
(d) \(m = 2\), \(|S_1| = |S_2| = 2\), \(s_1 = x\) and \(w = s_2\).

(Note that Case (a) means that the chain \(s_0, S_1, \ldots, S_m, s_m\) of plane blocks is trivial.) In either case, \(y \in V(G_0) \cup \bigcup_{i=1}^{l} V(R_i)\). So we also have the following five possibilities:

(1) \(y = v_1\) or \(y = v_2\).
(2) \(y \in V(G_0) - \{u_1, u_2, v_1, v_2\}\).
(3) \(l = 1\), \(|R_1| = 1\) and \(y = r_0 = r_l\).
(4) \(y = r_l \neq r_0\).
(5) \(y \in \bigcup_{i=1}^{l} V(R_i)\) and \(y \neq r_l\).

Case (1)–(4).

In these cases, we use the following paths; By Theorem 3, for each 1 \(\leq i \leq l\), \(R_i\) has a \(C_{R_i}\) Tutte path \(T_{R_i}\) from \(r_{i-1}\) to \(r_i\).

Case (1), (c-2) or (d-2).

\(^6\)In the rest of Section 6.2, we deal with the vertices \(x\) and \(y\) also as vertices in \(G_0\) and/or a vertex in \(\bigcup_{i=1}^{l} R_i \cup \bigcup_{j=1}^{m} S_j\). This situation is same for other vertices, for example, \(u_1, u_2, v_1, v_2, s_0, s_1\) and so on.
Since $y \neq x$, Case (a-1) does not occur. By the same reason (for Case (b-1) or (c-1)) or symmetry (for Case (d-1)), we may assume that $y = v_2$. (In Case (d-1), we do not use the vertex $w$ and show that $G$ satisfies (T1). Therefore we can indeed use the symmetry between $s_0$ and $s_2$.) If Case (c-2) or (d-2) occurs, then trivially $y \neq u_1, u_2, v_1, v_2$. In either case, we have $y \neq v_2, v_1$. Then by Lemma 8, $G_0$ has two disjoint paths $T_1$ and $T_2$ such that $T_1$ and $T_2$ connect $\{u_1, u_2\}$ and $\{v_1, y\}$. If $|T_1| + |T_2| \geq 5$ and $T_1 \cup T_2$ is a $C_1[u_1, v_1]$-Tutte subgraph in $G_0$. Let

$$T = \begin{cases} T_1 \cup \bigcup_{i=1}^{l} T_{R_i} \cup T_2 & \text{if Case (b-1), (c-1) or (c-2) occurs,} \\ \{x_0\} \cup T_1 \cup \bigcup_{i=1}^{l} T_{R_i} \cup T_2 & \text{if Case (d-1) or (d-2) occurs.} \end{cases}$$

Note that $T$ is a path in $G$ from $x$ to $y$ with $|T| \geq |T_1| + |T_2| - 1 \geq 4$.

Suppose that there exists no $(T_1 \cup T_2)$-bridge $H_0$ of $G_0$ such that $H_0$ contains $v_2$ as a non-attachment and $H_0$ has at least three attachments on $T_1 \cup T_2$. See Figure 13 for Case (d-1). (Note that in Case (1), we have $v_2 = y$, and hence such a $(T_1 \cup T_2)$-bridge $H_0$ cannot exist.) This condition implies that if $v_2 \not \in V(T_1 \cup T_2)$, then the $(T_1 \cup T_2)$-bridge containing $v_2$ has at most two attachments. Note that for every non-trivial $T$-bridge $B$ of $G$, either (i) $B$ is a $(T_1 \cup T_2)$-bridge of $G_0$ and $B$ does not contain $v_2$ as a non-attachment, or (ii) $B$ contains $v_2$ as a non-attachment, or (iii) $B$ is a $T_{R_i}$-bridge of $R_i$ for some $1 \leq i \leq l$. Suppose first that $B$ satisfies (i). Then since $T_1 \cup T_2$ is a $C_1[u_1, v_1]$-Tutte subgraph in $G_0$ and $B$ does not contain $v_2$ as a non-attachment, $B = B \cap G_0$ has at most three attachments on $T_1 \cup T_2$. Suppose next that $B$ satisfies (ii). Since $B$ contains $v_2$ as a non-attachment, it follows from the assumption of this paragraph that $B$ has at most three attachments one of which is $x$. If $B$ satisfies (iii), then $B$ has at most three attachments, since $T_{R_i}$ is a $C_{R_i}$-Tutte path in $R_i$ for $1 \leq i \leq l$. Thus, these imply that $T$ is an extended $C$-Tutte path in $G$, and $G$ satisfies (T1).

Suppose next that there exists a $(T_1 \cup T_2)$-bridge $H_0$ of $G_0$ such that $H_0$ contains $v_2$ as a non-attachment and $H_0$ has at least three attachments on $T_1 \cup T_2$. As mentioned in the second sentence of the previous paragraph, Case (c-2) or (d-2) must occur. See Figure 14 for Case (c-2). Hence we have $w = v_2$. Since $T_1 \cup T_2$ is a $C_1[u_1, v_1]$-Tutte subgraph in $G_0$, $H_0$ has exactly three attachments on $T_1 \cup T_2$. Moreover, one of them is on $C_1[v_2, v_2] - \{v_2\}$, say $a$, and another one is on $C_2[u_2, v_2] - \{v_2\}$, say $c$. Let $d$ be the other attachment of $H_0$ on $T_1 \cup T_2$. Let $H$ be the $T$-bridge of $G$ containing $H_0$. In other words, $H$ consists of $H_0$ and the edge $f = xw$. Let $b = x$, and let $P$ be the subpath of $C$ from $a$ to $b$ such that $E(P) \subset E(H)$. Note that $H$ is an extended $C$-flap of $G$ with attachments $a, b, c, d$ and base path $P$ such that $a, f, x, b$ appear in $P$ in this order. Furthermore, if $d = v_1$ and Case (c-2) occurs, then $H$ is of Type II; otherwise it is of Type I. By the same argument as above, we can check that $G$ satisfies (T2). This completes the proof of Cases (1), (c-2) and (d-2).

**Case (a-2), (b-2), (a-3), (b-3), (a-4) or (b-4).**

In this case, $w \in V(C_1[u_1, v_1])$, and hence $f \in E(C_1[u_1, v_1])$. Since $v_1$ corresponds to $x$ in this case, it follows from Claim 1 that $G_0 - v_1$ is 2-connected. It follows from Lemma 9 with respect to $G_0, C_1, C_2, u_1, v_1, u_2, v_2, y$ and $f$ that $G_0$ satisfies (T1’) or (T2’). Let $T_1$ and $T_2$ be two disjoint paths, and $H$ be an extended $C_1[u_1, v_1]$-flap with attachments $a, v_1, c, d$ and base path $P$ satisfying the conditions in (T1’) or (T2’) in Lemma 9. (If $G_0$ satisfies (T1’), then ignore the definitions on $H, a$, and so on.) Let

$$T = \begin{cases} T_1 \cup \bigcup_{i=1}^{l} T_{R_i} \cup T_2 & \text{if Case (a-2), (a-3) or (a-4) occurs,} \\ \{x_1\} \cup T_1 \cup \bigcup_{i=1}^{l} T_{R_i} \cup T_2 & \text{if Case (b-2), (b-3) or (b-4) occurs.} \end{cases}$$

Since $v_1 \not \in V(T_1 \cup T_2)$, $T$ is a path in $G$ connecting $x$ and $y$ with $|T| \geq |T_1| + |T_2| - 1 \geq 4$.

Suppose first that $G_0$ satisfies (T1’). Notice that every non-trivial $T$-bridge $B$ of $G$ is either (i) a $(T_1 \cup T_2 \cup \{v_1\})$-bridge of $G_0$, or (ii) a $T_{R_i}$-bridge of $R_i$ for some $1 \leq i \leq l$. Hence it follows from (T1’) and the choice of $T_{R_i}$ that $B$ has at most four attachments,
and at most three attachments if it contains an edge in $C_1[u_1, v_1] \cup C_2[v_2, u_2]$ or in $C_{R_1}$. Then by condition (G3), $B$ has at most three attachments if it contains an edge in $C$. Hence $T$ is an extended $C$-Tutte path, and $G$ satisfies (T1).

Suppose next that $G_0$ satisfies (T2'). Note that $H$ is an extended $C$-flap in $G$ with attachments $a, b, c, d$ and base path $P$, where $b = x$. See Figures 15 and 16 for Cases (a-2) and (b-4), respectively. Notice also that if Case (a-2), (a-3) or (a-4) occurs and $d = v_2$, then $H$ is of Type II; otherwise it is of Type I. Moreover, $a, f, x, b$ appear in $P$ in this order. By the same argument as above, $T$ is an extended $C$-Tutte path in $G - (V(H) - \{a, b, c, d\})$. Hence $G$ satisfies (T2), and this completes Cases (a-2), (b-2), (a-3), (b-3), (a-4) and (b-4).

Case (c-3) or (d-3).

