Plane triangulations without a spanning Halin subgraph : Counterexamples to the Lovász-Plummer conjecture on Halin graphs

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This paper is dedicated to Prof. Rudolf Halin who died on the 14th November, 2014.

Abstract

A *Halin graph* is a simple plane graph consisting of a tree without degree 2 vertices and a cycle induced by the leaves of the tree. In 1975, Lovász and Plummer conjectured that *every 4-connected plane triangulation has a spanning Halin subgraph.* In this paper, we construct an infinite family of counterexamples to the conjecture.

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1 Introduction

A graph is called *planar* if it can be embedded in the plane without edge-crossing, and such an embedding is called a *plane graph*. A *Halin graph* is a simple plane graph $H = T \cup C$ consisting of a tree T with no vertices of degree 2 and a cycle C induced by the leaves of the tree T. The family of Halin graphs is a natural generalization of wheels, where T is a star. In his study of *edge-minimum* 3*-connected plane graphs*, Halin [6] constructed this family of graphs, named *Halin graphs* by Lovász and Plummer [7]. Inspired by Tutte's well-known result that *every* 4*-connected plane graph contains a Hamilton cycle*, Lovász and Plummer [7] in 1975 conjectured that *every* 4*-connected plane triangulation has a spanning Halin subgraph*, where a plane triangulation is a plane graph such that all its faces are triangles.

It was proved that every Halin graph is Hamiltonian [3] (see also [2, 8]), and hence the existence of a Hamilton cycle is a necessary condition for a graph to have a spanning Halin subgraph. Also, by the definition of Halin graphs, a spanning tree with no vertices of degree 2 is needed for the existence of spanning Halin subgraphs. Albertson, Berman, Hutchinson and Thomassen [1] proved that every plane triangulation has such a spanning tree (see also [4, 5]). This result and Tutte's well-known result give some hope for the Lovász-Plummer conjecture. However, the main result of this paper disproves the conjecture based upon a certain family $\mathcal{G}$ of graphs constructed below:

Construction of a family $\mathcal{G}$: For any three integers $n_1, n_2, n_3 \geq 5$, we construct the graph G_{n_1,n_2,n_3} as follows. Let Q be a plane embedding of $K_{2,4}$ with bipartite partition $V = \{v_1, v_2\}$ and $W = \{w_1, w_2, w_3, w_4\}$. In its embedding, Q has exactly four faces $F_i = v_1 w_i v_2 w_{i+1} v_1$ for $i = 1, 2, 3, 4$, where w_5 is regard as w_1. In each face F_i for $i = 1, 2, 3$, we add a path P_i of length n_i connecting v_1 and v_2, and add all possible edges with one endvertex in $\{w_i, w_{i+1}\}$ and the other endvertex in $V(P_i)$. Denote by G_{n_1,n_2,n_3} the resulting graph. Note that all faces of G_{n_1,n_2,n_3} other than F_4 are triangular faces (see Figure 1, where solid lines present edges of Q). Let $G^*_{n_1,n_2,n_3}$ be a graph obtained from G_{n_1,n_2,n_3} by identifying w_1 and w_4 (and hence we identify edges $w_1 v_j$ and $w_4 v_j$ for $j = 1, 2$). Let $\mathcal{G} = \{G^*_{n_1,n_2,n_3} \mid n_1, n_2, n_3 \geq 5\}$. Also, let Q^* be $K_{2,3}$ of $G^*_{n_1,n_2,n_3}$ corresponding to Q in G_{n_1,n_2,n_3}.

Clearly, each graph $G^*_{n_1,n_2,n_3} \in \mathcal{G}$ has $n = n_1 + n_2 + n_3 + 2$ vertices, $3(n_1 + n_2 + n_3)$ edges, and $2(n_1 + n_2 + n_3)$ triangular faces. In particular, the smallest graph in $\mathcal{G}$ is $G^*_{5,5,5}$, which has 17 vertices. The following is the main result of this paper.

Theorem 1 *For any triple of integers $n_i \geq 5$ with $i = 1, 2, 3$, the graph $G^*_{n_1,n_2,n_3} \in \mathcal{G}$ is a* 4*-connected plane triangulation without a spanning Halin subgraph.*

The proof of Theorem 1 will be given in Section 3 based on some necessary conditions concerning graphs with a spanning Halin subgraph which are given in Section 2.

