Grünbaum colorings of even triangulations on surfaces

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Abstract

A *Grünbaum coloring* of a triangulation G is a map $c : E(G) \to \{1, 2, 3\}$ such that for each face f of G, the three edges of the boundary walk of f are colored by three distinct colors. In this paper we investigate the question whether each even (i.e. Eulerian) triangulation on a surface with representativity at least r has a Grünbaum coloring. We prove that, regardless of the representativity, every even triangulation on a surface $\mathbb{F}$ has a Grünbaum coloring as long as $\mathbb{F}$ is the projective plane, the torus, or the Klein bottle, and we observe that the same holds for the sphere and any surface with sufficiently large representativity. On the other hand, we construct even triangulations with no Grünbaum coloring and representativity $r = 1, 2$, and 3 for all but finitely many surfaces. In dual terms, our results imply that no snark admits an even map on the projective plane, the torus, or the Klein bottle, and that all but finitely many surfaces admit an even map of a snark with representativity at least 3.

Keywords: graph, map, embedding, triangulation, Grünbaum coloring, snark, polyhedral, even, Eulerian.
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1 Introduction

In 1968, B. Grünbaum conjectured the following generalization of the Four Color Theorem:

Grünbaum's conjecture [6]. *Every simple triangulation of an orientable surface has a Grünbaum coloring.*

However, the relationship between triangulations and Grünbaum colorings dates back much further—already in 1880 Tait showed that every 4-colorable planar triangulation has a Grünbaum coloring in an unsuccessful attempt to prove the Four Color Theorem. In the dual form, Grünbaum's conjecture states that no snark (i.e. a bridgeless 3-regular graph without proper 3-edge-coloring) has a polyhedral map in an orientable surface, where a map of a 3-regular graph is polyhedral if its dual is simple. More generally, *representativity* of a map, defined as the minimum number of intersections of an essential curve on the surface with the graph, is a measure of how much the map locally resembles a planar map. Polyhedral maps are then precisely the 3-connected maps with representativity at least 3. Since every graph admits a map in some orientable surface, Grünbaum's conjecture would imply that each map of a snark in an orientable surface has representativity at most 2. For nonorientable surfaces, the Petersen graph has a polyhedral map in the projective plane and therefore, the focus is on orientable maps of snarks. In fact, in the nonorientable case it is known that each nonorientable surface admits infinitely many polyhedral embeddings of snarks; see [11]. Over the years, embeddings of snarks and Grünbaum's conjecture attracted significant attention [3, 13, 15, 16], culminating in Kochol's disproval of the conjecture in 2009 for all orientable genera greater than 4; see [10]. However, embeddings of snarks remain rather poorly understood, with a particular open problem being when precisely a snark has a polyhedral map.

For the remaining cases of Grünbaum's conjecture, Albertson et al. [1] showed that every toroidal triangulation with chromatic number different from 5 has a Grünbaum coloring. Indeed, for a triangulation to have a Grünbaum coloring it is sufficient that its chromatic number is at most 4; see for example [1] for a short proof. Under the additional assumption that the triangulations are even (i.e. Eulerian), it is often possible to obtain good bounds on the chromatic number of the graph, or even a characterisation of the triangulations with a given chromatic number; see for instance [7, 12, 17]. The duals of even triangulations are even maps, where *even* maps are ones in which every face has even length. Motivated by the general lack of understanding of embeddings of snarks, we focus on the question whether a variant of Grünbaum's conjecture might be true when restricted to even triangulations. More precisely, for a surface $\mathbb{F}$ and representativity r, we investigate the following problem:

Is there an r-representative even triangulation on $\mathbb{F}$ without a Grünbaum coloring?

In the dual language, we ask in which surfaces snarks have even maps with a given representativity. It is not immediately obvious why even maps of snarks should exist at all—snarks are precisely the cubic graphs in which every 2-factor contains at least 2 odd cycles. However, every 'nontrivial' snark is upper-embeddable, that is, has an orientable map with one, respectively two faces (if its cycle rank is even, resp. odd). It follows that every nontrivial snark on $4k+2$ vertices has an even map with representativity 1 in some orientable surface; see Section 2 for details. Our main result is the following theorem.

Theorem 1. *Every even triangulation of the sphere, the projective plane, the torus or the Klein bottle admits a Grünbaum coloring.*

The Four Color Theorem implies that there is no planar snark, and thus also every planar even triangulation admits a Grünbaum coloring. Finally, we observe that known results on the colorings of even triangulations can be used to show that every even triangulation with sufficiently high representativity admits a Grünbaum coloring. On the negative side, we show how to construct a polyhedral even map of a snark in each orientable surface of genus at least 16 and each nonorientable surface of genus at least 4. Our results are summarized in Table 1.

surface $\setminus r$	1	2	3	high	surface $\setminus r$	1	2	3	high
$\mathbb{S}_1$	Y	Y	Y	Y	$\mathbb{N}_1$	Y	Y	Y	Y
$\mathbb{S}_2$	N	?	?	Y	$\mathbb{N}_2$	Y	Y	Y	Y
$\mathbb{S}_i$ $(3 \le i \le 15)$	N	N	?	Y	$\mathbb{N}_3$	N	N	?	Y
$\mathbb{S}_{\ge 16}$	N	N	N	Y	$\mathbb{N}_{\ge 4}$	N	N	N	Y

Table 1: Does every even triangulation on the given surface with representativity r have a Grünbaum coloring?

The paper is organized as follows. In Section 2 we collect all necessary definitions and basic results. Section 3, which is devoted to proving that even triangulations with either small genus or high representativity admit Grünbaum colorings, is further divided as follows. First we deal with the 6-regular case in Subsection 3.1 and then in Subsection 3.2 we prove Theorem 1. Finally, in Subsection 3.3 we show that high representativity guarantees that every even triangulation has a Grünbaum coloring. The negative results are contained in Section 4, where we provide constructions of even embeddings of snarks with representativity r for each $r \le 3$ in all but finitely many surfaces. We conclude the paper with a brief section highlighting some of the open problems in the area.

2 Preliminaries

We generally allow our graphs to contain both loops and multiple edges. If a graph has neither loops nor multiple edges, then it is *simple*. A *k-vertex* is a vertex of degree k. A graph G is *k-regular* if each vertex of G has degree k, and in particular, G is *cubic* if G is 3-regular. A closed walk is *even* if it is of even length; otherwise it is *odd*. In particular, a *k-cycle* is a cycle of length k.

A *surface* is a connected topological space in which every point has an open neighborhood homeomorphic to an open disk. So, throughout this paper, a surface is always connected and closed. Let $\mathbb{S}_i$ denote the orientable surface of (orientable) genus i, and let $\mathbb{N}_i$ denote the nonorientable surface of (nonorientable) genus i. A *map* G on a surface $\mathbb{F}$ is a 2-cell embedding of a graph on $\mathbb{F}$. A map G is a *triangulation* if each face of G is bounded by three distinct edges. A face f of G is *even* if f is bounded by an even closed walk; otherwise it is *odd*. For a vertex v of G, the *link* of v is the boundary walk of the region formed by v and all edges and faces incident to v. Let ℓ be a simple closed curve on a surface $\mathbb{F}$. If $\mathbb{F} - \ell$ is disconnected, then ℓ is *separating*. Otherwise it is *non-separating*. Moreover, ℓ is *contractible* if ℓ bounds a 2-cell region on $\mathbb{F}$. If ℓ does not bound a 2-cell region, then it is *essential*. An essential closed curve ℓ is 1-*sided* if a tubular neighborhood of ℓ is a Möbius band, and 2-*sided* otherwise. (In particular, ℓ is 2-sided if it is contractible.)