In this case, note that $w = v_2$. Let $G'_0 = G_0 - u_2$. Let $C'_2$ be the unique facial walk in $G'_0$ that is not facial in $G_0$. Then by applying Theorem 4 to the block of $G'_0$ containing $v_1$ and $u_1$, $G'_0$ has a $C'_2$-Tutte path $T_0$ from $v_1$ to $u_1$ with $|T_0| \geq 4$. (Note that the exceptional case in Theorem 4 does not occur if we take an appropriate vertex as $u$.) Let

$$T = \begin{cases} T_0 & \text{if Case (c-3) occurs}, \\ \{x_{s_0}\} \cup T_0 & \text{if Case (d-3) occurs}. \end{cases}$$

Note that $T$ is a path in $G$ from $x$ to $y$ with $|T| \geq |T_0| \geq 4$.

Suppose that there exists no $(T_0 \cup \{u_2\})$-bridge $H_0$ of $G_0$ such that $H_0$ contains $v_2$ as a non-attachment and $H_0$ has at least three attachments on $T_0$. Each non-trivial $T$-bridge $B$ of $G$ satisfies that either (i) $B$ is a $(T_0 \cup \{u_2\})$-bridge of $G_0$ that does not contain $v_2$ as a non-attachment, or (ii) $B$ contains $v_2$ as a non-attachment and $B \cap G_0$ is a $(T_0 \cup \{u_2\})$-bridge of $G_0$ having at most two attachments on $T_0$. By the same argument as in Case (1), (c-2) or (d-2), we can show that $T$ is an extended $C$-Tutte path in $G$. Hence $G$ satisfies (T1).

Suppose next that there exists a $(T_0 \cup \{u_2\})$-bridge $H_0$ of $G_0$ such that $H_0$ contains $v_2$ as a non-attachment and $H_0$ has at least three attachments on $T_0 \cup \{u_2\}$. See Figure 17.
for Case (d-3). Since $H_0$ contains $v_2$ as a non-attachment, $H_0 \cap G'_0$ contains an edge in $C'_2$. This implies that $H_0 \cap G'_0$ has at most two attachments on $T_0$, and hence $H_0$ has exactly three attachments on $T_0 \cup \{u_2\}$ one of which is $u_2$. Note that one of the attachments of $H_0 \cap G'_0$ on $T_0$ is on $C_2[u_2, v_2] - \{u_2, v_2\}$, say $a$. Let $b = x, c = u_2 = y$, and let $d$ be the attachments of $H_0 \cap G'_0$ on $T_0$ with $d \neq a, c$. Let $H$ be the $T$-bridge of $G$ containing $H_0$, and let $P$ be the subpath of $C$ from $a$ to $b$ such that $E(P) \subseteq E(H)$. In other words, $H$ consists of $H_0$ and $f$. In either case, $H$ is an extended $C$-flap of $G$ with attachments $a, b, c, d$ and base path $P$ such that $a, f, x, b$ appear in $P$ in this order. Notice also that when Case (c-3) occurs and $d = v_1$, then $H$ is of Type II; otherwise it is of Type I. By the same argument as above, we can check that $G$ satisfies (T2). This completes the proof of Cases (c-3) and (d-3).

Case (c-4) or (d-4).

In this case, note that $w = v_2$. Then it follows from Theorem 4 that $G_0$ has a $C_1$-Tutte path $T_0$ from $v_1$ to $u_2$ through $u_1$ with $|T_0| \geq 4$. (If the exceptional case in Theorem 4 occurs, then we can find an essential curve on the torus that hits $G$ at only $u_2$, contradicting that the representativity of $G$ is exactly 2.) Let

$$T = \begin{cases} T_0 & \text{if Case (c-4) occurs,} \\ \{x, s_0\} \cup T_0 & \text{if Case (d-4) occurs.} \end{cases}$$

Note that $T$ is a path in $G$ from $x$ to $y$ with $|T| \geq |T_0| \geq 4$.

Suppose that there exists no $T_0$-bridge $H_0$ of $G_0$ such that $H_0$ contains $v_2$ as a non-attachment and $H_0$ has at least three attachments on $T_0$. Note that for each non-trivial $T$-bridge $B$ of $G$, either (i) $B$ is a $T_0$-bridge of $G_0$ that does not contain $v_2$ as a non-attachment, or (ii) $B$ contains $v_2$ as a non-attachment and $B \cap G_0$ is a $T_0$-bridge of $G_0$ having at most two attachments on $T_0$, or (iii) $B = \bigcup_{i=1}^{j} R_i$ that is a $\{u_1, u_2\}$-bridge of $G$. By the same argument as in Case (1), (c-2) or (d-2), we can show that $T$ is an extended $C$-Tutte path. Hence $G$ satisfies (T1).

Suppose that there exists a $T_0$-bridge $H_0$ of $G_0$ such that $H_0$ contains $v_2$ as a non-attachment and $H_0$ has at least three attachments on $T_0$. See Figure 18 for Case (c-4). Since $H_0$ is a $T_0$-bridge of $G_0$ and $T_0$ is a $C_1$-Tutte subgraph in $G_0$, $H_0$ has exactly three attachments on $T_0$. Note that one of the attachments of $H_0$ on $T_0$ is on $C_2[u_2, v_2] - \{v_2\}$, say $a$, and another one is on $C_2[u_2, v_2] - \{v_2\}$, say $c$. Let $d$ be the other attachment of $H_0$ on $T_0$ and let $b = x$. Let $H$ be the $T$-bridge of $G$ containing $H_0$, and let $P$ be the subpath of $C$ from $a$ to $b$ such that $E(P) \subseteq E(H)$. In other words, $H$ consists of $H_0$ and $f$. In either case, $H$ is an extended $C$-flap of $G$ with attachments $a, b, c, d$ and base path $P$ such that $a, f, x, b$ appear in $P$ in this order. Notice also that when Case (c-4) occurs and $d = v_1$, then $H$ is of Type II; otherwise it is of Type I. By the same argument as above, we can check that $G$ satisfies (T2). This completes the proof of Cases (c-4) and (d-4).

Case (5).

If $y = r_0$, then let $p = 0$; Suppose otherwise, that is, suppose that $y \in \bigcup_{i=1}^{j} V(R_i) - \{r_0, r_1\}$. In this case, let $p$ be the integer with $1 \leq p \leq l$ such that $y \in V(R_p) - \{r_{p-1}\}$. By Theorem 3, for each $1 \leq i \leq p - 1$, $R_i$ has a $C_{R_i}$-Tutte path $T_{R_i}$ from $r_{i-1}$ to $r_i$. If $p < l$, then let $T_{R_p}$ be a $C_{R_p}$-Tutte path in $R_p$ from $r_{p-1}$ to $y$ through $r_p$. Otherwise, let $T_{R_p}$ be a path from $r_{p-1}$ to $y$ in $R_p$ with $r_p \notin V(T_{R_p})$ such that $T_{R_p} \cup \{r_p\}$ is a $C_{R_p}[r_p, r_{p-1}]$-Tutte subgraph of $R_p$. Such paths exist by Theorems 3 and 5.

Case (a-5).

Let $G'_0 = G_0 - v_2$. By Claim 1, $G'_0$ is 2-connected. Let $C'_2$ be the unique facial cycle in $G'_0$ that is not facial in $G_0$. Then it follows from Theorem 4 that $G'_0$ has a $C'_2$-Tutte path $T_0$ from $v_1$ to $u_1$ through $u_2$ with $|T_0| \geq 4$. (If the exceptional case in Theorem 4 occurs, then we can find an essential curve on the torus that hits $G$ at only $v_1$, contradicting that
Figure 19: Case (a-5) for the case $p < l$.

Figure 20: Case (b-5) for the case $p = l$.

the representativity of $G$ is exactly 2.) Let

$$T = T_0 \cup \bigcup_{i=1}^{p} T_{R_i}.$$  

See Figure 19 for the case $p < l$. Note that $T$ is a path in $G$ from $x$ to $y$ with $|T| \geq |T_0| \geq 4$. Notice also that for each non-trivial $T$-bridge $B$ of $G$, either (i) $B$ is a $(T_0 \cup \{v_2\})$-bridge of $G_0$, or (ii) $B$ is a $T_{R_i}$-bridge of $R_i$ for some $1 \leq i \leq p - 1$, or (iii) $B$ is a $(T_{R_i} \cup \{r_p\})$-bridge of $R_p$, or (iv) $B = \bigcup_{i=p+1}^{l} R_i$ that is an $\{r_p, u_2\}$-bridge of $G$. (Note that (iv) occurs only when $p < l$.) In either case, by the choice of $T_0$ and $T_{R_i}$, $B$ has at most three attachments. In fact, for (i), $B$ clearly has at most three attachments if $v_2$ is not an attachment; Otherwise, $B \cap G_0$ is a $T_0$-bridge of $G_0$ containing an edge of $C_2$, and hence $B$ also has at most three attachments one of which is $v_2$. Similarly we can show that for other cases. Hence $T$ is a Tutte path in $G$, in particular, an extended $C$-Tutte path in $G$. Therefore $G$ satisfies (T1). This completes Case (a-5).

Case (b-5), (c-5) or (d-5).

