Matching extension missing vertices and edges in triangulations of surfaces

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ABSTRACT

Let $G$ be a 5-connected triangulation of a surface $\Sigma$ different from the sphere, and let $\chi = \chi(\Sigma)$ be the Euler characteristic of $\Sigma$. Suppose that $V_0 \subseteq V(G)$ with $|V(G) - V_0|$ even and $M$ and $N$ are two matchings in $G - V_0$ of sizes $m$ and $n$ respectively such that $M \cap N = \emptyset$. It is shown that if the pairwise distance between any two elements of $V_0 \cup M \cup N$ is at least 5 and the face-width of the embedding of $G$ in $\Sigma$ is at least $\max\{20m - 8\chi - 23, 6\}$, then there is a perfect matching $M_0$ in $G - V_0$ containing $M$ such that $M_0 \cap N = \emptyset$.

Keywords: Triangulation, matching extension, representativity, face-width, genus

1 Introduction

A set $M$ of edges in a graph $G$ is a matching if no two members of $M$ share a vertex. A matching $M$ is perfect if every vertex of $G$ is covered by an edge of $M$. Let $n \geq 0$ be an integer. A connected graph $G$ is said to be 0-extendable if it has a perfect matching and, for $n \geq 1$, a connected graph $G$ having at least $2n + 2$ vertices is said to be $n$-extendable if $G$ has a perfect matching and every matching of size $n$ in $G$ extends to (i.e., is a subset of) a perfect matching in $G$. More generally, if $m$ and $n$ are two non-negative integers, then a connected graph $G$ is said to have property $E(m, n)$ if for every pair of matchings $M$ and $N$ with $|M| = m$ and $|N| = n$ such that $M \cap N = \emptyset$, there is a perfect matching $M_0$ in $G$ such that $M \subseteq M_0$ and $N \cap M_0 = \emptyset$.

A surface is a connected compact Hausdorff space that is locally homeomorphic to an open disc in the plane. We denote the Euler characteristic of a surface $\Sigma$ by $\chi = \chi(\Sigma)$. It is well-known that any graph $H$ with at least three vertices embedded in a surface $\Sigma$ with Euler characteristic $\chi$ satisfies that $|E(H)| \leq 3|V(H)| - 3\chi$, and furthermore $|E(H)| \leq 2|V(H)| - 2\chi$ if $H$ is bipartite. The face-width (or representativity) of a graph $G$ embedded in the surface $\Sigma$ different from the sphere is the smallest number $k$ such that

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Let $\Sigma$ contains a non-contractible closed curve $\gamma$ with $|G \cap \gamma| = k$, where $G \cap \gamma$ is the set of points on $\Sigma$ in which $\gamma$ intersects $G$. We shall denote the face-width of an embedded graph $G$ by $\text{fw}(G)$.

In [10] (and independently in [9]) it was shown that every 5-connected planar graph with an even number of vertices is 2-extendable. In [4] it was shown that any 5-connected triangulation with an even number of vertices of a surface of genus $g > 0$ with face-width sufficiently large is 2-extendable. In [7], the preceding result was generalized in three ways: (a) the assumption that $G$ is a triangulation was deleted; (b) the result was extended to non-orientable surfaces and (c) the bound on face-width required was reduced from exponential to linear.

Let $G$ be a graph. The distance between two vertices $x$ and $y$ in $G$, denoted by $d_G(x, y)$, is the length of a shortest path in $G$ connecting them. If there is no possibility of confusion, we simply write $d(x, y)$. Extending this notation, the distance between a vertex $x$ and an edge $e$ in $G$ is defined as the minimum of $d(x, y_1)$ and $d(x, y_2)$, where $y_1$ and $y_2$ are the endvertices of $e$, and the distance between two edges $e$ and $f$ in a graph $G$ is defined as the minimum of $\text{dist}_G(x, y)$ taken over all of the four pairs of vertices $x$ and $y$ such that $x$ is an endvertex of $e$ and $y$ is an endvertex of $f$.