The *representativity* $r(G)$ of a map G on a surface $\mathbb{F}$ is the minimum number of intersecting points of G and γ, where γ ranges over all essential simple closed curves on $\mathbb{F}$ (if $\mathbb{F}$ is the sphere, then we define $r(G) = \infty$). A map G on $\mathbb{F}$ is k-*representative* if $r(G) \geq k$. It is an important observation that G and its dual G^* have the same representativity (see [14, Proposition 5.5.5]), and that the representativity of a triangulation G coincides with the length of a shortest essential cycle of G. Since we consider only 2-cell maps, every triangulation is 1-representative. The following is a well-known characterization of 2- and 3-representative triangulations, which will play an important role in this paper.

Proposition 2 ([14, Propositions 5.5.11 and 5.5.12]). *Let G be a triangulation on a surface $\mathbb{F}$. Then*

(i) *G is 2-representative and 2-connected if and only if G does not have loops; and*

(ii) *G is 3-representative and 3-connected if and only if G is simple.*

A map $f : V(G) \to \{1, 2, \cdots, k\}$ in a graph G is a *vertex k-coloring*, and f is *proper* if $f(u) \neq f(v)$ for any $uv \in E(G)$. The *chromatic number* $\chi(G)$ of G is the minimum integer k such that G has a proper vertex k-coloring. We can similarly define an *edge k-coloring*. A *snark* is a 2-edge-connected cubic graph with no proper edge 3-coloring; the smallest snark is the Petersen graph. Usually, it can be assumed that the snark is cyclically 4-edge-connected, and we call such snarks *nontrivial*. (A graph G is *cyclically 4-edge-connected* if any edge cut of size at most four separates G into a graph and a forest, i.e., there exists at most one component having a cycle in the resulting graph.)

A *Grünbaum coloring* of a triangulation G is a map $c\colon E(G) \to \{1, 2, 3\}$ such that for each face f of G, the three edges of the boundary walk of f are colored by three distinct colors (see Figure 1). Note that a triangulation G has a Grünbaum coloring if and only if the dual graph G^* of G (i.e. a cubic graph) has a proper edge 3-coloring. Hence if H is a map of a snark on a surface, then its dual (i.e. a triangulation) has no Grünbaum coloring.

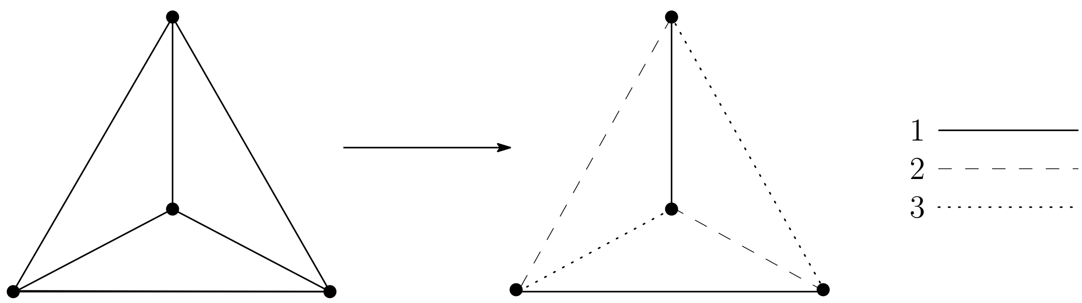

Figure 1: A triangulation G on the sphere and a Grünbaum coloring of G

The existence of Grünbaum colorings is related to vertex colorings by the following proposition, which was first proved for the planar case by Tait [19]. For a simple proof, see [1].

Proposition 3. *If a triangulation G has a proper vertex 4-coloring, then G has a Grünbaum coloring.* □

Note that the converse of Proposition 3 is true for the planar case, while false in general. For example, it is known that every triangulation on the torus with chromatic number 7 has a Grünbaum coloring and that such triangulations exist; see [1] for details.

In this paper, the *Four Color Theorem* plays an important role, where the Four Color Theorem states that every planar graph without loops is properly vertex 4-colorable. By Proposition 3, we have:

Lemma 4. *Let G be a triangulation of the sphere. If G has no loop, then G has a proper vertex 4-coloring and a Günbaum coloring.*

The *face subdivision* of a map G on a surface $\mathbb{F}$ is the map on $\mathbb{F}$ obtained from G by adding a single vertex into each face of G and joining it to all vertices on the corresponding boundary. In this case, if G is an even map, then the face subdivision of G is an even triangulation on $\mathbb{F}$.

The following lemma shows that it is easy to construct Grünbaum colorings of face subdivisions of even maps (the same fact is used in several papers; see [9] for example).

Lemma 5. *Let H be an even map on a surface $\mathbb{F}$, and let G be the face subdivision of H. Then G has a Grünbaum coloring.*

Proof. We first assign a color 1 to all edges of H. Secondly, since v has even degree, for each vertex $v \in V(G) \setminus V(H)$ we can give colors 2 and 3 to the edges incident to v alternately in the cyclic order around v. This gives a Grünbaum coloring of G. □

We finish this section with a short proof showing that there are infinitely many snarks with an even map. A graph is *upper-embeddable* if it has an orientable map with exactly one or two faces. (This is related to the *maximum genus* of a graph G, which is the maximum integer k such that G can be 2-cell embedded on $\mathbb{S}_k$; see [14, Section 4.5] for details.) It is known [21] that every cyclically 4-edge-connected graph is upper-embeddable, and thus every nontrivial snark is upper-embeddable. Since the number of faces of an orientable map of a graph G has the opposite parity as the cycle rank $|E(G)| - |V(G)| + 1$ (see [14, Page 122]), a nontrivial snark with n vertices has an embedding with exactly one face if and only if $n \equiv 2 \pmod 4$. It is easy to see that in any 1-face map of any graph, the length of the only face is always even which is twice $|E(G)|$. It follows that every nontrivial snark on n vertices with $n \equiv 2 \pmod 4$ has an even map in some surface. The duals of these maps are triangulations by bouquets of $3n/2$ circles (i.e. graphs with a single vertex and $3n/2$ loops incident to the vertex).

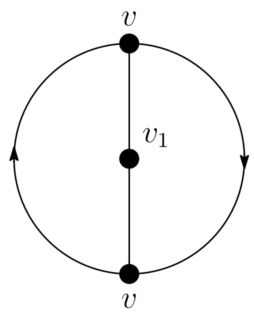

Figure 2: The even triangulation G_p on $\mathbb{N}_1$ with two vertices $\{v, v_1\}$, three edges $\{vv, vv_1, v_1v\}$ and two faces, both vv_1v.

3 Grünbaum colorings of even triangulations

We first propose the following lemma for even triangulations with loops. Let G_p be the even triangulation on the projective plane $\mathbb{N}_1$ shown in Figure 2.

