Plane triangulations without a spanning Halin subgraph: Counterexamples to the Lovász-Plummer conjecture on Halin graphs

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This paper is dedicated to Prof. Rudolf Halin who died on the 14th November, 2014.

Abstract

A Halin graph is a simple plane graph consisting of a tree without degree 2 vertices and a cycle induced by the leaves of the tree. In 1975, Lovász and Plummer conjectured that every 4-connected plane triangulation has a spanning Halin subgraph. In this paper, we construct an infinite family of counterexamples to the conjecture.

Keywords: Halin graph, spanning Halin subgraph, triangulation.

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1 Introduction

A graph is called planar if it can be embedded in the plane without edge-crossing, and such an embedding is called a plane graph. A Halin graph is a simple plane graph $H = T \cup C$ consisting of a tree $T$ with no vertices of degree 2 and a cycle $C$ induced by the leaves of the tree $T$. The family of Halin graphs is a natural generalization of wheels, where $T$ is a star. In his study of edge-minimum 3-connected plane graphs, Halin [6] constructed this family of graphs, named Halin graphs by Lovász and Plummer [7]. Inspired by Tutte’s well-known result that every 4-connected plane graph contains a Hamilton cycle, Lovász and Plummer [7] in 1975 conjectured that every 4-connected plane triangulation has a spanning Halin subgraph, where a plane triangulation is a plane graph such that all its faces are triangles.

It was proved that every Halin graph is Hamiltonian [3] (see also [2, 8]), and hence the existence of a Hamilton cycle is a necessary condition for a graph to have a spanning Halin subgraph. Also, by the definition of Halin graphs, a spanning tree with no vertices of degree 2 is needed for the existence of spanning Halin subgraphs. Albertson, Berman, Hutchinson and Thomassen [1] proved that every plane triangulation has such a spanning tree (see also [4, 5]). This result and Tutte’s well-known result give some hope for the Lovász-Plummer conjecture. However, the main result of this paper disproves the conjecture based upon a certain family $\mathcal{S}$ of graphs constructed below:

Construction of a family $\mathcal{S}$: For any three integers $n_1, n_2, n_3 \geq 5$, we construct the graph $G_{n_1,n_2,n_3}$ as follows. Let $Q$ be a plane embedding of $K_{2,4}$ with bipartite partition $V = \{v_1, v_2\}$ and $W = \{w_1, w_2, w_3, w_4\}$. In its embedding, $Q$ has exactly four faces $F_i = v_1w_1v_2w_{i+1}v_1$ for $i = 1, 2, 3, 4$, where $w_5$ is regarded as $w_1$. In each face $F_i$ for $i = 1, 2, 3$, we add a path $P_i$ of length $n_i$ connecting $v_1$ and $v_2$, and add all possible edges with one endvertex in $\{v_1, w_{i+1}\}$ and the other endvertex in $V(P_i)$. Denote by $G_{n_1,n_2,n_3}$ the resulting graph. Note that all faces of $G_{n_1,n_2,n_3}$ other than $F_1$ are triangular faces (see Figure 1, where solid lines present edges of $Q$). Let $G^*_{n_1,n_2,n_3}$ be a graph obtained from $G_{n_1,n_2,n_3}$ by identifying $w_1$ and $w_4$ (and hence we identify edges $w_1v_j$ and $w_4v_j$ for $j = 1, 2$). Let $\mathcal{S} = \{G^*_{n_1,n_2,n_3} \mid n_1, n_2, n_3 \geq 5\}$. Also, let $Q^*$ be $K_{2,3}$ of $G^*_{n_1,n_2,n_3}$ corresponding to $Q$ in $G_{n_1,n_2,n_3}$.

Clearly, each graph $G^*_{n_1,n_2,n_3} \in \mathcal{S}$ has $n = n_1 + n_2 + n_3 + 2$ vertices, $3(n_1 + n_2 + n_3)$ edges, and $2(n_1 + n_2 + n_3)$ triangular faces. In particular, the smallest graph in $\mathcal{S}$ is $G^*_{5,5,5}$, which has 17 vertices. The following is the main result of this paper.

**Theorem 1** For any triple of integers $n_i \geq 5$ with $i = 1, 2, 3$, the graph $G^*_{n_1,n_2,n_3} \in \mathcal{S}$ is a 4-connected plane triangulation without a spanning Halin subgraph.

