Matching extension missing vertices and edges in triangulations of surfaces

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ABSTRACT

Let G be a 5-connected triangulation of a surface Σ different from the sphere, and let $\chi = \chi(\Sigma)$ be the Euler characteristic of Σ. Suppose that $V_0 \subseteq V(G)$ with $|V(G) - V_0|$ even and M and N are two matchings in $G - V_0$ of sizes m and n respectively such that $M \cap N = \emptyset$. It is shown that if the pairwise distance between any two elements of $V_0 \cup M \cup N$ is at least 5 and the face-width of the embedding of G in Σ is at least $\max\{20m - 8\chi - 23, 6\}$, then there is a perfect matching M_0 in $G - V_0$ containing M such that $M_0 \cap N = \emptyset$.

Keywords: Triangulation, matching extension, representativity, face-width, genus

1 Introduction

A set M of edges in a graph G is a *matching* if no two members of M share a vertex. A matching M is *perfect* if every vertex of G is covered by an edge of M. Let $n \geq 0$ be an integer. A connected graph G is said to be 0-*extendable* if it has a perfect matching and, for $n \geq 1$, a connected graph G having at least $2n + 2$ vertices is said to be n-*extendable* if G has a perfect matching and every matching of size n in G extends to (i.e., is a subset of) a perfect matching in G. More generally, if m and n are two non-negative integers, then a connected graph G is said to have property $E(m, n)$ if for every pair of matchings M and N with $|M| = m$ and $|N| = n$ such that $M \cap N = \emptyset$, there is a perfect matching M_0 in G such that $M \subseteq M_0$ and $N \cap M_0 = \emptyset$.

A *surface* is a connected compact Hausdorff space that is locally homeomorphic to an open disc in the plane. We denote the *Euler characteristic* of a surface Σ by $\chi = \chi(\Sigma)$. It is well-known that any graph H with at least three vertices embedded in a surface Σ with Euler characteristic χ satisfies that $|E(H)| \leq 3|V(H)| - 3\chi$, and furthermore $|E(H)| \leq 2|V(H)| - 2\chi$ if H is bipartite. The *face-width* (or *representativity*) of a graph G embedded in the surface Σ different from the sphere is the smallest number k such that

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[2]This work was in part supported by JSPS KAKENHI Grant Number 25871053 and by Grant for Basic Science Research Projects from The Sumitomo Foundation.
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Σ contains a non-contractible closed curve γ with $|G \cap \gamma| = k$, where $G \cap \gamma$ is the set of points on Σ in which γ intersects G. We shall denote the face-width of an embedded graph G by $\mathrm{fw}(G)$.

In [10] (and independently in [9]) it was shown that every 5-connected planar graph with an even number of vertices is 2-extendable. In [4] it was shown that any 5-connected triangulation with an even number of vertices of a surface of genus $g > 0$ with face-width sufficiently large is 2-extendable. In [7], the preceding result was generalized in three ways: (a) the assumption that G is a triangulation was deleted; (b) the result was extended to non-orientable surfaces and (c) the bound on face-width required was reduced from exponential to linear.

Let G be a graph. The *distance* between two vertices x and y in G, denoted by $d_G(x, y)$, is the length of a shortest path in G connecting them. If there is no possibility of confusion, we simply write $d(x, y)$. Extending this notation, the *distance* between a vertex x and an edge e in G is defined as the minimum of $d(x, y_1)$ and $d(x, y_2)$, where y_1 and y_2 are the endvertices of e, and the *distance* between two edges e and f in a graph G is defined as the minimum of $\mathrm{dist}_G(x, y)$ taken over all of the four pairs of vertices x and y such that x is an endvertex of e and y is an endvertex of f.

Let us now call a matching M a *distance k matching* if the distance between any pair of edges in M is at least k. A graph G is said to be *distance k m-extendable* if every distance k matching of size m extends to a perfect matching in G. In [3] it was shown that if G is a 5-connected planar (or projective planar) triangulation of even order and if m is any non-negative integer with $m \leq (|V(G)| - 2)/2$, then G is distance 5 m-extendable. Also, if M is a distance 4 matching of size no greater than 7, then it too extends to a perfect matching. In [1] the same result was obtained for the torus and the Klein bottle. In [5] it was shown that if G is a 5-connected planar triangulation with an even number of vertices, then it is, in fact, distance 4 m-extendable for any $m \leq (|V(G)| - 2)/2$ and distance 3 m-extendable for all m with $0 \leq m \leq 9$. Moreover, these results are best possible.