Lemma 6. *Let G be a triangulation on a surface $\mathbb{F}$ with a loop ℓ incident to a vertex v.*

(i) *If ℓ is contractible, then there is a vertex of odd degree in the interior of ℓ such that the vertex is different from v.*

(ii) *If G is an even triangulation and ℓ is essential, then the degree of v is at least* 6 *unless $\mathbb{F}$ is the projective plane and G is isomorphic to G_p.*

Proof. (i) For contradictions, we suppose that G has a loop ℓ bounding a 2-cell region R and all vertices in the interior of R have even degree. Let T be the plane subgraph of G consisting of all vertices and edges in the interior of R and the edge ℓ. Since each inner vertex of T has even degree, its dual graph T^* is a bipartite graph with $|V(T^*)| - 1$ 3-vertices and exactly one 1-vertex u. We color the vertices of T^* by black and white so that u is white. Let B (resp., W) be the set of black (resp., white) vertices of T^*. Since every vertex in B has degree 3, we have $3|B| = |E(T^*)|$. On the other hand, $3(|W| - 1) + 1 = |E(T^*)|$ since each edge in T^* is incident to both a black vertex and white one. Hence, we have $3|B| = 3(|W| - 1) + 1 = 3|W| - 2$, a contradiction.

(ii) Suppose that G is an even triangulation and has an essential loop ℓ incident to a vertex v. For contradictions, we suppose that $\deg(v) = 2$ or 4. Observe that G has no loop incident to v other than ℓ. (For otherwise, we have $\deg(v) = 4$ and G must be a graph with a single vertex v with exactly two loops. It is easy to see that such a graph cannot triangulate any surface.) By the definition of a triangulation, G has no triangular face whose boundary walk contains ℓ twice. So we can take two distinct faces f_1 and f_2 incident to ℓ whose boundary walks are vvv_1 and vvv_2, respectively. By the assumption $\deg(v) \leq 4$, we must have $v_1 = v_2$ and $\deg(v_1) = 2$.

Therefore, if ℓ is 2-sided, then along ℓ, we can take a simple *essential* closed curve in the interior of f_2 (that is, the closed curve lies within the face f_2 and does not include vertices and edges), and so this contradicts that G is a 2-cell embedding on the surface. On the other hand, if ℓ is 1-sided, then it is easy to see that G is isomorphic to G_p and this completes the proof of the lemma. $\square$

Let v be a 4-vertex in an even triangulation with link $v_1v_2v_3v_4v_1$. The 4-*contraction* of v at $\{v_2, v_4\}$ is removing v, identifying v_2 and v_4, and replacing two pairs of multiple edges v_1v_2, v_1v_4 and v_3v_2, v_3v_4 with two single edges (see Figure 3). The inverse operation is a 4-*splitting*.

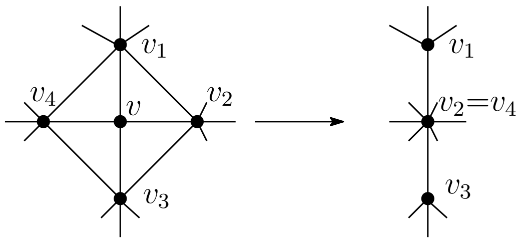

Figure 3: A 4-contraction

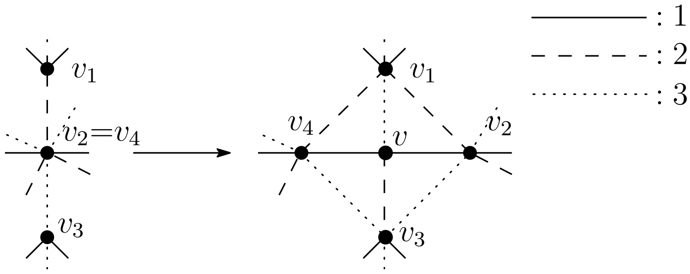

Figure 4: Extending a Grünbaum coloring by a 4-splitting

Lemma 7. *Let G be a triangulation on a surface $\mathbb{F}$ with a 4-vertex v, and let G' be a triangulation on $\mathbb{F}$ obtained from G by a 4-contraction of v. If G' has a Grünbaum coloring, then so does G.*

Proof. Let v be a 4-vertex in G with link $v_1v_2v_3v_4v_1$, and let G' be the triangulation obtained from G by the 4-contraction of v at $\{v_2, v_4\}$. Suppose that G' has a Grünbaum coloring $c\colon E(G') \to \{1, 2, 3\}$, in which there are two possibilities: one is $c(v_1v_2) \neq c(v_2v_3)$ and the other is $c(v_1v_2) = c(v_2v_3)$. In either case, we can get a Grünbaum coloring of G. (For example, see Figure 4 which shows the former case). □

3.1 Grünbaum colorings and 6-regular triangulations

In this section, we show that every 6-regular triangulation on the torus or the Klein bottle has a Grünbaum coloring.

Let $p \geq 1$ and $q \geq 0$ be non-negative integers. Let $L_{p,q}$ be a $p \times q$ grid graph with horizontal, vertical and slope-one diagonal edges. Let $H_{p,q}$ be the map on the annulus with $p \times (q + 1)$ vertices obtained from $L_{p+1,q+1}$ by identifying the top and the bottom, as shown in Figure 5.

For $u_{i,j} \in V(H_{p,q})$, we take i modulo $q+1$ and j modulo p. Let C_k be the cycle of $H_{p,q}$ passing only through $u_{k,0}, u_{k,1}, \ldots, u_{k,p-1}, u_{k,0}$ in this order, for $k \in \{0, 1, \ldots, q\}$. Now, if $q \geq 1$, then the vertices on C_0 and C_q have degree 4 and the others have 6 in $H_{p,q}$.

Let $0 \leq r \leq p - 1$ be an integer. Let $G[p \times q, r]$ be the 6-regular graph on the torus obtained from $H_{p,q}$ by identifying each vertex $u_{0,j}$ with $u_{q,j+r}$ and each edge $u_{0,j}u_{0,j+1}$ with $u_{q,j+r}u_{q,j+r+1}$ for $j \in \{0, 1, \ldots, p - 1\}$. Note that C_0 and C_q are identified and that $G[p \times q, r]$ has exactly $p \times q$ vertices.

Theorem 8 (Altshuler [2])**.** *Every 6-regular triangulation on the torus is isomorphic to $G[p \times q, r]$ for some integers $p, q > 0$ and $r \geq 0$.*

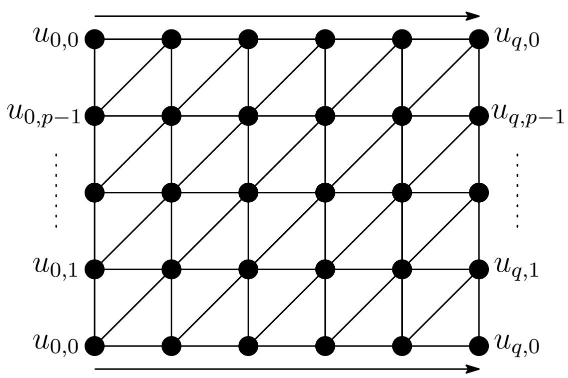

Figure 5: The graph $H_{p,q}$

On the other hand, the 6-regular triangulations on the Klein bottle were classified by Negami [18], as follows:

A 6-regular triangulation on the Klein bottle can be obtained from the annular map $H_{p,q}$ by identifying each vertex $u_{0,j}$ with $u_{q,-j}$ and each edge $u_{0,j}u_{0,j+1}$ with $u_{q,-j}u_{q,-j-1}$ for each $j \in \{0, 1, \ldots, p-1\}$. This is of *handle type* and denoted by $Kh(p, q)$.

