Decomposing plane cubic graphs

Kenta Ozeki*†‡ and Dong Ye§

Abstract

It was conjectured by Hoffmann-Ostenhof that the edge set of every cubic graph can be decomposed into a spanning tree, a matching and a family of cycles. We prove the conjecture for 3-connected cubic plane graphs and 3-connected cubic graphs on the projective plane. Our proof provides a polynomial time algorithm to find the decomposition for 3-connected cubic plane graphs.

1 Introduction

All graphs discussed in this paper are simple. A graph G consists of the vertex set $V(G)$ and the edge set $E(G)$. A graph G is *cubic* if every vertex v in G has degree 3. A graph without cycle is called an *acyclic* graph or a forest. A *spanning tree* of a graph G is a connected acyclic subgraph containing all vertices of G. A *matching* is a set of edges without common end vertices. A matching is *perfect* if it covers all vertices of G.

A *decomposition* of a graph G consists of pairwise edge-disjoint subgraphs whose union is G, that is, each edge in G belongs to exactly one of the subgraphs. The decompositions of graphs to forests and degree-bounded subgraphs have applications in graph coloring (cf. [5, 15]). In [8], Gonçalves proved that every plane graph has a decomposition into three forests one of which has degree at most 4, which was conjectured by Balogh, Kochol, Pluhar and Yu in [3]. Kleitman [18] proved that a plane graph with girth at least 6 has a decomposition into a forest, pairwise edge-disjoint paths and cycles. Further, a plane graph with large girth (at least 8) has a decomposition into a forest and a matching [5, 15, 28]. But a plane graph with smaller girth does not have these decompositions [18, 24]. The decomposition problem for sparse graphs also has been studied in [19, 24].

*National Institute of Informatics, 2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan. Email: ozeki@nii.ac.jp

†JST, ERATO, Kawarabayashi Large Graph Project, Japan.

‡This work was in part supported by JSPS KAKENHI Grant Number 25871053 and by Grant for Basic Science Research Projects from The Sumitomo Foundation.

§Department of Mathematical Sciences and Center for Computational Sciences, Middle Tennessee State University, Murfreesboro, TN 37132, USA. Email: dong.ye@mtsu.edu

For decompositions of cubic graphs with certain properties, the first result is the Vizing Theorem [27] on proper edge-coloring, which indicates that every cubic graph has a decomposition into four pairwise edge-disjoint matchings. Recently, Fouquet and Vanherpe studied the decomposition of cubic graphs into pairwise edge-disjoint paths with certain properties [12, 13]. As pointed out in [12], the decomposition problem of cubic graphs into paths is related to conjectures on cubic graphs, for example, the Fan-Raspaud conjecture [10] (which states that every 2-edge-connected cubic graph contains three perfect matchings with empty intersection). Note that every connected graph G with an even number of edges can be decomposed into pairwise edge-disjoint paths of length exactly 2. (To see this, consider the line graph $L(G)$ of G, which is a connected claw-free graph with an even number of vertices and hence has a perfect matching (see [21, 25]). A perfect matching of $L(G)$ corresponds to a desired decomposition of G).

A cubic graph does not have a decomposition into a forest and a matching because of the degree condition. But the Petersen Theorem implies that every 2-connected cubic graph can be decomposed into a forest (a perfect matching) and a family of cycles (a 2-factor). It seems also interesting to consider a decomposition of a cubic graph into a spanning tree and other subgraphs. A spanning tree T is called a *homeomorphically irreducible spanning tree* or shortly a HIST if T does not contain a vertex of degree 2 (see [2]). A cubic graph with a HIST is equivalent to having a decomposition into a spanning tree and a family of cycles. Malkevitch [22] investigated HIST in 3-polytopes and found infinitely many 3-connected cubic plane graphs without a HIST (see also examples on Page 81 in [9] or consider the prism over cycles). Albertson, Berman, Hutchinson and Thomassen [2] asked the following question: for each k, is there a cyclically k-edge-connected cubic graph without a HIST? Recently this question was shown by Hoffmann-Oftenhof and Ozeki [17] to be positive, (that is, for each $k \geq 4$, there is a cyclically k-edge-connected cubic graph without a HIST). Douglas [7] show that it is NP-complete to determine whether a graph with maximum degree 3 has a HIST or not. Instead of HIST, Hoffmann-Ostenhof made the following conjecture for all connected cubic graphs.

