A degree sum condition on the order, the connectivity and the independence number for Hamiltonicity

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Abstract

In [Graphs Combin. 24 (2008) 469–483.], the third author and the fifth author conjectured that if G is a k-connected graph such that $\sigma_{k+1}(G) \geq |V(G)| + \kappa(G) + (k-2)(\alpha(G)-1)$, then G contains a Hamiltonian cycle, where $\sigma_{k+1}(G)$, $\kappa(G)$ and $\alpha(G)$ are the minimum degree sum of $k+1$ independent

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vertices, the connectivity and the independence number of G, respectively. In this paper, we settle this conjecture. This is an improvement of the result obtained by Li: If G is a k-connected graph such that $\sigma_{k+1}(G) \geq |V(G)| + (k-1)(\alpha(G)-1)$, then G is Hamiltonian. The degree sum condition is best possible.

1 Introduction

1.1 Degree sum condition for graphs with high connectivity to be Hamiltonian

In this paper, we consider only finite undirected graphs without loops or multiple edges. For standard graph-theoretic terminology not explained, we refer the reader to [5].

A *Hamiltonian cycle* of a graph is a cycle containing all the vertices of the graph. A graph having a Hamiltonian cycle is called a *Hamiltonian graph*. The Hamiltonian problem has long been fundamental in graph theory. Since it is NP-complete, no easily verifiable necessary and sufficient condition seems to exist. Then instead of that, many researchers have investigated sufficient conditions for a graph to be Hamiltonian. In this paper, we deal with a degree sum type condition, which is one of the main stream of this study.

We introduce four invariants, including degree sum, which play important roles for the existence of a Hamiltonian cycle. Let G be a graph. The number of vertices of G is called its *order*, denoted by $n(G)$. A set X of vertices in G is called *an independent set in* G if no two vertices of X are adjacent in G. The *independence number* of G is defined by the maximum cardinality of an independent set in G, denoted by $\alpha(G)$. For two distinct vertices $x, y \in V(G)$, the *local connectivity* $\kappa_G(x, y)$ is defined to be the maximum number of internally-disjoint paths connecting x and y in G. A graph G is k*-connected* if $\kappa_G(x, y) \geq k$ for any two distinct vertices $x, y \in V(G)$. The *connectivity* $\kappa(G)$ of G is the maximum value of k for which G is k-connected. We denote by $N_G(x)$ and $d_G(x)$ the neighbor and the degree of a vertex x in G, respectively. If $\alpha(G) \geq k$, let

$$\sigma_k(G) = \min\Big\{ \sum_{x \in X} d_G(x) \colon X \text{ is an independent set in } G \text{ with } |X| = k \Big\};$$

otherwise let $\sigma_k(G) = +\infty$. If the graph G is clear from the context, we simply write n, α, κ and σ_k instead of $n(G)$, $\alpha(G)$, $\kappa(G)$ and $\sigma_k(G)$, respectively.

One of the main streams of the study of the Hamiltonian problem is, as mentioned above, to consider degree sum type sufficient conditions for graphs to have a Hamiltonian cycle. We list some of them below. (Each of the conditions is best possible in some sense.)

Theorem 1. *Let G be a graph of order at least three. If G satisfies one of the following, then G is Hamiltonian.*

(i) (Dirac [7]) *The minimum degree of G is at least $\frac{n}{2}$.*

(ii) (Ore [12]) $\sigma_2 \geq n$.

(iii) (Chvátal and Erdős [6]) $\alpha \leq \kappa$.

(iv) (Bondy [4]) *G is k-connected and* $\sigma_{k+1} > \dfrac{(k+1)(n-1)}{2}$.

(v) (Bauer, Broersma, Veldman and Li [2]) *G is 2-connected and* $\sigma_3 \geq n + \kappa$.

To be exact, Theorem 1 (iii) is not a degree sum type condition, but it is closely related. Bondy [3] showed that Theorem 1 (iii) implies (ii). The current research of this area is based on Theorem 1 (iii). Let us explain how to expand the research from Theorem 1 (iii): Let G be a k-connected graph, and suppose that one wants to consider whether G is Hamiltonian. If $\alpha \leq k$, then it follows from Theorem 1 (iii) that G is Hamiltonian. Hence we may assume that $\alpha \geq k + 1$, that is, G has an independent set of order $k + 1$. Thus, it is natural to consider a σ_{k+1} condition for a k-connected graph. Bondy [4] gave a σ_{k+1} condition of Theorem 1 (iv).

In this paper, we give a much weaker σ_{k+1} condition than that of Theorem 1 (iv).

Theorem 2. *Let k be an integer with $k \geq 1$ and let G be a k-connected graph. If*

$$\sigma_{k+1} \geq n + \kappa + (k - 2)(\alpha - 1),$$

then G is Hamiltonian.

Theorem 2 was conjectured by Ozeki and Yamashita [15], and has been proven for small integers k: The case $k = 2$ of Theorem 2 coincides Theorem 1 (v). The cases $k = 1$ and $k = 3$ were shown by Fraisse and Jung [8], and by Ozeki and Yamashita [15], respectively.

1.2 Best possibility of Theorem 2

In this section, we show that the σ_{k+1} condition in Theorem 2 is best possible in some senses.

We first discuss the lower bound of the σ_{k+1} condition. For an integer $l \geq 2$ and l vertex-disjoint graphs $H_1, \ldots, H_l$, we define the graph $H_1 + \cdots + H_l$ from the union of $H_1, \ldots, H_l$ by joining every vertex of H_i to every vertex of H_{i+1} for

$1 \leq i \leq l-1$. Fix an integer $k \geq 1$. Let κ, m and n be integers with $k \leq \kappa < m$ and $2m+1 \leq n \leq 3m-\kappa$. Let $G_1 = K_{n-2m} + \overline{K}_\kappa + \overline{K}_m + \overline{K}_{m-\kappa}$, where K_l denotes a complete graph of order l and $\overline{K}_l$ denotes the complement of K_l. Then $\alpha(G_1) = m+1$, $\kappa(G_1) = \kappa$ and

$$\begin{aligned} \sigma_{k+1}(G_1) &= (n-2m-1+\kappa) + km \\ &= n(G_1) + \kappa(G_1) + (k-2)(\alpha(G_1)-1) - 1. \end{aligned}$$

(Note that it follows from condition "$n \leq 3m-\kappa$" that $n-2m-1+\kappa < m$.) Since deleting all the vertices in $\overline{K}_\kappa$ and those in $\overline{K}_{m-\kappa}$ breaks G_1 into $m+1$ components, we see that G_1 has no Hamiltonian cycle. Therefore, the σ_{k+1} condition in Theorem 2 is best possible.

We next discuss the relation between the coefficient of κ and that of $\alpha - 1$. By Theorem 1 (iii), we may assume that $\alpha \geq \kappa + 1$. This implies that

$$n + \kappa + (k-2)(\alpha - 1) \geq n + (1+\varepsilon)\kappa + (k-2-\varepsilon)(\alpha-1)$$

for arbitrarily $\varepsilon > 0$. Then one may expect that the σ_{k+1} condition in Theorem 2 can be replaced with "$n + (1+\varepsilon)\kappa + (k-2-\varepsilon)(\alpha-1)$" for some $\varepsilon > 0$. However, the graph G_1 as defined above shows that it is not true: For any $\varepsilon > 0$, there exist two integers m and κ such that $\varepsilon(m-\kappa) \geq 1$. If we construct the above graph G_1 from such integers m and κ, then we have

$$\begin{aligned} \sigma_{k+1}(G_1) &= n + \kappa + (k-2)m - 1 \\ &= n + (1+\varepsilon)\kappa + (k-2-\varepsilon)m - 1 + \varepsilon(m-\kappa) \\ &\geq n(G_1) + (1+\varepsilon)\kappa(G_1) + (k-2-\varepsilon)\big(\alpha(G_1)-1\big), \end{aligned}$$

but G_1 is not Hamiltonian. This means that the coefficient 1 of κ and the coefficient $k-2$ of $\alpha - 1$ are, in a sense, best possible.

1.3 Comparing Theorem 2 to other results

In this section, we compare Theorem 2 to Theorem 1 (iv) and Ota's result (Theorem 3).

We first show that the σ_{k+1} condition of Theorem 2 is weaker than that of Theorem 1 (iv). Let G be a k-connected graph satisfying the σ_{k+1} condition of Theorem 1 (iv). Assume that $\alpha \geq (n+1)/2$. Let X be an independent set of order at least $(n+1)/2$. Then $|V(G) \setminus X| \leq (n-1)/2$ and $|V(G) \setminus X| \geq k$ since $V(G) \setminus X$ is a cut set. Hence $(n+1)/2 \geq k+1$, and we can take a subset Y of X with $|Y| = k+1$. Then $N_G(y) \subseteq V(G) \setminus X$ for $y \in Y$, and hence $\sum_{y \in Y} d_G(y) \leq (k+1)|V(G) \setminus X| \leq (k+1)(n-1)/2$. This contradicts the σ_{k+1}

condition of Theorem 1 (iv). Therefore $n/2 \geq \alpha$. Moreover, by Theorem 1 (iii), we may assume that $\alpha \geq \kappa + 1$. Therefore, the following inequality holds:

$$\begin{aligned}
\sigma_{k+1} &> \frac{(k+1)(n-1)}{2} \\
&= n-1+\frac{(k-1)(n-1)}{2} \\
&\geq n-1+\frac{(k-1)(2\alpha-1)}{2} \\
&\geq n-1+(k-1)(\alpha-1) \\
&\geq n+\kappa+(k-2)(\alpha-1)-1.
\end{aligned}$$

Thus, the σ_{k+1} condition of Theorem 1 (iv) implies that of Theorem 2.

We next compare Theorem 2 to the following Ota's result.

Theorem 3 (Ota [13]). *Let G be a 2-connected graph. If $\sigma_{l+1} \geq n + l(l-1)$ for all integers l with $l \geq \kappa$, then G is Hamiltonian.*

We first mention about the reason to compare Theorem 2 to Theorem 3. Li [10] proved the following theorem, which was conjectured by Li, Tian, and Xu [11]. (Harkat-Benhamadine, Li and Tian [9], and Li, Tian, and Xu [11] have already proven the case $k = 3$ and the case $k = 4$, respectively.)

