Extension to even triangulations

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Abstract

Extension of a graph G is to construct a new graph with certain properties by adding edges to some pairs of vertices in G. In this paper, we focus on extension of a quadrangulation of a surface to even triangulations, where a *quadrangulation* is a map on a surface with every face quadrangular and a triangulation is *even* if all the vertices have even degree.

Zhang and He [SIAM J. Comput. 34 (2005) 683–696] gave a formula for the exact number of distinct even triangulations extended from a given plane quadrangulation, and a lower bound of the number for the case of orientable non-spherical surfaces. They also posed the problem of finding the exact number for the latter case.

In this paper, using topological methods, we improve the results by Zhang and He in the following directions: (I) Extension of quadrangulations of a non-orientable surface and (II) Complete enumeration of even triangulations extended from a given quadrangulation of a non-spherical surface. Indeed, we completely solve the problem by Zhang and He.

Keywords. extension, even triangulation, quadrangulation

1 Introduction

Extension of a given graph G is to construct a new graph from G by adding edges between some pairs of vertices in G. One of popular topics in Algorithmic Graph Theory is to decide whether a given graph can be extended to graphs with certain properties. A typical example is, for a given graph G, the problem of finding the minimum number of edges whose addition makes G k-connected (or k-edge-connected), for example see [3, 4, 7]. In this paper, we consider extension of graphs on surfaces.

Let G be a *map* on a surface F^2, that is, a 2-cell embedding of a graph on F^2. It is easy to see that if a map G has no monogonal and no digonal faces, then G can be extended to a triangulation of F^2; We can add diagonals to a non-triangular face as long as such a face exists. (Note that this operation may introduce multiple edges or loops. Throughout the present paper, we allow them, but do not allow monogonal and digonal faces.) Hence any map on a surface can be extended to a triangulation, but if we require the resulting triangulation to have more

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properties, the problem might be difficult and interesting. In this paper, we consider extension to *even* triangulations (i.e., the one with each vertex of even degree).

As a classical result on even triangulations, it is well-known that a plane triangulation is even if and only if it is 3-colorable. (Maybe Kempe is the first to present this fact, see [8, page 200].) Hence when G is a plane map, the extension of G to even triangulations is equivalent to that to 3-colorable triangulations. Indeed, Hoffmann and Kriegel [5] first considered the extension to even triangulations, in order to prove some partial results for several *art gallery* and *prison guard* problems, which have been widely studied in Computational Geometry, see [12]. Although they [5] gave an algorithm to find extension of bipartite plane maps to even triangulations, their algorithm is complicated and takes $O(n^2)$ time, where n is the number of vertices of a given map. Zhang and He [14] gave an algorithm to do the same in $O(n)$ time, and moreover, their method can construct all distinct even triangulations extended from a given plane quadrangulation, where a *quadrangulation* is a map with every face quadrangular.

In this paper, we improve the results above in the following two directions (I) and (II).

1.1 (I) Extension of quadrangulations of a non-orientable surface

The algorithm in [14] can find extension of a given quadrangulation to some even triangulations. However, since the present method depends on a consistent rotation (clockwise or anticlockwise order) on a surface, it can work only for quadrangulations of an orientable surface. Indeed, no quadrangulations of a non-orientable surface have been considered, and it is even unknown whether every quadrangulation of a non-orientable surface can be extended to even triangulations. We will give a simpler description which can work also in non-orientable case, and give a positive answer to the problem. In fact, Theorem 1 is a direct corollary of Theorem 12, which is the main theorem of this paper.

Theorem 1 *Let F^2 be a surface and let G be a quadrangulation of F^2. Then G can be extended to an even triangulation.*

1.2 (II) Complete enumeration for a non-spherical case

The algorithm in [14] enumerates all distinct even triangulations extended from a given plane quadrangulation, however, it heavily depends on the planarity of an input map. Hence, using the idea, the authors only gave a lower bound on the number of even triangulations extended from a given quadrangulation of an orientable surface. Indeed, they left the following problem open:

Problem 2 (Zhang and He [14]) *For a given quadrangulation G of a non-spherical surface, determine the exact number of even triangulations extended from G.*

In Section 3, we give a complete answer to Problem 2. To do that, we need several definitions, which appear in Section 2.

2 Basic definitions

In this section, we give several definitions that play crucial roles in our proofs.

2.1 The dual of a map

Throughout the paper, we assume that a surface is always closed. Let $\mathbb{S}_k$ and $\mathbb{N}_k$ denote the orientable surface of genus k and the non-orientable surface of crosscap number k, respectively. For a surface F^2, let $g(F^2)$ denote the *Euler genus* of F^2, where $g(\mathbb{S}_k) = 2k$ and $g(\mathbb{N}_k) = k$. We assume that a map G is a *2-cell embedding* on a surface, that is, every face is homeomorphic to an open disk.

For a map G on a surface, the *dual* of G is denoted by G^*. Let G be a quadrangulation of a surface. Then every vertex of G^* has degree 4. For a vertex v of G^* with four edges e_1, e_2, e_3, e_4 incident to v in this cyclic order, we say that e_i is *opposite to* e_{i+2} *at* v. A walk W of G^* is a *straight walk*, or shortly an *S-walk*, of G^* if at each vertex v, the walk W passes through v from one edge to the opposite edge, and W uses any edge at most once. Note that it is possible that an S-walk may cross with itself. The edge set of G^* is uniquely partitioned into S-walks. The concept of S-walks can be found in [1, 5, 11, 14].

Subdividing each S-walk whenever it meets an edge in G, we obtain the *subdivided S-walk graph*, denoted by $\widehat{S}(G)$, of a quadrangulation G. To be exact, $\widehat{S}(G)$ is the graph such that

$$\begin{aligned} V\big(\widehat{S}(G)\big) &= E(G), \\ \text{and} \quad E\big(\widehat{S}(G)\big) &= \{e_1e_3 : e_1 \text{ and } e_3 \text{ are opposite at some vertex } v \text{ in } G^*\}. \end{aligned}$$

See Figure 1, for an example of the S-walks of G^* and the subdivided S-walk graph $\widehat{S}(G)$ of a plane quadrangulation G. As in Figure 1, we assume that $\widehat{S}(G)$ is drawn in the surface on which G is embedded. Hence some edges in $\widehat{S}(G)$ cross with each other at a vertex in G^*. Note that each vertex has degree 2 in $\widehat{S}(G)$, and hence each component of $\widehat{S}(G)$ is a cycle. Moreover, the number of components of $\widehat{S}(G)$ is exactly the same as the number of the S-walks of G^*.

