On the ratio of the domination number and the independent domination number in graphs

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Abstract

We let $\gamma(G)$ and $i(G)$ denote the domination number and the independent domination number of G, respectively. Recently, Rad and Volkmann conjectured that $i(G)/\gamma(G) \leq \Delta(G)/2$ for every graph G, where $\Delta(G)$ is the maximum degree of G. In this note, we construct counterexamples of the conjecture for $\Delta(G) \geq 6$, and give a sharp upper bound of the ratio $i(G)/\gamma(G)$ by using the maximum degree of G.

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1 Introduction

In this paper, we consider only finite, simple, undirected graphs with no loops and no multiple edges. Let G be a graph. We let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For $v \in V(G)$, we let $d_G(v)$, $N_G(v)$, and $N_G[v]$ denote the degree, the open neighborhood, and the closed neighborhood of v, respectively; thus $d_G(v) = |N_G(v)|$ and $N_G[v] = N_G(v) \cup \{v\}$. The maximum degree and the minimum degree of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively.

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For $X \subseteq V(G)$, we let $G[X]$ denote the subgraph of G induced by X. For terms and symbols not defined here, we refer the reader to [1].

Let again G be a graph. For two subsets X, Y of $V(G)$, we say that X *dominates* Y if $Y \subseteq \bigcup_{v \in X} N_G[v]$. A subset S of $V(G)$ is a *dominating set* of G if S dominates $V(G)$, and S is an *independent dominating set* of G if S is a dominating set of G and S is an independent set of G. The minimum cardinality of a dominating set (resp. an independent dominating set) of G is called the *domination number* (resp. the *independent domination number*) of G, and is denoted by $\gamma(G)$ (resp. $i(G)$). Various properties of the domination number and the independent domination number of graphs were explored in [2, 4].

Goddard, Henning, Lyle and Southey [3] focused on the ratio $i(G)/\gamma(G)$ for regular graphs G, and showed that $i(G)/\gamma(G) \leq 3/2$ for cubic graphs G. Furthermore, by Southey and Henning [6], it was improved that $i(G)/\gamma(G) \leq 4/3$ for connected cubic graphs G with $G \neq K_{3,3}$. Recently, Rad and Volkmann [5] studied the ratio $i(G)/\gamma(G)$ for graphs G which are not necessarily regular, and showed the following theorem.

Theorem A (Rad and Volkmann [5]) *For a graph G,*

$$i(G)/\gamma(G) \leq \begin{cases} \Delta(G)/2 & (3 \leq \Delta(G) \leq 5) \\ \Delta(G) - 3 + 2/(\Delta(G) - 1) & (\Delta(G) \geq 6) \end{cases}$$

They conjectured that it can be improved the case where $\Delta(G) \geq 6$ in Theorem A as follows.

Conjecture 1 (Rad and Volkmann [5]) *For a graph G, $i(G)/\gamma(G) \leq \Delta(G)/2$.*

However, there exist counterexamples of Conjecture 1. Let $\Delta \geq 1$ be a square integer, and write $\Delta = m^2$. Let G_Δ be the graph obtained from a complete graph of order m by adding $m^2 - m + 1$ pendant edges to each vertex (see Figure 1). Then we see that $\Delta(G_\Delta) = m^2 (= \Delta)$, $\gamma(G_\Delta) = m$ and $i(G_\Delta)/\gamma(G_\Delta) = (1 + (m-1)(m^2 - m + 1))/m = \Delta(G_\Delta) - 2\sqrt{\Delta(G_\Delta)} + 2$. In particular, for a square integer $\Delta \geq 9$, we have $i(G_\Delta)/\gamma(G_\Delta) > \Delta(G_\Delta)/2$. (For a fixed square number $\Delta \geq 9$, since nG_Δ ($n \geq 2$) also satisfies that $\Delta(nG_\Delta) = \Delta$ and $i(nG_\Delta)/\gamma(nG_\Delta) > \Delta(nG_\Delta)/2$ (where nG_Δ is the disjoint union of n copies of G_Δ), there exist infinitely many counterexamples G of Conjecture 1 satisfying $\Delta(G) = \Delta$).

