Spanning trees with vertices having large degrees

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Abstract

Let G be a connected simple graph, and let f be a mapping from $V(G)$ to the set of integers. This paper is concerned with the existence of a spanning tree in which each vertex v has degree at least $f(v)$. We show that if $\left|\Gamma_G(S)\right| - f(S) + |S| \geq 1$ for any nonempty subset $S \subseteq L$, then a connected graph G has a spanning tree such that $d_T(x) \geq f(x)$ for all $x \in V(G)$, where $\Gamma_G(S)$ is the set of neighbors v of vertices in S with $v \notin S$, $L = \{x \in V(G) : f(x) \geq 2\}$, and $d_T(x)$ is the degree of x in T. This is an improvement of several results, and the condition is best possible.

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1 Introduction

Because of its relationship to problems on Hamiltonian cycles, a spanning tree with degree constraint is one of the most important and attractive topics in graph theory and computer science. In particular, conditions for the existence of spanning trees satisfying upper bounds on degrees have widely been studied, and it is known that the toughness type necessary condition is "almost" sufficient [2, 6]. In contrast, little is known about spanning trees satisfying lower bounds on degrees. The purpose of this paper is to study a necessary and "almost" sufficient condition for the existence

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of spanning trees with such a property. As will be explained later, the existence of such a spanning tree is related to that of a Hamiltonian path with prescribed end vertices, which motivates the study of trees satisfying lower bounds.

Throughout this paper, we consider only simple graphs. Let G be a graph. For $x \in V(G)$, a vertex $v \in V(G) - \{x\}$ is called a *neighbor of x in G* if v is adjacent to x in G. We let $N_G(x)$ denote the set of neighbors of x in G. The degree of x is denoted by $d_G(x)$; thus $d_G(x) = |N_G(x)|$. For $S \subseteq V(G)$, let $N_G(S) := \bigcup_{x \in S} N_G(x)$ and $\Gamma_G(S) := N_G(S) - S$. Let $G[S]$ be the subgraph of G induced by S. We denote the number of components of G by $\omega(G)$.

Let f be a mapping from $V(G)$ to the set of integers. In this paper, we concentrate on the existence of a spanning tree T in G such that

$$d_T(x) \geq f(x) \text{ for all } x \in L, \tag{1}$$

where $L = \{x \in V(G) : f(x) \geq 2\}$. (In this context, the mapping f is referred to as a *lower capacity vector* or a *demand vector* in some papers.) Since any vertex in a tree has degree at least one, condition (1) is equivalent to the condition that "$d_T(x) \geq f(x)$ for all $x \in V(G)$".

Here we explain the relation between a spanning tree satisfying condition (1) and a Hamiltonian path with prescribed end vertices. Let G be a graph with two specified vertices u and v, and let f be a mapping from $V(G)$ to $\{1, 2\}$ such that $f(x) = 2$ for all $x \in V(G) - \{u, v\}$ and $f(u) = f(v) = 1$. Then we see that a spanning tree T in G satisfies condition (1) if and only if T is a Hamiltonian path in G with end vertices u and v. (In fact, if T is a Hamiltonian path with end vertices u and v, then T certainly satisfies condition (1). Conversely, if T satisfies condition (1), then both u and v must be leaves of T and T must be a path, because any tree of order at least two has at least two leaves and any vertex of degree at least three in the tree "creates" a new leaf.)

For the existence of a spanning tree satisfying (1), we have the following necessary condition; see, for example, [1, 4]. For $S \subseteq V(G)$, let $f(S) := \sum_{x \in S} f(x)$.

Proposition 1 *Let G be a connected graph, let f be a mapping from $V(G)$ to the set of integers, and let $L = \{x \in V(G) : f(x) \geq 2\}$. If there exists a spanning tree in G satisfying condition* (1), *then for any nonempty subset $S \subseteq L$,*

$$\bigl|\Gamma(S)\bigr| - f(S) + 2|S| - \omega\bigl(G[S]\bigr) \geq 1. \tag{2}$$

One might expect that the above necessary condition would also be sufficient. However, this is not the case: consider the graph G in Figure 1 with $f(x) = 2$ for $x = x_i$ with $i = 1, 2, 3, 4$ and $f(x) = 1$ otherwise. Then G satisfies condition (2), but has no spanning tree satisfying condition (1). Thus, condition (2) is not sufficient. On the other hand, the following theorem shows that condition (2) is "almost" sufficient.