Conjecture 1.1 (Hoffmann-Ostenhof) *Let G be a connected cubic graph. Then G has a decomposition into a spanning tree, a matching and a family of cycles.*

Conjecture 1.1 first appeared in [16] (see also [6, Problem BCC 22.12] and [20]). There are a few partial results known for Conjecture 1.1. Kostochka [20] noticed that the Petersen graph, the prisms over cycles, and many other graphs have a decomposition desired in Conjecture 1.1. Akbari [1] showed that Conjecture 1.1 is true for Hamiltonian cubic graphs.

In this paper, we prove Conjecture 1.1 for 3-connected cubic plane graphs. The following is our main theorem.

Theorem 1.2 *Let G be a 3-connected cubic plane graph. Then G can be decomposed into a spanning tree, a matching and a family of cycles.*

Note that a 3-connected cubic plane graph does not necessarily have a Hamiltonian cycle (see [26]) and a HIST (see the above). In the next section, we show a slightly stronger result (Theorem 2.1) than Theorem 1.2. The proof of Theorem 2.1 provides a polynomial time algorithm to find the decomposition. As another consequence of Theorem 2.1, we have the following result for cubic graphs on the projective plane. A proof of Theorem 1.3 is given in Section 3.

Theorem 1.3 *Let G be a 3-connected cubic graph embedded in the projective plane. Then G has a decomposition into a spanning tree, a matching and a family of cycles.*

2 Proof of Theorem 1.2

Let G be a connected plane graph. We denote the outer facial walk of G by ∂G. A facial cycle F of G is said to be *second outer* if F and ∂G shares at least one edge and $F \neq \partial G$. For two vertices u and v in F, we denote by $F[u, v]$ the subpath of F from u to v in the clockwise order. An edge e of a connected graph is called a *cut-edge* if $G - e$ has at least two components. An edge set S of size 2 is said to be a 2-*edge-cut* if $G - S$ has two components.

The following theorem is a stronger version of Theorem 1.2, since a 3-connected cubic plane graph trivially satisfies conditions (G1) and (G2).

Theorem 2.1 *Let G be a connected plane graph with maximum degree at most 3. Suppose that*

(G1) *all cut-edges of G are contained in ∂G, and*

(G2) *for all 2-edge-cuts S, both edges in S are contained in ∂G.*

Then G has a decomposition $\{T, M, H\}$ such that T is a spanning tree in G, M is a matching and H is a family of facial cycles of G.

Remark 1: A 2-connected plane graph satisfying condition (G2) is sometimes called a *circuit graph* or an *internally 3-connected graph*. The concept of circuit graphs was used to show, in 3-connected plane graphs, the existence of some structures, such as spanning 3-trees [4], spanning 2-walks [14], and so on.

Remark 2: It follows from conditions (G1) and (G2) that every vertex of degree at most 2 is contained in ∂G.

Remark 3: Let G be a graph with the properties stated in Theorem 2.1, and let v be a vertex of degree 2. Then at least one of the edges incident with v must be contained in T. This implies that no cycle in H can pass through v, and hence every facial cycle in H consists of edges joining two vertices of degree 3. Furthermore, if an edge incident with v is a cut-edge, then it is easy to see that both of the edges incident with v are contained in T.

Proof of Theorem 2.1. Assume that Theorem 2.1 does not hold, and let G be a minimum counterexample in the sense of the number of edges. If G has no cycle, that is, if G is a tree, then $\{G, \emptyset, \emptyset\}$ is a desired decomposition. Therefore, we may assume that G has at least one cycle, and in particular, at least one second outer facial cycle. It follows from condition (G1) that the boundaries of all faces, except ∂G, are cycles. First we show several claims.

Claim 1 *There exists no cut-edge in G.*

Proof. Suppose on the contrary that G has a cut-edge $e = v_1v_2$. Let D_1 and D_2 be the two components of $G - e$ such that $v_i \in D_i$. Note that $|D_i| < |G|$ and D_i is a connected plane graph with maximum degree at most 3 and satisfies conditions (G1) and (G2). Since G is a minimum counterexample, D_i has a decomposition, say $\{T_i, M_i, H_i\}$, as described in Theorem 2.1. Then T is a spanning tree of G, where $T := T_1 \cup T_2 \cup \{v_1v_2\}$. Moreover, $M_1 \cup M_2$ is a matching of G and $H_1 \cup H_2$ is a family of facial cycles of G. So G has the decomposition $\big\{T, M_1 \cup M_2, H_1 \cup H_2\big\}$ as described in Theorem 2.1, a contradiction. $\square$

Figure 1: A second outer facial cycle F in G (left) and the graph $G' := G - E(F)$ (right).