It follows from Theorem 4 that $G_0$ has a $C_2$-Tutte path $T_0$ from $v_1$ to $u_1$ through $u_2$ with $|T_0| \geq 4$. (If the exceptional case in Theorem 4 occurs, then we can find an essential curve on the torus that hits $G$ at only $v_1$, contradicting that the representativity of $G$ is exactly 2.) Let

$$T = \begin{cases} T_0 \cup \bigcup_{i=1}^{p} T_{R_i} & \text{if Case (b-5) or (c-5) occurs,} \\ \{x_0\} \cup T_0 \cup \bigcup_{i=1}^{p} T_{R_i} & \text{if Case (d-5) occurs.} \end{cases}$$

See Figure 20 for Case (b-5) with $p = l$. Note that $T$ is a path in $G$ from $x$ to $y$ with $|T| \geq |T_0| \geq 4$. Notice also that for each $T$-bridge $B$ of $G$, either (i) $B$ is a $T_0$-bridge of $G_0$ that does not contain $v_2$ as a non-attachment, or (ii) $B$ contains $v_2$ as a non-attachment and $B \cap G_0$ is a $T_0$-bridge of $G_0$ containing an edge of $C_2$, or (iii) $B$ is a $T_{R_i}$-bridge of $R_i$ for some $1 \leq i \leq p - 1$, or (iv) $B$ is a $(T_{R_p} \cup \{r_p\})$-bridge of $R_p$, or (v) $B = \bigcup_{i=p+1}^{l} R_i$ that is an $\{r_p, u_2\}$-bridge of $G$. (Note that (v) occurs only when $p < l$.) Hence $G$ satisfies (T1). This completes Cases (b-5), (c-5) and (d-5).

6.3 The case where the $(x, F)$-width is at least three and the $x$-width is exactly 2

In this case, we will show that $G$ actually satisfies (T1), and hence we can ignore the edge $f$. Since the $x$-width is exactly 2, there exists a face $F'$ of $G$ such that $x$ is contained in the boundary walk of $F'$ and the $(x, F')$-width is exactly 2. Since the $(x, F)$-width is at least three, we have $F' \neq F$. Let $C'$ be the boundary cycle of $F'$. By Lemma 10, $G$ can be decomposed into a 2-connected plane graph $G_0$ and two chains of plane blocks
occurs, then the same facial cycle in $G$ implies that there exist two attachments one of which is $C$ is a of $G$.

By Claim 1, we may assume that $G$ is 2-connected. If Case (a) occurs, then let $C'_2$ be the unique facial cycle in $G_0$ that is not facial in $G_0$; Otherwise, let $C'_2 = C_2$.

**Case (1) or (2).**

In either case, we use the following paths; By Theorem 3, for each $1 \leq i \leq l$, $R_i$ has a $C_{R_i}$-Tutte path $T_{R_i}$ from $r_{i-1}$ to $r_i$. By Lemma 8, $G_0'$ has two disjoint paths $T_1$ and $T_2$ such that $T_1$ and $T_2$ connect $\{u_1, u_2\}$ and $\{v_1, y\}$, $|T_1| + |T_2| \geq 5$ and $T_1 \cup T_2$ is a $C_1[u_1, v_1]$-Tutte subgraph in $G_0$. Let

$$T = T_1 \cup \bigcup_{i=1}^{l} T_{R_i} \cup T_2.$$  

See Figure 21 for Case (b-2). Note that $T$ is a path in $G$ from $x$ to $y$ with $|T| \geq |T_1| + |T_2| - 1 \geq 4$.

Note that for every non-trivial $T$-bridge $B$ of $G$, either (i) $B$ is also a $(T_1 \cup T_2)$-bridge of $G_0$, or (ii) $B$ is not a $(T_1 \cup T_2)$-bridge of $G_0$, $x$ is an attachment of $B$, and $B - x$ is a $(T_1 \cup T_2)$-bridge of $G_0$, or (iii) $B$ is a $T_{R_i}$-bridge of $R_i$ for some $i$ with $1 \leq i \leq l$.

Suppose first that $B$ satisfies (i). Then $B$ has at most three attachments, since $T_1 \cup T_2$ is a $C_1[u_1, v_1]$-Tutte subgraph in $G_0$.

Suppose next that $B$ satisfies (ii). In this case, $B$ has at most four attachments on $T$ one of which is $x$. Furthermore, suppose that $B$ contains an edge in $C$. This assumption implies that there exist two attachments $a_1$ and $a_2$ of $B$ on $T$ such that both $a_1$ and $a_2$ are contained in $C$. If Case (a-2) occurs, then $v_2$ and one of $a_1$ and $a_2$, say $a_1$, appear in the same facial cycle in $G_0'$. However, this implies the existence of an essential curve on the torus passing through $F$ and hitting $G$ at $a_1$ and $x$. On the other hand, if Case (b-2) occurs, then $B - x$ has three attachments on $T_1 \cup T_2$ in $G_0$, and hence one of $a_1$ and $a_2$,
say $a_1$, is also contained in $C_2$. This again implies the existence of an essential curve on the torus passing through $F$ and hitting $G$ at $a_1$ and $x$. In either case, this contradicts that $(x, F)$-width of $G$ is at least 3. Hence $B$ does not contain an edge in $C$.

Suppose finally that $B$ satisfies (iii). Then $B$ has at most three attachments since $T_{R_i}$ is a $C_{R_i}$-Tutte path in $R_i$ for $1 \leq i \leq l$.

Therefore, $T$ is an extended $C$-Tutte path in $G$, and $G$ satisfies (T1).

**Case (3).**

Let $G''_0 = G'_0 - u_1$, and let $C'_1$ be the unique facial walk in $G''_0$ that is not facial in $G'_0$. Then it follows from Theorem 4 that $G''_0$ has a $C'_1$-Tutte path $T$ from $v_1$ to $u_2$ with $|T| \geq 4$. (Note that the exceptional case in Theorem 4 does not occur if we take an appropriate vertex as $u$.) See Figure 22 for Case (a-3). Note that $T$ can be regarded also as a path in $G$ from $x$ to $y$.

Notice that for every non-trivial $T$-bridge $B$ of $G$, either (i) neither $v_2$ nor $u_1$ is an attachment of $B$ and $B$ is also a $T$-bridge of $G''_0$, or (ii) $B$ is not a $T$-bridge of $G_0$ and $v_2$ is not an attachment of $B$, (iii) $B$ is not a $T$-bridge of $G_0$ and $v_2$ is an attachment of $B$.

If $B$ satisfies (i), then $B$ has at most three attachments on $T$, since $T$ is a $C'_1$-Tutte path in $G''_0$.

Suppose next that $B$ satisfies (ii). Since $B$ is not a $T$-bridge of $G_0$ and $v_2$ is not an attachment of $B$, $u_1$ is an attachment of $B$ and $B \cap G''_0$ is a $T$-bridge of $G''_0$ containing an edge of $C'_1$. Hence $B$ has at most three attachments one of which is $u_1$.

Suppose finally that $B$ satisfies (iii). In this case, it follows from the same arguments as above that $B - v_2$ has at most three attachments on $T$. Then by the same arguments as in (ii) in Case (1) or (2), we see that $B$ has at most four attachments and contains no edge in $C$.

These imply that $T$ is an extended $C$-Tutte path in $G$, and $G$ satisfies (T1).

**Case (4).**

It follows from Theorem 4 that $G'_0$ has a $C_1$-Tutte path $T$ from $v_1$ to $u_2$ through $u_1$ with $|T| \geq 4$. (If the exceptional case in Theorem 4 occurs, then we can find an essential
curve on the torus that hits \( G \) at only \( u_2 \), contradicting that the representativity of \( G \) is exactly 2.) See Figure 23 for Case (b-4). Note that \( T \) can be regarded also as a path in \( G \) from \( x \) to \( y \).

Notice that for every non-trivial \( T \)-bridge \( B \) of \( G \), either (i) \( B \) is also a \( T \)-bridge of \( G_0 \), or (ii) \( B \) is not a \( T \)-bridge of \( G_0 \), \( x \) is an attachment of \( B \), and \( B - x \) is a \( T \)-bridge of \( G_0 \), or (iii) \( B = \bigcup_{i=1}^{l} R_i \) that is an \( \{r_0, r_l\} \)-bridge of \( G_0 \). By the same arguments as in Case (1) or (2), we see that if \( B \) satisfies (i), then \( B \) has at most three attachments, and if \( B \) satisfies (ii), then \( B \) has at most four attachments and contains no edge in \( C \). If \( B \) satisfies (iii), then \( B \) trivially has at most two attachments. Therefore \( T \) is an extended \( C \)-Tutte path in \( G \), and \( G \) satisfies (T1).