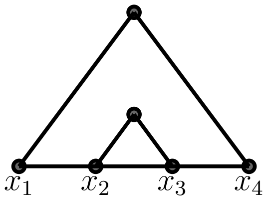

Figure 1: The graph G.

Theorem 2 (Singh and Lau [5]) *Let G be a connected graph, let f be a mapping from $V(G)$ to the set of integers, and let $L = \{x \in V(G) : f(x) \geq 2\}$. If for any nonempty subset $S \subseteq L$,*

$$\big|\Gamma(S)\big| - f(S) + |S| - \omega(G[S]) \geq 1,$$

then there exists a spanning tree in G satisfying condition (1).

Note that Theorem 2 is obtained as a corollary of the result in [5] that finds a minimum cost spanning tree satisfying certain degree constraints. It should also be noticed that the proof in [5] (and hence also the proof of Theorem 2) depends on arguments on the LP (linear programming) relaxation of an integer programming. Hence it is hard to find the ideas in the proof from the standpoint of graph theory. Also spanning trees satisfying (1) have a strong connection with Hamiltonian paths, which play an important role in various aspects of graph theory. Considering these situations, in this paper, we give the following result that improves Theorem 2, together with a graph theoretic proof of the result.

Theorem 3 *Let G be a connected graph, let f be a mapping from $V(G)$ to the set of integers, and let $L = \{x \in V(G) : f(x) \geq 2\}$. If for any nonempty subset $S \subseteq L$,*

$$\big|\Gamma(S)\big| - f(S) + |S| \geq 1,$$

then there exists a spanning tree in G satisfying condition (1).

Before proving Theorem 3 in Section 2, we introduce several results related to it. Frank and Gyárfás [3], and independently, Kaneko and Yoshimoto [4] showed that Theorem 3 holds in the case where L is an independent set. Moreover, the condition in Theorem 3 is best possible, since when L is independent, it is equivalent to necessary condition (2) in Proposition 1. (Note that $\omega\big(G[S]\big) = |S|$ when S is an independent set.) As a related result, the authors [1] recently showed that necessary condition (2) is also sufficient if $G[L]$ has no P_4 as an induced subgraph.

2 Proof of Theorem 3

In this section, we show the following theorem, which implies Theorem 3.

Theorem 4 *Let G be a graph and let $M \subseteq L \subseteq V(G)$. Let f be a mapping from L to the set of integers, and suppose that for any nonempty subset $S \subseteq L$,*

$$g(S; G, M, f) := \left|\Gamma_G(S) - M\right| - f(S) + |S| - \max\left\{|M \cap S|, 1\right\} \geq 0.$$

Then there exists a forest F in G such that

(L1) $V(F) = L \cup \Gamma_G(L)$,

(L2) $d_F(x) \geq f(x)$ *for each vertex* $x \in L$,

(L3) $d_F(v) \geq 1$ *for each vertex* $v \in \Gamma_G(L)$, *and*

(L4) *each component of F contains at most one vertex in M.*

Theorem 3 is obtained from Theorem 4 as follows. Suppose that Theorem 4 holds. Let G be a graph satisfying the condition in Theorem 3. Letting $M = \emptyset$, we see that G satisfies the condition in Theorem 4. Hence G has a forest F satisfying conditions (L1)–(L4). Let F' be the forest with $V(F') = V(G)$ and $E(F') = E(F)$ (that is to say, we obtain F' from F by adding all vertices in $V(G) - V(F)$ as isolated vertices). Adding edges connecting different components of F' until the resultant graph is connected, we obtain a spanning tree satisfying condition (1), by condition (L2) for F. This completes the proof of Theorem 3.

Thus it suffices to prove Theorem 4.

Proof of Theorem 4. We prove Theorem 4 by induction on $|L| + |E(G)|$. If $|L| = 0$ (in particular, if $|L| + |E(G)| = 0$), there is nothing to prove. Thus we may assume that $L \neq \emptyset$, which, in particular, implies that $|L| + |E(G)| \geq 1$.

For a nonempty subset $S \subseteq L$, we say that S is a *tight set* if $g(S; G, M, f) = 0$. We first show the following claim.

Claim 1 *If there exists no tight set L_0 with $L_0 \neq L$, then G has a forest satisfying conditions (L1)–(L4).*

Proof. Suppose that there exists no tight set L_0 with $L_0 \neq L$. This implies that for any nonempty subset $S \subseteq L$ with $S \neq L$, $g(S; G, M, f) \geq 1$.