By Claim 1, G is 2-edge-connected, and hence the boundary ∂G is a cycle.

Claim 2 *If F is a second outer facial cycle of G, then F contains a 2-edge-cut of G.*

Proof. Suppose on the contrary that there exists a second outer facial cycle F of G such that F does not contain a 2-edge-cut of G. Let $G' := G - E(F)$.

We first show that G' is connected. If not, let D'_1 and D'_2 be two components of G'. Let $D_1 = D'_1 - V(F)$ and $D_2 = D'_2 - V(F)$. If there is only one edge e_1 in G such that e_1 connects D_1 and F, then e_1 is a cut-edge of G, contradicting Claim 1. Hence by symmetry, there are at least two edges e_i and f_i in G between D_i and F for $i \in \{1, 2\}$. Let u_1 (resp. v_1) be the end vertex of e_1 (resp. f_1) such that u_1 (resp. v_1) is contained in F. By the planarity of G and symmetry, we can choose such edges e_1 and f_1 so that the path $F[u_1, v_1]$ contains all the end vertices of edges connecting D_1 and F, but does not contain any end vertices of edges connecting D_2 and F. Let h_u and h_v be two edges in F such that the edge h_u is incident with u_1, the edge h_v is incident with v_1, and $h_u, h_v \notin E(F[u_1, v_1])$. Since F is a facial cycle, it follows from the choice of h_u and

h_v that $\{h_u, h_v\}$ separates $V(D_1')$ from the other part, which contradicts that F does not contain a 2-edge-cut. The contradiction implies that G' is connected.

Hence G' is a connected plane graph with maximum degree at most 3. Since F is a second outer facial cycle, the outer facial walk $\partial G'$ of G' consists of $\partial G - E(F)$ together with $F' - E(F)$ over all facial cycles F' of G sharing edges with F. Hence it is easy to see that G' satisfies conditions (G1) and (G2) (see Figure 1) . Since G is a minimum counterexample and G' is smaller, G' has a decomposition $\{T', M', H'\}$ with desired properties in Theorem 2.1. Since H' is a family of facial cycles in $G' = G - E(F)$ and any cycles in H' and F are edge-disjoint. Thus, $H' \cup \{F\}$ is a family of edge-disjoint facial cycles in G. Then $\big\{T', M', H' \cup \{F\}\big\}$ is a desired decomposition of G, a contradiction. $\square$

The next claim plays a key role in the remaining part of the proof.

Claim 3 *There exists a matching M_0 in ∂G such that every second outer facial cycle of G contains exactly one edge in M_0.*

Proof. It follows from Claim 2 that each second outer facial cycle F contains a 2-edge-cut of G, say S_F. Let $e_1(F)$ and $e_2(F)$ be the edges in S_F. It follows from the choice and condition (G2) that both $e_1(F)$ and $e_2(F)$ are contained in $F \cap \partial G$. Let $\mathcal{P}$ be the cyclic order of the elements in

$$\big\{e_i(F) : \ F \text{ is a second outer facial cycle of } G \text{ and } i = 1, 2\big\}$$

along ∂G. We first show the following claim:

($*$) For any two second outer facial cycles F and F' of G, any of the two suborders of $\mathcal{P}$ between $e_1(F)$ and $e_2(F)$ contains both $e_1(F')$ and $e_2(F')$ or neither.

In fact, since $S_F = \big\{e_1(F), e_2(F)\big\}$ is a 2-edge-cut of G, F' is contained in one of the components of $G - S_F$, and hence so are $e_1(F')$ and $e_2(F')$. This directly shows the claim ($*$).