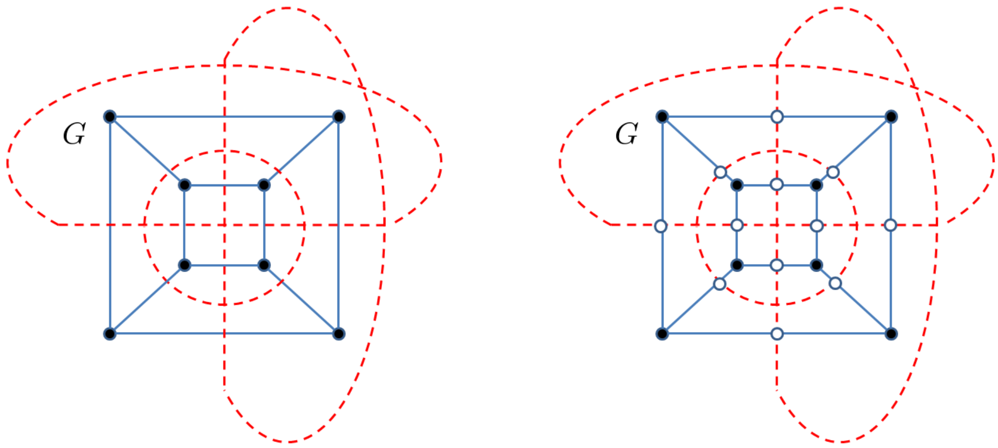

Figure 1: The dotted lines represent the S-walks of G^* (on the left side) and edges of the subdivided S-walk graph $\widehat{S}(G)$ (on the right side) of a plane quadrangulation G.

2.2 H-subdivided orientations and the parity condition

An *orientation* of a graph G is an assignment of directions to all edges of G. In this subsection, we introduce a special orientation of the subdivided S-walk graph $\widehat{S}(G)$ of a quadrangulation G, which is called an "H-subdivided orientation."

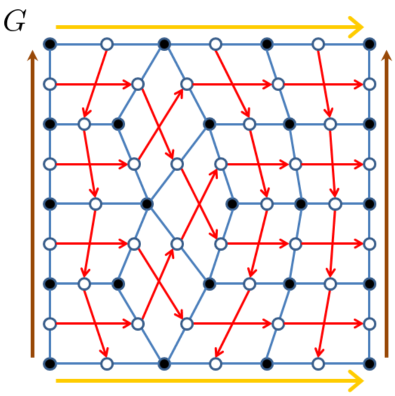

Figure 2: An ($\emptyset$-subdivided) orientation.

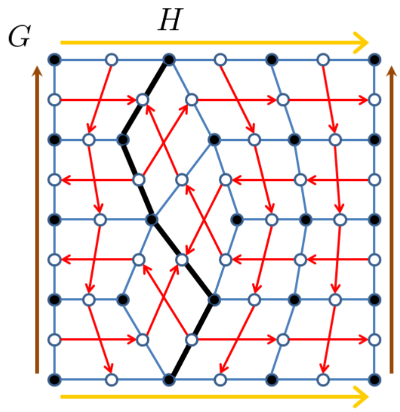

Figure 3: An H-subdivided orientation.

Definition 3 *Let H be a subgraph of a quadrangulation G of a surface F^2. An H-subdivided orientation of $\widehat{S}(G)$ is an orientation of $\widehat{S}(G)$ satisfying the following two conditions:*

(H1) *For every $e \in E(G) - E(H)$, the out-degree of e in $\widehat{S}(G)$ is exactly 1. (Hence the in-degree of e is also exactly 1.)*

(H2) *For every $e \in E(H)$, the out-degree of e is 0 or 2. (Hence the in-degree of e is also 0 or 2.)*

Figures 2 and 3 show quadrangulations on the torus and H-subdivided orientations of $\widehat{S}(G)$, where H is the empty graph, denoted by $\emptyset$, and the subgraph H of G with the thick edges indicated, respectively.

In Section 4, we construct even triangulations from a given quadrangulation by using H-subdivided orientations of $\widehat{S}(G)$ with certain properties of H. (The properties required for H are the parity condition, see below, and the evenness condition, see Lemma 18.) Note that Zhang and He [14] considered only $\emptyset$-subdivided orientations, which can induce even triangulations. However, not only $\emptyset$-subdivided orientations of $\widehat{S}(G)$ but also H-subdivided orientations can induce even triangulations. Especially if H satisfies some specific topological conditions, see Subsection 2.3 and Lemma 19 (ii), then there are even triangulations induced by H-subdivided orientations that are not induced by $\emptyset$-subdivided orientations. Indeed, our method can induce all the even triangulations, while the method in [14] could induce only some of them, see Section 3.

Next, we define an important notion, called the parity condition.

Definition 4 *Let H be a subgraph of a quadrangulation G of a surface F^2. We say that H satisfies the parity condition if every S-walk of G^* crosses with H an even number of times.*

By Definitions 3 and 4, one can easily prove the following lemmas:

Lemma 5 *Let H be a subgraph of a quadrangulation G of a surface F^2. Then $\widehat{S}(G)$ has an H-subdivided orientation if and only if H satisfies the parity condition.*

Lemma 6 *Let H be a subgraph of a quadrangulation G of a surface F^2, and let $\mathcal{O}$ and $\mathcal{O}'$ be two H-subdivided orientations. Then there exists a set of components $C_1, \ldots, C_q$ of $\widehat{S}(G)$ for some q such that $\mathcal{O}'$ is obtained from $\mathcal{O}$ by reversing the direction of all the edges in $C_1, \ldots, C_q$.*

Below, we introduce one more lemma. Let G be a quadrangulation of a surface F^2, and let $\mathcal{O}$ be an orientation of $\widehat{S}(G)$. Note that if we reverse the directions of all edges of $\widehat{S}(G)$, then we get another orientation, denoted by $\overline{\mathcal{O}}$, of $\widehat{S}(G)$. Two orientations $\mathcal{O}$ and $\overline{\mathcal{O}}$ are called an *orientation pair* of $\widehat{S}(G)$, defined in [14]. Since Lemma 7 directly follows from Definition 3, we omit its proof.

