Cyclic 4-colorings of graphs on surfaces

Atsuhiro Nakamoto*, Kenta Noguchi†, and Kenta Ozeki‡§

Abstract

To attack the Four Color Problem, in 1880, Tait gave a necessary and sufficient condition for plane triangulations to have a proper 4-vertex-coloring: a plane triangulation G has a proper 4-vertex-coloring if and only if the dual of G has a proper 3-edge-coloring. A cyclic coloring of a map G on a surface F^2 is a vertex-coloring of G such that any two vertices x and y receive different colors if x and y are incident with a common face of G. In this paper, we extend the result by Tait to two directions, that is, considering maps on a non-spherical surface and cyclic 4-colorings.

Keywords: Cyclic 4-colorings, 3-edge-colorings, the Four Color Theorem,

1 Introduction

The topic of colorings of plane graphs or graphs embedded on surfaces has been much attracted in Graph Theory. The most famous result is the Four Color Theorem, which states that every plane graph has a proper 4-vertex-coloring [2, 18]. In order to solve the Four Color Problem, Tait proved the following theorem, which implies his wrong "proof" of the Four Color Theorem. Although his proof was wrong, the following Theorem 1 gave an important direction to attack the Four Color Problem.

Theorem 1 (Tait [22]) *Let G be a plane triangulation. Then G has a proper 4-vertex-coloring if and only if the dual of G has a proper 3-edge-coloring.*

A *map* on a surface F^2 is a 2-cell embedding of a graph on F^2. A *cyclic coloring* of a map G on a surface is a vertex-coloring of G such that any two vertices x and y receive different colors if x and y are incident with a common face of G. Note that any cyclic coloring is a proper vertex-coloring, since any two adjacent vertices are incident

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with a common face. Ore and Plummer [17] defined the cyclic colorings of plane graphs and gave a conjecture on it, and many researchers have studied about cyclic colorings (see, for example, [6, 8, 19]).

In this paper, we focus on a *cyclic* 4*-coloring* of a map on a surface. As mentioned in Subsection 3.1, a cyclic 4-coloring of a map has been studied in several literatures, sometimes in different contexts. The purposes of this paper are (1) to extend Theorem 1 to maps on a non-spherical surface, and (2) to give a Tait-type relationship between cyclic 4-colorings of maps on surfaces and edge-colorings of the dual graph. Indeed, we will give a necessary and sufficient condition for maps on surfaces to have a cyclic 4-coloring (see Theorem 2). Here we need a few definitions.

In the rest of this paper, we use the term of a *coloring* of a graph G instead of a vertex-coloring of G. Note that in a coloring of G, we allow two adjacent vertices with the same color. A *proper coloring* of G is a coloring without such a pair of vertices. Similarly, we distinguish an *edge-coloring* and a *proper edge-coloring*. For an integer k, a k*-coloring* and a k*-edge-coloring* are a coloring and an edge-coloring with at most k colors, respectively.

It is clear that for a cyclic coloring of a map G on a surface, we need at least Δ^* colors, where Δ^* is the maximum size of faces of G. Hence for a cyclic 4-coloring, it is enough to consider only maps G on a surface F^2 in which all the faces have size 3 or 4. Such maps G are called *mosaics* of F^2. Borodin [4, 5] proved that every plane mosaic has a cyclic 6-coloring.

Throughout the present paper, we distinguish two terms, an *embedding* and a *drawing* of a graph on a surface; An *embedding* of a graph G on a surface F^2 is an injection from G to F^2, while a *drawing* of G on F^2 is not necessarily an injection. This means that a drawing of a graph G on a surface F^2 might have a crossing point of two edges of G. However, we assume that no three edges of G cross at a point on F^2 simultaneously.

An *essential closed curve* on a surface F^2 is one that does not bound a 2-cell on F^2. A closed curve γ on a surface F^2 is called *separating* if deleting γ makes F^2 disconnected. Throughout the present paper, we assume that a closed curve γ on a surface F^2 transversely intersects with a graph G drawn on F^2. For simplifying the arguments, we also assume that every closed curve γ on a surface F^2 passes through neither a vertex of G nor a crossing point of G, that is, γ intersects with G only at a point where exactly one edge of G is drawn. For a closed curve γ on F^2 and an edge set T of a graph G drawn on F^2, we denote by $T \cap \gamma$ the set of points on F^2 that are contained in both an edge in T and γ.