Let us now call a matching $M$ a distance $k$ matching if the distance between any pair of edges in $M$ is at least $k$. A graph $G$ is said to be distance $k$ $m$-extendable if every distance $k$ matching of size $m$ extends to a perfect matching in $G$. In [3] it was shown that if $G$ is a 5-connected planar (or projective planar) triangulation of even order and if $m$ is any non-negative integer with $m \leq (|V(G)| - 2)/2$, then $G$ is distance 5 $m$-extendable. Also, if $M$ is a distance 4 matching of size no greater than 7, then it too extends to a perfect matching. In [1] the same result was obtained for the torus and the Klein bottle. In [5] it was shown that if $G$ is a 5-connected planar triangulation with an even number of vertices, then it is, in fact, distance 4 $m$-extendable for any $m \leq (|V(G)| - 2)/2$ and distance 3 $m$-extendable for all $m$ with $0 \leq m \leq 9$. Moreover, these results are best possible.

5-connected triangulations embedded in surfaces with large face-width have received considerable attention in the past. Yu [11] showed that such graphs always have a Hamilton cycle and the first author [6] later obtained an even stronger result, namely, that such graphs are, in fact, Hamiltonian-connected. In the present paper, we present a distance 5 matching result for the property $E(m, n)$ for 5-connected triangulations embedded in a surface (orientable or non-orientable) with large face-width. Connectivity 5 is crucial here in that, using examples presented in [2], one can easily construct 4-connected even triangulations of genus $g$ having arbitrarily large face-width that contain two edges arbitrarily far apart that cannot be extended to a perfect matching.

In this paper, we consider the property $E(m, n)$ in the graph obtained by deleting some vertices from a 5-connected triangulation embedded in a surface with large face-width. We present an exact statement in the next section, but before that, we define some terminology used in this paper. For a graph $G$, a vertex set $V_0$ of $G$, and an edge set $N$ of $G$, we denote by $G - V_0 - N$ the graph obtained from $G$ by deleting all vertices in $V_0$ and all edges in $N$ (leaving the endvertices of each edge in $N$). We denote the set of endvertices of an edge in $N$ by $V(N)$.

2 The Main Theorem

**Theorem 2.1** Let $\Sigma$ be a surface different from the sphere with Euler characteristic $\chi = \chi(\Sigma)$, let $G$ be a 5-connected triangulation embedded in $\Sigma$, and suppose $V_0 \subseteq V(G)$ is
such that $|V(G) - V_0|$ is even. Suppose that $M$ and $N$ are two matchings in $G - V_0$ with $|M| = m$ and $|N| = n$ such that $M \cap N = \emptyset$ and the distance between any two distinct members of $V_0 \cup M \cup N$ is at least 5. Finally, suppose $fw(G) \geq \max\{20m - 8\chi - 23, 6\}$. Then $G - V_0$ contains a perfect matching $M_0$ that contains $M$ and such that $M_0 \cap N = \emptyset$.

For the proof of Theorem 2.1, we use the following lemma.

**Lemma 2.2** [8, Lemma 6] Let $G$ be a 5-connected graph embedded in a surface $\Sigma$. If $fw(G) \geq 5$, then $G$ does not contain $K_{2,3}$ as a subgraph.

**Proof of Theorem 2.1:** Let $G, V_0, M$ and $N$ be as in the hypothesis. Let $G' = G - V_0 - N$ and let $T = V_0 \cup N$.

**Claim 2.1:** Every non-contractible closed curve $\gamma$ in $\Sigma$ satisfies $|G \cap \gamma| \leq 2|G' \cap \gamma|$.