Lemma 7 *Let H be a subgraph of a quadrangulation G of a surface F^2, and let $\mathcal{O}$ be an orientation of $\widehat{S}(G)$. Then $\mathcal{O}$ is an H-subdivided orientation of $\widehat{S}(G)$ if and only if so is $\overline{\mathcal{O}}$.*

2.3 The cycle space

In this subsection, we give the definition of the cycle space of a graph G on a surface F^2. See [2, page 24] for more details about the cycle space.

For a closed walk D of a graph G, let $\widetilde{D}$ denote the subgraph of G consisting of the edges that appear in D an odd number of times. Recall that a graph H is *even* if all the vertices of H have even degree in H. In particular, the empty graph $\emptyset$ is regarded as an even subgraph. For two subgraphs H and H' of a graph G, we denote by $H \triangle H'$ the *symmetric difference* of H and H', which is the graph induced by the edges that are contained in exactly one of H and H'. We can easily see that if both H and H' are even graphs, then so is $H \triangle H'$.

For a graph G, let $\mathscr{C}(G)$ be the set of all even subgraphs of G. It is known that $\mathscr{C}(G)$ becomes a $\mathbb{Z}_2$-space if we take the symmetric difference as addition, the identity mapping as negation, and the empty graph as zero. A *base* $\mathcal{D}$ of $\mathscr{C}(G)$ is a set of even subgraphs of G with the following two conditions.

(i) **(Spanning condition)** For every even subgraph H of G, there exists a set of even subgraphs $D_1, \ldots, D_r$ in $\mathcal{D}$ such that H is obtained by the symmetric difference among D_i for $1 \leq i \leq r$.

(ii) **(Linearly independent condition)** No element D of $\mathcal{D}$ can be created by the symmetric difference among even subgraphs in a subset of $\mathcal{D} - \{D\}$.

In particular, a set $\mathcal{D}$ is *spanning* in $\mathscr{C}(G)$ if $\mathcal{D}$ satisfies the condition (i), and *linearly independent* if $\mathcal{D}$ satisfies the condition (ii), respectively. It is known that any base of $\mathscr{C}(G)$ has the same cardinality, called the *dimension* of $\mathscr{C}(G)$, and the dimension of $\mathscr{C}(G)$ is

$$\left|E(G)\right| - \left|V(G)\right| + 1.$$

If G is a plane graph, then for all but one faces $f_1, \ldots, f_t$ of G, the set $\left\{\widetilde{D_i} : 1 \leq i \leq t\right\}$ becomes a base of the cycle space $\mathscr{C}(G)$, where D_i is the facial walk of f_i. See [10, Theorem 2.4.5], which was shown by MacLane [9]. If a map G is embedded on a non-spherical surface, then the set $\left\{\widetilde{D_i} : 1 \leq i \leq t\right\}$ is not a base of $\mathscr{C}(G)$, but it has the following property.

Lemma 8 *Let G be a map on a surface F^2. Let $f_1, \ldots, f_t$ be all but one faces of G, and D_i be the facial walk of f_i for $1 \leq i \leq t$. Then the set $\left\{\widetilde{D_i} : 1 \leq i \leq t\right\}$ is linearly independent.*

By Euler's formula, for a map G on F^2, we have

$$t = |F(G)| - 1 = |E(G)| - |V(G)| - g + 1,$$

where g is the Euler genus of F^2. Let $\mathcal{D} = \left\{\widetilde{D_i} : 1 \leq i \leq t\right\}$. By Lemma 8, the set $\mathcal{D}$ is linearly independent. So $\mathcal{D}$ becomes a base of the subspace of $\mathscr{C}(G)$, which we call the *facial cycle space* of G. Note that for the exceptional face f of G, the facial walk D of f satisfies $\widetilde{D} = \widetilde{D_1} \triangle \cdots \triangle \widetilde{D_t}$. This implies that the facial cycle space of G does not depend on the choice of faces $f_1, \ldots, f_t$ of G.

Since $\mathcal{D}$ is linearly independent, $\mathcal{D}$ can be extended to a base of the cycle space $\mathscr{C}(G)$ of G, that is, there exists a set $\mathcal{W}$ of even subgraphs of G such that $\mathcal{D} \cup \mathcal{W}$ is a base of $\mathscr{C}(G)$. Since the dimension of $\mathscr{C}(G)$ is $|E(G)| - |V(G)| + 1$ and $\mathcal{D}$ has t elements, where $t = |E(G)| - |V(G)| - g + 1$, we have

$$|\mathcal{W}| = \big(|E(G)| - |V(G)| + 1\big) - t = g.$$

Note that $\mathcal{W}$ becomes a base of the orthogonal complement of the facial cycle space of G (on $\mathscr{C}(G)$), which is the $\mathbb{Z}_2$-*homology space* of G. Its dimension is exactly g.

Let $\mathcal{W} = \{W_1, \ldots, W_g\}$, and let H be any even subgraph of G. Since H is a member of the cycle space $\mathscr{C}(G)$, there exists a unique set of indices $i_1, \ldots, i_p$ and $j_1, \ldots, j_q$ with $1 \leq i_1 < \cdots < i_p \leq g$ and $1 \leq j_1 < \cdots < j_q \leq t$ such that

$$H = W_{i_1} \triangle \cdots \triangle W_{i_p} \triangle \widetilde{D_{j_1}} \triangle \cdots \triangle \widetilde{D_{j_q}}.$$

In particular, the indices $i_1, \ldots, i_p$ are uniquely determined by H. Note that the element $W_{i_1} \triangle \cdots \triangle W_{i_p}$ of the $\mathbb{Z}_2$-homology space of G is the *projection of* H to the $\mathbb{Z}_2$-homology space. Two even subgraphs H and H' of G are $\mathbb{Z}_2$-*homologous (on G)* if H and H' have the same projection to the $\mathbb{Z}_2$-homology space. By definition, we have the following:

Fact 9 *Let G be a map on a surface F^2, and let H and H' be two even subgraphs of G. Then H and H' are $\mathbb{Z}_2$-homologous on G if and only if*

($*$) *there exists a set $\{f_1, \ldots, f_r\}$ of faces of G such that $H' = H \triangle \widetilde{D_1} \triangle \cdots \triangle \widetilde{D_r}$,*

where D_i is the facial walk of f_i for $1 \leq i \leq r$.