5-connected triangulations embedded in surfaces with large face-width have received considerable attention in the past. Yu [11] showed that such graphs always have a Hamilton cycle and the first author [6] later obtained an even stronger result, namely, that such graphs are, in fact, Hamiltonian-connected. In the present paper, we present a distance 5 matching result for the property $E(m, n)$ for 5-connected triangulations embedded in a surface (orientable or non-orientable) with large face-width. Connectivity 5 is crucial here in that, using examples presented in [2], one can easily construct 4-connected even triangulations of genus g having arbitrarily large face-width that contain two edges arbitrarily far apart that cannot be extended to a perfect matching.

In this paper, we consider the property $E(m, n)$ in the graph obtained by deleting some vertices from a 5-connected triangulation embedded in a surface with large face-width. We present an exact statement in the next section, but before that, we define some terminology used in this paper. For a graph G, a vertex set V_0 of G, and an edge set N of G, we denote by $G - V_0 - N$ the graph obtained from G by deleting all vertices in V_0 and all edges in N (leaving the endvertices of each edge in N). We denote the set of endvertices of an edge in N by $V(N)$.

2 The Main Theorem

Theorem 2.1 *Let Σ be a surface different from the sphere with Euler characteristic $\chi = \chi(\Sigma)$, let G be a 5-connected triangulation embedded in Σ, and suppose $V_0 \subseteq V(G)$ is* *such that $|V(G) - V_0|$ is even. Suppose that M and N are two matchings in $G - V_0$ with $|M| = m$ and $|N| = n$ such that $M \cap N = \emptyset$ and the distance between any two distinct members of $V_0 \cup M \cup N$ is at least 5. Finally, suppose fw$(G) \geq \max\{20m - 8\chi - 23, 6\}$. Then $G - V_0$ contains a perfect matching M_0 that contains M and such that $M_0 \cap N = \emptyset$.*

For the proof of Theorem 2.1, we use the following lemma.

Lemma 2.2 **[8, Lemma 6]** *Let G be a 5-connected graph embedded in a surface Σ. If fw$(G) \geq 5$, then G does not contain $K_{2,3}$ as a subgraph.*

Proof of Theorem 2.1: Let G, V_0, M and N be as in the hypothesis. Let $G' = G - V_0 - N$ and let $T = V_0 \cup N$.

Claim 2.1: Every non-contractible closed curve γ in Σ satisfies $|G \cap \gamma| \leq 2|G' \cap \gamma|$.

Consider the face-chain in G' through which γ passes. Let F be any face in this face-chain. Since G is a triangulation, if there are at least two elements of T lying in the interior of F, then we can find two elements t_1 and t_2 of T with $d(t_1, t_2) \leq 2$, a contradiction. So each face F in the face-chain contains at most one element of T. Suppose that F_{i-1}, F_i, F_{i+1} is a subsequence of consecutive faces in the face-chain and that w_i is a vertex of attachment of F_{i-1} and F_i, while w_{i+1} is a vertex of attachment of F_i and F_{i+1}. Again, since G is a triangulation and F_i contains at most one element of T, it follows that $d(w_i, w_{i+1}) \leq 2$. Then we have a non-contractible face-chain in G of length no more than twice that of the face-chain in G', and the claim follows. ■

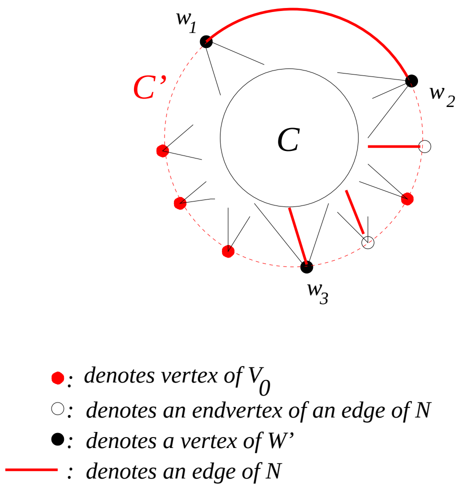

Figure 1:

Claim 2.2: G' is 4-connected.