**Case (5).**

If \( y = r_0 \), then let \( p = 0 \); Suppose otherwise, that is, suppose that \( y \in \bigcup_{i=1}^{p} V(R_i) - \{r_0, r_l\} \). In this case, let \( l \) be the integer with \( 1 \leq p \leq l \) such that \( y \in V(R_p) - \{r_{p-1}\} \). By Theorem 3, for each \( 1 \leq i \leq p - 1 \), \( R_i \) has a \( C_{R_i} \)-Tutte path \( T_{R_i} \) from \( r_{i-1} \) to \( r_i \). If \( p < l \), then let \( T_{R_p} \) be a \( C_{R_p} \)-Tutte path from \( r_{p-1} \) to \( y \) through \( r_p \) in \( R_p \); Otherwise, let \( T_{R_p} \) be a path from \( r_{p-1} \) to \( y \) in \( R_p \) with \( r_p \notin V(T_{R_p}) \) such that \( T_{R_p} \cup \{r_p\} \) is a \( C_{R_p}[r_p, r_{p-1}] \)-Tutte subgraph of \( R_p \). Such paths exist by Theorems 3 and 5.

It follows from Theorem 4 that \( G' \) has a \( C_{x} \)-Tutte path \( T_0 \) from \( v_1 \) to \( u_1 \) through \( u_2 \) with \( |T_0| \geq 4 \). (If the exceptional case in Theorem 4 occurs, then we can find an essential curve on the torus that hits \( G \) at only \( u_1 \), contradicting that the representativity of \( G \) is exactly 2.) Let

\[
T = T_0 \cup \bigcup_{i=1}^{p} T_{R_i}.
\]

See Figure 24 for Case (a-5) with \( p < l \). Note that \( T \) can be regarded also as a path in \( G \) from \( x \) to \( y \).

Notice that for every non-trivial \( T \)-bridge \( B \) of \( G \), either (i) \( B \) is a \( T_0 \)-bridge of \( G_0 \), or (ii) \( B \) is not a \( T_0 \)-bridge of \( G_0 \), \( x \) is an attachment of \( B \), and \( B - x \) is a \( T_0 \)-bridge of \( G_0 \), or (iii) \( B \) is a \( T_{R_1} \)-bridge of \( R_i \) for some \( 1 \leq i \leq p - 1 \), or (iv) \( B \) is a \( (T_{R_p} \cup \{r_p\}) \)-bridge of \( R_p \), or (v) \( B = \bigcup_{i=p+1}^{l} R_i \) that is an \( \{r_p, u_2\} \)-bridge of \( G \). (Note that (v) occurs only when \( p < l \).) By the same arguments as in Case (1) or (2) or in Case (a-5) in Section 6.2, we see that \( T \) is an extended \( C \)-Tutte path in \( G \), and \( G \) satisfies (T1).

### 6.4 The case where the \( x \)-width of \( G \) is at least three

Now we are ready to proceed to the main part of the proof of Theorem 2. Let \( G' = G - x \). Note that \( G' \) is also 2-connected by Claim 1 and the representativity of \( G' \) is at least two by the assumption of this case. Let \( F' \) be the face of \( G' \) containing \( F \), and let \( C' \) be the boundary cycle of \( F' \). Note that for every 2-separation \((G'_1, G'_2)\) of \( G' \), \( E(C') \cap E(G'_1) \neq \emptyset \) and \( E(C') \cap E(G'_2) \neq \emptyset \).

If \( xy \in E(C) \), then let \( w = y \); Otherwise, that is, if \( xy \notin E(C) \), then let \( w \) be the vertex in \( C \) such that \( f = xw \). In either case, let \( x' \) be the neighbor of \( x \) in \( C \) with \( x' \neq w \), and let \( f' \) be the edge of \( C' \) such that \( f' \) is incident with \( x' \) and \( f' \notin E(C) \). Let \( f' = x'w' \). We may assume that \( x' \), \( x \), \( w \) appear in \( C \) in this order. This order implies that \( x' \), \( f' \), \( w' \), \( w \) appear in \( C' \) in this order and \( E(C'|w, x') = E(C) - \{x'x, xw\} \). See Figure 25.\(^7\) By the induction hypothesis with respect to \( G', C', x', y \) and \( f' \), \( G' \) satisfies (T1) or (T2). Let \( T' \) be an extended \( C' \)-Tutte path, and \( H' \) be an extended \( C' \)-flap with attachments \( a', u', c', d' \) and base path \( P' \) satisfying the conditions in (T1) or (T2) in Theorem 2. (If \( G' \) satisfies

\(^7\)As figures in Section 6.1, in Figure 25, the bottom curve represents a part of \( C \), and the bold line represents the path \( C'[x', u] \) that is edge-disjoint from \( C \). This situation is same for all figures in this subsection.
(T1), then ignore the definitions on $H'$, $a'$ and so on.) We divide this case into three subcases.

**Case I.** $G'$ satisfies (T1).

Let $T = T' \cup \{x'x\}$.

Suppose that either $w \in V(T)$, or $w \notin V(T)$ and the $T$-bridge of $G$ containing $w$ has at most three attachments on $T$. See Figure 26. (Note that if $xy \in E(C)$, then this always occurs since $w = y \in V(T)$.) Let $B$ be a non-trivial $T$-bridge of $G$. Note that $B$ is a $(T' \cup \{x\})$-bridge of $G$. If $x$ is not an attachment of $B$, then $B$ is also a $T'$-bridge of $G'$.

Since $T'$ is an extended $C'$-Tutte subgraph in $G'$, $B$ has at most four attachments, and at most three attachments if $B$ contains an edge of $C'$. On the other hand, suppose that $x$ is an attachment of $B$. Note that $B - x$ is a $T'$-bridge of $G'$ and contains an edge in $E(C'[x', w])$, so $B - x$ has at most three attachments on $T'$. Hence $B$ has at most four attachments on $T' \cup \{x\}$ one of which is $x$. Assume further that $B$ contains an edge in $C$. If $B$ does not contain $w$ as a non-attachment, then $B$ contains an edge in $C[w, x']$, but this contradicts that $G'$ is 2-connected or the $(x, F)$-width is at least three. Therefore, $B$ contains $w$ as a non-attachment. Then it follows from the assumption of this paragraph that $B$ has at most three attachments on $T$. These imply that $T$ is an extended $C$-Tutte path in $G$. Therefore, $G$ satisfies (T1).

Suppose next that $w \notin V(T)$ and the $T$-bridge $H$ of $G$ containing $w$ has at least four attachments on $T$. See Figure 27. Note that $x$ is an attachment of $H$ since $H$ contains $w$ as a non-attachment and $x \in V(T)$. Note that $H - x$ is a $T'$-bridge of $G'$ containing an edge of $C'$, and hence $H - x$ has at most three attachments on $T'$. Notice also that one of the attachments of $H - x$ is on $C[w, x'] - \{w\}$, say $a$, and another one is on $C'[x', w] - \{w\}$, say $c$. Let $d$ be the other attachment of $H - x$ on $T'$, let $b = x$, and let $P$ be the subpath of $C$ between $a$ and $b$ with $E(P) \subseteq E(H)$. Hence $P = \{bw\} \cup C[w, a]$. This implies that $H$ is an extended $C$-flap (of Type I) with attachments $a, b, c, d$ and base path $P$. By the same argument as above, $T$ is an extended $C$-Tutte path in $G - (V(H) - \{a, b, c, d\})$. Hence $G$ satisfies (T2), and completes the proof of Case I.

**Case II.** $G'$ satisfies (T2), and $a'$ is placed in $C[w, x'] - \{x'\}$.

Since $a', f', x', b'$ appear in $P'$ in this order, $b'$ is contained in $C[a', x'] - \{a'\}$. Note that if $xy \in E(C)$, then $w = y \in V(T')$, and hence we have always $a' = w$. Let $H$ be the subgraph of $G$ induced by the edges in $H'$ and all edges incident with $x$ in $G$. Let $a = a'$,

Figure 27: Case I with $T$-bridge $H$ containing $w$ and having at least four attachments.
Figure 29: The extended $C'$-flap $H'$ in $G'$ and the extended $C$-flap $H$ in $G$ in Case II-2 with $f = xw$.

Figure 30: Case II-2 with $xy \in E(C)$.

$b = b', c = c', d = d'$ and $P$ be the subpath of $C$ from $a$ to $b$ with $E(P) \subset E(H)$, that is, $P = (P' - C'[x', w]) \cup \{x'x, xw\}$. Note that $H$ satisfies the conditions of an extended $C$-flap of $G$ with attachments $a, b, c, d$ and base path $P$ (Case II-2), unless $H - \{a, b, c, d\}$ is not connected (Case II-1). Indeed, in the exceptional case, $H$ is not an $\{a, b, c, d\}$-bridge of $G$. Conversely, if $H - \{a, b, c, d\}$ is connected, then $H$ is an extended $C$-flap satisfying the conditions in (T2). It is easy to see that the exceptional case happens only when $b' = x'$, $a' = w$ and $x'$ and $w$ are the only neighbors of $x$ in $G$. Thus, we further divide this case into two subcases, to distinguish the two situations above.

**Case II-1.** $b' = x'$, $a' = w$ and $x'$ and $w$ are the only neighbors of $x$ in $G$.

In this case, $H$ is not an $\{a, b, c, d\}$-bridge of $G$ since $H - \{a, b, c, d\}$ is not connected. See Figure 28. Then let $T = T' \cup \{x'x\}$. By the conditions, $H'$ is a $T$-bridge of $G$ having exactly four attachments. Since the $x$-width is at least 3, $H'$ contains no edges in $C$. Moreover, every non-trivial $T'$-bridge of $G$ that is not $H''$ is also a $T'$-bridge of $G'$. So, $T$ is an extended $C$-Tutte path in $G$, and hence $G$ satisfies (T1).