Extending the definition of $\mathbb{Z}_2$-homologous, we say that two (not necessarily even) subgraphs are $\mathbb{Z}_2$-*homologous* if they satisfy the property ($*$) in Fact 9.

Remark: Note that $\{W_1, \ldots, W_g\}$ is a generator set of the $\mathbb{Z}_2$-*homology group* $H_1(F^2; \mathbb{Z}_2)$ of F^2. Indeed, the projection of an even subgraph represents its $\mathbb{Z}_2$-homology class. See [13] for the $\mathbb{Z}_2$-homology group.

2.4 The intersection system

Let G be a quadrangulation of a surface F^2. Let $\mathcal{W} = \{W_1, \ldots, W_g\}$ be a base of the $\mathbb{Z}_2$-homology space of G, let $S_1, \ldots, S_l$ be the set of all S-walks in G^*, and let w'_{ij} denote the number of crossings between S_i and W_j. The *intersection matrix of G* (*with respect to the base $\mathcal{W}$ of the $\mathbb{Z}_2$-homology space*), denoted by $\mathrm{IM}(G)$, is an $(l \times g)$-matrix on $\mathbb{Z}_2$ with entry $w_{ij} \in \{0, 1\}$ such that $w_{ij} \equiv w'_{ij}$ (mod 2), see the following:

$$\mathrm{IM}(G) \;=\; \begin{array}{c} \\ S_1 \\ S_2 \\ \vdots \\ S_l \end{array} \!\!\begin{array}{c} \begin{array}{cccc} W_1 & W_2 & \cdots & W_g \end{array} \\ \begin{pmatrix} w_{11} & w_{12} & \ldots & w_{1g} \\ w_{21} & w_{22} & \ldots & w_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ w_{l1} & w_{l2} & \ldots & w_{lg} \end{pmatrix} \end{array}.$$

When F^2 is the sphere, the intersection matrix $\mathrm{IM}(G)$ of G is an $(l \times 0)$-matrix, that is, degenerated, and regard the rank of $\mathrm{IM}(G)$ as 0.

For a column vector $\vec{x} = (x_1, \ldots, x_g)^T$, we set the following system of linear equations, and call it the *intersection system* of G:

$$\mathrm{IM}(G)\ \vec{x} = \vec{0}. \qquad (1)$$

Note that the intersection system of G has the trivial solution $\vec{x} = \vec{0}$. It is known that all the solutions of a system $A\vec{x} = \vec{0}$ of linear equations spans a subspace (which is called the *solution space*) of dimension $m - \mathrm{rank}(A)$, where m is the number of columns of A. Hence for the intersection system (1) of a quadrangulation G, we see the following fact.

Fact 10 *Let G be a quadrangulation of a surface F^2. Then the intersection system (1) of G has exactly 2^k solutions, where $k = g - \mathrm{rank}\big(IM(G)\big)$ and g is the Euler genus of F^2.*

3 Solution of Problem 2

In this section, we give the complete answer to Problem 2. As mentioned in Section 1, Zhang and He [14] gave a lower bound on the number of even triangulations extended from a given quadrangulation as follows:

Theorem 11 ([14]) *Let G be a quadrangulation of an orientable surface F^2 and let l be the number of S-walks of G^*. Then G can be extended to at least $\max\{2^{l-1}, 2^{g-2}\}$ distinct even triangulations, where g is the Euler genus of F^2.*

We state that the exact number can be expressed by the intersection system, which was defined in Subsection 2.4, as follows:

Theorem 12 *Let G be a quadrangulation of a surface F^2 and let l be the number of S-walks of G^*. Then G can be extended to exactly 2^{l-1+k} distinct even triangulations, where $k = g - \mathrm{rank}\big(IM(G)\big)$ and g is the Euler genus of F^2.*

Since $\mathrm{IM}(G)$ is an $(l \times g)$-matrix, we have $\mathrm{rank}\big(\mathrm{IM}(G)\big) \leq \min\{l, g\}$. This implies that $\max\{l-1, g-2\} \leq l - 1 + g - \mathrm{rank}(\mathrm{IM}(G))$, and Theorem 12 is indeed an improvement of Theorem 11.

Remark: In this paper, we only consider the extension of quadrangulations. However, our method is strong enough to deal with extension of maps possibly having triangular faces to even triangulations. We call a map in which each face is triangular or quadrangular a *mosaic*. Here we briefly explain the modification of Theorem 12 for mosaics.

First, in the definition of the subdivided S-walk graph $\widehat{S}(G)$ of a mosaic G, we regard all the triangular faces of G as vertices of $\widehat{S}(G)$, and we connect a triangular face of G and an edge surrounding it by an edge of $\widehat{S}(G)$. Let $\{C_1, \ldots, C_{l'}\}$ be a base of the cycle space of $\widehat{S}(G)$. Then we define the *extended intersection matrix $IM'(G)$* of G in the same way as in Subsection 2.4, but with $C_1, \ldots, C_{l'}$ instead of $S_1, \ldots, S_l$. (That means that the (i,j)-entry of $\mathrm{IM}'(G)$ is the parity of the number of crossings between C_i and W_j.) Let $\vec{c} = (c_1, \ldots, c_{l'})^T$ be the column vector such that $c_i \in \{0,1\}$ is the parity of the number of triangular faces contained in C_i. Then define the *extended intersection system* as follows:

$$\mathrm{IM}'(G)\ \vec{x} = \vec{c}. \qquad (2)$$

Therefore, we can obtain the following theorem:

Theorem 13 *Let G be a mosaic of a surface F^2. Then G can be extended to an even triangulation if and only if the extended intersection system (2) has a solution. Moreover, if the extended intersection system (2) has a solution, then G can be extended to exactly 2^{m-1+k} distinct even triangulations, where m is the number of components of $\widehat{S}(G)$, $k = g - \text{rank}\left(IM'(G)\right)$, and g is the Euler genus of F^2.*

The proof of Theorem 13 is based on the same idea of that of Theorem 12, and we leave it to the reader.