Suppose not. Let W' be a minimum cut in G' and let C be a component of $G' - W'$. Note that $|W'| \leq 3$. Consider the graph $G - C$ and the region on Σ obtained by the deletion of C, and let C' be the boundary of the region. Since G is 5-connected, C' contains at least five vertices. If the distance 5 hypothesis is ignored, in Figure 1 we show the various ways the contractible cycle C' can interact with members of the set T (shown in red). We let $W' = \{w_1, w_2, w_3\}$.

Since $\text{fw}(G) \geq 6$, at least one of the components of C' contains at least two members of T. But regardless of how the members of T interact with the component, since G is a triangulation, there must be two members of T at distance no more than 2 from each other in G, a contradiction. This proves Claim 2.2. ■

Suppose now that G, V_0, M and N satisfy the hypotheses of the theorem, but that there is no perfect matching in G' containing M. Let $|M| = m$. Then by Tutte's theorem, if $G_0 = G' - V(M)$, there is a set $S_0 \subseteq V(G_0)$ such that $c_o(G_0 - S_0) \geq |S_0| + 2$. (Here $c_0(G_0 - S_0)$ denotes the number of odd components of $G_0 - S_0$.) Let $S = S_0 \cup V(M)$. So $|S| = |S_0| + 2m$.

By Claim 2.2, any odd component of $G' - S$ must have at least four vertices of attachment in S. We now partition the odd components of $G' - S$ into two sets $\mathcal{O}_4$ and $\mathcal{O}_5$, where $\mathcal{O}_4$ denotes the set of odd components that have exactly four neighbors in S and $\mathcal{O}_5$ denotes the set of remaining odd components, which have at least five neighbors in S.

Let $C \in \mathcal{O}_4$. Since G is 5-connected, C must have a neighbor that is either in V_0 or an endvertex of an edge in N. If a neighbor of C in G' is contained in $V(M)$, then together with the fifth vertex of attachment of C, it violates the distance 5 hypothesis. Therefore, all of the four neighbors of C in G' are contained in S_0. Since G is a triangulation and since $fw(G) \geq 6$, the five neighbors of C in G must form a cycle and this cycle must be contractible.

Now suppose C_1 and C_2 are two members of $\mathcal{O}_4$ that share at least one neighbor in S_0. Let x denote a common neighbor of C_1 and C_2 in S_0. We will show that x cannot be a neighbor of a third member of $\mathcal{O}_4$.

Suppose to the contrary that there is a third odd component C_3 belonging to $\mathcal{O}_4$ that has x as a vertex of attachment. Recall that for $i = 1, 2, 3$, the neighbors of C_i must form a 5-cycle, say C'_i, by the triangulation assumption on G and the distance hypothesis on T. Note that C'_i must contain precisely four vertices from S_0, whereas the fifth vertex, call it t_i, must belong to V_0 or else is an endvertex of an edge in N. Furthermore, if t_i is an endvertex of an edge in N, say e, then e must connect t_i and a vertex in C_i. Note that $t_i \neq x$. We then claim the following.

Claim 2.3: $t_1 = t_2 = t_3$.

Suppose not. In particular, suppose that $t_1 \neq t_2$. It is then clear that along with C'_1 and C'_2, we have $d_G(t_1, t_2) \leq 4$, and hence t_1 and t_2 violate the distance 5 hypothesis. The other cases can be handled using the same reasoning and hence Claim 2.3 holds. ■

As mentioned before, if t_i is an endvertex of an edge in N, then the edge must connect t_i and a vertex in C_i. Therefore, Claim 2.3 directly implies that t_i is a vertex in V_0. Let $t = t_1 = t_2 = t_3$.

Recall that for each $i \in \{1, 2, 3\}$, C'_i contains exactly five vertices, two of which are x and $t \in V_0$. Then x and t appear either *consecutively* in C'_i or *non-consecutively* in C'_i. Since G is a triangulation, if x and t appear consecutively in C'_i, then $xt \in E(G)$ and there exists a vertex w_i in C_i such that x, t and w_i form a triangular face of G. On the other hand, if x and t appear non-consecutively in C'_i, then there is a (unique) vertex in C'_i that appears exactly between x and t. We call the vertex w_i. In either case, w_i is a common neighbor of x and t.