For a map G on a surface, the *dual* of G is denoted by G^*. The *dual edge* e^* of an edge e of G is one in G^* that corresponds to e in a natural way. We simply write G and e for $(G^*)^*$ and $(e^*)^*$, respectively, which are well-defined. Note that when G is a mosaic, every vertex of G^* has degree 3 or 4. For a vertex v of degree 4 with four incident edges e_1, e_2, e_3, e_4 in this cyclic order around v, we say that e_1 is the *opposite edge of* e_3 *at* v. In the same manner, e_2 is the opposite edge of e_4 at v. For a map G^* on a surface such that every vertex of G^* has degree 3 or 4, a walk W of G^* is a *straight walk* if W satisfies one of the following conditions:

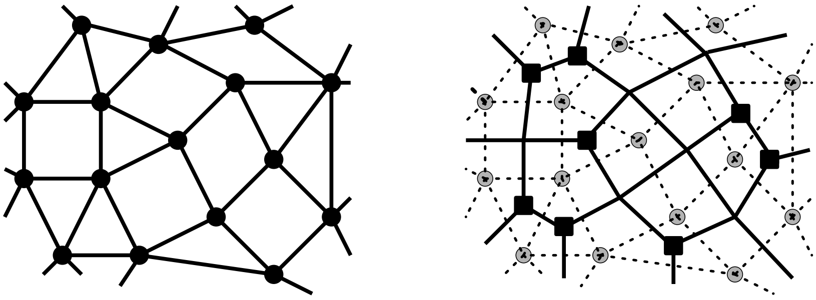

Figure 1: A mosaic G (the left side) and the straight walk dual $\widetilde{G}$ of G.

(i) W connects vertices of degree 3 in G^*, and for every internal vertex v of W, v has degree 4 in G^* and W passes through v from one edge to the opposite edge at v.

(ii) W is a closed walk, and for every vertex v of W, v has degree 4 and W passes through v from one edge to the opposite edge at v.

Note that W might intersect with itself.

Using the concept of straight walks, we define the *straight walk dual* $\widetilde{G}$ of a mosaic G of a surface F^2 as follows:

$$\begin{array}{rrcl} & V(\widetilde{G}) & = & \{F : F \text{ is a triangular face of } G\}, \\ \text{and} & E(\widetilde{G}) & = & \{W : W \text{ is a straight walk of } G^*\}, \end{array}$$

where each straight walk W corresponds to an edge of $\widetilde{G}$ connecting the two end vertices of W (if W satisfies (i)), or an edge having no vertex (if W satisfies (ii)). See Figure 1[1]. Note that $\widetilde{G}$ is 3-regular and might have multiple edges or loops. When G is a triangulation of F^2, then $\widetilde{G} = G^*$, and when G is a quadrangulation of F^2, then $\widetilde{G}$ has no vertices and consists of edges only. We assume that $\widetilde{G}$ is drawn on the surface F^2 in the natural way, that is, every edge in $\widetilde{G}$ is drawn on F^2 so that the corresponding straight walk of G^* is drawn on F^2. Hence $\widetilde{G}$ might have crossing edges, and moreover, an edge of $\widetilde{G}$ might intersect with itself.

Now we are ready to state our main theorem.

Theorem 2 *A mosaic G of a surface F^2 has a cyclic 4-coloring if and only if the straight walk dual $\widetilde{G}$ of G has a 3-edge-coloring $c : E(\widetilde{G}) \to \{1, 2, 3\}$ satisfying the following two conditions.*

(C1) *Any two crossing edges of $\widetilde{G}$ have different colors. (So, no edge intersects with itself.)*

(C2) *For every closed curve γ on F^2,*

$$|c^{-1}(1) \cap \gamma| \equiv |c^{-1}(2) \cap \gamma| \equiv |c^{-1}(3) \cap \gamma| \pmod 2. \tag{1}$$

[1] In Figure 1 black squares at the right side represent vertices of $\widetilde{G}$.

Here, for $i \in \{1,2,3\}$, $c^{-1}(i)$ denotes the set of edges $\widetilde{e}$ of $\widetilde{G}$ such that $c(\widetilde{e}) = i$, and $c^{-1}(i) \cap \gamma$ denotes the set of points on F^2 that are contained in both an edge in $c^{-1}(i)$ and γ.

This paper is organized as follows. In Section 2 we give some comments on the condition (C2). After giving some application of Theorem 2 in Section 3, we will prove Theorem 2 in Section 4.

2 Comments on the condition (C2)

2.1 Properness of a 3-edge-coloring c

In Theorem 2 we do not require an edge-coloring c of $\widetilde{G}$ to be proper, but the condition (C2) implies the properness of the 3-edge-coloring c.

Proposition 3 *Let $\widetilde{G}$ be a 3-regular graph drawn on a surface F^2. Suppose that $\widetilde{G}$ has a 3-edge-coloring $c : E(\widetilde{G}) \to \{1,2,3\}$. If c satisfies the condition (C2), then c is a proper 3-edge-coloring of $\widetilde{G}$.*

Proof. Let v be any vertex of $\widetilde{G}$, let $\widetilde{e_1}, \widetilde{e_2}$, and $\widetilde{e_3}$ be three edges that are incident with v, and let c be a 3-edge-coloring of $\widetilde{G}$. Suppose that c satisfies the condition (C2). Let γ be a non-essential closed curve on F^2 that surrounds v and intersects with each of $\widetilde{e_1}, \widetilde{e_2}$ and $\widetilde{e_3}$ exactly once. It follows from the equality (1) for γ that

$$|c^{-1}(1) \cap \gamma| \equiv |c^{-1}(2) \cap \gamma| \equiv |c^{-1}(3) \cap \gamma| \pmod 2,$$

which directly implies that $\widetilde{e_1}, \widetilde{e_2}$, and $\widetilde{e_3}$ are colored by three distinct colors. Hence c is a proper 3-edge-coloring. □

We point out that the equality (1) for separating closed curves corresponds to a well-known lemma, called *Parity Lemma* on a proper 3-edge-coloring (see, for example, [24, page 253]). Recall that an *edge-cut* of a connected graph H is an inclusionwise minimal set of edges whose removal makes H disconnected.