In particular, here we consider the case where F^2 is the sphere. In this case, the extended intersection matrix $\text{IM}'(G)$ is degenerated, and hence the extended intersection system (2) has a solution if and only if $\vec{c} = \vec{0}$. Then consider the *S-walk graph* of G, which is obtained from the subdivided S-walk graph $\widehat{S}(G)$ by suppressing all the vertices corresponding to an edge in G. (Thus, the vertex set of the S-walk graph is the set of triangular faces of G.) Note that $\vec{c} = \vec{0}$ if and only if the S-walk graph of G is bipartite. Therefore, we obtain the following corollary, which is a characterization obtained by Hoffmann-Ostenhof [6] before:

Corollary 14 *Let G be a mosaic of the sphere. Then G can be extended to an even triangulation if and only if the S-walk graph of G is bipartite.*

4 How to extend to even triangulations

In this section, using methods defined in Section 2, we discuss how to construct all distinct even triangulations extended from a given quadrangulation. The lemmas in Subsection 4.2 are obtained similarly to the ones in [14], but the lemmas in Subsections 4.1 and 4.3 are new and play important roles in the non-spherical cases.

4.1 The intersection system and even subgraphs satisfying the parity condition

In this subsection, we give a method to find even subgraphs satisfying the parity condition in a given quadrangulation of a surface. Moreover, we count the number of even subgraphs satisfying the parity condition such that they are pairwise not $\mathbb{Z}_2$-homologous. (When a surface F^2 is the sphere, the intersection matrix degenerates and the intersection system (1) has no solution, and hence skip this subsection.)

Let G be a quadrangulation of a surface F^2, and let $\vec{x} = (x_1, \ldots, x_g)^T$ be any solution of the intersection system (1) of G. Then $\vec{x}$ indicates the subgraph $H_{\vec{x}}$ of G as follows. Let $i_1, \ldots, i_p$ be the indices such that $x_r = 1$ if and only if $r \in \{i_1, \ldots, i_p\}$. Then define $H_{\vec{x}}$ as the empty graph (for simplicity we denote $H_{\vec{x}} = \emptyset$) if $\vec{x} = \vec{0}$; Otherwise let

$$H_{\vec{x}} = W_{i_1} \triangle \cdots \triangle W_{i_p}.$$

We say that the subgraph $H_{\vec{x}}$ is *produced by $\vec{x}$*. Note that $H_{\vec{x}}$ is an element of the $\mathbb{Z}_2$-homology space of G.

Lemma 15 *Let G be a quadrangulation of a surface F^2. Then all of the following hold:*

(i) *For any solution $\vec{x}$ of the intersection system (1) of G, the subgraph $H_{\vec{x}}$ produced by $\vec{x}$ is an even subgraph of G satisfying the parity condition.*

(ii) *For any two distinct solutions $\vec{x}$ and $\vec{x}'$ of the intersection system (1) of G, the subgraphs $H_{\vec{x}}$ and $H_{\vec{x}'}$ are not $\mathbb{Z}_2$-homologous.*

(iii) *For any even subgraph H of G satisfying the parity condition, there exists a unique solution $\vec{x}$ of the intersection system (1) of G such that H is $\mathbb{Z}_2$-homologous to $H_{\vec{x}}$.*

By Lemma 15, we can see the 1-to-1 correspondence between the solutions of the intersection system (1) of G and the elements of the $\mathbb{Z}_2$-homology space of G satisfying the parity condition.

Proof of Lemma 15. (i) Let $\vec{x} = (x_1, \ldots, x_g)^T$ be any solution of the intersection system (1) of G. Let $i_1, \ldots, i_p$ be the indices with $1 \leq i_1 < \cdots < i_p \leq g$ such that $x_r = 1$ if and only if $r \in \{i_1, \ldots, i_p\}$. Note that W_{i_j} is an even subgraph of G for $1 \leq j \leq p$, and hence $H_{\vec{x}}$ is an even subgraph of G.

Now we show that $H_{\vec{x}}$ satisfies the parity condition. It suffices to show that for any q with $1 \leq q \leq l$, if the S-walk S_q crosses with $H_{\vec{x}}$ exactly $s(q)$ times, then $s(q)$ is even. Indeed, since $\vec{x}$ is a solution of (1), we have

$$s(q) \equiv \sum_{1 \leq j \leq p} w_{q i_j} \equiv 0 \pmod{2}.$$

(ii) Since $\vec{x}$ and $\vec{x}'$ are two distinct solutions of (1) and $\mathcal{W}$ is a base of the $\mathbb{Z}_2$-homology space of G, the subgraphs $H_{\vec{x}}$ and $H_{\vec{x}'}$ are not $\mathbb{Z}_2$-homologous.

(iii) Let H be any even subgraph of G satisfying the parity condition. Consider the projection H' of H to the $\mathbb{Z}_2$-homology space of G, and let $H' = W_{i_1} \triangle \cdots \triangle W_{i_p}$ with $1 \leq i_1 < \cdots < i_p \leq g$. Let $\vec{x}$ be the column vector corresponding to H'. By the parity condition of H, the vector $\vec{x}$ is a solution of the intersection system (1) of G. $\square$

4.2 H-subdivided orientations and even triangulations

For the extension of a quadrangulation G to a triangulation, we need to add a diagonal in every face of G from the two choices of diagonals. In this subsection, we explain how to choose it, from a given H-subdivided orientation of $\widehat{S}(G)$.

Let G be a quadrangulation of a surface F^2 and let $\mathcal{O}$ be an orientation of $\widehat{S}(G)$. Let f be a face of G, and let $v_1v_2v_3v_4$ be the facial walk of f. For $i \in \{1, 2, 3, 4\}$, let $e_i = v_iv_{i+1}$. Suppose that in $\widehat{S}(G)$, the edge e_1e_3 (resp. e_2e_4) is directed from e_1 to e_3 (resp. from e_2 to e_4) in $\mathcal{O}$. In this case, the diagonal of f connecting v_1 and v_3 is called the $\mathcal{O}$-*primary diagonal* at f, defined in [14]. Figure 4 is an example of the $\mathcal{O}$-primary diagonal. Adding the $\mathcal{O}$-primary diagonal to every face of G, we get a triangulation T, which is *induced by the orientation* $\mathcal{O}$ of $\widehat{S}(G)$. (For example, see Figure 5.)