Then along the cyclic order $\mathcal{P}$, we choose every other edges and let M_0 be the set of chosen edges. We prove that the set M_0 satisfies the conditions desired in Claim 3. Since no two edges in M_0 are consecutive in $\mathcal{P}$, we see that M_0 is actually a matching in ∂G. Let F be a second outer facial cycle of G and suppose that $e_1(F)$ is not contained in M_0. It follows from the claim ($*$) that the suborder of $\mathcal{P}$ between $e_1(F)$ and $e_2(F)$ contains an even number of elements. Since we chose every other edges along $\mathcal{P}$, we can easily see that $e_2(F)$ is contained in M_0. The same argument implies that if $e_1(F)$ is contained in M_0, then $e_2(F)$ is not. Therefore, we see that every second outer facial cycle of G contains exactly one edge in M_0, and this competes the proof of Claim 3. $\square$.

Then let $G' = G - M_0$. Note that G' is a plane graph with maximum degree at most 3. Let $e(F)$ be the edge in $M_0 \cap F$, where F is a second outer facial cycle of G. Since G is 2-edge-connected, $G - e(F)$ is still connected. Furthermore, each edge in $F \cap \partial G - \{e(F)\}$ is now a cut-edge of $G - e(F)$, and all second outer facial cycles F' with $F' \neq F$ are still second outer facial in $G - e(F)$. Therefore, for any second outer facial cycles F' with $F' \neq F$, deleting the edges $e(F')$ in $M_0 \cap F'$ from $G - e(F)$ still keeps the connectedness and each edge in $F' \cap \partial G - \{e(F')\}$ is a cut-edge of $G - \{e(F), e(F')\}$. Performing this procedure inductively, we finally obtain that $G' = G - M_0$ is also connected and each edge in $\partial G - M_0$ is a cut-edge of G'. (Recall that since G is 2-edge-connected, every edge in ∂G is contained in F' for some second outer facial cycle F' of G.) In particular, all the end vertices of edges in M_0 have degree at most 2 in G' and are incident with a cut-edge of G'.

On the other hand, since all edges in M_0 are contained in ∂G, the obtained graph G' satisfies conditions (G1) and (G2). Hence by the minimality of G, G' has a decomposition $\{T', M', H'\}$ with desired properties. It follows from the properties of M_0 in the last of the previous paragraph (see also Remark 3 just after Theorem 1.2) that no edge in M' is incident with an end vertex in M_0, and hence $M_0 \cup M'$ is a matching. Therefore, we have the decomposition $\{T', M' \cup M_0, H'\}$ of G with desired properties. So the proof of Theorem 2.1 is complete. ■

Remark 4. In the proof of Theorem 2.1, we either partition G at a cut-edge (Claim 1), or remove a second outer facial cycle from G to H (Claim 2), or remove a matching from the boundary ∂G to M. It costs at most $O(n)$ steps to find a removable facial cycle or a desired matching M from ∂G. So the total steps to find the decomposition is $O(n^2)$. Hence the proof of Theorem 2.1 provides an $O(n^2)$-time algorithm to find a decomposition of a 3-connected cubic plane graph to a spanning tree, a matching and a family of facial cycles.

3 Proof of Theorem 1.3

In this section, we are going to prove Theorem 1.3 by using the following lemma. For undefined terminology, we refer the reader to [23].

Lemma 3.1 (Fiedler, Huneke, Richter and Robertson [11]) *Each 3-connected non-planar graph on the projective plane Σ has a spanning 2-connected subgraph bounded by a disc on Σ.*

Proof of Theorem 1.3. Let G be a 3-connected cubic graph on the projective plane Σ. Assume that G is not planar; otherwise, it has a desired decomposition by Theorem 1.2. Then G has a non-contractible cycle. We first claim that G has a non-contractible cycle C such that $G - E(C)$ is connected.

It follows from Lemma 3.1 that G has a spanning 2-connected subgraph H bounded by a disc. Assume that H is maximal, that is, all edges of $G - E(H)$ pass through the crosscap of Σ. Since G is 3-connected, every 2-cut is contained on the boundary ∂H of H that is a cycle. Since G is not planar, there exists an edge e in $G - E(H)$. Let x and y be two end vertices of e. Let $\partial H[x, y]$ be the path from x to y in clockwise direction along ∂H. Then every edge different from e through the crosscap of Σ has one end vertex in $\partial H[x, y]$ and the other in $\partial H[y, x]$. Since H is maximal and e passes through the crosscap, both $e \cup \partial H[x, y]$ and $e \cup \partial H[y, x]$ are non-contractible cycles of G. Let $C' = e \cup \partial H[x, y]$.