Lemma 4 (Parity Lemma) *Let H be a 3-regular graph with a proper 3-edge-coloring c by the colors $1, 2$ and 3. Then each edge-cut T of H satisfies $|c^{-1}(1) \cap T| \equiv |c^{-1}(2) \cap T| \equiv |c^{-1}(3) \cap T|$ (mod 2).*

Now we show how the equality (1) for separating closed curves is related to Lemma 4. For a closed curve γ on a surface F^2, let T_γ be the set of edges $\widetilde{e}$ of a graph $\widetilde{G}$ drawn on F^2 such that $\widetilde{e}$ intersects with γ an odd number of times. It is easy to see that γ is separating if and only if T_γ is the union of disjoint edge-cuts of $\widetilde{G}$ or $T_\gamma = \emptyset$. In this sense, separating closed curves on F^2 correspond to edge-cuts of $\widetilde{G}$, and we see the correspondence between the equality (1) and the equality in Lemma 4. Indeed, if c is a proper 3-edge-coloring, then every separating closed curve γ on F^2 satisfies the equality (1). Note that the condition (C2) requires the equality (1) for all closed curves γ on F^2, but the arguments above say that the equality (1) is automatically satisfied if c is a proper 3-edge-coloring and γ is a separaing closed curve.

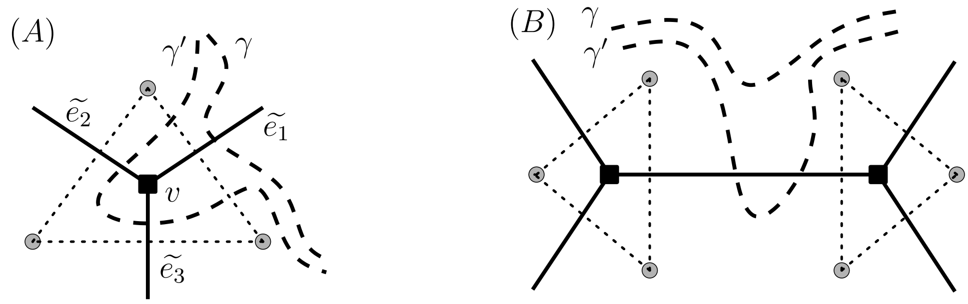

Figure 2: Two closed curves γ and γ' that are homotopic.

2.2 Checking the condition (C2) and the fundamental group of F^2

In this subsection we consider how to check the condition (C2) in Theorem 2. In order to check the condition (C2), we have to consider all closed curves on a surface F^2. However, it is not necessary to do that, and we will explain that if we assume the properness of the 3-edge-coloring c of $\widetilde{G}$, then it is enough to check only $g(F^2)$ appropriate non-separating closed curves on F^2. Recall that $g(F^2)$ is the *Euler genus* of a surface F^2, which is equal to $2k$ if F^2 is the orientable surface with genus k; Otherwise, that is, if F^2 is the non-orientable surface with crosscap number k, then it is equal to k.

To see this, we first look at two situations (A) and (B) in Figure 2. The both situations represent a part of two closed curves γ and γ' on F^2, and assume that the remaining parts of γ and γ' are exactly the same. Let c be a proper 3-edge-coloring of $\widetilde{G}$. In the situation (A), there is a vertex v of degree 3 in $\widetilde{G}$, and let $\widetilde{e}_1, \widetilde{e}_2$, and $\widetilde{e}_3$ be three edges of $\widetilde{G}$ incident with v. Since c is a proper 3-edge-coloring, we may assume that by symmetry, $c(\widetilde{e}_i) = i$ for $i \in \{1, 2, 3\}$. Note that $|c^{-1}(1) \cap \gamma'| = |c^{-1}(1) \cap \gamma| - 1$, $|c^{-1}(2) \cap \gamma'| = |c^{-1}(2) \cap \gamma| + 1$, and $|c^{-1}(3) \cap \gamma'| = |c^{-1}(3) \cap \gamma| + 1$, which directly implies that if γ satisfies the equality (1), then so does γ'. Similarly, in the situation (B), we can easily see that if γ satisfies the equality (1), then so does γ'. These two facts imply that for any two closed curves γ and γ' with the same homotopy type on F^2, if γ satisfies the equality (1), then so does γ', since γ' can be obtained from γ by a sequence of homotopic shifts as in the situation (A) or (B) in Figure 2.