The proof of Theorem 1 will be given in Section 3 based on some necessary conditions concerning graphs with a spanning Halin subgraph which are given in Section 2.
2 Preliminaries

In this section, we present a few properties of graphs having a spanning Halin subgraph, which will be used to prove certain graphs do not contain such a subgraph. We use $G$ for both a graph $G$ and the vertex set $V(G)$ if there is no confusion arises. Let $G$ be a plane graph having a spanning Halin subgraph $H = T \cup C$. By Jordan's closed curve theorem, the cycle $C$ separates the plane into two regions: the interior region $\text{Int}(C)$ and the exterior region $\text{Ext}(C)$. Since $T$ is a spanning tree of $G$, $T - V(C)$ is connected. Thus all vertices of $G - V(C)$ lie in one region. Throughout this paper, we always assume that $\text{Int}(C)$ contains the vertices of $G - V(C)$. Hence, the following statements hold.

**Lemma 2** If a plane graph $G$ contains a spanning Halin subgraph $H = T \cup C$, then $\text{Ext}(C)$ contains no vertices of $T - V(C)$ and each vertex on $C$ is adjacent to exactly one vertex in $T - V(C)$, which lies in $\text{Int}(C)$.

**Lemma 3** Let $G$ be a graph containing a spanning Halin subgraph $H = T \cup C$ and let $X$ be a vertex cut of $G$. If $D$ is a component of $G - X$ with $|D| \geq |X| - 1$, then $C \cap D \neq \emptyset$. Consequently, if every component of $G - X$ contains at least $|X| - 1$ vertices, then $|C \cap X| \geq 2$.

**Proof.** Suppose to the contrary that $D \cap C = \emptyset$, that is, $d_T(v) \geq 3$ for every $v \in D$. For $V(D) \subset V(T)$, the neighborhood $N_T(D)$ is the subset of $V(T) - V(D)$ each of whose vertices is adjacent to at least one member of $D$ in $T$. Since $T$ is a tree, we have

$$|N_T(D)| \geq \sum_{v \in D} d_T(v) - 2|E(T \cap D)| \geq 3|D| - 2(|D| - 1) \geq |D| + 2.$$ 

Hence, $|D| + 1 \geq |X| \geq |N_T(D)| \geq |D| + 2$, a contradiction. \qed

A vertex cut $X$ of a graph is called *minimal* if every proper subset of $X$ is no longer a vertex cut of the graph. The following is a well-known fact on a minimal vertex cut of plain triangulations (and hence we use it without a proof).
Fact 4 In a plane triangulation, every minimal vertex cut induces a cycle.

3 Proof of Theorem 1

First, we show that every graph $G$ in $\mathcal{G}$ is a 4-connected triangulation. By the construction, $G$ is a plane triangulation. So, it suffices to show the following claim.

Claim 3.1 Every graph $G \in \mathcal{G}$ is 4-connected.

Proof. Let $G \in \mathcal{G}$. Following the construction of $G$, every 3-cycle of $G$ contains two consecutive vertices of $P_i$ for some $1 \leq i \leq 3$, which in turn shows that no 3-cycle forms a vertex cut. So, $G$ is 4-connected by Fact 4. □

Now we show that each member in $\mathcal{G}$ contains no spanning Halin subgraph. Let $G := G^*_{n_1,n_2,n_3} \in \mathcal{G}$ with $n_i \geq 5$ for $1 \leq i \leq 3$. Assume, to the contrary, that $G$ has a spanning Halin subgraph $H = T \cup C$, where $V(C)$ consists of all leaves of $T$. We recall that $Q^*$ is a planar embedding of $K_{2,3}$ and $P_i$ is a $v_1$-$v_2$ path inside the quadrangle $v_1w_iw_{i+1}v_1$ for $i = 1, 2, 3$. For each $i = 1, 2, 3$, let $Q_i := P_i - \{v_1, v_2\}$ and let $X_i = V(F_i) = \{v_1, v_2, w_i, w_{i+1}\}$.

Claim 3.2 For each $i = 1, 2, 3$, $|C \cap X_i| \geq 2$ and $C \cap Q_i \neq \emptyset$.

Proof. From the construction of $G^*_{n_1,n_2,n_3}$, $G - X_i$ contains exactly two components with vertex sets $Q_i$ and $Q_{i-1} \cup Q_{i+1} \cup \{w_{i-1}\}$, respectively. Consequently, both components of $G - X_i$ contain at least $|Q_i| \geq 4 > |X_i| - 1$ vertices. By Lemma 3, $C \cap Q_i \neq \emptyset$ and $|C \cap X_i| \geq 2$. □

Claim 3.3 $|\{w_1, w_2, w_3\} \cap C| \geq 2$.