The following two lemmas are useful for counting the number of even triangulations extended from a given quadrangulation. Recall that $\overline{\mathcal{O}}$ is the orientation of $\widehat{S}(G)$ obtained from $\mathcal{O}$ by reversing the directions of all the edges.

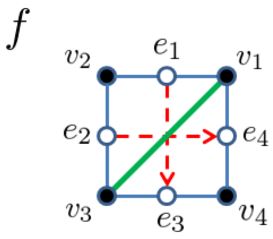

Figure 4: The $\mathcal{O}$-primary diagonal at a quadrangular face f.

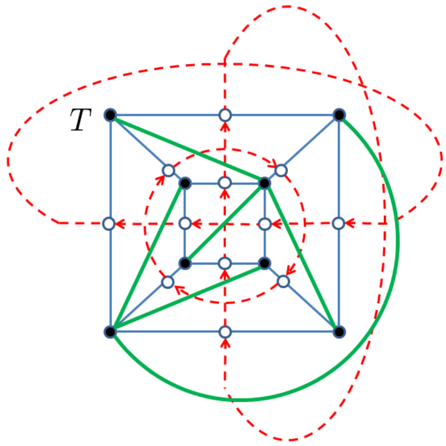

Figure 5: The triangulation T induced by the $\emptyset$-subdivided orientation $\mathcal{O}$.

Lemma 16 *Let G be a quadrangulation of a surface F^2, let H be a subgraph of G satisfying the parity condition, and let $\mathcal{O}$ and $\mathcal{O}'$ be H-subdivided orientations of $\widehat{S}(G)$. Then $\mathcal{O}$ and $\mathcal{O}'$ induce the same triangulation if and only if $\mathcal{O}' = \mathcal{O}$ or $\mathcal{O}' = \overline{\mathcal{O}}$.*

Proof of Lemma 16. If $\mathcal{O}' = \mathcal{O}$ or $\mathcal{O}' = \overline{\mathcal{O}}$, then by definition, for each face f of G, the $\mathcal{O}$-primary diagonal at f is the same as the $\mathcal{O}'$-primary diagonal at f. Hence the triangulation induced by $\mathcal{O}$ and the one induced by $\mathcal{O}'$ coincide.

Conversely, suppose that $\mathcal{O}' \neq \mathcal{O}$ and $\mathcal{O}' \neq \overline{\mathcal{O}}$. Let l be the number of S-walks in G^*. By Lemma 6, there exists a set of components $C_1, \ldots, C_q$ of $\widehat{S}(G)$ for some q with $0 \leq q \leq l$ such that $\mathcal{O}'$ is obtained from $\mathcal{O}$ by reversing the directions of all the edges in C_i for $1 \leq i \leq q$. Since $\mathcal{O}' \neq \mathcal{O}$ and $\mathcal{O}' \neq \overline{\mathcal{O}}$, we have $1 \leq q \leq l-1$. Since the dual G^* of G is connected, we see that there exist two components C_i and C_j of $\widehat{S}(G)$ with $1 \leq i \leq q$ and $q+1 \leq j \leq l$ such that some edge of C_i crosses with some edge of C_j at a face of G, say f. Then the $\mathcal{O}$-primary diagonal at f is different from the $\mathcal{O}'$-primary diagonal, which implies that the triangulation induced by $\mathcal{O}$ is different from the one induced by $\mathcal{O}'$. This completes the proof of Lemma 16. □

Lemma 17 *Let G be a quadrangulation of a surface F^2, and let T be a triangulation of F^2 extended from G. Then there exists an orientation $\mathcal{O}$ of $\widehat{S}(G)$ such that $\mathcal{O}$ induces T.*

Proof of Lemma 17. For each face f of G, we give a direction of the two edges of $\widehat{S}(G)$ inside f so that the $\mathcal{O}$-primary diagonal at f is contained in T. These give an orientation of $\widehat{S}(G)$ with the required property. □

We finally deal with the evenness of the triangulations induced by H-subdivided orientations. In fact, the property that H is an even subgraph plays an important role.

Lemma 18 *Let G be a quadrangulation of a surface F^2, and let $\mathcal{O}$ be an orientation of $\widehat{S}(G)$. Then $\mathcal{O}$ induces an even triangulation if and only if $\mathcal{O}$ is an H-subdivided orientation for some even subgraph H of G.*

Proof of Lemma 18. Let $\mathcal{O}$ be an orientation of $\widehat{S}(G)$. Note that each vertex of $\widehat{S}(G)$, that is, each edge of G has out-degree 0, 1, or 2 in $\widehat{S}(G)$. Let H be the subgraph of G induced by

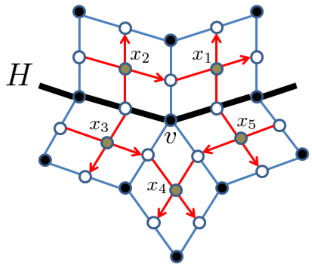

Figure 6: An example of the situation around v.

the edges whose out-degree in $\widehat{S}(G)$ is 0 or 2. Then $\mathcal{O}$ satisfies the conditions (H1) and (H2) in Definition 3, and hence $\mathcal{O}$ is an H-subdivided orientation of G. Let T be a triangulation induced by $\mathcal{O}$. For $v \in V(G)$, let $d = \deg_G(v)$ and let $D_v = x_1e_1x_2e_2 \cdots x_de_dx_1$, which is the alternate sequence of faces and edges in G such that all of them are incident with v in G and e_i is an edge between x_i and x_{i+1} for $1 \leq i \leq d$, where $x_{d+1} = x_1$. (Note that some edges or faces may appear in D_v twice or more times. In such a case, to simplify the arguments, here we regard them as the distinct ones.) Note that we can regard D_v as the cycle obtained from the boundary of the face v^* in G^* by subdividing all the edges, where v^* is the face of G^* corresponding to v. We give the direction to all the edges in D_v naturally from the orientation $\mathcal{O}$ in $\widehat{S}(G)$.