Suppose first that there exists an edge xy in $G[L]$. Let $G' := G - xy$. We will show that for any nonempty subset $S \subseteq L$, we have $g(S; G', M, f) \geq 0$. Let S be a nonempty subset of L. If S contains both x and y, or neither x nor y, then $\Gamma_{G'}(S) = \Gamma_G(S)$, and hence $g(S; G', M, f) = g(S; G, M, f) \geq 0$. On the other hand, if S contains exactly one of x and y, then $g(S; G', M, f) \geq 1$ (because $S \neq L$) and $|\Gamma_{G'}(S) - M| \geq |\Gamma_G(S) - M| - 1$, and hence $g(S; G', M, f) \geq g(S; G, M, f) - 1 \geq 0$. In either case, we have $g(S; G', M, f) \geq 0$. Hence by applying the induction hypothesis to G', L, M and f, we see that the graph G' (and hence G) has a forest satisfying conditions (L1)–(L4) in Theorem 4, and we are done.

Hence we may assume that there exists no edge in $G[L]$. Suppose next that $N_G(x) \cap N_G(y) \neq \emptyset$ for some $x, y \in L$ with $x \neq y$. Let $v \in N_G(x) \cap N_G(y)$, and let $G' := G - xv$. Arguing as above, we see that for any nonempty subset $S \subseteq L$, we have $g(S; G', M, f) \geq 0$. Hence by applying the induction hypothesis to G', L, M and f, we see that the graph G' (and hence G) has a forest satisfying conditions (L1)–(L4) in Theorem 4, and we are again done.

Thus, we may also assume that $N_G(x) \cap N_G(y) = \emptyset$ for all vertices $x, y \in L$ with $x \neq y$. Let F be the subgraph of G induced by all edges incident with a vertex in L. Since $G[L]$ has no edge and $N_G(x) \cap N_G(y) = \emptyset$ for all vertices $x, y \in L$, F is a forest in which each component is a star with center in L. Hence conditions (L1), (L3) and (L4) in Theorem 4 are satisfied. Moreover, for each $x \in L$, since $g(\{x\}; G, M, f) = |\Gamma_G(x) - M| - f(x) + |\{x\}| - \max\big\{|M \cap \{x\}|, 1\big\} \geq 0$, we have $d_F(x) = |\Gamma_G(x) - M| \geq f(x)$. Hence F also satisfies condition (L2). This completes the proof of Claim 1. $\square$

By Claim 1, we may assume that there exists a tight set L_0 with $L_0 \neq L$. Take a tight set L_0 so that $|L_0|$ is as large as possible under the condition that $L_0 \neq L$. Let $M_0 := M \cap L_0$, and $G_0 := G - (M - M_0)$. Note that for any nonempty subset $S \subseteq L_0$,

$$\begin{aligned} g(S; G_0, M_0, f) &= \big|\Gamma_{G_0}(S) - M_0\big| - f(S) + |S| - \max\big\{|M_0 \cap S|, 1\big\} \\ &= \big|\Gamma_G(S) - M\big| - f(S) + |S| - \max\big\{|M \cap S|, 1\big\} \\ &= g(S; G, M, f) \quad \geq \quad 0. \end{aligned}$$

Hence by applying the induction hypothesis to G_0, L_0, M_0 and f, we see that G_0 has a forest F_0 such that (L1) $V(F_0) = L_0 \cup \Gamma_{G_0}(L_0)$, (L2) $d_{F_0}(x) \geq f(x)$ for all $x \in L_0$, (L3) $d_{F_0}(v) \geq 1$ for each vertex $v \in \Gamma_{G_0}(L_0)$, and (L4) each component of F_0 contains at most one vertex in M_0.

Let $L_1 := L - L_0$, $M_1 := M - M_0 = M \cap L_1$, $M_1' := \Gamma_{G_0}(L_0) \cap L_1$, $\widetilde{M_1} := M_1 \cup M_1'$, $G_1 := G - \big(L_0 \cup \big(\Gamma_{G_0}(L_0) - M_1'\big)\big)$, and for each x in L_1, let

$$f_1(x) := \begin{cases} f(x) - 1 & \text{if } x \in M_1', \\ f(x) & \text{if } x \in L_1 - M_1'. \end{cases}$$

Now we show the following claim, which will play a crucial role in our proof.

Claim 2 *For any subset $S \subseteq L_1$,*

$$g(L_0 \cup S; G, M, f) \leq g(S; G_1, \widetilde{M_1}, f_1) - |M_1' \cap S| + 1.$$

Furthermore, if equality holds, then $M_0 \cap L_0 = \emptyset$ or $\widetilde{M_1} \cap S = \emptyset$.