When p is even, identify each vertex $u_{0,j}$ with $u_{0,j+p/2}$, each edge $u_{0,j}u_{0,j+1}$ with $u_{0,j+p/2}u_{0,j+p/2+1}$, each vertex $u_{q,j}$ with $u_{q,j+p/2}$ and each edge $u_{q,j}u_{q,j+1}$ with $u_{q,j+p/2}u_{q,j+p/2+1}$, in $H_{p,q}$ for each $j \in \{0, \ldots, p/2-1\}$, respectively. Then we obtain a 6-regular triangulation on the Klein bottle such that two cycles of length $p/2$ arise from C_0 and C_q. Now, suppose that $p = 2m+1$. Add a crosscap (a Möbius band) to each boundary of $H_{p,q-1}$ (that is, paste a boundary of a Möbius band to a boundary of the annulus), and for each $j \in \{0, \ldots, m\}$, join $u_{0,j}$ to $u_{0,j+m}$ and $u_{0,j+m+1}$, and join $u_{q-1,j}$ to $u_{q-1,j+m}$ and $u_{q-1,j+m+1}$ on the added crosscaps. The resulting 6-regular graph is of *crosscap type* and is denoted by $Kc(p, q)$, although the constructions of $Kc(p, q)$ are different, depending on the parity of p. Note that even if p is odd, the resulting graph $Kc(p, q)$ is an embedding on the Klein bottle since the surface has exactly two Möbius bands by the above construction.

Theorem 9 (Negami [18])**.** *Every* 6*-regular triangulation on the Klein bottle is isomorphic to precisely one of $Kh(p, q)$ and $Kc(p, q)$ for some integers $p, q > 0$.*

We are now ready to prove that every 6-regular triangulation of the torus or the Klein bottle has a Grünbaum coloring. Note that the result for the torus has already appeared in [1].

Theorem 10. *Every* 6*-regular triangulation on the torus or the Klein bottle has a Grünbaum coloring.*

Proof. Let G be a 6-regular triangulation on the torus or the Klein bottle. First observe that if G is on the torus or G is a handle type $Kh(p, q)$, then we can easily obtain a Grünbaum coloring of G by assigning color 1 to vertical edges, color 2 to horizontal edges, and color 3 to all other diagonal edges in $H_{p,q}$. Moreover, if G is a crosscap type $Kc(p, q)$ with p even, then we can also obtain a Grünbaum coloring of G by the above coloring.

We suppose that G is a crosscap type $Kc(p, q)$ with $p \geq 3$ odd (if $p = 1$, then G is not a triangulation by definition). Now G consists of two triangulated Möbius bands (crosscaps) M_1, M_2 and $H_{p,q-1}$. We only consider Grünbaum colorings of M_1 and $H_{p,q-1}$, since we can give the coloring of M_2 as well as one of M_1.

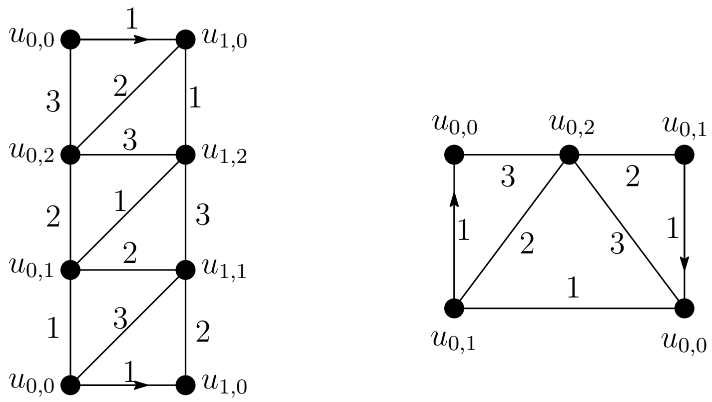

Figure 6: A Grünbaum coloring of $Kc(3, 2)$

First, observe that $Kc(3, 2)$ has a Grünbaum coloring as shown in Figure 6. (If $q = 1$, then we need not consider a Grünbaum coloring of $H_{p,q-1}$, since the annular part is degenerate.) Then we can extend the Grünbaum coloring of $Kc(3, 2)$ to those of $Kc(5, 2)$ and $Kc(7, 2)$ as shown in Figure 7. Note that colors with parentheses of Grünbaum colorings in Figure 7 are recurrent colors, that is, we can extend a Grünbaum coloring of $H_{p,1}$ to that of $H_{p+2,1}$ by using colors with parentheses repeatedly (based on the coloring of Figure 6). Moreover, we have a Grünbaum coloring of $H_{p,2}$, by pasting two copies of $H_{p,1}$ with a Grünbaum coloring so that edges with the same color are identified. Hence we can also extend the coloring of $H_{p,q}$ to that of $H_{p,q+1}$. Therefore, every crosscap type $Kc(p, q)$ with $p \geq 3$ odd has a Grünbaum coloring. $\square$

3.2 Surfaces with low genus

In this section, we shall prove Theorem 1. To prove the theorem, we introduce several lemmas. A *pinched surface* $\mathbb{F}_p$ is a non-smooth topological space such that except for exactly one point called a *pinched point*, all points have an open neighborhood homeomorphic to an open disk. Let G be a graph embedded on $\mathbb{F}_p$ so that some vertex v coincides with the pinched point, where v is a *pinched vertex* of G. Then a *surface splitting* at v is to split v into two copies of v on $\mathbb{F}_p$ to get a (pinched) surface or two (pinched) surfaces; for example, see Figure 8.

Lemma 11. *Let G be a triangulation on a pinched surface $\mathbb{F}_p$ and let v be a pinched vertex of G.*

(i) *Suppose that a surface splitting at v produces a triangulation G'. If G' has a Grünbaum coloring, then so does G.*

(ii) *Suppose that a surface splitting at v produces two triangulations H and H'. If both H and H' have Grünbaum colorings, then so does G.*