**Case II-2.** $b' \neq x'$ or $a' \neq w$ or $x$ has a neighbor in $G$ other than $x'$ and $w$.

As mentioned above, $H$ is an extended $C$-flap with attachments $a, b, c, d$ and base path $P$. If $f = xw$ (that is, if $xy \notin E(C)$), then $a, f, x, b$ appear in $P$ in this order. Let $T = T'$, and we can check that $G$ satisfies (T2) in Theorem 2. See Figure 29.

Therefore, we may assume that $xy \in E(C)$. Since $y \in V(T)$, we have $y = w = a'$. Let $H' = H'$, $b = b'$ and $d = d'$ if $H'$ is of Type I; otherwise let $H'$ be the plane graph and $b$ and $d$ be the vertices as in (H3) in the definition of an extended $C$-flap of Type II. Note that $H'$ is a plane graph. Let $\overrightarrow{C_H}$ be the outer facial walk in $H'$ containing $a', x', x, b, c'$ and $d$ in this clockwise order. It follows from Theorem 6 with $A' = \{a', c', d\}$ that $\overrightarrow{H'}$ has a path $T_H'$ from $x$ to $b$ with $V(T_H') \cap A' = \emptyset$ that every $(T_H' \cup A')$-bridge $B$ of $\overrightarrow{H'}$ has at most four attachments and at most two attachments if $B$ contains an edge of $\overrightarrow{C_H}$. See Figure 30. Let $T = T_H' \cup T'$. Note that every non-trivial $T'$-bridge $B$ of $G$ is either (i) a $T'$-bridge of $G' - \{V(H') - \{a', b', c', d'\}\}$, or (ii) a $(T_H' \cup A')$-bridge of $\overrightarrow{H'}$. Therefore, by the conditions for $T'$ and $T_H'$, we can check that $T$ is an extended $C$-Tutte path in $G$. Hence $G$ satisfies (T1).

**Case III.** $G'$ satisfies (T2), and $a'$ is placed in $C[x', w] - \{w\}$.

In this case, depending on the place of the vertex $b'$, we further divide the proof into two subcases; $b'$ is placed either (1) in $C'[w, x']$ or (2) in $C'[a', w] - \{a', w\}$. (If $xy \in E(C)$, then (1) must occur.) See the left side of Figures 31 and 32.

Let $\overrightarrow{H'} = H'$, $b = b'$ and $d = d'$ if $H'$ is of Type I; Otherwise let $\overrightarrow{H'}$ be the plane graph and $b$ and $d$ be the vertices as in (H3) of the definition of an extended $C$-flap of Type II. Let $\overrightarrow{H}$ be the graph obtained from $\overrightarrow{H'} \cup \{x\}$ by adding all edges of $G$ connecting $x$ and a vertex in $C'[x', a']$. Note that when (2) $b'$ is placed in $C'[a', w] - \{a', w\}$, then we do not add any edges connecting $x$ and a vertex in $C'[b', w]$. Notice also that $\overrightarrow{H}$ is a plane graph. Let $\overrightarrow{C_H}$ be the facial walk in $\overrightarrow{H}$ containing $x, x', b, c', d$ and $a'$ in this order. By Theorem
The graph $\tilde{H}$

![Image of Case III-(1)](image1)

**Figure 31:** Case III-(1).

The graph $\tilde{H}$

![Image of Case III-(2)](image2)

**Figure 32:** Case III-(2).

6 with $A' = \{a', c', d\}$, there exists a path $T_H$ in $\tilde{H}$ from $x$ to $b$ with $V(T_H) \cap A' = \emptyset$ such that every $(T_H \cup A')$-bridge $B$ has at most four attachments and at most two attachments if $B$ contains an edge of $\hat{C}_H[x, b]$. See the middle of Figures 31 and 32. Note that we can regard $T_H$ as a path also in $G$, and let $T = T' \cup T_H$.

Suppose first that either $w \in V(T)$, or $w \notin V(T)$ and the $T$-bridge of $G$ containing $w$ has at most three attachments on $T$. See the right side of Figure 32. By the same argument as in Case I, $T$ is an extended $C$-Tutte path in $G$, and hence $G$ satisfies $(T1)$.

Suppose next that $w \notin V(T)$ and the $T$-bridge of $G$ containing $w$ has at least four attachments on $T$. Let $H$ be the $T$-bridge of $G$ containing $w$. Note that $x$ is an attachment of $H$ since $H$ contains $w$ as a non-attachment and $x \in V(T)$. If $(2)$ $b'$ is placed in $C'[a', w] - \{a', w\}$, then $H - x$ is a $(T_H \cup A')$-bridge of $\tilde{H}$ containing an edge of $\hat{C}_H[x, b]$. This implies that $H - x$ has at most two attachments, and hence $H$ has at most three attachments one of which is $x$, a contradiction. Thus, we have that $(1)$ $b'$ is placed in $C'[w, x']$. Then $H$ is a $(T' \cup \{x\})$-bridge of $G - (V(H') - \{a', b', c', d'\})$. This implies that $H - x$ is a $T'$-bridge of $G' - (V(H') - \{a', b', c', d'\})$ containing an edge of $C'$, and hence $H - x$ has exactly three attachments in $G' - (V(H') - \{a', b', c', d'\})$. Note that one of the attachments of $H - x$ is on $C[w, x'] - \{w\}$, say $a$, and another one is on $C'[a', w] - \{w\}$, say $c$. Let $d$ be the other attachment of $H - x$ on $T'$, let $b = x$, and let $P$ be the subpath of $C$ from $a$ to $b$ such that $E(P) \subseteq E(H)$. Hence $P = \{bw\} \cup C[w, a]$. This implies that $H$ is an extended $C$-flap (of Type I) with attachments $a, b, c, d$ and base path $P$. Moreover, $a, f, x, b$ appear in $P$ in this order. See the right side of Figure 31.

Now we will show that $T$ is an extended $C$-Tutte path in $G - (V(H) - \{a, b, c, d\})$ satisfying the desired conditions. Let $B$ be a non-trivial $T$-bridge of $G - (V(H) - \{a, b, c, d\})$. Note that $B$ is either (i) a $(T' \cup \{x\})$-bridge of $G - (V(H') - \{a', b', c', d'\})$, or (ii) a $(T_H \cup A')$-bridge of $\tilde{H}$. If $B$ satisfies (i), then by the same argument as in Case I, we can show that $B$ has at most four attachments and at most three attachments if $B$ contains an edge of $C$. Next suppose that $B$ satisfies (ii). Recall that $B$ has at most four attachments on $T_H \cup A'$ and exactly two attachments if $B$ contains an edge of $\hat{C}_H[x, b]$. Since the $x$-width of $G$ is at least three, we have that $E(C) \cap E(\hat{H}) = E(C'[b', x]) = E(\hat{C}_H[x, b])$. So, if $B$ contains an edge of $C$, then $B$ contains an edges of $\hat{C}_H[x, b]$, and hence $B$ has exactly two attachments. These imply that $T$ is an extended $C$-Tutte path in $G - (V(H) - \{a, b, c, d\})$. 


Hence $G$ satisfies (T2) and this completes the proof of Theorem 2. \qed

7 Proofs of Lemmas

7.1 Proof of Lemma 8

Let $G$ be a 2-connected plane graph, let $C_1, C_2$ be facial cycles in $G$, let $u_1, v_1 \in V(C_1) - V(C_2)$ with $u_1 \neq v_1$, let $u_2 \in V(C_2) - V(C_1)$, and let $y \in V(G) - \{u_2,v_1\}$. We divide the proof into two cases.

Case I. There exists no path in $G - V(C_1[u_1,v_1])$ from $u_2$ to $y$.

In this case, either there exists a 2-separation $(G_1,G_2)$ of $G$ with $V(G_1) \cap V(G_2) \subseteq V(C_1[u_1,v_1])$ such that $G_1 - V(G_2)$ contains $u_2$ and $G_2 - V(G_1)$ contains $y$, or $y \in V(C_1[u_1,v_1])$. Suppose first that the former occurs. Then there exists a 2-separation $(G_1,G_2)$ of $G$, say $\{z_1,z_2\} = V(G_1) \cap V(G_2)$, such that $z_1,z_2 \in V(C_1[u_1,v_1])$, $z_1$ is closer to $u_1$ on $C_1[u_1,v_1]$ than $z_2$, $u_2 \in V(G_1) - V(G_2)$ and $y \in V(G_2) - \{z_2\}$. In this case, we take such a 2-separation $(G_1,G_2)$ so that $|G_2|$ is as large as possible. On the other hand, suppose otherwise. In this case, let $G_1 = G$, $z_1 = y$, $z_2$ be the neighbor of $y$ in $C_1[y,v_1]$, and $G_2$ be the graph consisting of only $y,z_2$.