Proof. If $S = \emptyset$, then we can easily see that the claim holds. Hence we may assume that $S \neq \emptyset$.

First note that

$$\begin{aligned} |\Gamma_G(L_0) - M| &= \big|\Gamma_G(L_0) - L\big| + \big|\Gamma_G(L_0) \cap (L - S) - M\big| + \big|\Gamma_G(L_0) \cap S - M\big|, \\ \text{and} \quad \big|\Gamma_{G_1}(S) - \widetilde{M_1}\big| &= \big|\Gamma_{G_1}(S) - L_1\big| + \big|\Gamma_{G_1}(S) \cap L_1 - \widetilde{M_1}\big|. \end{aligned}$$

$$\begin{aligned}
\text{Since} \quad & \left|\Gamma_G(L_0) - L\right| + \left|\Gamma_{G_1}(S) - L_1\right| \\
&= \left|\Gamma_G(L_0) - L\right| + \left|\Gamma_G(S) - (L_0 \cup \Gamma_G(L_0)) - L_1\right| \\
&= \left|\Gamma_G(L_0 \cup S) - L\right|, \\
& \left|\Gamma_G(L_0) \cap (L - S) - M\right| + \left|\Gamma_{G_1}(S) \cap L_1 - \widetilde{M}_1\right| \\
&= \left|\Gamma_G(L_0) \cap (L - S) - M\right| + \left|\Gamma_G(S) \cap L_1 - (L_0 \cup \Gamma_G(L_0)) - \widetilde{M}_1\right| \\
&= \left|\Gamma_G(L_0) \cap (L_1 - S) - M\right| + \left|\Gamma_G(S) \cap (L_1 - S) - \Gamma_G(L_0) - M\right| \\
&= \left|\Gamma_G(L_0 \cup S) \cap L - M\right|, \\
\text{and} \quad & \Gamma_G(L_0) \cap S - M = M_1' \cap S,
\end{aligned}$$

we have

$$\begin{aligned}
&|\Gamma_G(L_0) - M| + \left|\Gamma_{G_1}(S) - \widetilde{M}_1\right| \\
&\qquad = \left|\Gamma_G(L_0 \cup S) - L\right| + \left|\Gamma_G(L_0 \cup S) \cap L - M\right| + \left|M_1' \cap S\right| \\
&\qquad = \left|\Gamma_G(L_0 \cup S) - M\right| + \left|M_1' \cap S\right|. \qquad (3)
\end{aligned}$$

Note that

$$\begin{aligned}
f(L_0 \cup S) &= f(L_0) + f(S) \\
&= f(L_0) + f_1(S) + |M_1' \cap S|, \qquad (4) \\
\text{and} \quad & \max\left\{|M \cap (L_0 \cup S)|, 1\right\} \\
&= \max\left\{|M_0 \cap L_0| + |\widetilde{M}_1 \cap S| - |M_1' \cap S|, 1\right\} \\
&\geq \max\left\{|M_0 \cap L_0|, 1\right\} + \max\left\{|\widetilde{M}_1 \cap S|, 1\right\} - |M_1' \cap S| - 1. \qquad (5)
\end{aligned}$$

Hence by inequalities (3)–(5) and the fact that $g(L_0; G, M, f) = 0$,

$$\begin{aligned}
& g(L_0 \cup S; G, M, f) \\
&= \left|\Gamma_G(L_0 \cup S) - M\right| - f(L_0 \cup S) + |L_0 \cup S| - \max\left\{|M \cap (L_0 \cup S)|, 1\right\} \\
&\leq \left(\left|\Gamma_G(L_0) - M\right| + \left|\Gamma_{G_1}(S) - \widetilde{M}_1\right| - |M_1' \cap S|\right) \\
&\qquad - \left(f(L_0) + f_1(S) + |M_1' \cap S|\right) + \left(|L_0| + |S|\right) \\
&\qquad - \Big(\max\left\{|M \cap L_0|, 1\right\} + \max\left\{|\widetilde{M}_1 \cap S|, 1\right\} - |M_1' \cap S| - 1\Big) \\
&= \left(\left|\Gamma_G(L_0) - M\right| - f(L_0) + |L_0| - \max\left\{|M_0 \cap L_0|, 1\right\}\right) \\
&\qquad + \left(\left|\Gamma_{G_1}(S) - \widetilde{M}_1\right| - f_1(S) + |S| - \max\left\{|\widetilde{M}_1 \cap S|, 1\right\}\right) - |M_1' \cap S| + 1 \\
&= g(L_0; G, M, f) + g(S; G_1, \widetilde{M}_1, f_1) - |M_1' \cap S| + 1 \\
&= g(S; G_1, \widetilde{M}_1, f_1) - |M_1' \cap S| + 1.
\end{aligned}$$