Theorem 4 (Li [10]). *Let k be an integer with $k \geq 1$ and let G be a k-connected graph. If $\sigma_{k+1} \geq n + (k-1)(\alpha - 1)$, then G is Hamiltonian.*

In fact, Li showed Theorem 4 just as a corollary of Theorem 3. Note that Theorem 2 is, assuming Theorem 1 (iii), an improvement of Theorem 4. Therefore we should show that Theorem 2 cannot be implied by Theorem 3. (Ozeki, in his Doctoral Thesis [14], compared the relation between several theorems, including Theorem 1 (i), (ii), (iii) and (v), the case $k = 3$ of Theorems 2 and 4, and Theorem 3.)

Let κ, r, k, m be integers such that $4 \leq r$, $3 \leq k \leq \kappa - 2$ and $m = (k+1)(r-2)+4$. Let $G_2 = K_1 + \overline{K}_\kappa + K_{\kappa+m-r} + (\overline{K}_m + K_r)$. Then $n(G_2) = 2\kappa + 2m + 1$, $\kappa(G_2) = \kappa$ and $\alpha(G_2) = \kappa + m$. Since

$$\begin{aligned}
\kappa + k(\kappa+m) - (k+1)(\kappa+m-r+1) &= (k+1)(r-1) - m \\
&= (k+1)(r-1) - (k+1)(r-2) - 4 \\
&= k-3 \\
&\geq 0,
\end{aligned}$$

it follows that

$$\begin{aligned}
\sigma_{k+1}(G_2) &= \min\big\{\kappa + k(\kappa+m),\ (k+1)(\kappa+m-r+1)\big\} \\
&= \kappa + k(\kappa+m) - (k-3) \\
&= (2\kappa+2m+1) + \kappa + (k-2)(\kappa+m-1) \\
&= n(G_2) + \kappa(G_2) + (k-2)(\alpha(G_2)-1).
\end{aligned}$$

Hence the assumption of Theorem 2 holds. On the other hand, for $l = \alpha(G_2) - 1 = \kappa + m - 1$, we have

$$\begin{aligned} n(G_2) + l(l-1) - \sigma_{l+1}(G_2) &= (2\kappa + 2m + 1) + (\kappa + m - 1)(\kappa + m - 2) \\ &\quad -\big\{\kappa(\kappa + m - r + 1) + m(\kappa + m)\big\} \\ &= \kappa(r-2) - m + 3 \\ &= \kappa(r-2) - (k+1)(r-2) - 4 + 3 \\ &= (\kappa - k - 1)(r-2) - 1 \\ &\geq (r-2) - 1 \\ &> 0. \end{aligned}$$

Hence the assumption of Theorem 3 does not hold. These yield that for the graph G_2, we can apply Theorem 2, but cannot apply Theorem 3.

2 Notation and lemmas

Let G be a graph and H be a subgraph of G, and let $x \in V(G)$ and $X \subseteq V(G)$. We denote by $N_G(X)$ the set of vertices in $V(G) \setminus X$ which are adjacent to some vertex in X. We define $N_H(x) = N_G(x) \cap V(H)$ and $d_H(x) = |N_H(x)|$. Furthermore, we define $N_H(X) = N_G(X) \cap V(H)$. If there is no fear of confusion, we often identify H with its vertex set $V(H)$. For example, we often write $G - H$ instead of $G - V(H)$. For a subgraph H, a path P is called an *H-path* if both end vertices of P are contained in H and all internal vertices are not contained in H. Note that each edge of H is an H-path.

Let C be a cycle (or a path) with a fixed orientation in a graph G. For $x, y \in V(C)$, we denote by $C[x,y]$ the path from x to y along the orientation of C. The reverse sequence of $C[x,y]$ is denoted by $\overleftarrow{C}[y,x]$. We denote $C[x,y] - \{x,y\}$, $C[x,y] - \{x\}$ and $C[x,y] - \{y\}$ by $C(x,y)$, $C(x,y]$ and $C[x,y)$, respectively. For $x \in V(C)$, we denote the successor and the predecessor of x on C by x^+ and x^-, respectively. For $X \subseteq V(C)$, we define $X^+ = \{x^+ : x \in X\}$ and $X^- = \{x^- : x \in X\}$. Throughout this paper, we consider that every cycle has a fixed orientation.

In this paper, we extend the concept of *insertible*, introduced by Ainouche [1], which has been used for the proofs of the results on cycles.

Let G be a graph, and H be a subgraph of G. Let $X(H) = \{u \in V(G-H) : uv_1, uv_2 \in E(G)$ for some $v_1v_2 \in E(H)\}$, let $I(x;H) = \{v_1v_2 \in E(H) : xv_1, xv_2 \in E(G)\}$ for $x \in V(G-H)$, and let $Y(H) = \{u \in V(G-H) :\ d_H(u) \geq \alpha(G)\}$.

Lemma 1. *Let D be a cycle of a graph G. Let k be a positive integer and let $Q_1, Q_2, \dots, Q_k$ be paths of $G - D$ with fixed orientations such that $V(Q_i) \cap V(Q_j) = \emptyset$ for $1 \leq i < j \leq k$. If the following* (I) *and* (II) *hold, then $G[V(D \cup Q_1 \cup Q_2 \cup \cdots \cup Q_k)]$ is Hamiltonian.*

(I) *For* $1 \le i \le k$ *and* $a \in V(Q_i)$, $a \in X(D) \cup Y(Q_i(a, b_i] \cup D)$, *where* b_i *is the last vertex of* Q_i.

(II) *For* $1 \le i < j \le k$, $x \in V(Q_i)$ *and* $y \in V(Q_j)$, $I(x; D) \cap I(y; D) = \emptyset$.

Proof. We can easily see that $G[V(D \cup Q_1 \cup Q_2 \cup \cdots \cup Q_k)]$ contains a cycle D^* such that $V(D) \cup \big(X(D) \cap V(Q_1 \cup Q_2 \cup \cdots \cup Q_k)\big) \subseteq V(D^*)$. In fact, we can insert all vertices of $X(D) \cap V(Q_1)$ into D by choosing the following $u_1, v_1 \in V(Q_1)$ and $w_1w_1^+ \in E(D)$ inductively. Take the first vertex u_1 in $X(D) \cap V(Q_1)$ along the orientation of Q_1, and let v_1 be the last vertex in $X(D) \cap V(Q_1)$ on Q_1 such that $I(u_1; D) \cap I(v_1; D) \neq \emptyset$. Then we can insert all vertices of $Q_1[u_1, v_1]$ into D. To be exact, taking $w_1w_1^+ \in I(u_1; D) \cap I(v_1; D)$, $D_1^1 := w_1Q_1[u_1, v_1]D[w_1^+, w_1]$ is such a cycle. By the choice of u_1 and v_1, $w_1w_1^+ \notin I(x; D)$ for all $x \in V(Q_1 - Q_1[u_1, v_1])$, and $X(D) \cap V(Q_1 - Q_1[u_1, v_1])$ is contained in some component of $Q_1 - Q_1[u_1, v_1]$. Moreover, note that $E(D) \setminus \{w_1w_1^+\} \subseteq E(D_1^1)$. Hence by repeating this argument, we can obtain a cycle D_1^* of $G[V(D \cup Q_1)]$ such that $V(D) \cup \big(X(D) \cap V(Q_1)\big) \subseteq V(D_1^*)$ and $E(D) \setminus \bigcup_{x \in V(Q_1)} I(x; D) \subseteq E(D_1^*)$. Then by (II), $I(x; D) \subseteq E(D_1^*)$ for all $x \in V(Q_2 \cup \cdots \cup Q_k)$. Therefore $G[V(D \cup Q_1 \cup Q_2 \cup \cdots \cup Q_k)]$ contains a cycle D^* such that $V(D) \cup \big(X(D) \cap V(Q_1 \cup Q_2 \cup \cdots \cup Q_k)\big) \subseteq V(D^*)$.

We choose a cycle C of $G[V(D \cup Q_1 \cup Q_2 \cup \cdots \cup Q_k)]$ containing all vertices in $V(D) \cup \big(X(D) \cap V(Q_1 \cup Q_2 \cup \cdots \cup Q_k)\big)$ so that $|C|$ is as large as possible. Now, we change the "base" cycle from D to C, and use the symbol $(\cdot)^+$ for the orientation of C. Suppose that $V(Q_i - C) \neq \emptyset$ for some i with $i \in \{1, 2, \dots, k\}$. We may assume that $i = 1$. Let w be the last vertex in $V(Q_1 - C)$ along Q_1. Since C contains all vertices in $X(D) \cap V(Q_1)$, it follows from (I) that $w \in Y(Q_1(w, b_1] \cup D)$, that is, $|N_G(w) \cap V(Q_1(w, b_1] \cup D)| \ge \alpha(G)$. By the choice of w, we obtain $V(Q_1(w, b_1] \cup D) \subseteq V(C)$. Therefore $|N_C(w)^+ \cup \{w\}| \ge |N_G(w) \cap V(Q_1(w, b_1] \cup D)| + 1 \ge \alpha(G) + 1$. This implies that $N_C(w)^+ \cup \{w\}$ is not an independent set in G. Hence $wz^+ \in E(G)$ for some $z \in N_C(w)$ or $z_1^+z_2^+ \in E(G)$ for some distinct $z_1, z_2 \in N_C(w)$. In the former case, let $C' = wC[z^+, z]w$, and in the latter case, let $C' = w\overleftarrow{C}[z_1, z_2^+]C[z_1^+, z_2]w$. Then C' is a cycle of $G[V(D \cup Q_1 \cup Q_2 \cup \cdots \cup Q_k)]$ such that $V(C) \cup \{w\} \subseteq V(C')$, which contradicts the choice of C. Thus $V(Q_1 \cup Q_2 \cup \cdots \cup Q_k)$ are contained in C, and hence C is a Hamiltonian cycle of $G[V(D \cup Q_1 \cup Q_2 \cup \cdots \cup Q_k)]$. □

In the rest of this section, we fixed the following notation. Let C be a longest cycle in a graph G, and H_0 be a component of $G - C$. For $u \in N_C(H_0)$, let $u' \in N_C(H_0)$ be a vertex such that $C(u, u') \cap N_C(H_0) = \emptyset$, that is, u' is the successor of u in $N_C(H_0)$ along the orientation of C.