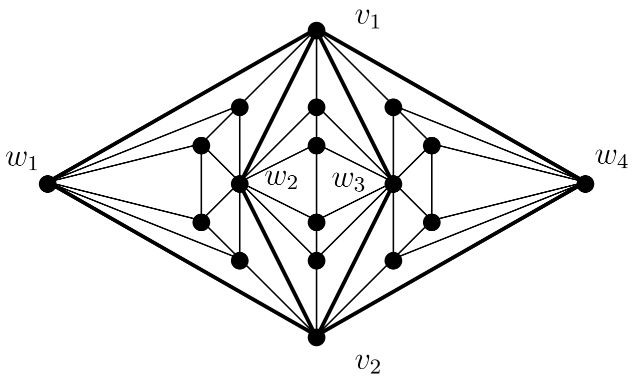

Figure 1: $G_{5,5,5}$

2 Preliminaries

In this section, we present a few properties of graphs having a spanning Halin subgraph, which will be used to prove certain graphs do not contain such a subgraph. We use G for both a graph G and the vertex set $V(G)$ if there is no confusion arises. Let G be a plane graph having a spanning Halin subgraph $H = T \cup C$. By Jordan's closed curve theorem, the cycle C separates the plane into two regions: the *interior region* $Int(C)$ and the *exterior region* $Ext(C)$. Since T is a spanning tree of G, $T - V(C)$ is connected. Thus all vertices of $G - V(C)$ lie in one region. Throughout this paper, we always assume that $Int(C)$ contains the vertices of $G - V(C)$. Hence, the following statements hold.

Lemma 2 *If a plane graph G contains a spanning Halin subgraph $H = T \cup C$, then $Ext(C)$ contains no vertices of $T - V(C)$ and each vertex on C is adjacent to exactly one vertex in $T - V(C)$, which lies in $Int(C)$.*

Lemma 3 *Let G be a graph containing a spanning Halin subgraph $H = T \cup C$ and let X be a vertex cut of G. If D is a component of $G - X$ with $|D| \geq |X| - 1$, then $C \cap D \neq \emptyset$. Consequently, if every component of $G - X$ contains at least $|X| - 1$ vertices, then $|C \cap X| \geq 2$.*

Proof. Suppose to the contrary that $D \cap C = \emptyset$, that is, $d_T(v) \geq 3$ for every $v \in D$. For $V(D) \subset V(T)$, the *neighborhood* $N_T(D)$ is the subset of $V(T) - V(D)$ each of whose vertices is adjacent to at least one member of D in T. Since T is a tree, we have

$$|N_T(D)| \geq \sum_{v \in D} d_T(v) - 2|E(T \cap D)| \geq 3|D| - 2(|D| - 1) \geq |D| + 2.$$

Hence, $|D| + 1 \geq |X| \geq |N_T(D)| \geq |D| + 2$, a contradiction. $\square$

A vertex cut X of a graph is called *minimal* if every proper subset of X is no longer a vertex cut of the graph. The following is a well-known fact on a minimal vertex cut of plain triangulations (and hence we use it without a proof).

Fact 4 *In a plane triangulation, every minimal vertex cut induces a cycle.*

3 Proof of Theorem 1

First, we show that every graph G in $\mathcal{G}$ is a 4-connected triangulation. By the construction, G is a plane triangulation. So, it suffices to show the following claim.

Claim 3.1 *Every graph $G \in \mathcal{G}$ is 4-connected.*

Proof. Let $G \in \mathcal{G}$. Following the construction of G, every 3-cycle of G contains two consecutive vertices of P_i for some $1 \leq i \leq 3$, which in turn shows that no 3-cycle forms a vertex cut. So, G is 4-connected by Fact 4. $\square$

Now we show that each member in $\mathcal{G}$ contains no spanning Halin subgraph. Let $G := G^*_{n_1,n_2,n_3} \in \mathcal{G}$ with $n_i \geq 5$ for $1 \leq i \leq 3$. Assume, to the contrary, that G has a spanning Halin subgraph $H = T \cup C$, where $V(C)$ consists of all leaves of T. We recall that Q^* is a planar embedding of $K_{2,3}$ and P_i is a v_1-v_2 path inside the quadrangle $v_1 w_i v_2 w_{i+1} v_1$ for $i = 1, 2, 3$. For each $i = 1, 2, 3$, let $Q_i := P_i - \{v_1, v_2\}$ and let $X_i = V(F_i) = \{v_1, v_2, w_i, w_{i+1}\}$.