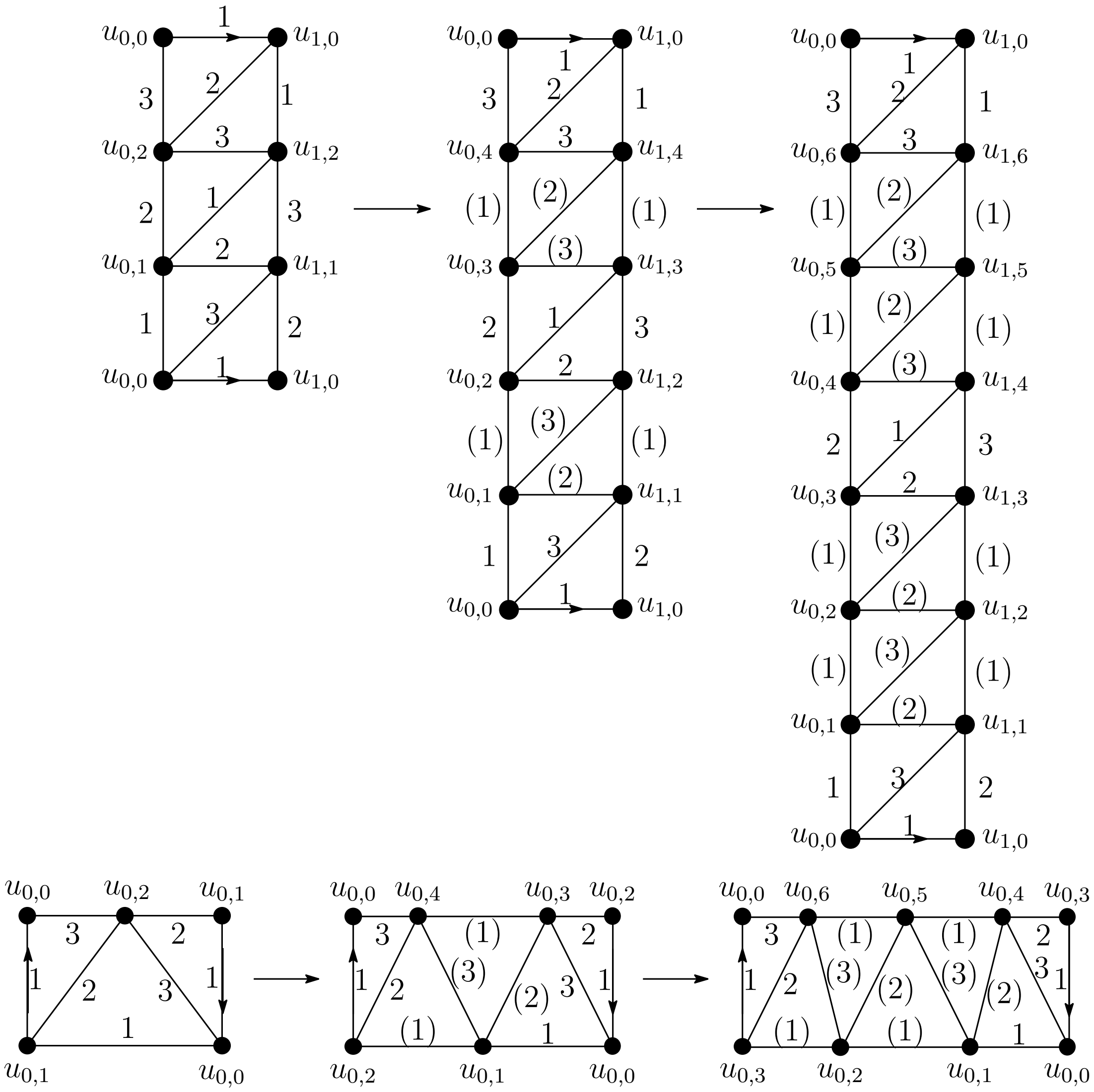

Figure 7: Extension of the Grünbaum coloring of $H_{3,1}$ to those of $H_{5,1}$ and $H_{7,1}$, and extension for the Möbius bands.

Proof. Every triangular face of G' (or of H and H') is also a face of G. Therefore, this implies that a Grünbaum coloring of G' (or of H and H') can be extended to that of G. $\square$

Lemma 12. *Let G be an even triangulation on a surface $\mathbb{F}$, let v be a 4-vertex of G with link $v_1v_2v_3v_4v_1$, and let G' be the graph obtained from G by a 4-contraction of v at $\{v_2, v_4\}$.*

(i) *If $v_2 \neq v_4$, then G' is an even triangulation on $\mathbb{F}$.*

(ii) *If $v_2 = v_4$ and the 2-cycle vv_2v is 2-sided, then the 4-contraction of v deforms G into an even triangulation on a pinched surface.*

(iii) *If $v_2 = v_4$ and the 2-cycle vv_2v is 1-sided, then G' is an even triangulation on a nonorientable surface (or the sphere) whose nonorientable genus is just one less than that of $\mathbb{F}$.*

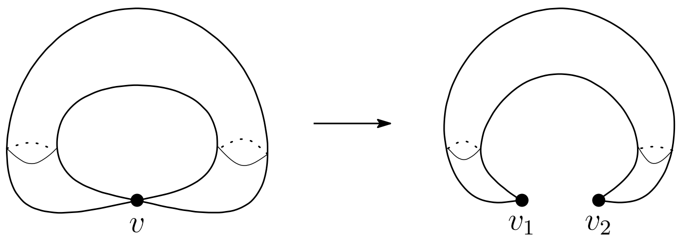

Figure 8: A surface splitting of the pinched torus

Proof. Since the statement of (i) clearly holds by the definition, it suffices to show only (ii) and (iii). Assume that $v_2 = v_4$, and let C be the 2-cycle vv_2v. In this case, the 4-contraction of v corresponds to the contraction of C into one vertex v', which breaks the topology of $\mathbb{F}$. We focus on a tubular neighborhood R_C of C and consider how it changes through the 4-contraction.

(ii) Suppose first that C is 2-sided. In this case, R_C is homeomorphic to an open annulus, and by the 4-contraction, it becomes a pinched surface obtained from two open disks (which correspond to the left and the right side of C, respectively) by identifying each one point of them corresponding to v' (i.e., the resulting pinched surface obtained from R_C is a double cone with the boundaries of R_C). For any points other than v', there exists an open neighborhood in the surface homeomorphic to an open disk, which is also an open disk in $\mathbb{F}$. Thus, the obtained topological space is a pinched surface.

(iii) Next suppose that C is 1-sided. In this case, R_C in $\mathbb{F}$ is homeomorphic to an open Möbius band, and it is still connected in the resultant topological space $\mathbb{F}'$ by the 4-contraction of v. So, R_C in $\mathbb{F}'$ is now homeomorphic to a disk, which is an open neighborhood of v'. Therefore, the resultant topological space $\mathbb{F}'$ is also a surface. Note that the 4-contraction of v reduces the number of vertices by one (recall that $v_2 = v_4$), the number of edges by six, and the number of faces by four. Thus, the Euler genus of $\mathbb{F}'$ is one less than that of $\mathbb{F}$. This directly implies that if $\mathbb{F}$ is the projective plane $\mathbb{N}_1$, then $\mathbb{F}'$ is the sphere. On the other hand, if $\mathbb{F}$ is the nonorientable surface $\mathbb{N}_i$ with $i \geq 2$, then a 1-sided closed curve on $\mathbb{F}$ that is not homotopic to C remains on $\mathbb{F}'$, and hence $\mathbb{F}'$ is still nonorientable. Hence the assertion follows. $\square$

To simplify the proof of Theorem 1, we also propose the following lemma that generalizes Lemma 7 to include the cases of Lemma 12. The proof of Lemma 7 naturally extends to these cases.

Lemma 13. *Let G be a triangulation on a surface $\mathbb{F}$ with a 4-vertex v and let G' be a triangulation obtained from G by a 4-contraction of v. If G' has a Grünbaum coloring, then so does G.* $\square$

We are now prepared to prove Theorem 1.

Proof of Theorem 1. Let G be an even triangulation on a surface $\mathbb{F}$ with loops and multiple edges allowed, where $\mathbb{F}$ is the sphere, the projective plane, the torus or the Klein bottle. By Lemmas 4 and 6 (i), if $\mathbb{F}$ is the sphere, then the theorem holds. So, it suffices to consider the projective plane, the torus and the Klein bottle.

Case 1. $\mathbb{F}$ is the projective plane.

If G is 2-representative, then we can use the following result proved by Mohar [12]: Every 2-representative even triangulation G on the projective plane is the face subdivision of some even map H on the projective plane, and hence, by Lemma 5, G has a Grünbaum coloring.

So suppose that G has representativity 1, that is, suppose that G has an essential loop ℓ incident to a vertex v, by Proposition 2 (i). In this case, we cut open the projective plane (where G is embedded) along ℓ to get a plane graph D such that all finite faces are triangular and such that the boundary of the infinite face is a 2-cycle C, where the two vertices of C are two copies of v. Removing one edge from C, we get a plane triangulation D'. Suppose that D has a loop ℓ'. Since ℓ' is contained in the disk bounded by C, the loop ℓ' is contractible on the projective plane $\mathbb{F}$. However, this contradicts Lemma 6 (i). Thus, there is no loop in D, and hence also in D'. So, by Lemma 4, D' has a Grünbaum coloring. A Grünbaum coloring of G can be naturally obtained from that of D', since any three edges of G forming a triangular face also bound a triangular face of D' and vice versa. This completes the proof of Case 1.