Proof. Suppose to the contrary that $|\{w_1, w_2, w_3\} \cap C| < 2$. Then we may assume that $w_i \notin V(C)$ for $i = 1, 2$. By Claim 3.2, $Q_1 \cap C \neq \emptyset$. Since $C$ is a cycle and $P_i$ is an induced path in $G - \{w_1, w_2\}, C \supseteq P_i$. Thus, $w_1$ and $w_2$ lie in two different regions of the plane divided by the Jordan curve $C$, which contradicts the fact that $w_1, w_2 \in \text{Int}(C)$. □

Claim 3.4 Let $xy \in E(Q_i)$ for some $i = 1, 2, 3$. If $\{x, y\} \subseteq V(G) - V(C)$, then $xy \in E(T)$, and either $\{xw_i, yw_{i+1}\} \subseteq E(T)$ or $\{xw_{i+1}, yw_i\} \subseteq E(T)$.

Proof. Since $x, y \notin V(C)$, both $d_T(x) \geq 3$ and $d_T(y) \geq 3$ hold. Assume without loss of generality that $wxyz$ is a subpath of $P_i$, i.e., we assume that, along $P_i$, $w$ is the predecessor
of $x$, $x$ is the predecessor of $y$ and $z$ is the successor of $y$. Then, we have

$$N_G(x) = \{w, w_{i+1}, w, y\} \text{ and } N_G(y) = \{w, w_{i+1}, x, z\}.$$ 

If $xy \notin E(T)$, then $xw, xw_{i+1}, yw, yw_{i+1} \in E(T)$, which in turn shows that $T$ contains a 4-cycle $xw, yw_{i+1}, x$, a contradiction. If both $xw, xw_{i+1} \in E(T)$, then $yw, yw_{i+1} \notin E(T)$, which in turn gives $d_T(y) \leq 2$, a contradiction. So, either $\{xw, yw_{i+1}\} \subseteq E(T)$ or $\{xw_{i+1}, yw\} \subseteq E(T)$. □

The following is an immediate consequence of Claim 3.4.

**Claim 3.5** For each $i = 1, 2, 3$, among any three consecutive vertices along path $Q_i$, at least one of them is on $C$.

We consider two cases to finish the proof.

**Case 1.** Suppose $\{v_1, v_2\} \cap V(C) = \emptyset$, i.e., $d_T(v_i) \geq 3$ for $i = 1, 2$.
For each $1 \leq i \leq 3$, since $|C \cap X_i| \geq 2$, $C \cap X_i = \{w, w_{i+1}\}$. Consequently, $w_i \in V(C)$ for each $i = 1, 2, 3$. By Claim 3.2, $C \cap Q_i \neq \emptyset$ for each $i = 1, 2, 3$. So $V(C) = \{w_1, w_2, w_3\} \cup (\bigcup_{i=1}^3 V(C \cap Q_i))$ separates $v_1$ and $v_2$, which contradicts the fact that $T - V(C)$ is connected.

**Case 2.** Suppose $\{v_1, v_2\} \cap V(C) \neq \emptyset$.
Assume without loss of generality that $v_1 \in V(C)$. By Claim 3.3 and symmetry, we may assume that both $w_1, w_2 \in V(C)$. Let $P_1 = v_1u_1u_2 \ldots u_{n_1-1}v_2$. So, $v_1, w_1$ and $w_2$ are in $C$. Since $N_G(u_1) = \{w_1, w_2, v_1, u_2\}$ and $u_1$ must be adjacent to a vertex in $T - V(C)$, we have $u_2 \notin V(C)$. Since $Q_1$ is a component of $G - \{v_1, w_1, v_2, w_2\}$ and $T - V(C)$ is connected, $u_2u_3 \ldots u_{n_1}v_2$ must be a subtree of $T - V(C)$. On the other hand, by Claim 3.5 and the fact that $n_1 \geq 5$, at least one of $u_2, u_3$ and $u_4$ must be in $C$, a contradiction.

We note that Case 2 is the only place in the proof where we use the assumption $n_i \geq 5$ for each $i = 1, 2, 3$. In the proof of Claim 3.2 we only require $n_i \geq 4$. □

**References**