Consider the face-chain in $G'$ through which $\gamma$ passes. Let $F$ be any face in this face-chain. Since $G$ is a triangulation, if there are at least two elements of $T$ lying in the interior of $F$, then we can find two elements $t_1$ and $t_2$ of $T$ with $d(t_1, t_2) \leq 2$, a contradiction. So each face $F$ in the face-chain contains at most one element of $T$. Suppose that $F_{i-1}, F_i, F_{i+1}$ is a subsequence of consecutive faces in the face-chain and that $w_i$ is a vertex of attachment of $F_{i-1}$ and $F_i$, while $w_{i+1}$ is a vertex of attachment of $F_i$ and $F_{i+1}$. Again, since $G$ is a triangulation and $F_i$ contains at most one element of $T$, it follows that $d(w_i, w_{i+1}) \leq 2$. Then we have a non-contractible face-chain in $G$ of length no more than twice that of the face-chain in $G'$, and the claim follows.

![Figure 1](image.png)

**Figure 1:**

**Claim 2.2:** $G'$ is 4-connected.

Suppose not. Let $W'$ be a minimum cut in $G'$ and let $C$ be a component of $G' - W'$. Note that $|W'| \leq 3$. Consider the graph $G - C$ and the region on $\Sigma$ obtained by the deletion of $C$, and let $C'$ be the boundary of the region. Since $G$ is 5-connected, $C'$ contains at least five vertices. If the distance 5 hypothesis is ignored, in Figure 1 we show the various ways the contractible cycle $C'$ can interact with members of the set $T$ (shown in red). We let $W' = \{w_1, w_2, w_3\}$.
Since \( \text{fw}(G) \geq 6 \), at least one of the components of \( C' \) contains at least two members of \( T \). But regardless of how the members of \( T \) interact with the component, since \( G \) is a triangulation, there must be two members of \( T \) at distance no more than 2 from each other in \( G \), a contradiction. This proves Claim 2.2. ■

Suppose now that \( G, V_0, M \) and \( N \) satisfy the hypotheses of the theorem, but that there is no perfect matching in \( G' \) containing \( M \). Let \( |M| = m \). Then by Tutte’s theorem, if \( G_0 = G' - V(M) \), there is a set \( S_0 \subseteq V(G_0) \) such that \( c_0(G_0 - S_0) \geq |S_0| + 2 \). (Here \( c_0(G_0 - S_0) \) denotes the number of odd components of \( G_0 - S_0 \).) Let \( S = S_0 \cup V(M) \). So \( |S| = |S_0| + 2m \).

By Claim 2.2, any odd component of \( G' - S \) must have at least four vertices of attachment in \( S \). We now partition the odd components of \( G' - S \) into two sets \( O_4 \) and \( O_5 \), where \( O_4 \) denotes the set of odd components that have exactly four neighbors in \( S \) and \( O_5 \) denotes the set of remaining odd components, which have at least five neighbors in \( S \).

Let \( C \in O_4 \). Since \( G \) is 5-connected, \( C \) must have a neighbor that is either in \( V_0 \) or an endvertex of an edge in \( N \). If a neighbor of \( C \) in \( G' \) is contained in \( V(M) \), then together with the fifth vertex of attachment of \( C \), it violates the distance 5 hypothesis. Therefore, all of the four neighbors of \( C \) in \( G' \) are contained in \( S_0 \). Since \( G \) is a triangulation and since \( \text{fw}(G) \geq 6 \), the five neighbors of \( C \) in \( G \) must form a cycle and this cycle must be contractible.

Now suppose \( C_1 \) and \( C_2 \) are two members of \( O_4 \) that share at least one neighbor in \( S_0 \). Let \( x \) denote a common neighbor of \( C_1 \) and \( C_2 \) in \( S_0 \). We will show that \( x \) cannot be a neighbor of a third member of \( O_4 \).

Suppose to the contrary that there is a third odd component \( C_3 \) belonging to \( O_4 \) that has \( x \) as a vertex of attachment. Recall that for \( i = 1, 2, 3 \), the neighbors of \( C_i \) must form a 5-cycle, say \( C'_i \), by the triangulation assumption on \( G \) and the distance hypothesis on \( T \). Note that \( C'_i \) must contain precisely four vertices from \( S_0 \), whereas the fifth vertex, call it \( t_i \), must belong to \( V_0 \) or else is an endvertex of an edge in \( N \). Furthermore, if \( t_i \) is an endvertex of an edge in \( N \), say \( e \), then \( e \) must connect \( t_i \) and a vertex in \( C_i \). Note that \( t_i \neq x \). We then claim the following.