If $G - E(C')$ is connected, then let $C = C'$ and the claim holds. So assume that $G - E(C')$ is not connected, and let $D_0, D_1, \ldots, D_r$ be all components of $G - E(C')$ such that D_0 contains the path $\partial H[y, x]$ (see Figure 2). For each i with $1 \leq i \leq r$, $\partial H[x, y]$ contains at least three vertices of D_i because G is 3-connected. Choose u_i and v_i from $V(D_i) \cap V\big(\partial H[x, y]\big)$ such that $V(D_i) \cap V\big(\partial H[x, y]\big) \subseteq \partial H[u_i, v_i] \subset \partial H[x, y]$.

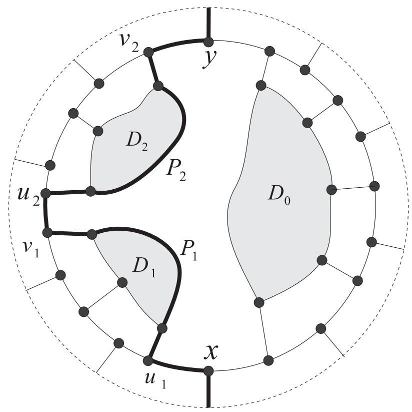

Figure 2: The cycle C is illustrated in bold lines.

Let P_i be a path of D_i connecting u_i and v_i such that the disc of Σ bounded by the cycle $P_i \cup \partial H[u_i, v_i]$ contains as many faces of G as possible (see Figure 2). Then all vertices of D_i belong to the disc bounded by the cycle $P_i \cup \partial H[u_i, v_i]$. Let $Q_i := D_i \cup \partial H[u_i, v_i]$. Then Q_i is plane graph and has boundary $P_i \cup \partial H[u_i, v_i]$.

If $D_i - E(P_i)$ has two components, then Q_i contains a face F of G that intersects P_i twice. Then $F \cap P_i$ contains a 2-cut of G, contradicting that G is 3-connected. Hence $D_i - E(P_i)$ is connected. So is $Q_i - E(P_i)$. Note that $\{u_i, v_i\}$ is a 2-cut of H separating vertices of $Q_i \backslash \{u_i, v_i\}$ from vertices of D_0. Since G is 3-connected, then $\{u_i, v_i\}$ is not a 2-cut of G. Hence $\partial H[u_i, v_i]$ contains a vertex joined to a vertex of $\partial H[y, x]$ by an edge through the crosscap of Σ. So the connected graph $Q_i - E(P_i)$ is connected to D_0 by an edge through the crosscap.

Let $D_0' := D_0 \cup \big(\bigcup_{i=1}^{r} \big(Q_i - E(P_i)\big)\big)$. The above arguments imply that D_0' is connected. Let

$$C := \Big(C' - \bigcup_{i=1}^{r} \partial H[u_i, v_i]\Big) \cup \bigcup_{i=1}^{r} P_i.$$

Since $P_i \cup \partial H[u_i, v_i]$ bounds a disc on Σ for $1 \leq i \leq r$, P_i is homotopic to $\partial H[u_i, v_i]$. Therefore, $(C' - \partial H[u_i, v_i]) \cup P_i$ is homotopic to C'. Note that C' is non-contractible. It follows that $(C' - \partial H[u_i, v_i]) \cup P_i$ is non-contractible. Similarly, C is homotopic to C' and hence is non-contractible. Moreover, since $G - E(C) = D'_0$, the graph $G - E(C)$ is connected. Therefore, the claim holds.

Let $G' := G - E(C)$. Since Σ is the projective plane, Σ contains a closed curve ℓ homotopic to C such that $\Sigma - \ell$ is an open disc (cutting the surface along ℓ). Clearly, G' is embedded in $\Sigma - \ell$, which is a plane embedding. Since G is 3-connected, G' satisfies conditions (G1) and (G2) in Theorem 2.1. Hence G' has a desired decomposition $\{T, M, H\}$ as in Theorem 1.2. Then $\{T, M, H \cup C\}$ is a desired decomposition of G. This completes the proof. ■

Acknowledgments

The authors are grateful to Professor Ota and the referees for their suggestion to shorten the proof of Theorem 1.2.

References

[1] S. Akbari, personal communication, 2010.

[2] M.O. Albertson, D.M. Berman, J.P. Hutchinson and C. Thomassen, Graphs with homeomorphically irreducible spanning trees, *J. Graph Theory* **14** (1990) 247–258.