We prove the remaining cases by an inductive argument on the number of vertices, to reduce to the planar or projective planar case. We first prove the following.

Claim 14. *Let G be an even triangulation on a surface containing a 2-cell region R bounded by two distinct edges e_1, e_2, and let H be a graph obtained from G by removing the interior of R and identifying e_1 and e_2. If H has a Grünbaum coloring, then so does G and e_1 and e_2 receive the same color.*

Proof. Let T be the plane subgraph of G consisting of all vertices and edges in the interior of R and the edge e_1. By Lemma 6 (i), T is a triangulation of the sphere without loops, and hence Lemma 4 implies that T has a Grünbaum coloring. We may choose colors so that e_1 in the Grünbaum coloring of T is assigned the same color as e_1 is in H. Then combining those Grünbaum colorings, we obtain a Grünbaum coloring of G. □

It follows that for the remaining cases of the proof, we may suppose that G has no 2-cell region such as described in Claim 14. If G has a 2-vertex x, then the link of x consists of two edges, which form a 2-cell region as in Claim 14. Thus, in particular, we may assume that G has no 2-vertex.

Case 2. $\mathbb{F}$ is the torus.

By Euler's formula, G is 6-regular or has a 4-vertex, since G is an even triangulation. If G is 6-regular, then G has a Grünbaum coloring by Theorem 10. So we may suppose that G has a 4-vertex, say v with link $v_1v_2v_3v_4v_1$. If $v_2 \neq v_4$, then the 4-contraction of v at $\{v_2, v_4\}$ produces a smaller even triangulation G' on $\mathbb{F}$, and we obtain a Grünbaum coloring of G from a Grünbaum coloring of G' by Lemmas 12 (i) and 13. Therefore, we may assume $v_2 = v_4$.

Let C be the 2-cycle vv_2v. If C is contractible, then the 2-cell region bounded by C is one described in Claim 14 (with $e_1 = vv_2$ and $e_2 = v_2v$), a contradiction. So, we may also assume that C is essential. By Lemma 12 (ii), the 4-contraction of v at $\{v_2, v_4\}$ gives an even triangulation, say G', on a pinched surface $\mathbb{F}_p$. Let v' be the vertex obtained by the 4-contraction. Since $\mathbb{F}$ is the torus and C is essential, $\mathbb{F}_p$ is the pinched torus with pinched vertex v'. (See the direction from left to center in Figure 9.) Then we apply the surface splitting at v' of G', and the resulting graph G'' is a triangulation on the sphere. If G'' does not contain a loop, then Lemma 4 guarantees the existence of a Grünbaum coloring of G'', and also in G' and G by Lemmas 11 (i) and 13 respectively. Therefore, we may assume that G'' contains a loop ℓ. By Lemma 6 (i), both the interior and the exterior of ℓ contain a vertex of odd degree in G''. Since G' is an even triangulation of the pinched torus $\mathbb{F}_p$, those vertices must be the two copies of v'. Thus, ℓ is homotopic to C on $\mathbb{F}$.

In this case, we consider the 4-contraction of v at $\{v_1, v_3\}$. By the same argument as in the first paragraph of Case 2, we may assume that $v_1 = v_3$. By the existence of the loop ℓ that is homotopic to C, we see that ℓ is incident to the vertex v_1. In this case, because of the loop ℓ and the cycle C, there is no loop in G homotopic to the cycle vv_1v on $\mathbb{F}$. Then as in the previous paragraph, by the 4-contraction of v at $\{v_1, v_3\}$ and the surface splitting at the obtained pinched vertex, we obtain a triangulation $\widehat{G}$ on the sphere without any loops. By Lemma 4, $\widehat{G}$ has a Grünbaum coloring, and so does G by Lemmas 11 (i) and 13.

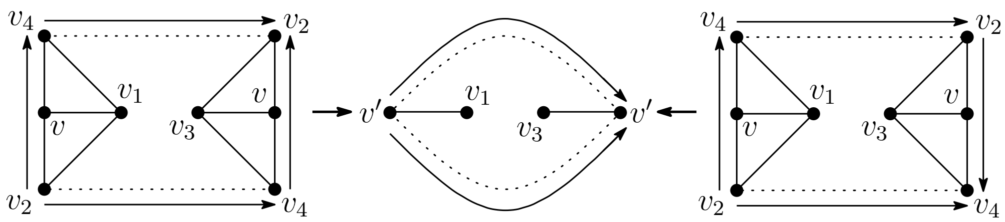

Figure 9: Contracting a 2-sided non-separating cycle $C = vv_2(= vv_4)$ in G to have a pinched vertex; the left is the toroidal case and the right is the Klein-bottle case.

Case 3. $\mathbb{F}$ is the Klein bottle.

As in the toroidal case, if G is a 6-regular triangulation, then we can obtain a Grünbaum coloring of G by Theorem 10. So we may suppose that G has a 4-vertex, say v with link $v_1v_2v_3v_4v_1$. If $v_2 \neq v_4$, then we can apply a 4-contraction of v at $\{v_2, v_4\}$ and use the same argument as in the toroidal case. So, we may assume that $v_2 = v_4$. In this case, we must be careful in the case when the topology of the surface is broken by a 4-contraction. Let C be the 2-cycle vv_2v. By Claim 14, C is essential. Then it suffices to prove the following three cases.

Case (i). C is 2-sided and non-separating. By Lemma 12 (ii), the contraction of C into a single vertex v' produces an even triangulation, say G', on a pinched surface $\mathbb{F}_p$. (See the direction from right to center in Figure 9.) Since C is 2-sided and non-separating, $\mathbb{F}_p$ is the pinched torus. Then by the same way as in the toroidal case, we can show that G has a Grünbaum coloring.

Case (ii). C is 2-sided and separating. Again by Lemma 12 (ii), the contraction of C into a single vertex v' produces an even triangulation, say G', on a pinched surface $\mathbb{F}'_p$. Since C is 2-sided and separating, by the surface splitting at v' in $\mathbb{F}'_p$, we have two triangulations on the projective plane. Note that both of those triangulations must be even by the Handshaking Lemma. As we have already shown in Case 1, since every even triangulation on the projective plane has a Grünbaum coloring, so does G by Lemmas 11 (ii) and 13.

Case (iii). C is 1-sided. By Lemma 12 (iii), the contraction of C into a single vertex v' produces an even triangulation, say G', on the projective plane. Thus, G' has a Grünbaum coloring by Case 1, and hence so does G by Lemma 13.

Therefore, since G has a Grünbaum coloring in either case, the theorem holds. □

3.3 Locally planar even triangulations

We conclude this section by showing that all even triangulations with sufficiently high representativity admit Grünbaum colorings. We begin with two results on coloring of even triangulations.

Theorem 15 (Hutchinson et al. [7]). *For any orientable surface $\mathbb{S}_g$, there exists a positive integer $N(\mathbb{S}_g)$ such that every even triangulation G on $\mathbb{S}_g$ with $r(G) \geq N(\mathbb{S}_g)$ is properly vertex 4-colorable.* □

Theorem 16 (Nakamoto [17]). *For any nonorientable surface $\mathbb{N}_k$, there exists a positive integer $N(\mathbb{N}_k)$ such that every even triangulation G on $\mathbb{N}_k$ with $r(G) \geq N(\mathbb{N}_k)$ is properly vertex 5-colorable. Furthermore, $\chi(G) = 5$ if and only if G is the face subdivision of some even map H containing a 4-chromatic quadrangulation H' as a subgraph.* □

Note that Thomassen [20] had showed the first statement of Theorem 16 before, with more general form. Using Lemma 5 and Theorems 15 and 16, it is now easy to show that locally planar even triangulations admit Grünbaum colorings.