On the other hand, let $[\gamma_1]$ and $[\gamma_2]$ be two generators of the fundamental group of F^2. (Note that $[\gamma_1]$ and $[\gamma_2]$ are two homotopy classes of the set of closed curves on F^2, and γ_1 and γ_2 are representatives of them, respectively.) It is easy to see that if both γ_1 and γ_2 satisfy the equality (1), (and hence if every closed curve on F^2 homotopic to γ_1 or γ_2 satisfies the equality (1),) then any closed curve γ contained in the homotopy class $[\gamma_1] * [\gamma_2]$ or $[\gamma_1]^{-1}$ also satisfies the equality (1), where $[\gamma_1] * [\gamma_2]$ is the product of $[\gamma_1]$ and $[\gamma_2]$ on the fundamental group of F^2 and $[\gamma_1]^{-1}$ is the homotopy class containing γ_1^{-1}.

Since any homotopy class of the set of closed curves on F^2 is obtained by the products of generators of the fundamental group of F^2, these arguments, together with Proposition 3, imply that the following Theorem 5 is equivalent to Theorem 2. Indeed, since there are exactly $g(F^2)$ generators in the fundamental group of F^2, it is enough to check only $g(F^2)$ appropriate non-separating closed curves on F^2.

Theorem 5 *A mosaic G of a surface F^2 has a cyclic 4-coloring if and only if the straight walk dual $\widetilde{G}$ has a proper 3-edge-coloring $c : E(\widetilde{G}) \to \{1, 2, 3\}$ satisfying the condition (C1) and the following condition:*

(C2′) *for every generator $[\gamma]$ of the fundamental group of F^2, a representative γ of $[\gamma]$ satisfies the equality (1).*

3 Application of Theorems 2 and 5

3.1 The plane case and colorings equivalent to cyclic 4-colorings

We consider the plane case in Theorem 5. (It should be noticed that as in Theorem 5, the arguments in this subsection can be extended to maps on non-spherical surfaces.)

By the Jordan curve theorem, the plane does not admit a non-separating closed curve, and hence in this case we do not need the condition (C2′) in Theorem 5. Moreover, if a map G is a plane triangulation, then the straight walk dual $\widetilde{G}$ of G coincides with the dual G^*, and hence the condition (C1) trivially holds. Indeed, Theorem 1 is a corollary of Theorem 5.

A graph G is called 1-*plane* if G is drawn on the plane in such a way that each edge of G is crossed by at most one other edge. A 1-plane graph G is *maximal* if any edge cannot be added into G keeping the 1-planarity of G. It is easy to see that from every maximal 1-plane graph G, we can obtain a 1-plane graph G' (with possibly multiple edges) by adding some edges, such that the graph $(G')^-$ is a plane mosaic, where $(G')^-$ is obtained from G' by removing all the edges that are crossed by other edges. (See [21, page 1535], in which a mosaic is called a *triquadrangulation*. Note that the statement in the paper contains a small omission and the above is correct, since the author did not care about the simpleness of graphs. However, we can allow multiple edges in this paper, and hence here no issues occur.) Indeed, the result of Borodin [4, 5] asserting that every plane mosaic has a cyclic 6-coloring was first obtained in a context of 1-plane graphs. It is easy to see that c is a proper coloring of a maximal 1-plane graph G if and only if c is a cyclic coloring of the graph $(G')^-$. Thus, Theorem 5 also gives a necessary and sufficient condition for maximal 1-plane graphs to have a proper 4-coloring.

On the other hand, for the case where G is a plane quadrangulation, Theorem 5 gives the following corollary, which was shown by Berman and Shank [3]. (They called a cyclic coloring of a plane quadrangulation a *full-coloring*.)

Corollary 6 ([3]) *Let G be a plane quadrangulation. Then G has a cyclic 4-coloring if and only if the straight walk dual $\widetilde{G}$ has a proper 3-edge-coloring satisfying the condition (C1).*

This corollary is also related to a *polychromatic* coloring of quadrangulations. A k-coloring of a map of a surface is called a *polychromatic k-coloring* if each face receives all k colors on the boundary vertices. We can find several papers on polychromatic k-colorings (see, for example, [1, 12]). Note that when G is a quadrangulation of a surface, a polychromatic 4-coloring of G is equivalent to a cyclic 4-coloring of G. Hence Theorem 5 also gives a necessary and sufficient condition for a quadrangulation of a surface to have a polychromatic 4-coloring.

It was shown in [1] that the problem of deciding whether a given plane quadrangulation has a cyclic 4-coloring is NP-complete ([1, Proposition 29]). Theorem 5 gives another proof to this result as follows.

Any graph H can be represented as an intersection graph of closed curves on the plane (Stockmeyer [20]). In particular, we may assume that no closed curve passes through a crossing point of other two closed curves, and every closed curve transversely intersects with other closed curves; Otherwise we can easily modify those closed curves so that they satisfy these conditions (with possibly increasing the number of crossing points). Now consider a plane graph G' by regarding crossing points of those closed curves in the representation of H as vertices and parts of closed curves connecting two consecutive crossing points as edges. Since no closed curve passes through a crossing point of other two closed curves, G' is 4-regular. Then the dual $G = (G')^*$ of G' is a plane quadrangulation and each straight walk of G^* is a closed curve corresponding to some vertex of H. Therefore, the straight walk dual $\widetilde{G}$ of G is the representation of H. Note that the vertex set of H is the set of straight walks of G^* and two straight walks are adjacent in H if and only if they have a common vertex in G^*. By the construction of G, H has a proper 3-coloring if and only if $\widetilde{G}$ has a proper 3-edge-coloring satisfying the condition (C1). Hence by Theorem 5, H has a proper 3-coloring if and only if G has a cyclic 4-coloring. Thus, the problem of proper 3-coloring of a graph H can be reduced to the problem of cyclic 4-coloring of a plane quadrangulation G. Since the problem of deciding whether a given graph has a proper 3-coloring is NP-complete (Karp [14]), so is the problem of deciding whether a given plane quadrangulation has a cyclic 4-coloring.