Therefore the first statement of Claim 2 holds. Moreover, if equality in the above inequality holds, then so does the one in inequality (5), which implies that $|M_0 \cap L_0| = 0$ or $|\widetilde{M}_1 \cap S| = 0$. This completes the proof of Claim 2. □

Using Claim 2, we show that G_1 almost satisfies the condition in Theorem 4.

Claim 3 *For any nonempty subset $S \subseteq L_1$ with $S \neq L_1$, we have $g(S; G_1, \widetilde{M_1}, f_1) \geq 0$.*

Proof. Suppose that there exists a nonempty subset $S \subseteq L_1$ such that $S \neq L_1$ and $g(S; G_1, \widetilde{M_1}, f_1) \leq -1$. Then by Claim 2, we have

$$g(L_0 \cup S; G, M, f) \leq g(S; G_1, \widetilde{M_1}, f_1) - |M_1' \cap S| + 1 \leq 0.$$

On the other hand, by the assumption of Theorem 4, $g(L_0 \cup S; G, M, f) \geq 0$, which implies that $g(L_0 \cup S; G, M, f) = 0$. Hence $L_0 \cup S$ is also a tight set. However, since $S \neq L_1$, we have $L_0 \cup S \neq L$, which contradicts the maximality of L_0. This completes the proof of Claim 3. □

Now we divide the proof into two cases, depending on the value of $g(L_1; G_1, \widetilde{M_1}, f_1)$.

Case 1. $g(L_1; G_1, \widetilde{M_1}, f_1) \geq 0$.

By Claim 3 and the assumption of Case 1, for any nonempty subset $S \subseteq L_1$, we have $g(S; G_1, \widetilde{M_1}, f_1) \geq 0$. Hence by applying the induction hypothesis to G_1, L_1, $\widetilde{M_1}$ and f_1, we see that G_1 has a forest F_1 such that (L1) $V(F_1) = L_1 \cup \Gamma_{G_1}(L_1)$, (L2) $d_{F_1}(x) \geq f_1(x)$ for all $x \in L_1$, (L3) $d_{F_1}(v) \geq 1$ for each vertex $v \in \Gamma_{G_1}(L_1)$, and (L4) each component of F_1 contains at most one vertex in $\widetilde{M_1}$.

Let F be the subgraph of G with $V(F) = V(F_0) \cup V(F_1)$ and $E(F) = E(F_0) \cup E(F_1)$. Note that

$$V(F) = \big(L_0 \cup \Gamma_{G_0}(L_0)\big) \cup \big(L_1 \cup \Gamma_{G_1}(L_1)\big) = L \cup \Gamma_G(L),$$

and hence F satisfies condition (L1). Let $x \in L$. If $x \in L_0$, then $d_F(x) = d_{F_0}(x) \geq f(x)$; if $x \in L_1 - M_1'$, then $d_F(x) = d_{F_1}(x) \geq f_1(x) = f(x)$; otherwise, that is, if $x \in M_1'$, then $d_F(x) = d_{F_0}(x) + d_{F_1}(x) \geq f_1(x) + 1 = f(x)$. Thus $d_{F_0}(x) \geq f(x)$ for all $x \in L$, which implies that F satisfies condition (L2). By condition (L3) for F_0 and F_1, we see that F also satisfies condition (L3). Let T be a component of F. We show that T is a tree containing at most one vertex in M. If T consists of only edges in F_0 or only edges in F_1, then T is a component of F_0 or F_1, and hence T is a tree containing at most one vertex in M. Therefore, we may assume that T contains edges in both F_0 and F_1. Since $V(F_0) \cap V(F_1) = M_1'$, this implies that T contains a vertex in M_1'. By condition (L4) for F_1, each component of $T \cap F_1$ contains at most one vertex in M_1'. Since T is connected, this implies that $T \cap F_0$ consists of a single component of F_0, and each component of $T \cap F_1$ contains precisely one vertex in M_1'. Since F_1 satisfies (L4) with M replaced by M_1, it follows that $T \cap F_1$ contains no vertex in M_1. Since F_0 satisfies (L4), it also follows that $T \cap F_0$ contains at most one vertex in M_0. Therefore, T is a tree containing at most one vertex in M. Consequently, F is a forest in G satisfying condition (L4). Hence F is a forest with desired properties. □

Case 2. $g(L_1; G_1, \widetilde{M_1}, f_1) \leq -1$.