Claim 2.3: \( t_1 = t_2 = t_3 \).

Suppose not. In particular, suppose that \( t_1 \neq t_2 \). It is then clear that along with \( C'_1 \) and \( C'_2 \), we have \( d_G(t_1,t_2) \leq 4 \), and hence \( t_1 \) and \( t_2 \) violate the distance 5 hypothesis. The other cases can be handled using the same reasoning and hence Claim 2.3 holds. ■

As mentioned before, if \( t_i \) is an endvertex of an edge in \( N \), then the edge must connect \( t_i \) and a vertex in \( C_i \). Therefore, Claim 2.3 directly implies that \( t_i \) is a vertex in \( V_0 \). Let \( t = t_1 = t_2 = t_3 \).

Recall that for each \( i \in \{1, 2, 3\} \), \( C'_i \) contains exactly five vertices, two of which are \( x \) and \( t \in V_0 \). Then \( x \) and \( t \) appear either consecutively in \( C'_i \) or non-consecutively in \( C'_i \). Since \( G \) is a triangulation, if \( x \) and \( t \) appear consecutively in \( C'_i \), then \( xt \in E(G) \) and there exists a vertex \( w_i \) in \( C_i \) such that \( x, t, w_i \) form a triangular face of \( G \). On the other hand, if \( x \) and \( t \) appear non-consecutively in \( C'_i \), then there is a (unique) vertex in \( C'_i \) that appears exactly between \( x \) and \( t \). We call the vertex \( w_i \). In either case, \( w_i \) is a common neighbor of \( x \) and \( t \).

If \( w_1, w_2 \) and \( w_3 \) are all distinct, then we can find \( K_{2,3} \) in \( G \) as a subgraph, contradicting Lemma 2.2. Therefore, at least two of them coincide, and by symmetry, we may assume
that \( w_1 = w_2 \). By the definition of \( w_i \), we see that \( x \) and \( t \) appear non-consecutively in \( C'_1 \) and \( C'_2 \), and hence \( w_1 = w_2 \in V(C'_1) \cap V(C'_2) \). Thus, we have the situation in Figure 2. However, since \( t_3 = t \), no matter how \( C'_3 \) is configured with respect to \( C'_1 \) and \( C'_2 \), there must be a 3-cut, a 4-cut, a non-contractible 3-cycle, or a non-contractible 4-cycle. This contradicts either the assumption that \( G \) is 5-connected or the assumption that \( fw(G) \geq 6 \).

![Figure 2](image)

So we have shown that

each vertex in \( S_0 \) is a neighbor of at most two components in \( O_4 \). \( (*) \)

We now construct a bipartite graph \( H \) with bipartition \( \{ S, O_4 \cup O_5 \} \) such that \( x \in S \) and \( C \in O_4 \cup O_5 \) are adjacent in \( H \) if and only if \( x \) is a neighbor of \( C \) in \( G' \). In other words, \( H \) is obtained from \( G' \) by contracting each odd component in \( O_4 \cup O_5 \) to a separate single vertex, and deleting any even components of \( G' - S \), all edges spanned by \( S \) and any multiple edges thus formed. In particular, \( H \) is still embedded in the surface \( \Sigma \) (but not necessarily as a 2-cell embedding).