For $u \in N_C(H_0)$, a vertex $v \in C(u, u')$ is *insertible* if $v \in X(C[u', u]) \cup Y(C(v, u])$. A vertex in $C(u, u')$ is said to be *non-insertible* if it is not insertible.

Lemma 2. *There exists a non-insertible vertex in* $C(u, u')$ *for* $u \in N_C(H_0)$.

Proof. Let $u \in N_C(H_0)$, and suppose that every vertex in $C(u, u')$ is insertible. Let P be a C-path joining u and u' with $V(P) \cap V(H_0) \neq \emptyset$. Let $D = C[u', u]P[u, u']$ and $Q = C(u, u')$. Let $v \in V(Q)$. Since v is insertible, it follows that $v \in X(C[u', u]) \cup Y(C(v, u])$. Since $C[u', u]$ is a subpath of D, we have $v \in X(D) \cup Y(Q(v, u') \cup D)$. Hence, by Lemma 1, $G[V(D \cup Q)]$ is Hamiltonian, which contradicts the maximality of C. □

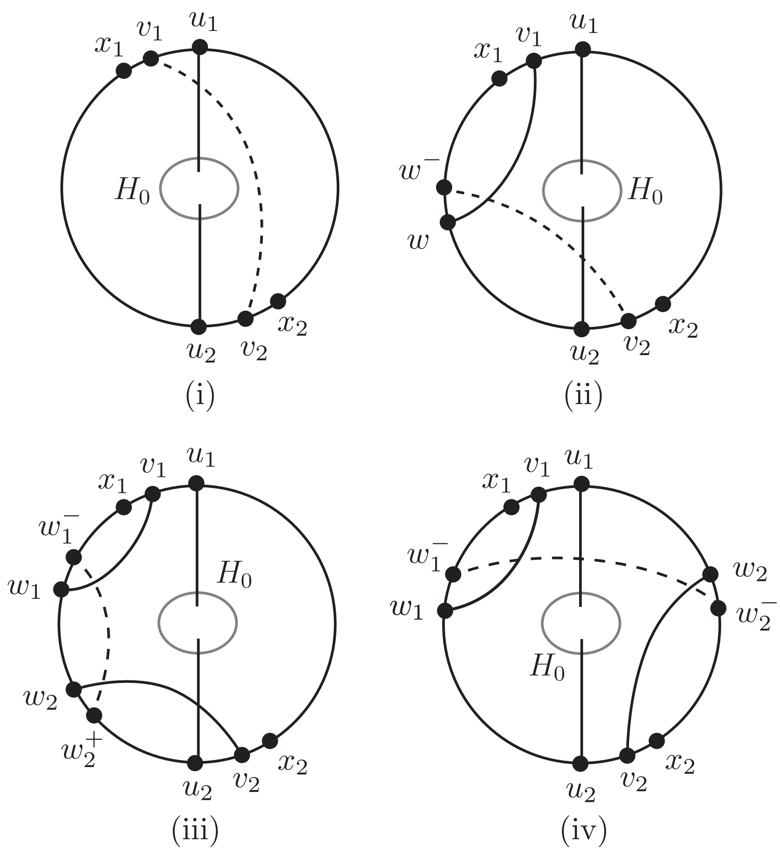

Figure 1: Lemma 3

Lemma 3. *Let $u_1, u_2 \in N_C(H_0)$ with $u_1 \neq u_2$, and let x_i be the first non-insertible vertex along $C(u_i, u_i')$ for $i \in \{1, 2\}$. Then the following hold (see Figure 1).*

(i) *There exists no C-path joining $v_1 \in C(u_1, x_1]$ and $v_2 \in C(u_2, x_2]$. In particular, $x_1x_2 \notin E(G)$.*

(ii) *If there exists a C-path joining $v_1 \in C(u_1, x_1]$ and $w \in C(v_1, u_2]$, then there exists no C-path joining $v_2 \in C(u_2, x_2]$ and w^-.*

(iii) *If there exist a C-path joining $v_1 \in C(u_1, x_1]$ and $w_1 \in C(v_1, u_2)$ and a C-path joining $v_2 \in C(u_2, x_2]$ and $w_2 \in C[w_1, u_2)$, then there exists no C-path joining w_1^- and w_2^+.*

(iv) *If for each* $i \in \{1,2\}$*, there exists a* C*-path joining* $v_i \in C(u_i, x_i]$ *and* $w_i \in C(v_i, u_{3-i}]$*, then there exists no* C*-path joining* w_1^- *and* w_2^-*.*

Proof. Let P_0 be a C-path which connects u_1 and u_2, and $V(P_0) \cap V(H_0) \neq \emptyset$. We first show (i) and (ii). Suppose that the following (a) or (b) holds for some $v_1 \in C(u_1, x_1]$ and some $v_2 \in C(u_2, x_2]$: (a) There exists a C-path P_1 joining v_1 and v_2. (b) There exist disjoint C-paths P_2 joining v_l and w, and P_3 joining v_{3-l} and w^- for some $l \in \{1,2\}$ and some $w \in C(v_l, u_{3-l}]$. We choose such vertices v_1 and v_2 so that $|C[u_1, v_1]| + |C[u_2, v_2]|$ is as small as possible. Without loss of generality, we may assume that $l = 1$ if (b) holds. Since $N_C(H_0) \cap \{v_1, v_2\} = \emptyset$, $(V(P_1) \cup V(P_2) \cup V(P_3)) \cap V(P_0) = \emptyset$. Therefore, we can define a cycle

$$D = \begin{cases} P_1[v_1, v_2]C[v_2, u_1]P_0[u_1, u_2]\overleftarrow{C}[u_2, v_1] & \text{if (a) holds,} \\ P_2[v_1, w]C[w, u_2]\overleftarrow{P_0}[u_2, u_1]\overleftarrow{C}[u_1, v_2]P_3[v_2, w^-]\overleftarrow{C}[w^-, v_1] & \text{otherwise.} \end{cases}$$

For $i \in \{1,2\}$, let $Q_i = C(u_i, v_i)$. By Lemma 2, we can obtain the following statement (1), and by the choice of v_1 and v_2, we can obtain the following statements (2)–(5):

(1) $N_G(x) \cap P_0(u_1, u_2) = \emptyset$ for $x \in V(Q_1 \cup Q_2)$.

(2) $N_G(x) \cap (P_1(v_1, v_2) \cup P_2(v_1, w) \cup P_3(v_2, w^-)) = \emptyset$ for $x \in V(Q_1 \cup Q_2)$.

(3) $xy \notin E(G)$ for $x \in V(Q_1)$ and $y \in V(Q_2)$.

(4) $I(x; C) \cap I(y; C) = \emptyset$ for $x \in V(Q_1)$ and $y \in V(Q_2)$.

(5) If (b) holds, then $w^-w \notin I(x; C)$ for $x \in V(Q_1 \cup Q_2)$.

Let $a \in V(Q_i)$ for some $i \in \{1,2\}$. Note that each vertex of Q_i is insertible, that is, $a \in X(C[u_i', u_i]) \cup Y(C(a, u_i])$. We show that $a \in X(D) \cup Y(Q_i(a, v_i) \cup D)$. If $a \in X(C[u_i', u_i])$, then the statements (3) and (5) yield that $a \in X(D)$. Suppose that $a \in Y(C(a, u_i])$. By (3), $N_G(a) \cap C(a, u_i] \subseteq N_G(a) \cap \big(Q_i(a, v_i) \cup D\big)$. This implies that $a \in Y(Q_i(a, v_i) \cup D)$. By (1), (2) and (4), $I(x; D) \cap I(y; D) = \emptyset$ for $x \in V(Q_1)$ and $y \in V(Q_2)$. Thus, by Lemma 1, $G[V(D \cup Q_1 \cup Q_2)]$ is Hamiltonian, which contradicts the maximality of C.

By using similar argument as above, we can also show (iii) and (iv). We only prove (iii). Suppose that for some $v_1 \in C(u_1, x_1]$ and $v_2 \in C(u_2, x_2]$, there exist disjoint C-paths $P_1[v_1, w_1]$, $P_2[v_2, w_2]$ and $P_3[w_1^-, w_2^+]$ with $w_1 \in C(v_1, u_2)$ and $w_2 \in C[w_1, u_2)$. We choose such v_1 and v_2 so that $|C[u_1, v_1]| + |C[u_2, v_2]|$ is as small as possible. Let $Q_i = C(u_i, v_i)$ for $i \in \{1,2\}$. Then by Lemma 3 (i), $xy \notin E(G)$ for $x \in V(Q_1)$ and $y \in V(Q_2)$. By the choice of v_1 and v_2 and Lemma 3 (ii), $w_1w_1^-, w_2w_2^+ \notin I(x; C[v_1, u_1]) \cup I(y; C[v_2, u_2])$ for $x \in V(Q_1)$ and $y \in V(Q_2)$. By

Lemma 3 (i) and (ii), $I(x; C[v_1, u_2] \cup C[v_2, u_1]) \cap I(y; C[v_1, u_2] \cup C[v_2, u_1]) = \emptyset$ for $x \in V(Q_1)$ and $y \in V(Q_2)$. Hence by applying Lemma 1 as

$$D = P_1[v_1, w_1]C[w_1, w_2]\overleftarrow{P_2}[w_2, v_2]C[v_2, u_1]P_0[u_1, u_2]\overleftarrow{C}[u_2, w_2^+]\overleftarrow{P_3}[w_2^+, w_1^-]\overleftarrow{C}[w_1^-, v_1],$$