If w_1, w_2 and w_3 are all distinct, then we can find $K_{2,3}$ in G as a subgraph, contradicting Lemma 2.2. Therefore, at least two of them coincide, and by symmetry, we may assume that $w_1 = w_2$. By the definition of w_i, we see that x and t appear non-consecutively in C'_1 and C'_2, and hence $w_1 = w_2 \in V(C'_1) \cap V(C'_2)$. Thus, we have the situation in Figure 2. However, since $t_3 = t$, no matter how C'_3 is configured with respect to C'_1 and C'_2, there must be a 3-cut, a 4-cut, a non-contractible 3-cycle, or a non-contractible 4-cycle. This contradicts either the assumption that G is 5-connected or the assumption that $\mathrm{fw}(G) \geq 6$.

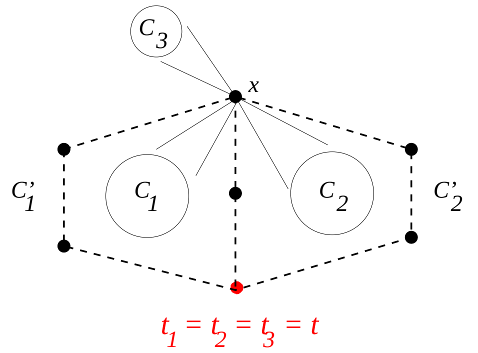

Figure 2:

So we have shown that

each vertex in S_0 is a neighbor of at most two components in $\mathcal{O}_4$. $\quad (*)$

We now construct a bipartite graph H with bipartition $\{S, \mathcal{O}_4 \cup \mathcal{O}_5\}$ such that $x \in S$ and $C \in \mathcal{O}_4 \cup \mathcal{O}_5$ are adjacent in H if and only if x is a neighbor of C in G'. In other words, H is obtained from G' by contracting each odd component in $\mathcal{O}_4 \cup \mathcal{O}_5$ to a separate single vertex, and deleting any even components of $G' - S$, all edges spanned by S and any multiple edges thus formed. In particular, H is still embedded in the surface Σ (but not necessarily as a 2-cell embedding).

Let H_0 result from H by deleting all vertices in $V(M)$ and all vertices corresponding to components in $\mathcal{O}_5$. Thus, H_0 is a bipartite graph with bipartition $\{S_0, \mathcal{O}_4\}$. Since each component in $\mathcal{O}_4$ has exactly four neighbors in S_0, we have $|E(H_0)| = 4|\mathcal{O}_4|$. On the other hand, it follows from $(*)$ that each vertex in S_0 can be adjacent to at most two components in $\mathcal{O}_4$. This implies $|E(H_0)| \leq 2|S_0|$. Hence $4|\mathcal{O}_4| \leq 2|S_0|$, or

$$|\mathcal{O}_4| \;\leq\; \frac{1}{2}|S_0|. \tag{2.1}$$

Recall that $\mathcal{O}_4 \cup \mathcal{O}_5$ is the set of all odd components of $G' - V(M) - S_0$ and $c_o(G' - V(M) - S_0) \geq |S_0| + 2$, so we have

$$|\mathcal{O}_4| + |\mathcal{O}_5| \;\geq\; |S_0| + 2. \tag{2.2}$$

Back in the graph H, the number of edges joining S and all the odd components in $\mathcal{O}_4 \cup \mathcal{O}_5$ is at least $4|\mathcal{O}_4| + 5|\mathcal{O}_5|$, as viewed from each component in $\mathcal{O}_4 \cup \mathcal{O}_5$. Now $|V(H)| = |S_0| + 2m + |\mathcal{O}_4| + |\mathcal{O}_5|$. Noting that H is a bipartite graph embedded on the surface Σ, by Euler's Formula,

$$|E(H)| \;\leq\; 2|V(H)| - 2\chi \;=\; 2\big(|S_0| + 2m + |\mathcal{O}_4| + |\mathcal{O}_5|\big) - 2\chi.$$