In this case, it follows from Claim 2 that

$$0 \leq g(L; G, M, f) \leq g(L_1; G_1, \widetilde{M_1}, f_1) - |M_1' \cap L_1| + 1 \leq -|M_1' \cap L_1|,$$

and hence $\Gamma_{G_0}(L_0) \cap L_1 = M_1' \cap L_1 = \emptyset$ and equality holds. (So $\widetilde{M_1} = M_1$, and $f_1(x) = f(x)$ for all $x \in L_1'$.) Hence

$$\begin{aligned} -1 &= g(L_1; G_1, \widetilde{M_1}, f_1) \\ &= \left|\Gamma_{G_1}(L_1) - \widetilde{M_1}\right| - f_1(L_1) + |L_1| - \max\left\{|\widetilde{M_1} \cap L_1|, 1\right\} \\ &= \left|\Gamma_{G_1}(L_1) - M_1\right| - f(L_1) + |L_1| - \max\left\{|M_1 \cap L_1|, 1\right\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} 0 &\leq g(L_1; G, M, f) \\ &= \left|\Gamma_G(L_1) - M\right| - f(L_1) + |L_1| - \max\left\{|M \cap L_1|, 1\right\} \\ &= \left|\Gamma_G(L_1) - M\right| - f(L_1) + |L_1| - \max\left\{|M_1 \cap L_1|, 1\right\}, \end{aligned}$$

which implies that

$$\begin{aligned} &\left|\Gamma_G(L_1) - M\right| - \left|\Gamma_{G_1}(L_1) - M_1\right| \\ &\geq \left(f(L_1) - |L_1| + \max\left\{|M_1 \cap L_1|, 1\right\}\right) - \left(f(L_1) - |L_1| + \max\left\{|M_1 \cap L_1|, 1\right\} - 1\right) \\ &= 1. \end{aligned}$$

Hence there exists a vertex v such that $v \in \Gamma_G(L_1) - M$ and $v \notin \Gamma_{G_1}(L_1) - M_1$. Let $G_2 := G[V(G_1) \cup \{v\}]$. Let $S \subseteq L_1$ be a nonempty subset. If $S \neq L_1$, then by Claim 3, we have $g(S; G_2, M_1, f_1) \geq g(S; G_1, M_1, f_1) \geq 0$. On the other hand, if $S = L_1$, then $g(S; G_2, M_1, f_1) = g(S; G_1, M_1, f_1) + 1 = 0$. In either case, $g(S; G_2, M_1, f_1) \geq 0$. Hence by applying the induction hypothesis to G_2, L_1, M_1 and f_1, G_2 has a forest F_2 such that (L1) $V(F_2) = L_1 \cup \Gamma_{G_2}(L_1)$, (L2) $d_{F_2}(x) \geq f_1(x)$ for all $x \in L_1$, (L3) $d_{F_2}(v) \geq 1$ for each vertex $v \in \Gamma_{G_2}(L_1)$, and (L4) each component of F_2 contains at most one vertex in M_1.

Let F be the subgraph of G with $V(F) = V(F_0) \cup V(F_2)$ and $E(F) = E(F_0) \cup E(F_2)$. Since $\Gamma_{G_0}(L_0) \cap L_1 = \emptyset$, we have $V(F_0) \cap V(F_2) = \{v\}$, and hence F is a forest. Note that

$$V(F) = V(F_0) \cup V(F_2) = \left(L_0 \cup \Gamma_{G_0}(L_0)\right) \cup \left(L_1 \cup \Gamma_{G_2}(L_1)\right) = L \cup \Gamma_G(L),$$

and hence condition (L1) holds. It follows from conditions (L2) and (L3) for F_0 and F_2 that $d_F(x) \geq f(x)$ for each vertex $x \in L$, and $d_F(v) \geq 1$ for each vertex $v \in \Gamma_G(L)$, and hence F satisfies conditions (L2) and (L3). By the second statement of Claim 2, $M_0 \cap L_0 = M_0 = \emptyset$ or $\widetilde{M_1} \cap L_1 = M_1 = \emptyset$. Therefore, since F_0 and F_2 satisfy condition (L4) and $V(F_0) \cap V(F_2) = \{v\}$, F also satisfies condition (L4). This completes the proof of Theorem 4. $\square$

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