Let \( H_0 \) result from \( H \) by deleting all vertices in \( V(M) \) and all vertices corresponding to components in \( O_5 \). Thus, \( H_0 \) is a bipartite graph with bipartition \( \{ S_0, O_4 \} \). Since each component in \( O_4 \) has exactly four neighbors in \( S_0 \), we have \( |E(H_0)| = 4|O_4| \). On the other hand, it follows from \( (*) \) that each vertex in \( S_0 \) can be adjacent to at most two components in \( O_4 \). This implies \( |E(H_0)| \leq 2|S_0| \). Hence \( 4|O_4| \leq 2|S_0| \), or

\[
|O_4| \leq \frac{1}{2}|S_0|. \quad (2.1)
\]

Recall that \( O_4 \cup O_5 \) is the set of all odd components of \( G' - V(M) - S_0 \) and \( c_o(G' - V(M) - S_0) \geq |S_0| + 2 \), so we have

\[
|O_4| + |O_5| \geq |S_0| + 2. \quad (2.2)
\]

Back in the graph \( H \), the number of edges joining \( S \) and all the odd components in \( O_4 \cup O_5 \) is at least \( 4|O_4| + 5|O_5| \), as viewed from each component in \( O_4 \cup O_5 \). Now \( |V(H)| = |S_0| + 2m + |O_4| + |O_5| \). Noting that \( H \) is a bipartite graph embedded on the surface \( \Sigma \), by Euler’s Formula,

\[
|E(H)| \leq 2|V(H)| - 2\chi = 2(|S_0| + 2m + |O_4| + |O_5|) - 2\chi.
\]

Recall that \( \chi = \chi(\Sigma) \) is the Euler characteristic of \( \Sigma \). So

\[
4|O_4| + 5|O_5| \leq 2|S_0| + 4m + 2|O_4| + 2|O_5| - 2\chi.
\]


Thus, using inequalities (2.1) and (2.2), we obtain
\[
2|S_0| + 4m - 2\chi \geq 2|O_4| + 3|O_5|
\]
\[
= 3(|O_4| + |O_5|) - |O_4|
\]
\[
\geq 3(|S_0| + 2) - \frac{1}{2}|S_0|
\]
\[
= \frac{5}{2}|S_0| + 6,
\]
or
\[
|S_0| \leq 8m - 4\chi - 12.
\]
Hence
\[
|S| = |S_0| + 2m \leq 10m - 4\chi - 12. \quad (2.3)
\]

Now again consider the bipartite graph \(H\) with bipartition \(\{S, O_4 \cup O_5\}\), where \(S_0 = S - V(M)\) is a Tutte set of \(G' - V(M)\).

**Claim 2.4:** If \(H\) is non-planar, then there must exist a non-contractible curve \(\gamma\) in the surface \(\Sigma\) such that \(\gamma \cap H \subseteq S\).

Since \(H\) is non-planar, we let \(\gamma\) be a non-contractible closed curve such that \(\gamma \cap H \subseteq V(H)\). Suppose that \(\gamma\) passes through a vertex \(u\) of \(H\) that is not in \(S\). Let \(f_1\) and \(f_2\) be two faces such that \(\gamma\) passes through \(f_1, u\) and \(f_2\) consecutively in this order. Then we can modify \(\gamma\) to become a new curve \(\gamma'\) so that \(\gamma'\) passes through neighbors of \(u\) when passing from \(f_1\) to \(f_2\). (See Figure 3.)

![Figure 3](image-url)

Repeating this process for all vertices passed through by \(\gamma\) that are not in \(S\) and call the resulting curve \(\gamma_0\). This curve \(\gamma_0\) serves to prove the Claim 2.4.

Let \(M' = \{f \in M : f\) is incident with at least two components in \(O_4 \cup O_5\}\),
and \(V_0' = \{v \in V_0 : v\) is a neighbor of at least one component in \(O_4 \cup O_5\}\),
where \(f \in M\) is incident with a component in \(O_4 \cup O_5\) if at least one of endvertices of \(f\) is a neighbor of the component.

Then consider the bipartite graph \(H'\) with bipartition \(\{S_0 \cup V(M') \cup V_0', O_4 \cup O_5\}\). Note that \(H - V(M - M') = H' - V_0'\). Similarly to the case of \(H, H'\) is embedded in the surface \(\Sigma\) (not necessarily as a 2-cell embedding).