Q_1 and Q_2, we see that there exits a longer cycle than C, a contradiction. □

3 Proof of Theorem 2

Proof of Theorem 2. The cases $k = 1$, $k = 2$ and $k = 3$ were shown by Fraisse and Jung [8], by Bauer et al. [2] and by Ozeki and Yamashita [15], respectively. Therefore, we may assume that $k \geq 4$. Let G be a graph satisfying the assumption of Theorem 2. By Theorem 1 (iii), we may assume $\alpha(G) \geq \kappa(G) + 1$. Let C be a longest cycle in G. If C is a Hamiltonian cycle of G, then there is nothing to prove. Hence we may assume that $G - V(C) \neq \emptyset$. Let $H = G - V(C)$ and $x_0 \in V(H)$. Choose a longest cycle C and x_0 so that

$$d_C(x_0) \text{ is as large as possible.}$$

Let H_0 be the component of H such that $x_0 \in V(H_0)$. Let

$$N_C(H_0) = U = \{u_1, u_2, \ldots, u_m\}.$$

Note that $m \geq \kappa(G) \geq k$. Let

$$M_0 = \{0, 1, \ldots, m\} \text{ and } M_1 = \{1, 2, \ldots, m\}.$$

Let u_i' be the vertex in $N_C(H_0)$ such that $C(u_i, u_i') \cap N_C(H_0) = \emptyset$. By Lemma 2, there exists a non-insertible vertex in $C(u_i, u_i')$. Let $x_i \in C(u_i, u_i')$ be the first non-insertible vertex along the orientation of C for each $i \in M_1$, and let

$$X = \{x_1, x_2, \ldots, x_m\}.$$

Note that $d_C(x_0) \leq |U| = |X|$. Let

$$D_i = C(u_i, x_i) \text{ for each } i \in M_1, \text{ and } D = \bigcup_{i \in M_1} D_i.$$

We check the degree of x_i in C and H. Since x_i is non-insertible, we can see that

$$d_C(x_i) \leq |D_i| + \alpha(G) - 1 \quad \text{for } i \in M_1. \tag{1}$$

By the definition of x_i, we clearly have $N_{H_0}(x_i) = \emptyset$ for $i \in M_1$. Moreover, by Lemma 3 (i), $N_H(x_i) \cap N_H(x_j) = \emptyset$ for $i, j \in M_1$ with $i \neq j$. Thus we obtain

$$\sum_{i \in M_0} d_H(x_i) \leq |H| - 1, \tag{2}$$

and

$$\sum_{i\in M_1} d_H(x_i) \le |H| - |H_0|. \tag{3}$$

We check the degree sum in C of two vertices in X. Let i and j be distinct two integers in M_1. In this paragraph, we let $C_i = C[x_i, u_j]$ and $C_j = C[x_j, u_i]$. By Lemma 3 (ii), we have $N_{C_i}(x_i)^- \cap N_{C_i}(x_j) = \emptyset$ and $N_{C_j}(x_j)^- \cap N_{C_j}(x_i) = \emptyset$. By Lemma 3 (i), $N_{C_i}(x_i)^- \cup N_{C_i}(x_j) \subseteq C_i \setminus D$, $N_{C_j}(x_j)^- \cup N_{C_j}(x_i) \subseteq C_j \setminus D$ and $N_{D_i}(x_j) = N_{D_j}(x_i) = \emptyset$. Thus, we obtain

$$d_C(x_i) + d_C(x_j) \le |C| - \sum_{h\in M_1\setminus\{i,j\}} |D_h| \ \text{ for } i,j \in M_1 \text{ with } i \ne j. \tag{4}$$

By Lemma 3 (i) and since $N_{H_0}(x_i) = \emptyset$ for $i \in M_1$, we obtain the following.

Claim 1. *$X \cup \{x_0\}$ is an independent set, and hence $|X| \le \alpha(G) - 1$.*

Claim 2. $|X| \ge \kappa(G) + 1$.

Proof. Let s and t be distinct two integers in M_1. By the inequality (4), we have

$$d_C(x_s) + d_C(x_t) \le |C| - \sum_{i\in M_1\setminus\{s,t\}} |D_i|.$$

Let I be a subset of M_0 such that $|I| = k + 1$ and $\{0, s, t\} \subseteq I$. By Claim 1, $\{x_i : i \in I\}$ is an independent set. By the inequality (1), we deduce

$$\sum_{i\in I\setminus\{0,s,t\}} d_C(x_i) \ \le \ \sum_{i\in I\setminus\{0,s,t\}} |D_i| + (k-2)(\alpha(G) - 1).$$

By the inequality (2) and the definition of I, we obtain

$$\sum_{i\in I} d_H(x_i) \ \le \ |H| - 1.$$

Thus, it follows from these three inequalities that

$$\sum_{i\in I} d_G(x_i) \le n + (k-2)(\alpha(G) - 1) - 1 + d_C(x_0).$$

Since $\sigma_{k+1}(G) \ge n + \kappa(G) + (k-2)(\alpha(G)-1)$, we have $|X| \ge d_C(x_0) \ge \kappa(G) + 1$. □

Let S be a cut set with $|S| = \kappa(G)$, and let $V_1, V_2, \dots, V_p$ be the components of $G - S$. By Claim 2, we may assume that

$$\text{there exists an integer } l \text{ such that } C[u_l, u_l') \subseteq V_1.$$

By Lemma 3 (i), we obtain

$$d_C(x_l) \le |C \cap (V_1 \cup S)| - |(\bigcup_{i\in M_1\setminus\{l\}} D_i \cup X) \cap (V_1 \cup S)|. \tag{5}$$

By replacing the labels x_2 and x_3 if necessary, we may assume that x_1, x_2 and x_3 appear in this order along the orientation of C. In this paragraph, the indices are taken modulo 3. From now we let

$$C_i = C[x_i, u_{i+1}]$$

and

$$W_i := \{w \in V(C_i) : w^+ \in N_{C_i}(x_i) \text{ and } w^- \in N_{C_i}(x_{i+1})\}$$

for each $i \in \{1,2,3\}$, and let $W := W_1 \cup W_2 \cup W_3$ (see Figure 2). Note that $W \cap (U \cup \{x_1, x_2, x_3\}) = \emptyset$, by the definition of C_i and W_i and by Lemma 3 (i).

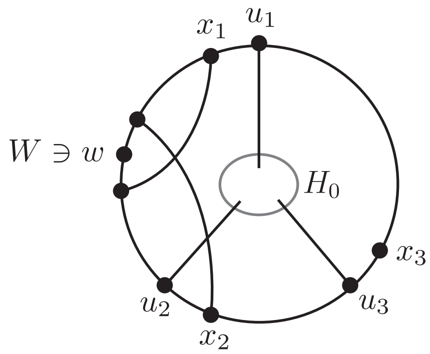

Figure 2: The definition of W.

Claim 3. *$D \cup X \cup W \cup H \subseteq V_1 \cup S$. In particular, $x_0 \in V_1 \cup S$.*

Proof. We first show that $D \cup X \cup W \subseteq V_1 \cup S$. Suppose not. Without loss of generality, we may assume that there exists an integer h in $M_1 \setminus \{l\}$ such that $\big(D_h \cup \{x_h\} \cup (W \cap C(x_h, u'_h))\big) \cap V_2 \neq \emptyset$, say $v \in \big(D_h \cup \{x_h\} \cup (W \cap C(x_h, u'_h))\big) \cap V_2$. Since $v \in V_2$, it follows from Lemma 3 (i) and (ii) that

$$d_C(v) \leq |C \cap (V_2 \cup S)| - |(\bigcup_{i \in M_1 \setminus \{h\}} D_i \cup X) \cap (V_2 \cup S)|.$$

Let I be a subset of $M_0 \setminus \{h\}$ such that $|I| = k$ and $\{0, l\} \subseteq I$. By Claim 1 and Lemma 3 (i) and (ii), $\{x_i : i \in I\} \cup \{v\}$ is an independent set of order $k+1$. By the above inequality and the inequality (5), we obtain

$$\begin{aligned}
& d_C(x_l)+d_C(v)\\
\le\ & |C\cap(V_1\cup V_2\cup S)|+|C\cap S|-|(\bigcup_{i\in M_1\setminus\{l,h\}} D_i\cup X)\cap(V_1\cup V_2\cup S)|\\
=\ & |C|+|C\cap S|-|C\cap(\bigcup_{3\le j\le p} V_j)|-|(\bigcup_{i\in M_1\setminus\{l,h\}} D_i\cup X)\cap(V_1\cup V_2\cup S)|\\
\le\ & |C|+|C\cap S|-|(\bigcup_{i\in M_1\setminus\{l,h\}} D_i\cup X)\cap(\bigcup_{3\le j\le p} V_j)|\\
& -|(\bigcup_{i\in M_1\setminus\{l,h\}} D_i\cup X)\cap(V_1\cup V_2\cup S)|\\
\le\ & |C|+\kappa(G)-\sum_{i\in M_1\setminus\{l,h\}}|D_i\cap(\bigcup_{1\le j\le p} V_j\cup S)|-|X\cap(\bigcup_{1\le j\le p} V_j\cup S)|\\
\le\ & |C|+\kappa(G)-\sum_{i\in I\setminus\{0,l\}}|D_i|-|X|\\
\le\ & |C|+\kappa(G)-\sum_{i\in I\setminus\{0,l\}}|D_i|-d_C(x_0).
\end{aligned}$$

On the other hand, the inequality (1) yields that

$$\sum_{i\in I\setminus\{0,l\}} d_C(x_i)\le\sum_{i\in I\setminus\{0,l\}}|D_i|+(k-2)(\alpha(G)-1).$$

By the above two inequalities, we deduce

$$\sum_{i\in I} d_C(x_i)+d_C(v)\le|C|+\kappa(G)+(k-2)(\alpha(G)-1).$$

Recall that $\{x_i : i\in I\}\cup\{v\}$ is an independent set, in particular, $x_0\notin\bigcup_{i\in I} N_H(x_i)\cup N_H(v)$. Since $N_H(x_i)\cap N_H(x_j)=\emptyset$ for $i,j\in I$ with $i\ne j$ and $(\bigcup_{i\in I} N_H(x_i))\cap N_H(v)=\emptyset$ by Lemma 3 (i) and (ii), it follows that $\sum_{i\in I} d_H(x_i)+d_H(v)\le|H|-1$. Combining this inequality with the above inequality, we get $\sum_{i\in I} d_G(x_i)+d_G(v)\le n+\kappa(G)+(k-2)(\alpha(G)-1)-1$, a contradiction.