Theorem 17. *For any surface $\mathbb{F}$, there is a positive integer $N(\mathbb{F})$ such that every even triangulation G on $\mathbb{F}$ with $r(G) \geq N(\mathbb{F})$ has a Grünbaum coloring.*

Proof. Let G be an even triangulation on a surface $\mathbb{F}$ with high representativity. By Theorems 15 and 16, every even triangulation on $\mathbb{F}$ with high representativity is either properly vertex 4-colorable or the face subdivision of some even map. Therefore, G has a Grünbaum coloring by Proposition 3 or Lemma 5. □

4 Even triangulations with no Grünbaum coloring

In this section, we construct even triangulations with no Grünbaum coloring. Our constructions require a restricted dual form of the *Parity Lemma* (see [4, 8] for example), as follows:

Lemma 18. *Let G be a triangulation on a surface $\mathbb{F}$ and let C be a separating 3-cycle of G. If G has a Grünbaum coloring, then the three edges of C are colored by three distinct colors.*

We next introduce a *Fisk triangulation*, which is one with exactly two odd vertices u and v such that u and v are adjacent. It is known that every Fisk triangulation has no proper vertex 4-coloring [5]. Thus the sphere admits no Fisk triangulation, by the Four Color Theorem. We will soon need the fact that the torus and the projective plane each admit a simple Fisk triangulation, as shown in the left and the right of Figure 10, respectively.

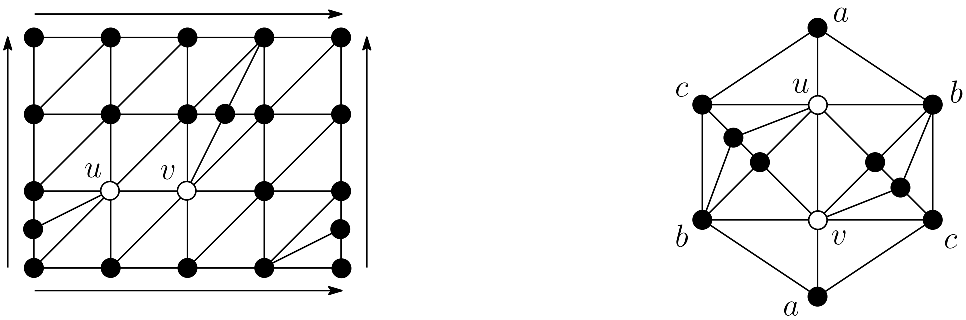

Figure 10: Simple Fisk triangulations on the torus (left) and the projective plane (right); the odd vertices are white.

Proposition 19. *For any integer $g \geq 16$, there exists a simple even triangulation on $\mathbb{S}_g$ with no Grünbaum coloring, and for any integer $k \geq 4$, there exists a simple even triangulation on $\mathbb{N}_k$ with no Grünbaum coloring.*

Proof. Let T be a simple Fisk triangulation on the torus with two odd vertices u and v. (Such a triangulation actually exists, as in the left of Figure 10.) Let G be the triangulation on $\mathbb{S}_5$ dual to the snark shown in Figure 11. (In the figure, each square hole corresponds to a handle of the surface. We get handles by identifying opposite arrows.) Then G has no Grünbaum coloring. Moreover, by Proposition 2 (ii), we see that G is simple since Kochol's snark shown in Figure 11 is 3-connected and has representativity 3.

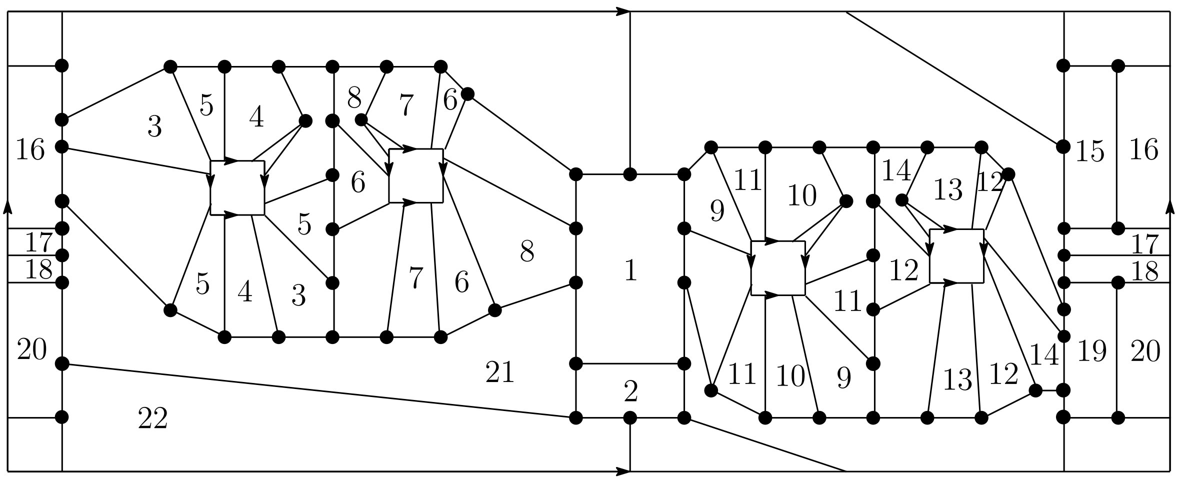

Figure 11: The snark on $\mathbb{S}_5$ constructed by Kochol [10].

Observe that G has exactly twenty-two odd vertices $v_1, v_2, \ldots, v_{22}$ corresponding to the odd faces labeled in Figure 11, and there is a matching $M = \{e_1, e_2, \ldots, e_{11}\}$ of G such that each edge $e_i \in M$ connects two odd vertices v_{2i-1} and v_{2i} for $i \in \{1, 2, \dots, 11\}$. Now identify a face containing the edge uv in T with a face in G containing an edge $xy \in M$, so that uv and xy are identified. In this case, the resulting graph H is a simple triangulation on $\mathbb{S}_6$ with exactly twenty odd vertices and no Grünbaum coloring by Lemma 18. Thus, by repeating these operations, we produce an even triangulation on $\mathbb{S}_{16}$ with no Grünbaum coloring. (Clearly, the triangulation is simple.)

For the nonorientable case, note that K_6 has six odd vertices that form a perfect matching. By identifying a face of a simple Fisk triangulation on $\mathbb{N}_1$ with a face of K_6 on $\mathbb{N}_1$ and iterating as above, we finally have a simple even triangulation on $\mathbb{N}_4$ with no Grünbaum coloring.

By identifying two faces of two distinct simple even triangulations on surfaces of genus g_1 and g_2, we get a simple even triangulation on the surface of genus $g_1 + g_2$, and so, for any integer $g \geq 16$ (resp., $k \geq 4$), we can construct a simple even triangulation on $\mathbb{S}_g$ (resp., $\mathbb{N}_k$) with no Grünbaum coloring from the one on $\mathbb{S}_{16}$ (resp., $\mathbb{N}_4$) and any simple even triangulation of a surface of an appropriate (nonorientable) genus. □

Proposition 20. *For any integer $g \geq 3$, there exists a 2-representative even triangulation on $\mathbb{S}_g$ with no Grünbaum coloring, and for any integer $k \geq 3$, there exists a 2-representative even triangulation on $\mathbb{N}_k$ with no Grünbaum coloring.*

Proof. To prove the proposition, we construct a snark on $\mathbb{S}_g$ (resp., $\mathbb{N}_k$) with representativity 2 for integers $g \geq 3$ (resp., $k \geq 3$).