3.2 Locally planar graphs and the condition (C2′)

Here we focus on proper colorings of maps with large *representativity* on non-spherical surfaces. For a map G on a surface F^2, the *representativity* of G is the minimum of $|G \cap \gamma|$ taken over all essential closed curves γ on F^2. The representativity of G is also called the *face-width*, and a graph with large enough representativity (usually comparing with the Euler genus of F^2) is called *locally planar*.

Thomassen [23] showed that for every non-spherical surface F^2, there exists an integer k such that every map on F^2 with representativity at least k has a proper 5-coloring. Moreover, he showed that for any surface F^2 and any integer k, there are infinitely many maps on F^2 with representativity at least k and no proper 4-coloring.

On the other hand, Robertson posed the following conjecture on proper 3-edge-colorings of locally planar 3-regular graphs. See [16, page 153] for Conjecture 7 below.

Conjecture 7 (Robertson) *For any non-spherical surface F^2, there exists an integer k such that every 3-regular map on F^2 with representativity at least k has a proper 3-edge-coloring.*

Note that originally, Grünbaum [10] conjectured that $k = 3$ is enough for Conjecture 7 when F^2 is an orientable surface, but Kochol [15] gave a counterexample to the case of orientable surfaces with genus at least 5. Conjecture 7, which is a weaker version of the conjecture by Grünbaum, is still open even for orientable surfaces with high genus.

If Conjecture 7 is true, then for any locally planar triangulation G of a surface F^2, the dual G^* has a proper 3-edge-coloring c. Since G is a triangulation, the edge-coloring c trivially satisfies the condition (C1). Hence by Theorem 5, if c satisfies the condition (C2′), then G has a proper 4-coloring. However, as mentioned above, there are infinitely many maps on F^2 with large representativity and no proper 4-coloring. This means that even if Conjecture 7 is true, there are infinitely many triangulations G of a surface F^2 with large representativity such that no proper 3-edge-colorings of $\widetilde{G}$ satisfy the condition (C2′). Such triangulations cannot be properly 4-colorable because of the condition (C2′).

As mentioned above, we have no hope to extend the result of Thomassen on proper 5-colorings, even if we assume large representativity. However, one might be able to consider a *circular coloring* instead of an ordinary proper coloring. For a positive real number r, a *circular r-coloring* of a graph G is a mapping $f : V(G) \to [0, r)$ such that for any edge xy of G, $1 \leq |f(x) - f(y)| \leq r - 1$. It is known that for any graph G, if G has a proper r-coloring, then G has a circular r-coloring, and if G has a circular r-coloring, then G has a proper $\left(\lceil r \rceil\right)$-coloring (see, for example [25, 26]). Goddyn [9] posed a conjecture concerning a circular coloring (see [26, page 539]).

Conjecture 8 (Goddyn [9]) *For any $\varepsilon > 0$ and any orientable surface F^2, there exists an integer k such that every triangulation of F^2 with representativity at least k has a circular $(4 + \varepsilon)$-coloring.*

Conjecture 8 means that we can expect that all locally planar triangulations of an orientable surface are "close" to be properly 4-colorable in the sense of a circular coloring. However, the gap between the existence of a proper 4-coloring and the existence of a circular $(4 + \varepsilon)$-coloring still exists. Theorem 5 suggests that the gap comes from the condition (C2′).

It is shown in [7] that Conjecture 7 is stronger than Conjecture 8. Hence, roughly speaking, if all locally planar 3-regular maps G^* on an orientable surface have a proper 3-edge-coloring, then all locally planar triangulations G have a circular $(4+\varepsilon)$-coloring. Indeed, a proper 3-edge-coloring of G^* satisfying the condition (C2′) indicates a proper 4-coloring of G by Theorem 5, while a proper 3-edge-coloring without satisfying the condition (C2′) indicates a circular $(4 + \varepsilon)$-coloring of G (see the proof of Theorem 1.1 in [7]).

3.3 Weakly cyclic colorings and the condition (C1)

Now we will consider the role of the condition (C1). We need the condition (C1) in Theorem 2 for a cyclic 4-coloring, but without the condition (C1) we can also find an interesting coloring that is analogous to cyclic 4-colorings. A *weakly cyclic coloring* of a map G on a surface F^2 is a proper coloring of G such that for each face F of G, the number of colors that F receives on the boundary vertices has the same parity as the size of F. It is easy to see that a cyclic k-coloring of a map G on a surface is definitely a weakly cyclic k-coloring of G, but generally the converse does not hold. Note that in a weakly cyclic coloring of a mosaic, each triangular face receives exactly three colors and each quadrangular face receives two or four colors.