We next show that $H-H_0\subseteq V_1\cup S$. Suppose not. Without loss of generality, we may assume that there exists a vertex $y\in(H-H_0)\cap V_2$. Let H_y be a component of H with $y\in V(H_y)$. Note that $H_y\ne H_0$. Suppose that $N_C(H_y)\cap(D_h\cup\{x_h\})\ne\emptyset$ for some $h\in M_1\setminus\{l\}$. Then Lemma 3 (i) yields that

$$d_C(y)\le|C\cap(V_2\cup S)|-|(\bigcup_{i\in M_1\setminus\{h\}} D_i\cup X)\cap(V_2\cup S)|.$$

Hence, by the same argument as above, we can obtain a contradiction. Thus we may assume that $N_C(H_y)\cap(D_i\cup\{x_i\})=\emptyset$ for all $i\in M_1\setminus\{l\}$. Then, since $y\in V_2$ and

$D_l \cup \{x_l\} \subseteq V_1$, we have

$$d_C(y) \leq |C \cap (V_2 \cup S)| - |(\bigcup_{i \in M_1} D_i \cup X) \cap (V_2 \cup S)|.$$

Let I be a subset of M_0 such that $|I| = k$ and $\{0, l\} \subseteq I$. Since $x_l \in V_1$, $y \in V_2$, $H_y \neq H_0$ and $N_C(H_y) \cap (D_i \cup \{x_i\}) = \emptyset$ for all $i \in M_1 \setminus \{l\}$, it follows from Claim 1 that $\{x_i : i \in I\} \cup \{y\}$ is an independent set of order $k+1$. By the above inequality and the inequality (5), we obtain

$$\begin{aligned} & d_C(x_l) + d_C(y) \\ \leq \ & |C \cap (V_1 \cup V_2 \cup S)| + |C \cap S| - |(\bigcup_{i \in M_1 \setminus \{l\}} D_i \cup X) \cap (V_1 \cup V_2 \cup S)| \\ \leq \ & |C| + |C \cap S| - \sum_{i \in I \setminus \{0, l\}} |D_i| - d_C(x_0). \end{aligned}$$

Therefore, by the above inequality and the inequality (1), we obtain

$$\sum_{i \in I} d_C(x_i) + d_C(y) \leq |C| + |C \cap S| + (k-2)(\alpha(G) - 1).$$

Since $H_0 \neq H_y$ and $N_C(H_y) \cap (D_i \cup \{x_i\}) = \emptyset$ for all $i \in M_1 \setminus \{l\}$, it follows that $(\bigcup_{i \in I \setminus \{l\}} N_H(x_i)) \cap V(H_y) = \emptyset$. Since $x_l \in V_1$ and $y \in V_2$, we have $N_H(x_l) \cap N_H(y) \subseteq H \cap S$. Therefore, we obtain

$$\sum_{i \in I} d_H(x_i) + d_H(y) \leq |H| + |H \cap S| - 2.$$

Combining the above two inequalities, $\sum_{i \in I} d_G(x_i) + d_G(y) \leq n + \kappa(G) + (k-2)(\alpha(G) - 1) - 2$, a contradiction.

We finally show that $H_0 \subseteq V_1 \cup S$. Suppose not. Without loss of generality, we may assume that there exists a vertex $y_0 \in H_0 \cap V_2$. Then

$$d_G(y_0) \leq |U \cap (V_2 \cup S)| + |H_0| - 1.$$

Since $u_l \in V_1$, we have $H_0 \cap S \neq \emptyset$. Note that by the above argument, $X \subseteq V_1 \cup S$. Therefore, by Claim 2, $|X \cap V_1| = |X| - |X \cap S| \geq \kappa(G) + 1 - (|S| - |H_0 \cap S|) \geq \kappa(G) + 1 - (\kappa(G) - 1) = 2$. Let $x_s \in X \cap V_1$ with $x_s \neq x_l$. Let I be a subset of M_1 such that $|I| = k$ and $\{l, s\} \subseteq I$. Then $\{x_i : i \in I\} \cup \{y_0\}$ is an independent set of order $k+1$. By Lemma 3 (i), we have $N_C(x_l)^- \cap (U \setminus \{u_l\}) = \emptyset$ and $N_C(x_s)^- \cap (U \setminus \{u_s\}) = \emptyset$. Since $x_l, x_s \in V_1$, it follows that $(N_C(x_l) \cup N_C(x_s)) \cap (U \cap V_2) = \emptyset$. Therefore, we can improve the inequality (4) as follows:

$$d_C(x_l) + d_C(x_s) \leq |C| - \sum_{i \in I \setminus \{l, s\}} |D_i| - |U \cap V_2|.$$

By the inequality (1) and the inequality (3),

$$\sum_{i\in I\setminus\{l,s\}} d_C(x_i) \le \sum_{i\in I\setminus\{l,s\}} |D_i| + (k-2)(\alpha(G)-1) \quad \text{and} \quad \sum_{i\in I} d_H(x_i) \le |H| - |H_0|.$$

Hence, by the above four inequalities, we deduce $d_G(y_0) + \sum_{i\in I} d_G(x_i) \le n + \kappa(G) + (k-2)(\alpha(G)-1) - 1$, a contradiction. □

By Claim 3,

$$\text{there exists an integer } r \text{ such that } C(x_r, u'_r] \cap \textstyle\bigcup_{i=2}^{p} V_i \ne \emptyset,$$

say

$$v_2 \in C(x_r, u'_r] \cap \bigcup_{i=2}^{p} V_i.$$

Choose r and v_2 so that $v_2 \ne u'_r$ if possible. Without loss of generality, we may assume that $v_2 \in V_2$. Note that

$$d_G(v_2) \le |V_2 \cup S| - 1. \tag{6}$$

Claim 4. $d_C(w) \le d_C(x_0) \le |X| \le \alpha(G) - 1$ *for each* $w \in W$.

Proof. Let $w \in W$. Without loss of generality, we may assume that $w \in W_1$. Then by applying Lemma 1 as $Q_1 = D_1$, $Q_2 = D_2$ and

$$D = x_1 C[w^+, u_2] P[u_2, u_1] \overleftarrow{C}[u_1, x_2] \overleftarrow{C}[w^-, x_1],$$

where $P[u_2, u_1]$ is a C-path passing through some vertex of H_0, we can obtain a cycle C' such that $V(C) \setminus \{w\} \subseteq V(C')$ and $V(C') \cap V(H_0) \ne \emptyset$ (note that (I) and (II) of Lemma 1 hold, by Lemma 3 (i) and (ii) and the definition of insertible and D_i). Note that by the maximality of $|C|$, $|C'| = |C|$. Note also that $d_{C'}(w) \ge d_C(w)$. By the choice of C and x_0, we have $d_{C'}(w) \le d_C(x_0)$, and hence by Claim 1 and the fact that $d_C(x_0) \le |X|$, we obtain $d_C(w) \le d_C(x_0) \le |X| \le \alpha(G) - 1$. □

By Lemma 3 and Claim 3, we have

$$\sum_{i\in M_0} d_H(x_i) + \sum_{w\in W} d_H(w) \le |H| - |\{x_0\}| = |H \cap (V_1 \cup S)| - 1. \tag{7}$$

Moreover, by Lemma 3 and Claim 1, the following claim holds.

Claim 5. $X \cup W \cup \{x_0\}$ *is an independent set.*

We now check the degree sum of the vertices x_1, x_2 and x_3 in C. In this paragraph, the indices are taken modulo 3. By Lemma 3 (ii), $(N_{C_i}(x_i)^- \cup N_{C_i}(x_{i+1})^+) \cap N_{C_i}(x_{i+2}) = \emptyset$ for $i \in \{1, 2, 3\}$. Clearly, $N_{C_i}(x_i)^- \cap N_{C_i}(x_{i+1})^+ = W_i$ and $N_{C_i}(x_i)^- \cup$

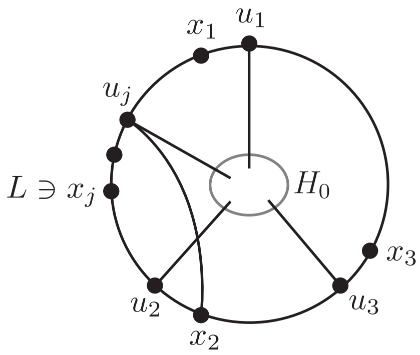

Figure 3: The definition of L.

$N_{C_i}(x_{i+1})^+ \cup N_{C_i}(x_{i+2}) \subseteq C_i \cup \{u_{i+1}^+\}$. By Lemma 3 (i), $(N_{C_i}(x_i)^- \cup N_{C_i}(x_{i+2})) \cap D_j = \emptyset$ for $i \in \{1,2,3\}$ and $j \in M_1$. For $i \in \{1,2,3\}$, let

$$L_i = \left\{x_j \in X \setminus \{x_{i+1}\} : N_{C_i}(x_{i+1})^+ \cap D_j \neq \emptyset\right\}$$

and let $L = \bigcup_{i \in \{1,2,3\}} L_i$ (see Figure 3).

Note that $L \cap \{x_1, x_2, x_3\} = \emptyset$ and $W \cap L = \emptyset$ by Lemma 3 (i). Therefore the following inequality holds:

$$d_{C_i}(x_1) + d_{C_i}(x_2) + d_{C_i}(x_3) \leq |C_i| + |W_i| + 1 - \sum_{j \in M_1} |C_i \cap D_j| + |L_i|$$

for $i \in \{1,2,3\}$. By Lemma 3 (i), we have $N_C(x_i) \cap D_j = \emptyset$ for $i, j \in M_1$ with $i \neq j$, and hence

$$d_{D_i}(x_1) + d_{D_i}(x_2) + d_{D_i}(x_3) \leq |D_i|$$

for $i \in \{1,2,3\}$. Let I be a subset of M_0 such that $I \cap \{1,2,3\} = \emptyset$. Let $L_I = L \cap \{x_i : i \in I\}$. Note that $|L \cap \{x_i\}| - |D_i| \leq 0$ for each $i \in M_1 \setminus \{1,2,3\}$. Thus, we deduce

$$\begin{aligned} d_C(x_1) + d_C(x_2) + d_C(x_3) &\leq \sum_{i=1}^{3}(|C_i| + |W_i| + |L_i| + 1 - \sum_{j \in M_1} |C_i \cap D_j| + |D_i|) \\ &= |C| + |W| + |L| - \sum_{i \in M_1 \setminus \{1,2,3\}} |D_i| + 3 \\ &\leq |C| + |W| + |L_I| - \sum_{i \in I \setminus \{0\}} |D_i| + 3 \qquad (8) \\ &\leq |C| + |W| + 3. \qquad (9) \end{aligned}$$

Claim 6. $|W| + |L| \geq \kappa(G) - 2 \geq 1$.