[3] J. Balogh, M. Kochol, A. Pluhar, and X. Yu, Covering planar graphs with forests, *J. Combin. Theory Ser. B* **94** (2005) 147–158.

[4] D.W. Barnette, Trees in polyhedral graphs, *Canad. J. Math.* **18** (1966) 731–736.

[5] O.V. Borodin, A.V. Kostochka, N.N. Sheikh, and G. Yu. Decomposing a planar graph with girth 9 into a forest and a matching, *European J. Combin.* **29** (2008) 1235–1241.

[6] P.J. Cameron, Research problems from the BCC22, *Discrete Math.* **311** (2011) 1074–1083.

[7] R.J. Douglas, NP-completeness and degree restricted spanning trees, *Discrete Math.* **105** (1992) 41-47.

[8] D. Gonçalves, Covering planar graphs with forests, one having bounded maximum degree, *J. Combin. Theory Ser. B* **99** (2009) 314–322.

[9] M.N. Ellingham, Spanning paths, cycles and walks for graphs on surfaces, *Congr. Numer.* **115** (1996) 55–90.

[10] G. Fan and A. Raspaud, Fulkerson's conjecture and circuit covers, *J. Comb. Theory Ser. B* **61** (1994) 133–138.

[11] J.R. Fiedler, J.P. Huneke, R.B. Richter and N. Robertson, Computing the orientable genus of projective graphs, *J. Graph Theory* **20** (1995) 297–208.

[12] J.L. Fouquet and J.M. Vanherpe, On compatible normal odd partitions in cubic graphs, *J. Graph Theory* **72** (2013) 440–461.

[13] J.L. Fouquet and J.M. Vanherpe, On normal odd partitions in cubic graphs, *Discuss. Math. Graph Theory* **29** (2009) 293–312.

[14] Z. Gao and R.B. Richter, 2-walks in circuit graphs, *J. Combin. Theory Ser. B* **62** (1994) 259–267.

[15] W. He, X. Hou, K.-W. Lih, J. Shao, W. Wang, and X. Zhu, Edgede decompositions of planar graphs and their game coloring numbers, *J. Graph Theory* **41** (2002) 307—-317.

[16] A. Hoffmann-Ostenhof, Nowhere-zero flows and structures in cubic graphs, Ph.D. dissertation, Universität Wien, 2011.

[17] A. Hoffmann-Ostenhof and K. Ozeki, On HISTs in cubic graphs, submitted. `http://arxiv.org/abs/1507.07689`

[18] D. J. Kleitman, Partitioning the edges of a girth 6 planar graph into those of a forest and those of a set of disjoint paths and cycles, Manuscript, 2006.

[19] S. Kim, A.V. Kostochka, D. B. West, H. Wu and X. Zhu, Decomposition of sparse graphs into forests and a graph with bounded degree, *J. Graph Theory* **74** (2013) 369–391.

[20] A. Kostochka, Spanning trees in 3-regular graphs, REGS in Combinatorics, University of Illinois at Urbana-Champaign, (2009). `http://www.math.uiuc.edu/~west/regs/span3reg.html`

[21] M. Las Vergnas, A note on matchings in graphs, *Cahiers Centre Etudes Rech. Oper.* **17** (1975) 257–260.

[22] J. Malkevitch, Spanning trees in polytopal graphs, Ann. New York Acad. Sci. **319** (1979) 362-367.

[23] B. Mohar and C. Thomassen, Graphs on Surfaces, Johns Hopkins Univ. Press, Baltimore, 2001.

[24] M. Montassier, P. Ossona de Mendez, A. Raspaud and X. Zhu, Decomposing a graph into forests, *J. Combin. Theory Ser. B* **102** (2012) 38–52.

[25] D. Sumner, 1-factors and anti-factor sets, *J. London Math. Soc.* **13** (1976) 351–359.

[26] W.T. Tutte, On Hamiltonian circuits, *J. London Math. Soc.* **2** (1946) 98–101.

[27] V.G. Vizing, On an estimate of the chromatic class of a p-graph, *Metody Diskret. Analiz.* **3** (1964) 25–30. (in Russian)

[28] Y. Wang and Q. Zhang, Decomposing a planar graph with girth 8 into a forest and a matching, *Discrete Math.* **311** (2011) 844–849.