Focusing on a weakly cyclic 4-coloring of a mosaic of a surface, we get the following theorem.

Theorem 9 *A mosaic G of a surface F^2 has a weakly cyclic 4-coloring if and only if the straight walk dual $\widetilde{G}$ has a 3-edge-coloring c satisfying the condition (C2).*

It is easy to see that a cyclic 4-coloring of G is a weakly cyclic 4-coloring of G without a 2-colored quadrangular face. Hence comparing Theorems 2 and 9, we can see that the condition (C1) plays a role to forbid a 2-colored quadrangular face. Note that Hoffmann-Ostenhof [11] gave a necessary and sufficient condition for a plane mosaic to have a weakly cyclic 4-coloring, which is a corollary of Theorem 9. (He called such a coloring a *q_4-coloring*.)

Note that as we obtain the equivalence between Theorems 2 and 5, Theorem 9 is equivalent to the statement that a mosaic G of a surface F^2 has a weakly cyclic 4-coloring if and only if the straight walk dual $\widetilde{G}$ has a proper 3-edge-coloring c satisfying the condition (C2′). In this case, if G is a plane quadrangulation, then the 3-edge-coloring c of $\widetilde{G}$ always satisfies the condition (C2′). This implies that any plane quadrangulation has a weakly cyclic 4-coloring. In fact, it is well-known that any plane quadrangulation G is bipartite, and hence G has a proper 2-coloring, which can be regarded as a weakly cyclic 2-coloring of G.

We can prove Theorem 9 similarly to Theorem 2 (see the remark after the proof of Theorem 9 in Section 4).

4 Proof of Theorem 2

To prove Theorem 2, we prepare one notation. Let H be a graph embedded on a surface F^2. (Note that here H is not necessarily 2-cell embedded on F^2.) A *region* of H is an arcwise connected component of $F^2 \setminus H$, and a *proper region-coloring* of H is an assignment of colors to the set of regions of H such that any two regions receive different colors if their boundaries share an edge of H.

Proof of Theorem 2. First, we show the "only if" part of Theorem 2. Let G be a mosaic of a surface F^2 and suppose that G has a cyclic 4-coloring f. We regard the

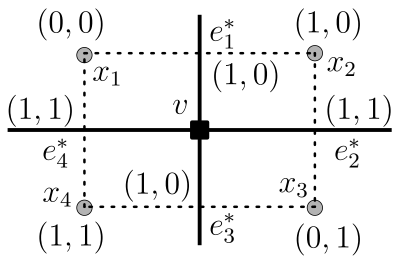

Figure 3: An example of the situation around v and the edge-coloring ∂f^* of G^*.

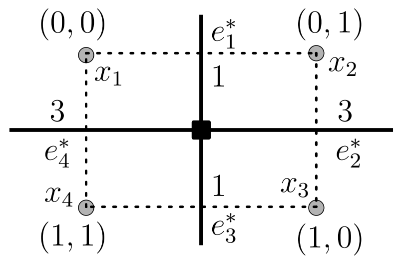

Figure 4: An example of the situation of the face $x_1x_2x_3x_4x_1$ and the coloring f of G.

colors of f as the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$, so we use the four colors $(0,0), (1,0), (0,1)$ and $(1,1)$. We construct a (not necessarily proper) edge-coloring ∂f of G as follows: for an edge $e = xy$ of G, define the color $\partial f(e) = f(x) + f(y)$, where $+$ is the sum of $\mathbb{Z}_2 \times \mathbb{Z}_2$. The edge-coloring ∂f is sometimes called the *coboundary* of f (see [7]). Since f is a proper coloring, every edge of G receives the color $(0,1), (1,0)$ or $(1,1)$ by ∂f. Let ∂f^* be the edge-coloring of G^* obtained from ∂f in a natural way; $\partial f^*(e^*) = \partial f(e)$ for any edge e^* of G^*. We show the following claim concerning the edge-coloring ∂f^*.

Claim 1 *Let v be a vertex of degree 4 in G^*, and let e_1^*, e_2^*, e_3^* and e_4^* be the four edges incident with v in this cyclic order around v. Then $\partial f^*(e_1^*) = \partial f^*(e_3^*) \neq \partial f^*(e_2^*) = \partial f^*(e_4^*)$.*

Proof. Let $x_1x_2x_3x_4$ be the face of G corresponding to v, where x_i is a vertex of G for $i \in \{1,2,3,4\}$. By symmetry, we may assume that $e_i = x_ix_{i+1}$ for $i \in \{1,2,3,4\}$, where $x_5 = x_1$. See Figure 3 for an example[2]. Since f is a cyclic 4-coloring of G, the four vertices x_1, x_2, x_3 and x_4 receive four distinct colors by f. Then it is easy to check that $f(x_1) \neq f(x_3)$ and $f(x_1) + f(x_2) = f(x_3) + f(x_4)$. Therefore,

$$\begin{array}{rcccccl} & \partial f^*(e_1^*) & = & f(x_1) + f(x_2) & = & f(x_3) + f(x_4) & = & \partial f^*(e_3^*), \\ \text{and} & \partial f^*(e_1^*) & = & f(x_1) + f(x_2) & \neq & f(x_2) + f(x_3) & = & \partial f^*(e_2^*). \end{array}$$