Proof. Let I be a subset of M_0 such that $|I| = k - 2$ and $I \cap \{1,2,3\} = \emptyset$. Suppose that $|W| + |L_I| \leq \kappa(G) - 3$. By Claim 5, $\{x_i : i \in I\} \cup \{x_1, x_2, x_3\}$ is an independent set of order $k+1$. By the inequality (8), we obtain

$$d_C(x_1) + d_C(x_2) + d_C(x_3) \leq |C| + \kappa(G) - \sum_{i \in I \setminus \{0\}} |D_i|.$$

Therefore, this inequality, the inequalities (1) and (2) and Claim 4 yield that

$$\sum_{i=1}^{3} d_G(x_i) + \sum_{i \in I} d_G(x_i) \leq n + \kappa(G) + (k-2)(\alpha(G)-1) - 1,$$

a contradiction. Therefore $|W| + |L| \geq |W| + |L_I| \geq \kappa(G) - 2$. □

Claim 7. $d_C(x_0) = |U| = |X| = \alpha(G) - 1$. *In particular,* $N_C(x_0) = U$.

Proof. Suppose that $d_C(x_0) \leq \alpha(G) - 2$. In this proof, we assume $x_l = x_1$ (recall that l is an integer such that $C[u_l, u_l') \subseteq V_1$, see the paragraph below the proof of Claim 2). We divide the proof into two cases.

Case 1. $|W| \geq k - 3$.

Subclaim 7.1. $|W| \leq \kappa(G) + k - 5$.

Proof. Suppose that $|W| \geq \kappa(G) + k - 4$. By Claim 3, we obtain

$$\begin{aligned} |(W \cup \{x_0, x_1, x_2, x_3\}) \cap V_1| &= |W \cup \{x_0, x_1, x_2, x_3\}| - |(W \cup \{x_0, x_1, x_2, x_3\}) \cap S| \\ &\geq (\kappa(G) + k - 4 + 4) - \kappa(G) = k. \end{aligned}$$

Let W' be a subset of $(W \cup \{x_0, x_1, x_2, x_3\}) \cap V_1$ such that $|W'| = k$ and $x_1 \in W'$. Since $W' \subseteq V_1$ and $v_2 \in V_2$, it follows from Claim 5 that $W' \cup \{v_2\}$ is an independent set of order $k+1$. By the inequality (5) and Claims 3 and 4, we obtain

$$\begin{aligned} d_C(x_1) &\leq |C \cap (V_1 \cup S)| - \sum_{i \in M_1 \setminus \{1\}} |(D_i \cap (V_1 \cup S)| - |X \cap (V_1 \cup S)| \\ &\leq |C \cap (V_1 \cup S)| - \sum_{i \in \{2,3\}} |D_i| - |X| \\ &\leq |C \cap (V_1 \cup S)| - \sum_{i \in \{2,3\}} |D_i| - d_C(w_0), \end{aligned}$$

where $w_0 \in W' \setminus \{x_1, x_2, x_3\}$ (note that $|W'| = k \geq 4$). By the inequality (1) and Claim 4,

$$\sum_{x \in W' \cap \{x_2, x_3\}} d_C(x) + \sum_{w \in W' \setminus \{w_0, x_1, x_2, x_3\}} d_C(w) \leq \sum_{i \in \{2,3\}} |D_i| + (k-2)(\alpha(G)-1).$$

By the above two inequalities, we obtain

$$\sum_{w \in W'} d_C(w) \leq |C \cap (V_1 \cup S)| + (k-2)(\alpha(G)-1).$$

Therefore, since $\sum_{w\in W'} d_H(w) \le |H \cap (V_1 \cup S)| - 1$ by the inequality (7), it follows that

$$\sum_{w\in W'} d_G(w) \le |V_1 \cup S| + (k-2)(\alpha(G)-1) - 1.$$

Summing this inequality and the inequality (6) yields that $\sum_{w\in W'} d_G(w) + d_G(v_2) \le n + \kappa(G) + (k-2)(\alpha(G)-1) - 2$, a contradiction. □

By the assumption of Case 1, we can take a subset W^* of $W \cup \{x_0\}$ such that $|W^*| = k-2$. By Claim 5, $W^* \cup \{x_1, x_2, x_3\}$ is independent. Moreover, by Claim 4 and the assumption that $d_C(x_0) \le \alpha(G) - 2$, we have

$$\sum_{w\in W^*} d_C(w) \le (k-2)(\alpha(G)-2).$$

By Subclaim 7.1, summing this inequality and the inequality (9) yields that

$$\begin{aligned}
&\sum_{i=1}^{3} d_C(x_i) + \sum_{w\in W^*} d_C(w) \\
\le\ & |C| + |W| + 3 + (k-2)(\alpha(G)-2) \\
\le\ & |C| + (\kappa(G) + k - 5) + 3 - (k-2) + (k-2)(\alpha(G)-1) \\
=\ & |C| + \kappa(G) + (k-2)(\alpha(G)-1).
\end{aligned}$$

Therefore, since $\sum_{i=1}^{3} d_H(x_i) + \sum_{w\in W^*} d_H(w) \le |H| - 1$ by the inequality (7), we obtain $\sum_{i=1}^{3} d_G(x_i) + \sum_{w\in W^*} d_G(w) \le n + \kappa(G) + (k-2)(\alpha(G)-1) - 1$, a contradiction.

Case 2. $|W| \le k-4$.

By Claim 6, we can take a subset L^* of L such that $|L^*| = k - 3 - |W|$. Let $I = \{i : x_i \in L^*\}$. By Claim 5, $W \cup L^* \cup \{x_0, x_1, x_2, x_3\}$ is an independent set of order $k+1$. By the inequality (8), we have

$$\begin{aligned}
d_C(x_1) + d_C(x_2) + d_C(x_3) &\le |C| + |W| + |L^*| - \sum_{i\in I} |D_i| + 3 \\
&= |C| + k - 3 - \sum_{i\in I} |D_i| + 3 \\
&\le |C| + \kappa(G) - \sum_{i\in I} |D_i|.
\end{aligned}$$

On the other hand, it follows from Claim 4, the assumption $d_C(x_0) \le \alpha - 2$ and the inequality (1) that

$$\begin{aligned}
\sum_{w\in W\cup\{x_0\}} d_C(w) + \sum_{x\in L^*} d_C(x) &\le (|W|+1)(\alpha(G)-2) + \sum_{i\in I} |D_i| + |L^*|(\alpha(G)-1) \\
&= (k-2)(\alpha(G)-1) - |W| - 1 + \sum_{i\in I} |D_i| \\
&\le (k-2)(\alpha(G)-1) + \sum_{i\in I} |D_i| - 1.
\end{aligned}$$

Thus, we deduce

$$\sum_{i=1}^{3} d_C(x_i) + \sum_{w\in W\cup\{x_0\}} d_C(w) + \sum_{x\in L^*} d_C(x) \leq |C| + \kappa(G) + (k-2)(\alpha(G)-1) - 1.$$

By the inequality (7), we obtain

$$\sum_{i=1}^{3} d_H(x_i) + \sum_{w\in W\cup\{x_0\}} d_H(w) + \sum_{x\in L^*} d_H(x) \leq |H| - 1.$$

Summing the above two inequalities yields that $\sum_{i=1}^{3} d_G(x_i) + \sum_{w\in W\cup\{x_0\}} d_G(w) + \sum_{x\in L^*} d_G(x) \leq n + \kappa(G) + (k-2)(\alpha(G)-1) - 2$, a contradiction.

By Cases 1 and 2, we have $d_C(x_0) \geq \alpha(G) - 1$. Since $|U| = |X|$, it follows from Claim 4 that $d_C(x_0) = |U| = |X| = \alpha(G) - 1$. In particular, $N_C(x_0) = U$ because $N_C(x_0) \subseteq N_C(H_0) = U$. This completes the proof of Claim 7. □

Claim 8. *$W \subseteq X$.*

Proof. If $W \setminus X \neq \emptyset$, then by Claim 5, we have $d_C(x_0) \leq |X| \leq \alpha(G) - 2$, which contradicts Claim 7. □

Claim 9. *If there exist distinct two integers s and t in M_1 such that $u_s \in N_C(x_t)$, then $N_C(x_s) \cap C[u_t, u_s] \subseteq U$.*

Proof. Suppose that there exists a vertex $z \in N_C(x_s) \cap C[u_t, u_s]$ such that $z \notin U$. We show that $X \cup \{x_0, z^+\}$ is an independent set of order $|X| + 2$. By Claim 5, we only show that $z^+ \notin X$ and $z^+ \notin N_C(x_i)$ for each $x_i \in X \cup \{x_0\}$. Since $z \notin U$, it follows from Lemma 3 (i) that $z^+ \notin X$. Suppose that $z^+ \in N_C(x_h)$ for some $x_h \in X \cup \{x_0\}$. Since x_s is a non-insertible vertex, it follows that $x_h \neq x_s$. Let z_s be the vertex in $C(u_s, x_s]$ such that $z \in N_G(z_s)$ and $z \notin N_G(v)$ for all $v \in C(u_s, z_s)$. By Lemma 3 (ii), we obtain $x_h \notin C[u'_s, z]$. Therefore, $x_h \in C(z, u_s] \cup \{x_0\}$. If $x_h \in C(z, u_s]$, then we let z_h be the vertex in $C(u_h, x_h]$ such that $z^+ \in N_G(z_h)$ and $z^+ \notin N_G(v)$ for all $v \in C(u_h, z_h)$. We define the cycle C^* as follows (see Figure 4):

$$C^* = \begin{cases} z_s \overleftarrow{C}[z, x_t] \overleftarrow{C}[u_s, z_h] C[z^+, u_h] x_0 \overleftarrow{C}[u_t, z_s] & \text{if } x_h \in C(z, u_s], \\ z_s \overleftarrow{C}[z, x_t] \overleftarrow{C}[u_s, z^+] x_h \overleftarrow{C}[u_t, z_s] & \text{if } x_h = x_0. \end{cases}$$