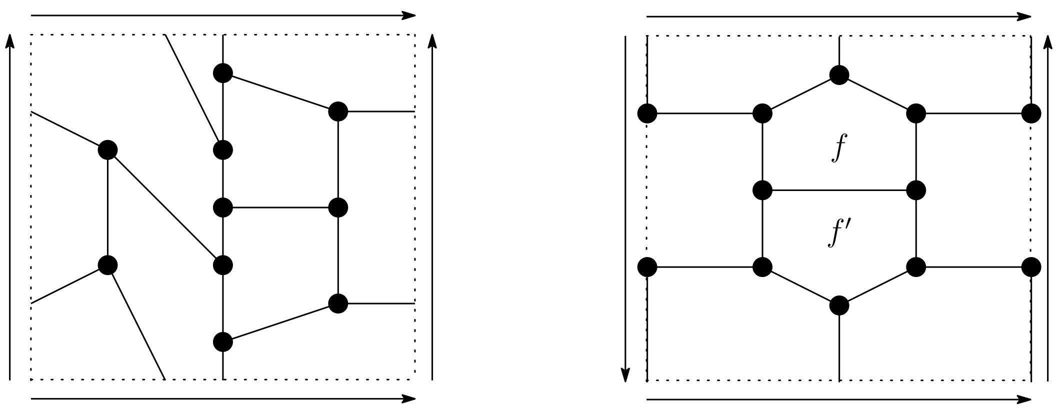

Figure 12: Embeddings of the Petersen graph on the torus and the Klein bottle.

We first consider the orientable case. Embed the Petersen graph on the torus as shown at the left in Figure 12. In the embedding, there are four odd faces, so that its dual is a triangulation with exactly four odd vertices. Thus, as in the proof of Proposition 19, we produce an even triangulation G on $\mathbb{S}_g$ for $g \geq 3$ with no Grünbaum coloring and $r(G) = 2$ by using Fisk triangulations on the torus.

Next, we prove the nonorientable case. Embed the Petersen graph on the Klein bottle as shown at the right in Figure 12. Since there are exactly two adjacent odd faces f and f' in the embedding, its dual is a Fisk triangulation on the Klein bottle. Therefore, following the above case, we have an even triangulation G on $\mathbb{N}_k$ with no Grünbaum coloring and $r(G) = 2$ for any $k \geq 3$, using Fisk triangulations on the Klein bottle. □

Proposition 21. *For any integer $g \geq 2$, there exists a 1-representative even triangulation on $\mathbb{S}_g$ with no Grünbaum coloring, and for any integer $k \geq 3$, there exists a 1-representative even triangulation on $\mathbb{N}_k$ with no Grünbaum coloring.*

Proof. First we see that the Petersen graph has 1-representative embeddings on $\mathbb{S}_2$ and $\mathbb{N}_3$ so that the boundary walk of each face has even length (see Figure 13). So, by Proposition 2, the dual graphs are 1-representative even triangulations on $\mathbb{S}_2$ and $\mathbb{N}_3$ with no Grünbaum coloring. Therefore, as in the proof of Propositions 19 and 20, we get even triangulations G on $\mathbb{S}_g$ (resp., $\mathbb{N}_k$) for any $g \geq 2$ (resp., $k \geq 3$) with no Grünbaum coloring and $r(G) = 1$. $\square$

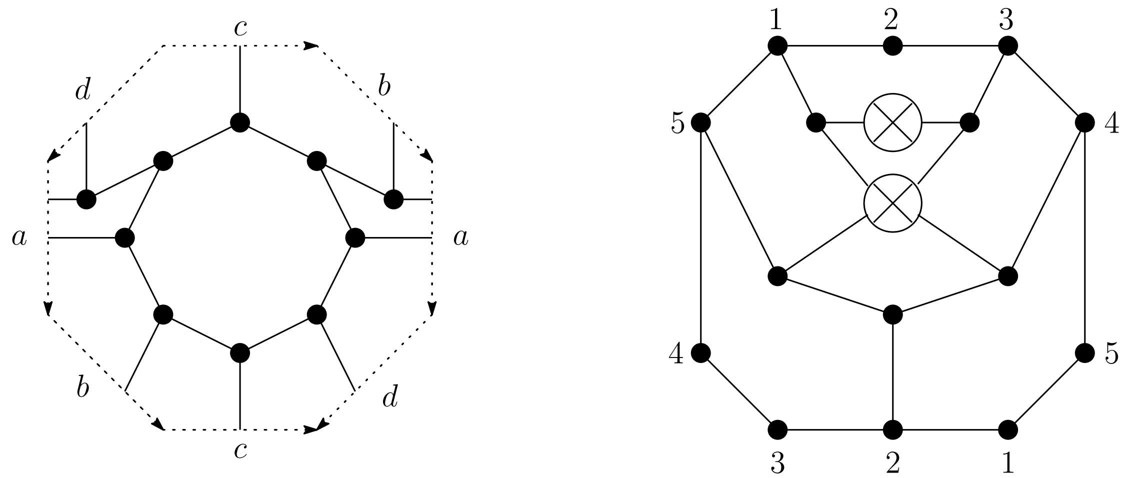

Figure 13: Embeddings of the Petersen graphs on $\mathbb{S}_2$ and $\mathbb{N}_3$ with representativity 1.

5 Concluding Remarks

We have investigated the existence of Grünbaum colorings of even triangulations with a given representativity, or equivalently, the existence of maps of snarks with all faces of even length and a given representativity. As our main results we show that there are no polyhedral even embeddings of snarks in $\mathbb{S}_1$, $\mathbb{N}_1$, and $\mathbb{N}_2$, and that for each surface $\mathbb{F}$ there is a constant $N(\mathbb{F})$ such that no snark embeddable in $\mathbb{F}$ as an even map can have representativity $N(\mathbb{F})$. On the other hand, we provide a method for decreasing the number of odd vertices in a polyhedral triangulation, which enables us to prove that almost all surfaces admit a polyhedral even embedding of some snark. Embeddings, and in particular polyhedral embeddings, of snarks are still far from being well understood and there are several open problems that seem worth investigating.

Despite the results of Albertson et al. [1] and Kochol [10], Grünbaum's conjecture is still open for $\mathbb{S}_2, \mathbb{S}_3$, and $\mathbb{S}_4$, and for toroidal triangulations with chromatic number 5. On the other hand, Theorem 1 implies that every toroidal even triangulation admits a Grünbaum coloring. Therefore, if there is a toroidal 3-representative triangulation without a Grünbaum coloring, then it must have chromatic number 5 and a vertex of odd degree.

Hutchinson et al. [7, Page 226] asked whether there is an absolute constant c such that every even triangulation on an orientable surface with representativity at least c is 4-colorable, and in particular, whether $c = 100$ has the property. An affirmative answer to this question would imply that even maps of snarks cannot have arbitrarily high representativity. On the other hand, it is not known whether there is any map, even or not, of a snark with representativity at least 4 in any surface, orientable or not. Indeed, Robertson posed a problem whether there is an absolute constant c' such that every triangulation on a surface with representativity at least c' has a Grünbaum coloring; see [14, Problem 5.5.17].

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