Let $G' = G - V(C_1[u_1,z_1])$ and let $C'_1$ be the unique facial walk in $G'$ that is not facial in $G$. Since $G$ is a 2-connected plane graph and $u_2 \neq v_1$, it follows from the maximality of $|G_2|$ that $u_2,v_1$ and $z_2$ are contained in the same block of $G'$. Then by Theorem 4, $G'$ has a $C'_1$-Tutte path $T_1$ from $u_2$ to $v_1$ through $z_2$ with $|T_1| \geq 4$. (To be exact, if $z_2 = v_1$, then such a path $T_1$ exists by taking an appropriate vertex as $u$ in Theorem 4. If $z_2 \neq v_1$ and the exceptional case in Theorem 4 occurs, then $v_1$ is contained in both $C_1$ and $C_2$, a contradiction.)

By Theorem 5 or trivially, $G_2$ has a $C_1[z_1,z_2]$-Tutte subgraph consisting of $z_2$ and a path $T_{G_2}$ from $z_1$ to $y$ with $z_2 \notin V(T_{G_2})$.

Let $B$ be the set of $(T_1 \cup C_1[u_1,z_1])$-bridges $B$ of $G$ such that $B$ has at least two attachments on $C_1[u_1,z_1]$. For each $B \in B$, let $\alpha_B$ and $\beta_B$ be the attachments of $B$ on $C_1[u_1,z_1]$ such that $\alpha_B$ is as close to $u_1$ on $C_1[u_1,z_1]$ as possible and $\beta_B$ is as close to $z_1$ on $C_1[u_1,z_1]$ as possible. Since $B$ has at least two attachments on $C_1[u_1,z_1]$, we have $\alpha_B \neq \beta_B$ for any $B \in B$. For $B,B' \in B$, we write $B' \leq B$ if either (i) $B = B'$ or (ii) $B'$ is contained in the disk bounded by $P \cup C_1[\alpha_B,\beta_B]$, where $P$ is a path in $B$ connecting $\alpha_B$ and $\beta_B$. Since $G$ is a plane graph, $\leq$ is a partial order on $B$. Let $\mathcal{B}$ be the set of maximal elements of $B$ with respect to the partial order $\leq$. By the planarity, for any $B,B' \in \mathcal{B}$, $C_1[\alpha_B,\beta_B]$ and $C_1[\alpha_B',\beta_B']$ are disjoint.

Let $B \in \mathcal{B}$, and let $A_B = V(B) \cap V(T_2)$. Since $B$ has at least two attachments on
$C_1[u_1, z_1], B \cap G'$ is a $T_1$-bridge of $G'$ containing an edge of $C_1'$, and hence $|A_B| \leq 2$. Let $B^*$ be the subgraph of $G$ induced by the union of all elements $B' \in \mathcal{B}$ such that $B' \subseteq B$, together with $C_1[\alpha_B, \beta_B]$. Note that $C_1[\alpha_B, \beta_B]$ is a subpath of the unique facial walk in $B^*$ that is not facial in $G$, and $V(C_1[\alpha_B, \beta_B]) \cap A_B = \emptyset$. Then by Theorem 6, $B^*$ has a path $T_B$ from $\alpha_B$ to $\beta_B$ such that $V(T_B) \cap A_B = \emptyset$ and $T_B \cup A_B$ is a $C_1[\alpha_B, \beta_B]$-Tutte subgraph. Let

$$T_2 = \left( C_1[u_1, z_1] - \bigcup_{B \in \mathcal{B}} C_1[\alpha_B, \beta_B] \right) \cup \bigcup_{B \in \mathcal{B}} T_B \cup T_{G_2}.$$

Note that $T_2$ is a path in $G$ from $u_1$ to $y$ such that $T_2$ is disjoint from $T_1$. Note that $|T_1| + |T_2| \geq 5$. See Figure 33. We will show that $T_1 \cup T_2$ is a $C_1[u_1, v_1]$-Tutte subgraph in $G$.

Let $D$ be a non-trivial ($T_1 \cup T_2$)-bridge of $G$. By the choice of $\mathcal{B}$, $D$ is either (i) a $(T_1 \cup C_1[u_1, z_1])$-bridge of $G$ having at most one attachment on $C_1[u_1, z_1]$, or (ii) a $(T_{G_2} \cup \{z_2\})$-bridge of $G_2$, or (iii) a $(T_B \cup A_B)$-bridge of $B^*$ for some $B \in \mathcal{B}$.

Suppose first that $D$ satisfies (i). If $D$ has no attachment on $C_1[u_1, z_1]$, then $D$ is also a $T_1$-bridge of $G'$, and hence $D$ has at most three attachments and at most two attachments if $D$ contains an edge in $C_1'$. Note that $C_1[z_2, v_1]$ is a subpath of $C_1'$ and $G' \cap C_1[u_1, v_1] = C_1[z_2, v_1]$, and hence if $D$ contains an edge in $C_1[u_1, v_1]$, then $D$ has at most two attachments. So we may assume that $D$ has an attachment on $C_1[u_1, z_1]$. This implies that $D \cap G'$ is a $T_1$-bridge of $G'$ containing an edge of $C_1'$, and hence $D \cap G'$ has at most two attachments on $T_1$. Hence $D$ has at most three attachments one of which is on $C_1[u_1, z_1]$. Furthermore, since $v_1, z_2 \in V(T_1)$, $D$ cannot contain an edge in $C_1[z_2, v_1]$.

Suppose that $D$ satisfies (ii). Recall that $T_{G_2} \cup \{z_2\}$ is a $C_1[z_1, z_2]$-Tutte subgraph in $G_2$. Thus, $D$ has at most three attachments on $T_{G_2} \cup \{z_2\}$ and at most two attachments if $D$ contains an edge in $C_1[z_1, z_2]$. Note that $C_1[u_1, v_1] \cap G_2 = C_1[z_1, z_2]$.

Suppose that $D$ satisfies (iii). Recall that $T_B \cup A_B$ is a $C_1[\alpha_B, \beta_B]$-Tutte subgraph of $B^*$. Thus, $D$ has at most three attachments on $T_B \cup A_B$ and at most two attachments if $D$ contains an edge in $C_1[\alpha_B, \beta_B]$. Note that $C_1[u_1, v_1] \cap B^* = C_1[\alpha_B, \beta_B]$.

Therefore, $T_1 \cup T_2$ is a $C_1[u_1, v_1]$-Tutte subgraph in $G$, and this completes the proof of Case I.

**Case II.** There exists a path in $G - V(C_1[u_1, v_1])$ from $u_2$ to $y$.

Let $G' = G - V(C_1[u_1, v_1])$, and let $C_1'$ be the unique facial walk in $G'$ that is not facial in $G$. Note that $C_1[u_1, v_1] - \{u_1, v_1\}$ is contained in $C_1'$. Let $w$ be a neighbor of $u_1$ in $C_1$ with $w \in V(C_1[v_1, u_1])$, except for the case when $C_1[v_1, u_1]$ consists of only an edge $v_1 u_1$. In the exceptional case, such a neighbor $w$ does not exist, and then let $w$ be any neighbor of $u_1$ in $C_1'$. Let $e$ be an edge of $C_1'$ such that $G'$ has a path from $u_2$ to $y$ through $e$, and we take such an edge $e$ so that it is as close to $w$ on $C_1'$ as possible. Moreover, we can take such an edge $e$ so that $e \neq u_2 y$. It follows from Theorem 3 that $G'$ has a $C_1'$-Tutte path $T_1$ from $u_2$ to $y$ through $e$. It is easy to see that $|T_1| \geq 3$.

Now we deal with $(T_1 \cup C_1[u_1, v_1])$-bridges of $G$ as in Case I, but this case is slightly more complicated because of the existence of a special bridge, defined below. Let $\mathcal{B}$ be the set of $(T_1 \cup C_1[u_1, v_1])$-bridges $B$ of $G$ such that $B$ has at least two attachments on $C_1[u_1, v_1]$. An element $H$ in $\mathcal{B}$ is said to be special if both $u_1$ and $v_1$ are attachments of $H$ and $C_1[u_1, v_1] \cup P$ bounds a region containing $C_2$, where $P$ is a path in $H$ from $u_1$ to $v_1$; Otherwise it is non-special.

For each $H \in \mathcal{B}$ that is special, let $\alpha_H$ and $\beta_H$ be the attachments of $H$ on $C_1[u_1, v_1]$ such that all attachments of $H$ on $C_1[u_1, v_1]$ appear in $C_1[u_1, \alpha_H] \cup C_1[\beta_H, v_1]$ and $C_1[\alpha_H, \beta_H] \cup P$ bounds a region containing $C_2$, where $P$ is a path in $H$ from $\alpha_H$ to $\beta_H$. Possibly $\alpha_H = u_1$ and/or $\beta_H = v_1$. For each $B \in \mathcal{B}$ that is not special, let $\alpha_B$ and $\beta_B$ be the attachments of $B$ on $C_1[u_1, v_1]$ such that $\alpha_B$ is as close to $u_1$ on $C_1[u_1, v_1]$ as possible and $\beta_B$ is as close to $v_1$ on $C_1[u_1, v_1]$ as possible. Note that $\alpha_B \neq \beta_B$ for any $B \in \mathcal{B}$.