For $1 \leq i \leq d$, the vertex x_i in D_v is *of Type 1* if the two edges of D_v incident with x_i are directed both away from or both towards x_i in $\mathcal{O}$; Otherwise, x_i is *of Type 2* in D_v. For $j \in \{1, 2\}$, let $N_j(v)$ be the number of vertices of Type j in D_v. (For example, see Figure 6.) Note that $N_1(v) + N_2(v) = \deg_G(v)$. Along the cycle D_v, the directions with respect to $\mathcal{O}$ is changed an even number of times, and the changes happen at all the vertices of Type 1 that correspond to a face of G or all the vertices that correspond to an edge in H. Hence we see that $N_1(v) + \deg_H(v)$ is even. Note that the number of $\mathcal{O}$-primary diagonals incident to v is $N_2(v)$. Then

$$\deg_T(v) \;=\; \deg_G(v) + N_2(v) \;=\; 2\deg_G(v) - N_1(v) \;\equiv\; \deg_H(v) \pmod 2.$$

Hence, T is an even triangulation if and only if $\deg_H(v)$ is even for every vertex v of H, that is, H is an even subgraph of G. □

4.3 $\mathbb{Z}_2$-homology classes and distinct triangulations

In this subsection, we show a lemma, which plays a crucial role when we count the number of even triangulations extended from a given quadrangulation. Let G be a quadrangulation of a surface F^2 and let H be a subgraph of G. Define

$$\mathscr{T}_H := \{T : T \text{ is a triangulation induced by an } H\text{-subdivided orientation of } \widehat{S}(G)\}.$$

By Lemma 5, $\mathscr{T}_H \neq \emptyset$ if and only if H satisfies the parity condition.

Lemma 19 *Let G be a quadrangulation of a surface F^2, and let H and H' be two subgraphs of G. Then both of the following hold.*

(i) *If H and H' are $\mathbb{Z}_2$-homologous, then $\mathscr{T}_H = \mathscr{T}_{H'}$.*

(ii) *If H and H' are not $\mathbb{Z}_2$-homologous, then $\mathscr{T}_H \cap \mathscr{T}_{H'} = \emptyset$.*

Proof of Lemma 19.

(i) We show the following claim.

> $(**)$ For any subgraph H of G and any facial walk D of G, we have $\mathscr{T}_H = \mathscr{T}_{H \triangle \widetilde{D}}$.

Indeed, this fact shows Lemma 19 (i) as follows. Since H and H' are $\mathbb{Z}_2$-homologous, it follows from Fact 9 that there exists a set $\{f_1, \ldots, f_t\}$ of faces of G such that $H' = H \triangle \widetilde{D_1} \triangle \cdots \triangle \widetilde{D_t}$, where D_i is the facial walk of f_i for $1 \leq i \leq t$. If $(**)$ holds, then using it t times, we obtain that $\mathscr{T}_H = \mathscr{T}_{H'}$.

So it suffices to show $(**)$. If neither an H-subdivided orientation nor an $(H \triangle \widetilde{D})$-subdivided orientation of $\widehat{S}(G)$ exists, then $\mathscr{T}_H = \emptyset = \mathscr{T}_{H \triangle \widetilde{D}}$. Hence we may assume that there exists an H-subdivided orientation or an $(H \triangle \widetilde{D})$-subdivided orientation of $\widehat{S}(G)$. Since $(H \triangle \widetilde{D}) \triangle \widetilde{D} = H$, we can use the symmetry between H and $H \triangle \widetilde{D}$, and hence we may assume that there exists an H-subdivided orientation of $\widehat{S}(G)$.

Let T be a triangulation induced by some H-subdivided orientation of $\widehat{S}(G)$, say $\mathcal{O}$. Let f be the face of G bounded by D. Note that inside f, there are exactly two edges of $\widehat{S}(G)$. Moreover, reversing the directions of the two edges inside f, we obtain the new orientation of $\widehat{S}(G)$, say $\mathcal{O}_D$. By the definition of an H-subdivided orientation, the orientation $\mathcal{O}_D$ is an $(H \triangle \widetilde{D})$-subdivided orientation of $\widehat{S}(G)$. Let T_D be the triangulation of F^2 that is induced by $\mathcal{O}_D$. Note that $\mathcal{O}_D$ and $\mathcal{O}$ are equivalent except for the edges inside f. Then for every face f' of G other than f, the directions of the two edges of $\widehat{S}(G)$ inside f' are the same, and hence the $\mathcal{O}$-primary diagonal at f' is the same as the $\mathcal{O}_D$-primary diagonal at f'. Even for f, both of the directions of the two edges inside f are reversed, and hence the $\mathcal{O}$-primary diagonal at f is also the same as the $\mathcal{O}_D$-primary diagonal at f. Then we see that $T_D = T$. (For example, see Figure 7.)

Therefore for each $T \in \mathscr{T}_H$, we have $T = T_D \in \mathscr{T}_{H \triangle \widetilde{D}}$, which implies that $\mathscr{T}_H \subset \mathscr{T}_{H \triangle \widetilde{D}}$. Since $(H \triangle \widetilde{D}) \triangle \widetilde{D} = H$, we also have $\mathscr{T}_{H \triangle \widetilde{D}} \subset \mathscr{T}_H$. Thus, $\mathscr{T}_H = \mathscr{T}_{H \triangle \widetilde{D}}$, and $(**)$ holds.

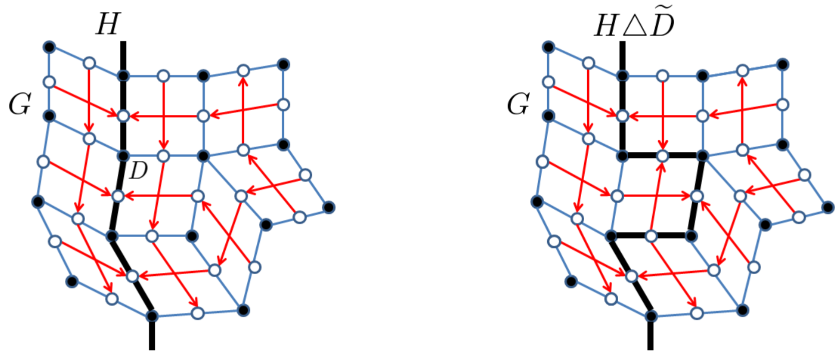

Figure 7: Examples of an H-subdivided orientation $\mathcal{O}$ and an $(H \triangle \widetilde{D})$-subdivided orientation $\mathcal{O}_D$.