The above (in)equalities and the symmetry directly show Claim 1. $\square$

Now we define the edge-coloring c of $\widetilde{G}$ as $c(W) = \partial f^*(e^*)$ for each edge W of $\widetilde{G}$, where e^* is an edge of G^* contained in the straight walk W. By Claim 1, this definition does not depend on the choice of an edge e^*, and hence the edge-coloring c is well-defined. Moreover, the edge-coloring c of $\widetilde{G}$ satisfies the condition (C1).

To show that c satisfies the condition (C2), we first need the fact that the spanning subgraph G_1^* of G^* induced by all the edges colored by $(1,0)$ or $(1,1)$ by ∂f^* has a proper 2-region-coloring. The definition of G_1^*, together with the construction of the edge-coloring ∂f^*, implies that the dual edge e of an edge e^* of G_1^* connects two vertices

[2]In Figure 3, the four vertices x_1, x_2, x_3 and x_4 have the colors $(0,0), (1,0), (0,1)$ and $(1,1)$ by f, respectively. Then the edge-coloring ∂f is defined as $\partial f(e_1) = \partial f^*(e_1^*) = (0,0) + (1,0) = (1,0)$, $\partial f(e_2) = (1,0) + (0,1) = (1,1)$, $\partial f(e_3) = (0,1) + (1,1) = (1,0)$, and $\partial f(e_4) = (1,1) + (0,0) = (1,1)$, respectively.

of G, one of which has the color $(0,0)$ or $(0,1)$ by f and the other has the color $(1,0)$ or $(1,1)$. This means that two sides of the edge e^* correspond to two distinct regions of G_1^*, and a region of G_1^* on one side of e^* contains regions of G^* corresponding to vertices of G with colors having 0 in the first coordinate by f, and that on the other side contains regions of G^* corresponding to vertices of G with colors having 1 in the first coordinate. Then depending on the first coordinate of the color by f, we can color each region of G_1^* by the two colors, 0 or 1.

Then each closed curve γ on F^2 has to pass through regions of G_1^* with color 0 and ones with color 1 alternatively. This directly implies that

$$\big|(\partial f^*)^{-1}\big((1,0)\big) \cap \gamma\big| + \big|(\partial f^*)^{-1}\big((1,1)\big) \cap \gamma\big| \equiv 0 \pmod 2.$$

By the definition of the edge-coloring c of $\widetilde{G}$, we have

$$\big|c^{-1}\big((1,0)\big) \cap \gamma\big| + \big|c^{-1}\big((1,1)\big) \cap \gamma\big| \equiv 0 \pmod 2. \tag{2}$$

Since we can use the same argument as above for the second coordinate of the colors of f, we also obtain

$$\big|c^{-1}\big((0,1)\big) \cap \gamma\big| + \big|c^{-1}\big((1,1)\big) \cap \gamma\big| \equiv 0 \pmod 2. \tag{3}$$

These equalities (2) and (3) imply that γ satisfies the equality (1). Hence the condition (C2) also holds, and this completes the proof of the "only if" part of Theorem 2.

Next, we show the "if" part of Theorem 2.

Let G be a mosaic of a surface F^2 and suppose that the straight walk dual $\widetilde{G}$ has a 3-edge-coloring $c : E(\widetilde{G}) \to \{1,2,3\}$ satisfying the conditions (C1) and (C2). By Proposition 3, c is a proper 3-edge-coloring. Let c^* be an edge-coloring of G such that $c^*(e) = c(W)$ for each edge e of G, where W is the straight walk of G^* containing e^*.

First, let G_1 be the graph obtained from G by contracting all the edges in $(c^*)^{-1}(1)$ and deleting all the loops with color 1 (but remaining all the multiple edges and the loops with color 2 or 3). We will show that G_1 is bipartite. Let S_1 be a cycle of G_1. Note that any edge of S_1 have the color 2 or 3 in G. Furthermore, it follows from the construction of G_1 that all the edges of S_1 form a set of disjoint paths in G, and by adding some appropriate edges in $(c^*)^{-1}(1)$, we obtain a cycle S in G. Then $E(S_1) = E(S) \cap \big((c^*)^{-1}(2) \cup (c^*)^{-1}(3)\big)$ and $E(S) - E(S_1) \subseteq (c^*)^{-1}(1)$. Since S is a cycle embedded on F^2, S can be regarded as a closed curve on F^2. Thus, it follows from the condition (C2) that S satisfies the equality (1), and hence

$$|c^{-1}(2) \cap S| \;+\; |c^{-1}(3) \cap S| \;\equiv\; 0 \pmod 2.$$

Note that $|c^{-1}(2) \cap S| = |E(S) \cap (c^*)^{-1}(2)|$ and $|c^{-1}(3) \cap S| = |E(S) \cap (c^*)^{-1}(3)|$. Therefore, we obtain

$$\begin{aligned} |E(S_1)| &= \big|E(S) \cap \big((c^*)^{-1}(2) \cup (c^*)^{-1}(3)\big)\big| \\ &= |c^{-1}(2) \cap S| \;+\; |c^{-1}(3) \cap S| \;\equiv\; 0 \pmod 2. \end{aligned}$$