For $B, B' \in \mathcal{B}$, we write $B' \leq B$ if either (i) $B = B'$, or (ii) $B$ is special and $B'$ is contained in a disk bounded by $P \cup C_1[\beta_B, \alpha_B]$; or (iii) $B$ is non-special and $B'$ is contained
in a disk bounded by $P \cup C_1[\alpha_B, \beta_B]$, where $P$ is a path in $B$ connecting $\alpha_B$ and $\beta_B$. Since $G$ is a plane graph, $\preceq$ is a partial order on $B$. Let $\widehat{B}$ be the set of maximal elements $B$ of $B$ with respect to the partial order $\preceq$. Note that at most one element in $\widehat{B}$ is special. For any $B, B' \in \widehat{B}$ such that both $B$ and $B'$ are non-special, $C_1[\alpha_B, \beta_B]$ and $C_1[\alpha_B, \beta_B]$ are disjoint.

Let $B$ be a non-special element in $\widehat{B}$, and let $A_B = V(B) \cap V(T_1)$. Since $B$ has at least two attachments on $C_1[u_1, v_1]$, $B \cap G'$ is a $T_1$-bridge of $G'$ containing an edge of $C_1'$, and hence $|A_B| \leq 2$. Let $B^*$ be the subgraph of $G$ induced by the union of all elements $B' \in \widehat{B}$ such that $B' \preceq B$, together with $C_1[\alpha_B, \beta_B]$. Note that $C_1[\alpha_B, \beta_B]$ is a subpath of the unique facial walk in $B^*$ that is not facial in $G$. By Theorem 6, $B^*$ has a path $T_B$ from $\alpha_B$ to $\beta_B$ such that $V(T_B) \cap A_B = \emptyset$ and $T_B \cup A_B$ is a $C_1[\alpha_B, \beta_B]$-Tutte subgraph.

We divide the rest of the proof into two cases, depending on whether $G$ has a special $(T_1 \cup C_1[u_1, v_1])$-bridge.

**Case II-1.** There exists no special $(T_1 \cup C_1[u_1, v_1])$-bridge of $G$.

Let

$$T_2 = \left( C_1[u_1, v_1] - \bigcup_{B \in \widehat{B}} C_1[\alpha_B, \beta_B] \right) \cup \bigcup_{B \in \widehat{B}} T_B.$$  

Note that $T_2$ is a path of $G$ from $u_1$ to $v_1$ that is disjoint from $T_1$. See Figure 34. We will show that $T_1 \cup T_2$ is a $C_1[u_1, v_1]$-Tutte subgraph in $G$.

Let $D$ be a non-trivial $(T_1 \cup T_2)$-bridge of $G$. By the choice of $T_1$ and the assumption of Case II-1, $D$ is either (i) a non-special $(T_1 \cup C_1[u_1, v_1])$-bridge of $G$ having at most one attachment on $C_1[u_1, v_1]$ or (ii) a $(T_B \cup A_B)$-bridge of $B^*$ for some $B \in \widehat{B}$. Suppose first that $D$ satisfies (i). If $D$ has no attachment on $C_1[u_1, v_1]$, then $D$ is also a $T_1$-bridge of $G'$, and hence $D$ has at most three attachments on $T_1$ and no attachment on $T_2$. So we may assume that $D$ has an attachment on $C_1[u_1, v_1]$. This implies that $D \cap G'$ is a $T_1$-bridge of $G'$ containing an edge of $C_1'$, and hence $D \cap G'$ has at most two attachments on $T_1$. Since $D$ has only one attachment on $C_1[u_1, v_1]$, $D$ has no edge in $C_1[u_1, v_1]$. If $D$ satisfies (ii), then $D$ has at most three attachments since $T_B \cup A_B$ is a $C_1[\alpha_B, \beta_B]$-Tutte subgraph of $B^*$. In either case, $D$ has at most three attachments, and hence $T_1 \cup T_2$ is a $C_1[u_1, v_1]$-Tutte subgraph in $G$. This completes Case II-1.

**Case II-2.** There exists a special $(T_1 \cup C_1[u_1, v_1])$-bridge of $G$.

In this case, there exists the unique element, say $H$, in $\widehat{B}$ such that $H$ is special. See Figure 35.\footnote{In this figure, the special $(T_1 \cup C_1[u_1, v_1])$-bridge $H$ is also represented by rising diagonals stroke from bottom left to top right.} Let $H^*$ be the subgraph of $G$ induced by the union of all elements $B' \in \widehat{B}$ such that $B' \preceq H$, together with $C_1[u_1, \alpha_H] \cup C_1[\beta_H, v_1]$. Note that $w \in V(H^*) \cap V(C_1')$, and $e \not\in E(H^*)$ by the choice of $e$. Let $A_H$ be the set of attachments of $H^*$ on $T_1$. If $|A_H| \geq 2$,
then \( H^* \cap C'_1 \) has an edge \( e' \) such that \( G' \) has a path from \( u_2 \) to \( y \) through \( e' \). (Consider the path obtained by the concatenation of the subpath of \( T_1 \) from \( u_2 \) to a vertex in \( A_H \), the path in \( H^* \) connecting two vertices in \( A_H \) through \( e' \), and the subpath of \( T_1 \) from the other vertex in \( A_H \) to \( y \).) Since \( w \in V(H) \cap V(C'_1) \), \( e' \) is closer to \( w \) than \( e \), contradicting the choice of \( e \). Therefore, we have \( |A_H| \leq 1 \).

If \( A_H \neq \emptyset \) then let \( \{z\} = A_H \); otherwise let \( z \) be a new vertex, so \( z \notin V(H^*) \). Let \( H^{**} \) be the graph obtained from \( H^* \cup \{z\} \) by adding an edge \( f_1 \) connecting \( \alpha_H \) and \( z \) and an edge \( f_2 \) connecting \( z \) and \( \beta_H \) so that \( C_1[u_1, \alpha_H] \cup \{f_1, f_2\} \cup C_1[\beta_H, v_1] \cup C_1[v_1, u_1] \) forms the outer facial walk in \( H^{**} \). By Theorem 7, \( H^{**} \) has a \((C_1[u_1, \alpha_H] \cup \{f_1, f_2\} \cup C_1[\beta_H, v_1])\)-Tutte path \( T_H \) from \( u_1 \) to \( v_1 \) with \( f_1, f_2 \in E(T_H) \). Deleting the vertex \( z \) together with the two edges \( f_1 \) and \( f_2 \) from \( T_H \), we obtain the two disjoint paths \( T_\alpha \) and \( T_\beta \) such that \( T_\alpha \) connects \( u_1 \) and \( \alpha_H \), and \( T_\beta \) connects \( \beta_H \) and \( v_1 \).

Let \( \mathcal{B}' = \mathcal{B} - \{H\} \). Note that for any \( B \in \mathcal{B}' \), we have \( C_1[\alpha_B, \beta_B] \subset C_1[\alpha_H, \beta_H] \). Let

\[
T_2 = T_\alpha \cup \left( C_1[\alpha_H, \beta_H] - \bigcup_{B \in \mathcal{B}\setminus\mathcal{B}'} C_1[\alpha_B, \beta_B] \right) \cup \bigcup_{B \in \mathcal{B}'} T_B \cup T_{\beta}.
\]

Note that \( T_2 \) is a path of \( G \) from \( u_1 \) to \( v_1 \) that is disjoint from \( T_1 \).

Let \( D \) be a \((T_1 \cup T_2)\)-bridge of \( G \). By the choice of \( T_2 \), \( D \) is either (i) a non-special \((T_1 \cup C_1[u_1, v_1])\)-bridge of \( G \) having at most one attachment on \( C_1[u_1, v_1] \), or (ii) a \((T_B \cup A_B)\)-bridge of \( B^* \) for some \( B \in \mathcal{B} \), or (iii) a \((T_H \cup A_H)\)-bridge of \( H^{**} \).

By the same argument as in Case I-1 or by the fact that \( T_H \) is a \((C_1[u_1, \alpha_H] \cup \{f_1, f_2\} \cup C_1[\beta_H, v_1])\)-Tutte path in \( H^{**} \), in either case, \( D \) has at most three attachments, and hence \( T_1 \cup T_2 \) is a \((C_1[u_1, v_1])\)-Tutte subgraph in \( G \). This completes the proof of Lemma 8. \( \square \)

### 7.2 Proof of Lemma 9

Let \( G \) be a 2-connected plane graph, let \( C_1, C_2 \) be facial cycles in \( G \), let \( u_1, v_1 \in V(C_1) - V(C_2) \) with \( u_1 \neq v_1 \), let \( u_2, v_2 \in V(C_2) - V(C_1) \) with \( u_2 \neq v_2 \), let \( y \in V(G) - \{u_1, v_1, v_2\} \), and let \( f \) be an edge of \( C_1[u_1, v_1] \) incident with \( v_1 \). Suppose that \( G - v_1 \) is also 2-connected.