(ii) Suppose that $\mathscr{T}_H \cap \mathscr{T}_{H'} \neq \emptyset$, and let $T \in \mathscr{T}_H \cap \mathscr{T}_{H'}$. Then T is induced by an H-subdivided orientation of $\widehat{S}(G)$, say $\mathcal{O}$, and an H'-subdivided orientation of $\widehat{S}(G)$, say $\mathcal{O}'$. Let $\{f_1, \ldots, f_t\}$ be the set of faces of G such that for $1 \leq i \leq t$, there is at least one edge e of $\widehat{S}(G)$ inside f_i such that the direction of e in $\mathcal{O}$ is different from that of e in $\mathcal{O}'$. Since both $\mathcal{O}$ and $\mathcal{O}'$ induce the same triangulation T, both of the two edges of f_i in $\mathcal{O}'$ have different directions from those in $\mathcal{O}$. Thus, both of the two edges of $\widehat{S}(G)$ inside f_i have different directions in $\mathcal{O}$ and $\mathcal{O}'$.

This implies that $\mathcal{O}'$ is obtained from $\mathcal{O}$ by reversing the directions of all edges of $\widehat{S}(G)$ inside f_i for $1 \leq i \leq t$. It is easy to see that for $1 \leq i \leq t$, the orientation of $\widehat{S}(G)$ obtained from $\mathcal{O}$ by reversing the directions of both of the two edges inside f_i is an $(H \triangle \widetilde{D_i})$-subdivided orientation, where D_i is the facial walk of f_i. Using this fact repeatedly, we have $H' = H \triangle \widetilde{D_1} \triangle \cdots \triangle \widetilde{D_t}$. Hence by Fact 9, H and H' are pairwise $\mathbb{Z}_2$-homologous. $\square$

5 Proof of Theorem 12

Let

$$s = \sum_{\vec{x} \text{ is a solution of (1)}} |\mathscr{T}_{H_{\vec{x}}}|,$$

and we first show the claim that there are exactly s even triangulations extended from G.

Let $\vec{x}$ be a solution of the intersection system (1) of G. It follows from Lemma 15 (i) that $\vec{x}$ produces the even subgraph $H_{\vec{x}}$ of G satisfying the parity condition. By Lemmas 5 and 18, there exists an $H_{\vec{x}}$-subdivided orientation, and each $H_{\vec{x}}$-subdivided orientation induces an even triangulation. Hence each triangulation in $\mathscr{T}_{H_{\vec{x}}}$ is even. By Lemma 15 (ii), for any distinct solutions $\vec{x}$ and $\vec{x}'$ of (1), the subgraphs $H_{\vec{x}}$ and $H_{\vec{x}'}$ are not $\mathbb{Z}_2$-homologous, and hence it follows from Lemma 19 (ii) that $\mathscr{T}_{H_{\vec{x}}} \cap \mathscr{T}_{H_{\vec{x}'}} = \emptyset$ for any distinct solutions $\vec{x}$ and $\vec{x}'$. Then we see that there are at least s even triangulations extended from G.

On the other hand, by Lemmas 17 and 18, we see that for any even triangulation T extended from G, there exists an even subgraph H satisfying the parity condition such that $T \in \mathscr{T}_H$. It follows from Lemma 15 (iii) that there exists a unique solution $\vec{x}$ such that H is $\mathbb{Z}_2$-homologous to $H_{\vec{x}}$, and by Lemma 19 (i), we have $T \in \mathscr{T}_H = \mathscr{T}_{H_{\vec{x}}}$. Then we see that there are at most s even triangulations extended from G. Therefore, the claim holds.

Now we count $|\mathscr{T}_{H_{\vec{x}}}|$ for each solution $\vec{x}$. By Lemma 6, there exist precisely 2^l H-subdivided orientations of $\widehat{S}(G)$, and by Lemma 7, there exist precisely 2^{l-1} H-subdivided orientation pairs of $\widehat{S}(G)$. Hence it follows from Lemma 16 that $|\mathscr{T}_{H_{\vec{x}}}| = 2^{l-1}$. By Fact 10, the intersection system (1) of G has exactly 2^k solutions, where $k = g - \operatorname{rank}\big(\mathrm{IM}(G)\big)$ and g is the Euler genus of F^2. Therefore, the number of even triangulations extended from G is exactly

$$s = \sum_{\vec{x} \text{ is a solution of (1)}} |\mathscr{T}_{H_{\vec{x}}}| \;=\; 2^{l-1+k}. \qquad \square$$

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References

[1] K.A. Berman, and H. Shank, Full 4-colorings of 4-regular maps, *J. Graph Theory* **3** (1979) 291–294.

[2] R. Diestel, Graph Theory, 4th edition (Graduate Texts in Mathematics **173**), Springer (2010).

[3] K.P. Eswaran, and R.E. Tarjan, Augmentation problems, *SIAM J. Comput.* **5** (1976) 653–665.

[4] A. Frank, Augmenting graphs to meet edge-connectivity requirements, *SIAM J. Discrete Math.* **5** (1992) 25–53.

[5] F. Hoffmann, and K. Kriegel, A graph coloring result and its consequences for polygon-guarding problems, *SIAM J. Discrete Math.* **9** (1996) 210–224.

[6] A. Hoffmann-Ostenhof, private communication.

[7] T. Jordán, On the optimal vertex-connectivity augmentation, *J. Combin. Theory Ser. B* **63** (1995) 8–20.

[8] A.B. Kempe, On the geographical problem of the four colours, *Amer. J. Math.* **2** (1879) 193–200.

[9] S. MacLane, A combinatorial condition for planar graphs, *Fund. Math.* **28** (1937) 22–32.

[10] B. Mohar, and C. Thomassen, Graphs on Surfaces, Johns Hopkins University Press, Baltimore, MD (2001).

[11] A. Nakamoto, K. Noguchi, and K. Ozeki, Cyclic 4-colorings of graphs on surfaces, *J. Graph Theory*, in press.

[12] J. O'Rourke, Art gallery theorems and algorithms, Oxford University Press, New York (1987).

[13] E.H. Spanier, Algebraic Topology, Springer (1991).

[14] H. Zhang, and X. He, On even triangulations of 2-connected embedded graphs, *SIAM J. Comput.* **34** (2005) 683–696.