So, S_1 has even length. This implies that G_1 is bipartite, and hence G_1 has a proper 2-coloring f_1 by two colors, say $(0, 0)$ and $(1, 0)$. Here we regard f_1 as a (not necessarily proper) 2-coloring of G such that $f_1(x) = f_1(x')$ for each vertex x in G, where x' is the vertex in G_1 corresponding to x. Then for an edge $e = xy$ of G, $f_1(x) = f_1(y)$ if and only if $c^*(e) = 1$.

Let G_2 be the graph obtained from G by contracting all the edges in $(c^*)^{-1}(2)$. By the same argument as above, G_2 has a proper 2-coloring f_2 by two colors, say $(0, 0)$ and $(0, 1)$, and we also regard f_2 as a (not necessarily proper) 2-coloring of G. Then For an edge $e = xy$ of G, $f_2(x) = f_2(y)$ if and only if $c^*(e) = 2$.

Then we define the coloring f of G as follows: for each vertex x of G, $f(x) = f_1(x) + f_2(x)$. We show that f is a cyclic 4-coloring of G.

For an edge $e = xy$ of G, e is contained in at least one of G_1 and G_2, which implies that x and y receive the colors by f that are different in the first and/or the second coordinate. Thus, for any edge xy of G, x and y have distinct colors by f, that is, f is a proper 4-coloring of G. In particular, each triangular face of G receives three distinct colors by f. We prove the following claim.

Claim 2 *For any quadrangular face of G, say $x_1x_2x_3x_4x_1$, we have $f(x_1) \neq f(x_3)$ and $f(x_2) \neq f(x_4)$.*

Proof. Let e_i be the edge of G connecting x_i and x_{i+1} for $i \in \{1, 2, 3, 4\}$, where $x_5 = x_1$. See Figure 4 for an example[3]. By the condition (C1), $c^*(e_1) \neq c^*(e_2)$, and hence without loss of generality, we may assume that $c^*(e_1) = 1$ and $c^*(e_2) \neq 1$. Recall that for an edge $e = xy$ of G, $f_1(x) = f_1(y)$ if and only if $c^*(e) = 1$, and hence

$$f_1(x_1) \;=\; f_1(x_2) \;\neq\; f_1(x_3).$$

Therefore, x_1 and x_3 receive distinct values in the first coordinate by f_1, and also by f. This implies that $f(x_1) \neq f(x_3)$, and similarly, $f(x_2) \neq f(x_4)$. □

Then for any quadrangular face of G, say $x_1x_2x_3x_4x_1$, $f(x_1) \neq f(x_2)$ and $f(x_1) \neq f(x_4)$ since f is a proper 4-coloring, and $f(x_1) \neq f(x_3)$ by Claim 2. Hence x_1 receives a color different from the colors of x_2, x_3 and x_4, and, by symmetry, the four vertices in a quadrangular face receive four distinct colors by f. Thus, f is a cyclic 4-coloring of G. This completes the proof of the "if" part, and the proof of Theorem 2. □

Remark: As we explained in Subsection 3.3, we can prove Theorem 9 similarly to Theorem 2. We here briefly explain the difference. In Claim 1 in the proof of the "only if" part of Theorem 2, the conclusion "$\partial f^*(e_1^*) \neq \partial f^*(e_2^*)$" might not necessarily hold for a weakly cyclic 4-coloring, which will violate the condition (C1) for the obtained edge-coloring c of $\widetilde{G}$. On the other hand, in Claim 2 in the proof of the "if" part, we

[3]In Figure 4, the four edges e_1, e_2, e_3 and e_4 have the colors $1, 3, 1$ and 3 by c^*, respectively. Then the edge-colorings f_1 and f_2 are defined as $f_1(x_1) = f_1(x_2) = (0, 0)$, $f_1(x_3) = f_1(x_4) = (1, 0)$, $f_2(x_1) = f_2(x_3) = (0, 0)$, and $f_2(x_2) = f_2(x_4) = (0, 1)$, respectively. Then finally, we obtain the coloring f as indicated to each vertex in Figure 4.

cannot obtain the conclusion "$f(x_1) \neq f(x_3)$ and $f(x_2) \neq f(x_4)$" without assuming the condition (C1). However, instead we obtain that "if $f(x_1) = f(x_3)$, then $f(x_2) = f(x_4)$" without assuming the condition (C1), which implies that the coloring f of G is a weakly cyclic 4-coloring of G. The above parts are the only difference between the proof of Theorem 2 and that of Theorem 9. So we leave it for the reader to work out the missing details of the proof.

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