Let \( G' = G - v_1 \), and let \( C'_1 \) be the unique facial walk in \( G' \) that is not facial in \( G \). Then it follows from applying Lemma 8 with changing the roles of the index \( i \) for \( C_i, u_i \) and \( v_i \) that \( G' \) has two disjoint paths \( T_1 \) and \( T_2 \) such that \( T_1 \) and \( T_2 \) connect \( \{u_1, u_2\} \) and \( \{v_2, y\} \), and \( T_1 \cup T_2 \) is a \((C_2[v_2, u_2])\)-Tutte subgraph in \( G' \). Note that \( T_1 \) and \( T_2 \) are also paths in \( G \) with \( v_1 \notin V(T_1 \cup T_2) \).

Suppose first that there exists no \((T_1 \cup T_2 \cup \{v_1\})\)-bridge of \( G \) containing an edge of \( C_1[u_1, v_1] \) and having at least four attachments on \( T_1 \cup T_2 \cup \{v_1\} \). By the choice of \( T_1 \) and \( T_2 \), each \((T_1 \cup T_2 \cup \{v_1\})\)-bridge \( B \) of \( G \) has at most four attachments. Suppose that \( B \) contains an edge of \( C_1[u_1, v_1] \cup C_2[v_2, u_2] \). If \( B \) contains an edge of \( C_1[u_1, v_1] \), then it follows from the assumption of this paragraph that \( B \) has at most three attachments. Then we may assume that \( B \) contains an edge of \( C_2[v_2, u_2] \). Since \( T_1 \cup T_2 \) is a \((C_2[v_2, u_2])\)-Tutte subgraph in \( G' \), it has at most two attachments, and hence \( B \) has at most three attachments. Therefore, \( T_1 \cup T_2 \cup \{v_1\} \) is an extended \((C_1[u_1, v_1] \cup C_2[v_2, u_2])\)-Tutte subgraph in \( G \). Thus, \( G \) satisfies \((T1')\).

Suppose next that there exists a \((T_1 \cup T_2 \cup \{v_1\})\)-bridge \( H \) of \( G \) containing an edge of \( C_1[u_1, v_1] \) and having at least four attachments on \( T_1 \cup T_2 \cup \{v_1\} \). Since \( H \cap G' \) is a \((T_1 \cup T_2)\)-bridge of \( G' \), \( H \cap G' \) has at most three attachments on \( T_1 \cup T_2 \), and hence \( H \) has exactly four attachments one of which is \( v_1 \). Let \( a \) be the attachment of \( H - v_1 \) on \( C_1[u_1, v_1] - \{v_1\} \) and let \( c \) and \( d \) be the attachments of \( H - v_1 \) on \( T_1 \cup T_2 \) such that \( a, v_1, c, d \) appear in a facial walk of \( H \) in this order. Let \( P \) be the subpath of \( C_1[u_1, v_1] \) from \( a \) to \( v_1 \). Then \( H \) is an extended \((C_1[u_1, v_1])\)-flap with attachments \( a, v_1, c, d \) and base path \( P \) such that \( a, c, d \in V(T_1 \cup T_2) \). By the same argument as the previous case, \( T_1 \cup T_2 \cup \{v_1\} \) is an extended \((C_1[u_1, v_1] \cup C_2[v_2, u_2])\)-Tutte subgraph in \( G - (V(H) - \{a, v_1, c, d\}) \). Moreover, we can check that \( G \) satisfies \((T2')\).
This completes the proof of Lemma 9. □

7.3 Proof of Lemma 10

Let $G$ be a 2-connected graph on the torus with the representativity exactly 2. Let $F$ be a face of $G$, let $C$ be the boundary cycle of $F$ and let $x \in V(C)$. Suppose that the $(x,F)$-width of $G$ is exactly 2.

Since the $(x,F)$-width of $G$ is exactly 2, there exists an essential closed curve $\gamma$ on the torus that passes through $F$ and hits $G$ at exactly two vertices, which are vertices in $C$, say $u$, and $x$. Cutting open the torus along $\gamma$, we obtain a connected plane graph $G'$ with two facial walks $C'_1$ and $C'_2$ and four distinct vertices $u'_1$, $u'_2$, $x'_1$ and $x'_2$ with $u'_1, x'_1 \in V(C'_1)$, $u'_2, x'_2 \in V(C'_2)$ and the following property: $G$ is obtained from $G'$ by identifying $u'_1$ and $u'_2$ into $u$, and $x'_1$ and $x'_2$ into $x$. Moreover, we may assume that $C'_1[u'_1, x'_1] \cup C'_2[x'_2, u'_2]$ becomes $C$ after these identifications.

Suppose that $G'$ has a 1-separation $(K_1, K_2)$. If $|V(K_1) \cap \{u'_1, x'_1, u'_2, x'_2\}| \geq 2$ for each $i = 1, 2$, then there exists an essential curve on the torus which passes through $G$ only at $V(K_1 \cap K_2)$, contradicting that the representativity of $G$ is at least 2. Thus, by symmetry, we may assume that $|V(K_1) \cap \{u'_1, x'_1, u'_2, x'_2\}| \leq 1$. On the other hand, if $V(K_1 - K_2) \cap \{u'_1, x'_1, u'_2, x'_2\} = \emptyset$, then $(K_1, K_2)$ is a 1-separation of $G$, a contradiction, where $K_1 = G - V(K_1 - K_2)$. Therefore, $|V(K_1 - K_2) \cap \{u'_1, x'_1, u'_2, x'_2\}| = 1$. Then we may assume that for each 1-separation $(K_1, K_2)$ of $G'$, $|V(K_1 - K_2) \cap \{u'_1, x'_1, u'_2, x'_2\}| = 1$.

If $G'$ has at least one 1-separation, then let

$$G_0 = \bigcap_{(K_1, K_2): 1 \text{-separation of } G'} K_2;$$

otherwise, that is, if $G'$ does not have any 1-separations, then let $G_0 = G'$. Note that $G_0$ is 2-connected and has both edges in $C'_1$ and $C'_2$. For $i = 1, 2$, let $C_i$ be the facial cycle in $G_0$ with $E(C_i) \subset E(C'_i)$. If there exists a 1-separation $(K_1, K_2)$ of $G'$ with $\{u'_1\} = V(K_1 - K_2) \cap \{u'_1, x'_1, u'_2, x'_2\}$, then let $u_1 \in V(G_0)$ such that $u_1$ separates $u'_1$ from $G_0$ in $G'$; otherwise let $u_1 = u'_1$. If there exists a 1-separation $(K_1, K_2)$ of $G'$ with $\{x'_1\} = V(K_1 - K_2) \cap \{u'_1, x'_1, u'_2, x'_2\}$, then let $v_1 \in V(G_0)$ such that $v_1$ separates $x'_1$ from $G_0$ in $G'$; otherwise let $v_1 = x'_1$. Similarly, we define $v_2$ and $v_2$, respectively. Note that $u_1, v_1 \in V(C_1)$ and $u_2, v_2 \in V(C_2)$. If $u_1 \in V(C_2)$, then there exists an essential curve on the torus that hits $G$ only at $u_1$, and hence the representativity of $G$ is exactly one, a contradiction. Thus, we have that $u_1 \notin V(C_2)$. Similarly, $v_1 \notin V(C_2)$ and $u_2, v_2 \notin V(C_1)$.

Hence condition (G1) holds.

Note that $G' - (V(G_0) - \{u_1, v_1, u_2, v_2\})$ has exactly four components each of which contains exactly one vertex in $\{u'_1, x'_1, u'_2, x'_2\}$. Hence each of the components is a chain of plane blocks. Then identifying $u'_1$ and $u'_2$, and $x'_1$ and $x'_2$, respectively, we obtain two chains of plane blocks, say $r_0, R_1, r_1, R_2, \ldots, r_{n-1}, R_l, r_l, s_0, S_1, \ldots, s_{m-1}, S_m, s_m$, which form $G - (V(G_0) - \{u_1, v_1, u_2, v_2\})$. By symmetry, we may assume that $G$ is obtained from $G_0$ and the two chains of plane blocks by identifying $u_1$ and $u_0, r_0, u_2$ and $r_1, v_1$ and $s_0$, and $v_2$ and $s_m$, respectively. Then condition (G2) also holds.

For each $1 \leq i \leq l$ and each $1 \leq j \leq m$, let $C_{R_i}$ (resp. $C_{S_j}$) be the facial walk in $R_i$ (resp. $S_j$) that is not facial in $G$. Since $C'_1$ and $C'_2$ become $C$, by symmetry, we may also assume that $E(C) = E(C_1[u_1, v_1]) \cup E(C_2[2v_2, u_2]) \cup \bigcup_{i=1}^{m} E(C_{R_i}[r_i, r_{i-1}]) \cup \bigcup_{j=1}^{m} E(C_{S_j}[s_j-1, s_j])$. Hence condition (G3) holds. Since $G$ is obtained from $G'$ by identifying $u'_1$ and $u'_2$ to $u$ and $x'_1$ and $x'_2$ to $x$, by symmetry, we may assume that $x = s_j$ for some $0 \leq j \leq m$, which implies condition (G4).

This completes the proof of Lemma 10. □
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References