Then, by similar argument in the proof of Lemma 3, we can obtain a longer cycle than C by inserting all vertices of $V(C \setminus C^*)$ into C^*. This contradicts that C is longest. Hence $z^+ \notin N_C(x_h)$ for each $x_h \in X \cup \{x_0\}$. Thus, by Claim 7, $X \cup \{x_0, z^+\}$ is an independent set of order $|X| + 2 = \alpha(G) + 1$, a contradiction. □

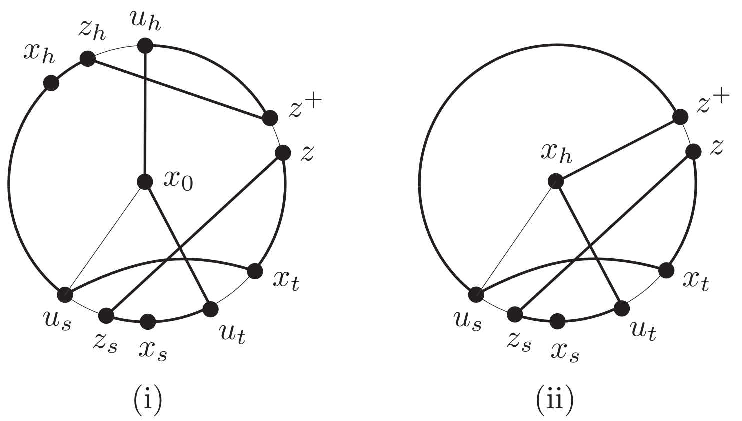

Figure 4: Claim 9

We divide the rest of the proof into two cases.

Case 1. $v_2 \notin U$**.**

Let $Y = N_G(v_2) \cap X$, and let $\gamma = |X| - \kappa(G) - 1$. Note that $|X| = \kappa(G) + \gamma + 1 \geq k + \gamma + 1$ and $x_l \notin Y$ since $x_l \in V_1$.

Claim 10. $|Y| \geq \gamma + 3$.

Proof. Suppose that $|Y| \leq \gamma + 2$. By the assumption of Case 1, we have $x_0 v_2 \notin E(G)$. Since $|M_0| = |X| + 1 \geq k + \gamma + 2$ and $|Y| \leq \gamma + 2$, there exists a subset I of $M_0 \setminus \{i : x_i \in Y\}$ such that $|I| = k$ and $\{0, l\} \subseteq I$. Then $\{x_i : i \in I\} \cup \{v_2\}$ is an independent set of order $k + 1$. By the inequality (5) and Claims 3 and 7, we obtain

$$\begin{aligned} d_C(x_l) &\leq |C \cap (V_1 \cup S)| - \sum_{i \in I \setminus \{0,l\}} |D_i| - |X| \\ &= |C \cap (V_1 \cup S)| - \sum_{i \in I \setminus \{0,l\}} |D_i| - d_C(x_0). \end{aligned}$$

Therefore it follows from the inequality (1) that

$$\sum_{i \in I} d_C(x_i) \leq |C \cap (V_1 \cup S)| + (k - 2)(\alpha(G) - 1).$$

By the inequality (7), $\sum_{i \in I} d_H(x_i) \leq |H \cap (V_1 \cup S)| - 1$. Summing these two inequalities and the inequality (6) yields that

$$\sum_{i \in I} d_G(x_i) + d_G(v_2) \leq n + \kappa(G) + (k - 2)(\alpha(G) - 1) - 2,$$

a contradiction. □

Recall that r is an integer such that $v_2 \in C(x_r, u'_r] \cap V_2$ (see the paragraph below the proof of Claim 3). In the rest of Case 1, we assume that $l = 1$. If $u'_r \neq u_1$, then let $r = 2$ and $u_3 = u'_2$; otherwise, let $r = 3$ and let u_2 be the vertex with $u'_2 = u_3$.

By Claim 8, we have $W \subseteq X$. Hence we obtain $Y \cup W \cup L \subseteq X \setminus \{x_1\}$. Recall that $W \cap L = \emptyset$. Therefore, by Claims 6 and 10, we obtain

$$\begin{aligned} |Y \cap (W \cup L)| &= |Y| + |W| + |L| - |Y \cup (W \cup L)| \\ &\geq \gamma + 3 + \kappa(G) - 2 - |X \setminus \{x_1\}| \\ &= \gamma + 3 + \kappa(G) - 2 - ((\kappa(G) + \gamma + 1) - 1) = 1. \end{aligned}$$

Hence there exists a vertex $x_h \in Y \cap (W \cup L)$, that is, $v_2 \in N_C(x_h) \setminus U$. Since $C(x_2, x_3) \cap X = \emptyset$ and $C(x_3, x_1) \cap X = \emptyset$ if $r = 3$, either $u_h \in N_C(x_1)$ and $u_h \in C(x_3, u_1)$ or $u_h \in N_C(x_2)$ and $u_h \in C(x_1, u_2)$ holds (especially, if $r = 3$ then $u_h \in N_C(x_2)$ and $u_h \in C(x_1, u_2)$ holds) (see Figure 5).

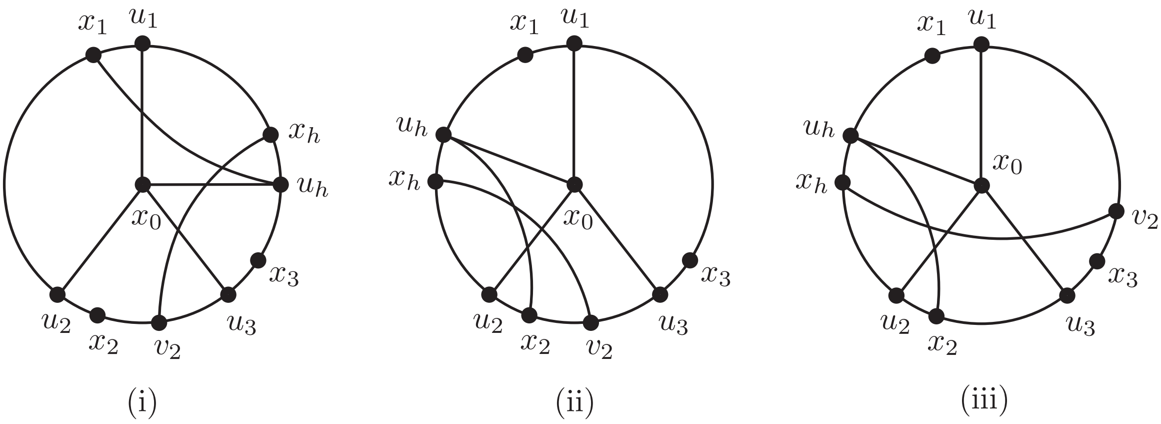

Figure 5: The case $r = 2$ and the case $r = 3$.

If $r = 2$ and $u_h \in N_C(x_1)$, then $v_2 \in C[u_1, u_h]$ (see Figure 5 (i)). If $r = 2$ and $u_h \in N_C(x_2)$, then $v_2 \in C[u_2, u_h]$ (see Figure 5 (ii)). If $r = 3$, then $u_h \in N_C(x_2)$ and $v_2 \in C[u_2, u_h]$ (see Figure 5 (iii)). In each case, we obtain a contradiction to Claim 9.

Case 2. $v_2 \in U$.

We rename $x_i \in X$ for $i \geq 1$ as follows (see Figure 6): Rename an arbitrary vertex of X as x_1. For $i \geq 1$, we rename $x_{i+1} \in X$ so that $u_{i+1} \in N_C(x_i) \cap (U \setminus \{u_i\})$ and $|C[u_{i+1}, x_i)|$ is as small as possible. (For $x_i \in X$, let x'_i and x''_i be the successors of x_i and x'_i in X along the orientation of C, respectively. Then by applying Claim 6 as $x_1 = x_i$, $x_2 = x'_i$ and $x_3 = x''_i$, it follows that $W \cup L \neq \emptyset$. By the definition of x'_i, x''_i and Claim 8, we have $W_1 = W_2 = \emptyset$ (note that $W \cap \{x_1, x_2, x_3\} = \emptyset$). By the definitions of x'_i, x''_i, L_1 and L_2, we also have $L_1 = L_2 = \emptyset$. Thus $W_3 \cup L_3 \neq \emptyset$. By Lemma 3 (i) and since $W \cup L \subseteq X$, this implies that $N_C(x_i) \cap (U \setminus \{u_i\}) \neq \emptyset$.) Let

h be the minimum integer such that $x_{h+1} \in C(x_h, x_1]$. Note that this choice implies $h \geq 2$. We rename h vertices in X as $\{x_1, x_2, \ldots, x_h\}$ as above, and $m-h$ vertices in $X \setminus \{x_1, x_2, \ldots, x_h\}$ as $\{x_{h+1}, x_{h+2}, \ldots, x_m\}$ arbitrarily. Let

$$A_1 = A_{h+1} = C[x_1, x_h) \text{ and } A_i = C[x_i, x_{i-1}) \text{ for } 2 \leq i \leq h.$$

Let

$$U_1 = \{u_i \in U : x_i \in X \cap V_1\}.$$

If possible, choose x_1 so that $A_2 \cap U_1 = \emptyset$.

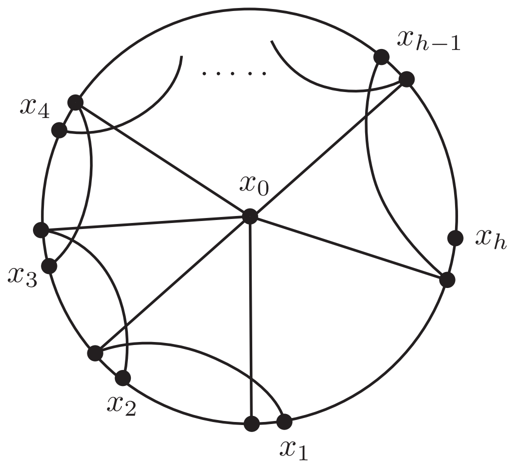

Figure 6: The choice of $\{x_1, \ldots, x_h\}$.