Recall that $\chi = \chi(\Sigma)$ is the Euler characteristic of Σ. So

$$4|\mathcal{O}_4| + 5|\mathcal{O}_5| \;\leq\; 2|S_0| + 4m + 2|\mathcal{O}_4| + 2|\mathcal{O}_5| - 2\chi.$$

Thus, using inequalities (2.1) and (2.2), we obtain

$$\begin{aligned} 2|S_0| + 4m - 2\chi &\geq 2|\mathcal{O}_4| + 3|\mathcal{O}_5| \\ &= 3\big(|\mathcal{O}_4| + |\mathcal{O}_5|\big) - |\mathcal{O}_4| \\ &\geq 3\big(|S_0| + 2\big) - \frac{1}{2}|S_0| \\ &= \frac{5}{2}|S_0| + 6, \end{aligned}$$

or

$$|S_0| \leq 8m - 4\chi - 12.$$

Hence

$$|S| = |S_0| + 2m \leq 10m - 4\chi - 12. \tag{2.3}$$

Now again consider the bipartite graph H with bipartition $\{S, \mathcal{O}_4 \cup \mathcal{O}_5\}$, where $S_0 = S - V(M)$ is a Tutte set of $G' - V(M)$.

Claim 2.4: If H is non-planar, then there must exist a non-contractible curve γ in the surface Σ such that $\gamma \cap H \subseteq S$.

Since H is non-planar, we let γ be a non-contractible closed curve such that $\gamma \cap H \subseteq V(H)$. Suppose that γ passes through a vertex u of H that is not in S. Let f_1 and f_2 be two faces such that γ passes through f_1, u and f_2 consecutively in this order. Then we can modify γ to become a new curve γ' so that γ' passes through neighbors of u when passing from f_1 to f_2. (See Figure 3.)

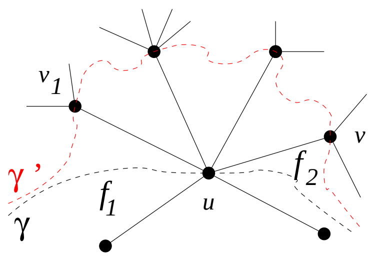

Figure 3:

Repeating this process for all vertices passed through by γ that are not in S and call the resulting curve γ_0. This curve γ_0 serves to prove the Claim 2.4. ■

$$\begin{aligned} \text{Let} \quad M' &= \{f \in M : f \text{ is incident with at least two components in } \mathcal{O}_4 \cup \mathcal{O}_5\}, \\ \text{and} \quad V_0' &= \{v \in V_0 : v \text{ is a neighbor of at least one component in } \mathcal{O}_4 \cup \mathcal{O}_5\}, \end{aligned}$$

where $f \in M$ is *incident* with a component in $\mathcal{O}_4 \cup \mathcal{O}_5$ if at least one of endvertices of f is a neighbor of the component.

Then consider the bipartite graph H' with bipartition $\{S_0 \cup V(M') \cup V_0', \mathcal{O}_4 \cup \mathcal{O}_5\}$. Note that $H - V(M - M') = H' - V_0'$. Similarly to the case of H, H' is embedded in the surface Σ (not necessarily as a 2-cell embedding).

Assume that H' is non-planar. Then there must exist a non-contractible closed curve γ in Σ such that $\gamma \cap H' \subseteq V(H')$. Then deleting all vertices in V_0' and adding some edges in M' if necessary, we can regard the non-contractible closed curve γ as satisfying $\gamma \cap H \subseteq V(H)$. Then by Claim 2.4, there exists a non-contractible closed curve γ_0 in Σ such that $\gamma_0 \cap H \subseteq S$. Now reinflate each vertex in H corresponding to $\mathcal{O}_4 \cup \mathcal{O}_5$ to its original odd component of $G' - S$ so as to recover the original G' from $H' - V_0'$, retaining the curve γ_0 in the process. So $\gamma_0 \cap G' \subseteq S$. Then Claim 2.1 and the inequality (2.3) imply that

$$|\gamma_0 \cap G| \;\le\; 2|\gamma_0 \cap G'| \;\le\; 2|S| \;\le\; 20m - 8\chi - 24.$$

But this contradicts the assumption on $\mathrm{fw}(G)$.