In this paper, we give a sharp upper bound of the ratio $i(G)/\gamma(G)$ for graphs G which improves the case where $\Delta(G) \geq 6$ in Theorem A.

Theorem 1.1 *For a graph G, $i(G)/\gamma(G) \leq \Delta(G) - 2\sqrt{\Delta(G)} + 2$.*

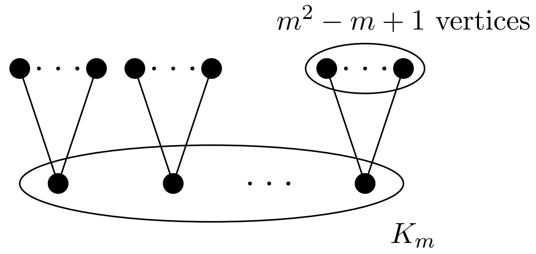

Figure 1: Graph G_Δ

By considering graphs G_Δ (or nG_Δ), we see that the bound in Theorem 1.1 is best possible.

2 Proof of Theorem 1.1

In this section, we show Theorem 1.1.

By tedious calculations, we have the following lemma (and we omit its detail).

Lemma 2.1 *Let $\Delta \geq 0$ be an integer, and let $f(x) = \frac{\Delta - 1 + x^2}{x+1}$ be a function. Then for $x \in \mathbb{R}$ with $x \geq 0$, $f(x) \geq f(\sqrt{\Delta} - 1) = 2\sqrt{\Delta} - 2$.*

Proof of Theorem 1.1. Let G be a graph , and let D be a minimum dominating set of G. We recursively define a graph G_i and a vertex x_i as follows: Let $G_1 = G[D]$, and let $x_1 \in V(G_1)$ be a vertex with $d_{G_1}(x_1) = \delta(G_1)$. For each $i > 1$, if $V(G_{i-1}) - N_{G_{i-1}}[x_{i-1}] \neq \emptyset$, let $G_i = G_{i-1} - N_{G_{i-1}}[x_{i-1}]$, and let $x_i \in V(G_i)$ be a vertex with $d_{G_i}(x_i) = \delta(G_i)$; if $V(G_{i-1}) - N_{G_{i-1}}[x_{i-1}] = \emptyset$, let $k = i - 1$ and end the recursive definitions. Let $X = \{x_i \mid 1 \leq i \leq k\}$. Note that X is an independent dominating set of $G[D]$. Since $\{N_{G_i}[x_i] \mid 1 \leq i \leq k\}$ is a partition of D,

$$\sum_{1 \leq i \leq k} (d_{G_i}(x_i) + 1) = |D| = \gamma(G) \tag{2.1}$$

Let I $(\subseteq V(G) - D)$ be an independent dominating set of $G - \bigcup_{1 \leq i \leq k} N_G[x_i]$. Then $X \cup I$ is an independent dominating set of G, and so

$$i(G) \leq |X| + |I| = k + |I| \tag{2.2}$$

Since D is a dominating set of G and $I \subseteq V(G) - D$, $I = \bigcup_{v \in D - X}(N_G(v) \cap I)$ $(= \bigcup_{1 \leq i \leq k}(\bigcup_{v \in N_{G_i}(x_i)}(N_G(v) \cap I)))$. For $1 \leq i \leq k$ and $v \in N_{G_i}(x_i)$, since $d_{G_i}(x_i) \leq d_{G_i}(v)$ by the choice of x_i, $|N_G(v) \cap I| \leq d_G(v) - d_{G_i}(v) \leq \Delta(G) - d_{G_i}(x_i)$.