We divide the proof of Case 2 according to whether $h \leq k$ or $h \geq k+1$.

Case 2.1. $h \leq k$.

By the choice of $\{x_1, \ldots, x_h\}$, we have

$$N_{A_{i+1}}(x_i) \cap U \subseteq \{u_i\} \text{ for } 1 \leq i \leq h. \tag{10}$$

By Claim 9 and (10), we obtain

$$N_{C \setminus A_i}(x_i) \subseteq (U \setminus (A_i \cup A_{i+1})) \cup D_i \cup \{u_i\} \text{ for } 2 \leq i \leq h. \tag{11}$$

By Lemma 3 (i) and (ii), $N_{A_i}(x_i)^- \cap N_{A_i}(x_1) = \emptyset$ for $2 \leq i \leq h$. By Lemma 3 (i), we have $N_{A_i}(x_i)^- \cup N_{A_i}(x_1) \subseteq A_i \setminus D$ for $3 \leq i \leq h$. Thus, it follows from (11) that for $3 \leq i \leq h$

$$d_C(x_i) \leq (|U| - |(A_i \cup A_{i+1}) \cap U| + |D_i| + 1) + (|A_i| - |A_i \cap D| - d_{A_i}(x_1)).$$

By Lemma 3 (i) and (10), we have $N_{A_2}(x_2)^- \cup N_{A_2}(x_1) \subseteq (A_2 \setminus (U \cup D)) \cup D_1 \cup \{u_1\}$. Thus, by (11), we have

$$\begin{aligned} d_C(x_2) &\leq (|U| - |(A_2 \cup A_3) \cap U| + |D_2| + 1) \\ &\quad + (|A_2| - |A_2 \cap (U \cup D)| + |D_1| + 1 - d_{A_2}(x_1)). \end{aligned}$$

Since $|A_1 \cap X| = |A_1 \cap U|$, it follows from Lemma 3 (i) that

$$\begin{aligned} d_{A_1}(x_1) &\leq |A_1| - |A_1 \cap D| - |A_1 \cap X| \\ &= |A_1| - |A_1 \cap D| - |A_1 \cap U|. \end{aligned}$$

By Claim 7, $d_C(x_0) = |U| = \alpha(G) - 1$. Thus, since $h \leq k$, we obtain

$$\begin{aligned} \sum_{0 \leq i \leq h} d_C(x_i) &\leq \sum_{1 \leq i \leq h} |A_i| + h|U| - 2\sum_{1 \leq i \leq h} |A_i \cap U| + h + \sum_{1 \leq i \leq h} |D_i| - \sum_{1 \leq i \leq h} |A_i \cap D| \\ &= |C| + (h-2)|U| + h + \sum_{1 \leq i \leq h} |D_i| - |D| \\ &\leq |C| + k + (h-2)(\alpha(G) - 1) + \sum_{1 \leq i \leq h} |D_i| - |D|. \end{aligned}$$

Let I be a subset of M_0 such that $|I| = k+1$ and $\{0, 1, \ldots, h\} \subseteq I$. By Claim 5, $\{x_i : i \in I\}$ is an independent set of order $k+1$. By the above inequality and the inequality (1), we have

$$\sum_{i \in I} d_C(x_i) \leq |C| + k + (k-2)(\alpha(G) - 1)$$

By the inequality (2), $\sum_{i \in I} d_H(x_i) \leq |H| - 1$. Hence $\sum_{i \in I} d_G(x_i) \leq |G| + \kappa(G) + (k-2)(\alpha(G) - 1) - 1$, a contradiction.

Case 2.2. $h \geq k+1$.

By Claims 3 and 7, the assumption of Case 2 and the choice of r and v_2, we have $\bigcup_{i=2}^{p} V_i \subseteq U = N_C(x_0)$. Since $x_0 \in V_1 \cup S$ by Claim 3, this implies that $x_0 \in S$.

Claim 11. $|X \cap V_1| \leq k-1$.

Proof. Suppose that $|X \cap V_1| \geq k$. Let I be a subset of M_1 such that $|I| = k$ and $I \subseteq \{i : x_i \in X \cap V_1\}$. Then $\{x_i : i \in I\} \cup \{v_2\}$ is an independent set of order $k+1$. Let s and t be integers in I. Since $x_s, x_t \in V_1$, $D \subseteq V_1 \cup S$ and $\bigcup_{i=2}^{p} V_i \subseteq U$, the similar argument as that of the inequality (4) implies that

$$d_C(x_s) + d_C(x_t) \leq |C \cap (V_1 \cup S)| - \sum_{i \in I \setminus \{s,t\}} |D_i|.$$

By the inequalities (1) and (7), we have $\sum_{i \in I \setminus \{s,t\}} d_C(x_i) \leq \sum_{i \in I \setminus \{s,t\}} |D_i| + (k-2)(\alpha(G) - 1)$ and $\sum_{i \in I} d_H(x_i) \leq |H \cap (V_1 \cup S)| - 1$, respectively. On the other hand, we obtain $d_G(v_2) \leq |V_2 \cup S| - 1$. By these four inequalities, $\sum_{i \in I} d_G(x_i) + d_G(v_2) \leq n + \kappa(G) + (k-2)(\alpha(G) - 1) - 2$, a contradiction. Therefore $|X \cap V_1| \leq k-1$. $\square$

Recall $U_1 = \{u_i \in U : x_i \in X \cap V_1\}$. By Claim 11, we have $|U_1| \le k-1$. By the assumption of Case 2.2 and the choice of x_1, we obtain $A_2 \cap U_1 = \emptyset$, and hence we can take a subset I of $\{2, 3, \ldots, h\}$ such that $|I| = k$ and $\{i : A_{i+1} \cap U_1 \neq \emptyset\} \subseteq I$. Let

$$X_I = \{x_i : i \in I\}.$$

By Claim 5, $X_I \cup \{x_0\}$ is an independent set of order $k+1$. Let

$$B_1 = B_{h+1} = C(u_1, u_h) \ \text{ and } \ B_i = C(u_i, u_{i-1}) \ \text{ for } 2 \le i \le h.$$

Then, since $|C[u_i, u_i')| \ge 2$ for $i \in M_1 \setminus I$, the following inequality holds:

$$\begin{aligned} |C| &\ge \sum_{i \in I} |B_i \cup \{u_i\}| + 2\Big(|U| - \sum_{i \in I} |(B_i \cup \{u_i\}) \cap U|\Big) \\ &= \sum_{i \in I} |B_i| + 2\Big(|U| - \sum_{i \in I} |B_i \cap U|\Big) - k. \end{aligned}$$

If $x_i \in X_I \cap S$, then it follows from Lemma 3 (i) and Claim 9 that

$$\begin{aligned} d_C(x_i) &\le \Big(|U| - |B_i \cap U| - |B_{i+1} \cap U_1|\Big) + \Big(|B_i| - |\{x_i\}| - |(B_i \cap U)^+|\Big) \\ &= |U| + |B_i| - 2|B_i \cap U| - |B_{i+1} \cap U_1| - 1. \end{aligned}$$

If $x_i \in X_I \cap V_1$, then, by Lemma 3 (i) and Claim 9,

$$\begin{aligned} d_C(x_i) &\le \Big(|U| - |B_i \cap U| - |B_{i+1} \cap U_1| - |(U \cap V_2) \setminus B_i| + |B_{i+1} \cap U_1 \cap V_2|\Big) \\ &\quad + \Big(|B_i| - |\{x_i\}| - |(B_i \cap U)^+| - |U \cap V_2 \cap B_i|\Big) \\ &= |U| + |B_i| - 2|B_i \cap U| - |B_{i+1} \cap U_1| - 1 - \Big(|U \cap V_2| - |B_{i+1} \cap U_1 \cap V_2|\Big). \end{aligned}$$

Since $U \cap V_2 \neq \emptyset$, we obtain $|U \cap V_2| - |B_{i+1} \cap U_1 \cap V_2| \ge 1$ for all $i \in I$ except for at most one, and hence

$$\sum_{i \in I \,:\, x_i \in X_I \cap V_1} \Big(|U \cap V_2| - |B_{i+1} \cap U_1 \cap V_2|\Big) \ge |X_I \cap V_1| - 1.$$

By the choice of I, we have

$$|U_1| = \sum_{i \in I} |A_{i+1} \cap U_1| = \sum_{i \in I} |B_{i+1} \cap U_1| + \Big|\{u_i : x_i \in X_I \cap V_1\}\Big|.$$

On the other hand, since $x_0 \in S$, it follows from Claim 3 that

$$|U_1| = |X \cap V_1| = |X \setminus S| \ge |X| - (\kappa(G) - 1).$$

Moreover, by Claim 7,

$$d_C(x_0) = |U| = |X| = \alpha(G) - 1.$$

Thus, we deduce

$$\begin{aligned}\sum_{i\in I\cup\{0\}} d_C(x_i) &\le (k+1)|U| + \sum_{i\in I}|B_i| - 2\sum_{i\in I}|B_i\cap U| \\ &\quad - \sum_{i\in I}|B_{i+1}\cap U_1| - k - (|X_I\cap V_1| - 1) \\ &= \Big(\sum_{i\in I}|B_i| + 2\big(|U| - \sum_{i\in I}|B_i\cap U|\big) - k\Big) + (k-1)|U| \\ &\quad - \Big(\sum_{i\in I}|B_{i+1}\cap U_1| + \big|\{u_i : x_i\in X_I\cap V_1\}\big|\Big) + 1 \\ &\le |C| + (k-1)|U| + \kappa(G) - |X| \\ &= |C| + \kappa(G) + (k-2)(\alpha(G)-1).\end{aligned}$$

By the inequality (2), $\sum_{i\in I\cup\{0\}} d_H(x_i) \le |H| - 1$. Hence $\sum_{i\in I\cup\{0\}} d_G(x_i) \le |G| + \kappa(G) + (k-2)(\alpha(G)-1) - 1$, a contradiction. □

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