So H' must be planar. Let $m' = |M'|$ and $\ell = |\mathcal{O}_4| + |\mathcal{O}_5|$. Note that $\ell \ge |S_0| + 2$. Then $|E(H')| \ge 5\ell$ (since G is 5-connected) and $|E(H')| \le 2(|S_0| + 2m' + |V_0'| + \ell) - 4$ (by the Euler formula), so $3(|S_0| + 2) \le 3\ell \le 2|S_0| + 4m' + 2|V_0'| - 4$, and hence

$$|S_0| \;\le\; 4m' + 2|V_0'| - 10. \tag{2.4}$$

For each $f \in M'$ and $v \in V_0'$, let

$$\begin{aligned} Q(f) &= \{s \in S_0 : d_G(f, s) \le 2\ \}, \\ \text{and} \qquad Q(v) &= \{s \in S_0 : d_G(v, s) \le 2\ \}. \end{aligned}$$

By the distance 5 hypothesis, we see that $Q(f) \cap Q(f') = \emptyset$, $Q(f) \cap Q(v) = \emptyset$, and $Q(v) \cap Q(v') = \emptyset$ for any $f, f' \in M'$ with $f \ne f'$ and $v, v' \in V_0'$ with $v \ne v'$. We next show that

$$|Q(f)| \ge 4 \text{ for any } f \in M', \text{ and } |Q(v)| \ge 2 \text{ for any } v \in V_0'.$$

In fact, this then directly implies $|S_0| \ge 4m' + 2|V_0'|$, a contradiction of inequality (2.4), and the proof of Theorem 2.1 is complete.

Let $f \in M'$, and let C_1 and C_2 be two components in $\mathcal{O}_4 \cup \mathcal{O}_5$ that are incident with f. By planarity and since G is a triangulation, the neighbors of C_1 surround C_1 in cyclic order C_1'. Then consider C_1' forward and backward with distance at most two from endvertices of f. Note that the distance 5 hypothesis implies that all of these vertices are contained in S_0. Therefore, if there exist at least four such vertices in C_1', then $|Q(f)| \ge 4$ and we are done. (See the left diagram in Figure 4. The vertices in squares belong to $Q(f)$.) Thus, we may assume that there exist only at most three such vertices in C_1', which directly implies that f is an edge in C_1' and C_1' is a 5-cycle. The structure surrounding f for C_2 must be similar, and hence the structure surrounding f must have the appearance shown in the right in Figure 4. This contradicts the assumption that G is 5-connected.

Let $v \in V_0'$ and let C_1 be a component in $\mathcal{O}_4 \cup \mathcal{O}_5$ of which v is a neighbor. Similarly (or in fact even more simply), we can find two vertices in C_1' that are adjacent with v. This shows $|Q(v)| \ge 2$, and we are done. $\square$

Acknowledgments

The authors would like to thank the anonymous referees for their helpful comments, which considerably improved the readability of this paper.

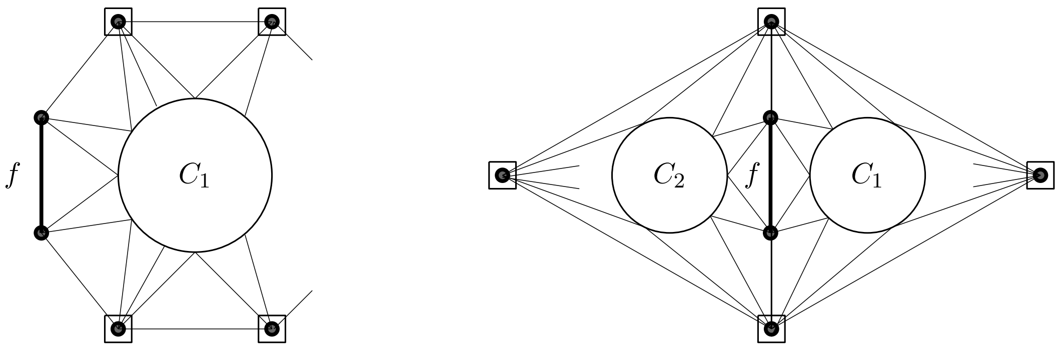

Figure 4: Structures surrounding f.

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