Assume that \(H'\) is non-planar. Then there must exist a non-contractible closed curve \(\gamma\) in \(\Sigma\) such that \(\gamma \cap H' \subseteq V(H')\). Then deleting all vertices in \(V_0'\) and adding some
edges in $M'$ if necessary, we can regard the non-contractible closed curve $\gamma$ as satisfying $\gamma \cap H \subseteq V(H)$. Then by Claim 2.4, there exists a non-contractible closed curve $\gamma_0$ in $\Sigma$ such that $\gamma_0 \cap H \subseteq S$. Now reinflate each vertex in $H$ corresponding to $O_4 \cup O_5$ to its original odd component of $G' - S$ so as to recover the original $G'$ from $H' - V_0'$, retaining the curve $\gamma_0$ in the process. So $\gamma_0 \cap G' \subseteq S$. Then Claim 2.1 and the inequality (2.3) imply that
\[
|\gamma_0 \cap G| \leq 2|\gamma_0 \cap G'| \leq 2|S| \leq 20m - 8\chi - 24.
\]
But this contradicts the assumption on $\text{fw}(G)$.

So $H'$ must be planar. Let $m' = |M'|$ and $\ell = |O_4| + |O_5|$. Note that $\ell \geq |S_0| + 2$. Then $|E(H')| \leq 5\ell$ (since $G$ is 5-connected) and $|E(H')| \leq 2(|S_0| + 2m' + |V_0'| + \ell) - 4$ (by the Euler formula), so $3(|S_0| + 2) \leq 3\ell \leq 2|S_0| + 4m' + 2|V_0'| - 4$, and hence
\[
|S_0| \leq 4m' + 2|V_0'| - 10. \quad (2.4)
\]

For each $f \in M'$ and $v \in V_0'$, let
\[
Q(f) = \{s \in S_0 : d_G(f, s) \leq 2\},
\]
and
\[
Q(v) = \{s \in S_0 : d_G(v, s) \leq 2\}.
\]
By the distance 5 hypothesis, we see that $Q(f) \cap Q(f') = \emptyset$, $Q(f) \cap Q(v) = \emptyset$, and $Q(v) \cap Q(v') = \emptyset$ for any $f, f' \in M'$ with $f \neq f'$ and $v, v' \in V_0'$ with $v \neq v'$. We next show that
\[
|Q(f)| \geq 4 \text{ for any } f \in M', \text{ and } |Q(v)| \geq 2 \text{ for any } v \in V_0'.
\]
In fact, this then directly implies $|S_0| \geq 4m' + 2|V_0'|$, a contradiction of inequality (2.4), and the proof of Theorem 2.1 is complete.

Let $f \in M'$, and let $C_1$ and $C_2$ be two components in $O_4 \cup O_5$ that are incident with $f$. By planarity and since $G$ is a triangulation, the neighbors of $C_1$ surround $C_1$ in cyclic order $C_1'$. Then consider $C_1'$ forward and backward with distance at most two from endvertices of $f$. Note that the distance 5 hypothesis implies that all of these vertices are contained in $S_0$. Therefore, if there exist at least four such vertices in $C_1'$, then $|Q(f)| \geq 4$ and we are done. (See the left diagram in Figure 4. The vertices in squares belong to $Q(f)$.) Thus, we may assume that there exist only at most three such vertices in $C_1'$, which directly implies that $f$ is an edge in $C_1'$ and $C_1'$ is a 5-cycle. The structure surrounding $f$ for $C_2$ must be similar, and hence the structure surrounding $f$ must have the appearance shown in the right in Figure 4. This contradicts the assumption that $G$ is 5-connected.

Let $v \in V_0'$ and let $C_1$ be a component in $O_4 \cup O_5$ of which $v$ is a neighbor. Similarly (or in fact even more simply), we can find two vertices in $C_1'$ that are adjacent with $v$. This shows $|Q(v)| \geq 2$, and we are done. \qed

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Figure 4: Structures surrounding $f$.

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