Claim 3.2 *For each $i = 1, 2, 3$, $|C \cap X_i| \geq 2$ and $C \cap Q_i \neq \emptyset$.*

Proof. From the construction of $G^*_{n_1,n_2,n_3}$, $G - X_i$ contains exactly two components with vertex sets Q_i and $Q_{i-1} \cup Q_{i+1} \cup \{w_{i-1}\}$, respectively. Consequently, both components of $G - X_1$ contain at least $|Q_i| \geq 4 > |X_i| - 1$ vertices. By Lemma 3, $C \cap Q_i \neq \emptyset$ and $|C \cap X_i| \geq 2$. $\square$

Claim 3.3 $|\{w_1, w_2, w_3\} \cap C| \geq 2$.

Proof. Suppose to the contrary that $|\{w_1, w_2, w_3\} \cap C| < 2$. Then we may assume that $w_i \notin V(C)$ for $i = 1, 2$. By Claim 3.2, $Q_1 \cap C \neq \emptyset$. Since C is a cycle and P_1 is an induced path in $G - \{w_1, w_2\}$, $C \supseteq P_1$. Thus, w_1 and w_2 lie in two different regions of the plane divided by the Jordan curve C, which contradicts the fact that $w_1, w_2 \in Int(C)$. $\square$

Claim 3.4 *Let $xy \in E(Q_i)$ for some $i = 1, 2, 3$. If $\{x, y\} \subseteq V(G) - V(C)$, then $xy \in E(T)$, and either $\{xw_i, yw_{i+1}\} \subseteq E(T)$ or $\{xw_{i+1}, yw_i\} \subseteq E(T)$.*

Proof. Since $x, y \notin V(C)$, both $d_T(x) \geq 3$ and $d_T(y) \geq 3$ hold. Assume without loss of generality that $wxyz$ is a subpath of P_i, i.e., we assume that, along P_i, w is the predecessor of x, x is the predecessor of y and z is the successor of y. Then, we have

$$N_G(x) = \{w_i, w_{i+1}, w, y\} \quad \text{and} \quad N_G(y) = \{w_i, w_{i+1}, x, z\}.$$

If $xy \notin E(T)$, then $xw_i, xw_{i+1}, yw_i, yw_{i+1} \in E(T)$, which in turn shows that T contains a 4-cycle $xw_iyw_{i+1}x$, a contradiction. If both $xw_i, xw_{i+1} \in E(T)$, then $yw_i, yw_{i+1} \notin E(T)$, which in turn gives $d_T(y) \leq 2$, a contradiction. So, either $\{xw_i, yw_{i+1}\} \subseteq E(T)$ or $\{xw_{i+1}, yw_i\} \subseteq E(T)$. □

The following is an immediate consequence of Claim 3.4.

Claim 3.5 *For each $i = 1, 2, 3$, among any three consecutive vertices along path Q_i, at least one of them is on C.*

We consider two cases to finish the proof.

Case 1. Suppose $\{v_1, v_2\} \cap V(C) = \emptyset$, i.e., $d_T(v_i) \geq 3$ for $i = 1, 2$.
For each $1 \leq i \leq 3$, since $|C \cap X_i| \geq 2$, $C \cap X_i = \{w_i, w_{i+1}\}$. Consequently, $w_i \in V(C)$ for each $i = 1, 2, 3$. By Claim 3.2, $C \cap Q_i \neq \emptyset$ for each $i = 1, 2, 3$. So $V(C) = \{w_1, w_2, w_3\} \cup (\bigcup_{i=1}^{3} V(C \cap Q_i))$ separates v_1 and v_2, which contradicts the fact that $T - V(C)$ is connected.

Case 2. Suppose $\{v_1, v_2\} \cap V(C) \neq \emptyset$.
Assume without loss of generality that $v_1 \in V(C)$. By Claim 3.3 and symmetry, we may assume that both $w_1, w_2 \in V(C)$. Let $P_1 = v_1u_1u_2 \ldots u_{n_1-1}v_2$. So, v_1, w_1 and w_2 are in C. Since $N_G(u_1) = \{w_1, w_2, v_1, u_2\}$ and u_1 must be adjacent to a vertex in $T - V(C)$, we have $u_2 \notin V(C)$. Since Q_1 is a component of $G - \{v_1, w_1, w_2, v_2\}$ and $T - V(C)$ is connected, $u_2u_3 \ldots u_mv_2$ must be a subtree of $T - V(C)$. On the other hand, by Claim 3.5 and the fact that $n_1 \geq 5$, at least one of u_2, u_3 and u_4 must be in C, a contradiction.

We note that Case 2 is the only place in the proof where we use the assumption $n_i \geq 5$ for each $i = 1, 2, 3$. In the proof of Claim 3.2 we only require $n_i \geq 4$. □

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