Consequently,

$$\begin{aligned}
|I| &\le \sum_{1\le i\le k}\Big(\sum_{v\in N_{G_i}(x_i)} |N_G(v)\cap I|\Big)\\
&\le \sum_{1\le i\le k}\Big(\sum_{v\in N_{G_i}(x_i)} (\Delta(G)-d_{G_i}(x_i))\Big)\\
&= \sum_{1\le i\le k}\Big(\sum_{v\in N_{G_i}(x_i)} \Delta(G)\Big) - \sum_{1\le i\le k}\Big(\sum_{v\in N_{G_i}(x_i)} d_{G_i}(x_i)\Big)\\
&= (|D|-k)\Delta(G) - \sum_{1\le i\le k} d_{G_i}(x_i)^2\\
&= \Delta(G)\gamma(G) - k\Delta(G) - \sum_{1\le i\le k} d_{G_i}(x_i)^2. \qquad (2.3)
\end{aligned}$$

By (2.1)–(2.3),

$$\begin{aligned}
i(G) &\le k+|I|\\
&\le k+\Delta(G)\gamma(G)-k\Delta(G)-\sum_{1\le i\le k} d_{G_i}(x_i)^2\\
&= \Delta(G)\gamma(G) - \sum_{1\le i\le k}(\Delta(G)-1+d_{G_i}(x_i)^2)\\
&= \Delta(G)\gamma(G) - \sum_{1\le i\le k}\frac{\Delta(G)-1+d_{G_i}(x_i)^2}{d_{G_i}(x_i)+1}\cdot(d_{G_i}(x_i)+1). \qquad (2.4)
\end{aligned}$$

By Lemma 2.1, for each $1\le i\le k$,

$$\frac{\Delta(G)-1+d_{G_i}(x_i)^2}{d_{G_i}(x_i)+1} \ge 2\sqrt{\Delta(G)}-2. \qquad (2.5)$$

By (2.4) and (2.5),

$$\begin{aligned}
i(G) &\le \Delta(G)\gamma(G) - \sum_{1\le i\le k}\frac{\Delta(G)-1+d_{G_i}(x_i)^2}{d_{G_i}(x_i)+1}\cdot(d_{G_i}(x_i)+1)\\
&\le \Delta(G)\gamma(G) - \sum_{1\le i\le k}(2\sqrt{\Delta(G)}-2)(d_{G_i}(x_i)+1)\\
&= \Delta(G)\gamma(G) - (2\sqrt{\Delta(G)}-2)\sum_{1\le i\le k}(d_{G_i}(x_i)+1)\\
&= \Delta(G)\gamma(G) - (2\sqrt{\Delta(G)}-2)\gamma(G).
\end{aligned}$$

Therefore $i(G)/\gamma(G) \le \Delta(G)-2\sqrt{\Delta(G)}+2$.

This completes the proof of Theorem 1.1. □

3 Concluding remarks

By inspection, Theorem 1.1 is weaker than Theorem A for the case where $\Delta(G)\in\{3,5\}$. However, in such case, we can also show Theorem A by using the proof technique of Theorem 1.1.

Let G be a graph with $\Delta(G) \in \{3, 5\}$, and let D, G_i and x_i be as in the proof of Theorem 1.1. Then (2.4) holds. By simple calculations, for a function $f(x) = \frac{\Delta(G)-1+x^2}{x+1}$ and an integer $x \geq 0$, we have

$$f(x) \geq \min\{f(\lfloor\sqrt{\Delta(G)}\rfloor - 1), f(\lceil\sqrt{\Delta(G)}\rceil - 1)\}.$$

If $\Delta(G) = 3$, then $f(\lfloor\sqrt{\Delta(G)}\rfloor - 1) = f(0) = 2$ and $f(\lceil\sqrt{\Delta(G)}\rceil - 1) = f(1) = 3/2$ (i.e. $f(x) \geq 3/2$ for $x \in \mathbb{N}$); if $\Delta(G) = 5$, then $f(\lfloor\sqrt{\Delta(G)}\rfloor - 1) = f(1) = 5/2$ and $f(\lceil\sqrt{\Delta(G)}\rceil - 1) = f(2) = 8/3$ (i.e. $f(x) \geq 5/2$ for $x \in \mathbb{N}$). In either case, we have $f(x) \geq \Delta(G)/2$ for $x \in \mathbb{N}$, and hence $\frac{\Delta(G)-1+d_{G_i}(x_i)^2}{d_{G_i}(x_i)+1} \geq \Delta(G)/2$ for each $1 \leq i \leq k$. This together with (2.4) leads to $i(G) \leq \Delta(G)\gamma(G)/2$. Consequently, we get another proof of the case where $\Delta(G) \in \{3, 5\}$ in Theorem A.

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