

# A relationship between Thomassen's conjecture and Bondy's conjecture

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## Abstract

In 1986, Thomassen posed the following conjecture: every 4-connected line graph has a Hamiltonian cycle. As a possible approach to the conjecture, many researchers have considered statements that are equivalent or related to it. One

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of them is the conjecture by Bondy: there exists a constant  $c_0$  with  $0 < c_0 \leq 1$  such that every cyclically 4-edge-connected cubic graph  $H$  has a cycle of length at least  $c_0|V(H)|$ . It is known that Thomassen's conjecture implies Bondy's conjecture, but nothing about the converse has been shown. In this paper, we show that Bondy's conjecture implies a slightly weaker version of Thomassen's conjecture: every 4-connected line graph with minimum degree at least 5 has a Hamiltonian cycle.

Keywords: Hamiltonian cycles, line graphs, Thomassen's conjecture, dominating cycles, Bondy's conjecture,

## 1 Introduction

The motivation of this paper is the following well-known conjecture due to Thomassen.

**Conjecture 1 (Thomassen [21])** *Every 4-connected line graph has a Hamiltonian cycle.*

As a possible approach to Conjecture 1, many researchers have considered statements that are equivalent or related to it. For example, Ryjáček [19] showed that Conjecture 1 is equivalent to the conjecture by Matthews and Sumner [18] stating that every 4-connected claw-free graph has a Hamiltonian cycle. The conjecture by Ash and Jackson [1] stating that every cyclically 4-edge-connected cubic graph has a dominating cycle, is also known to be equivalent to Conjecture 1, see the paper by Fleischner and Jackson [9]. Recall that for an integer  $k$ , a graph  $G$  is called *cyclically  $k$ -edge-connected* if deleting any  $k - 1$  edges from  $G$  does not create two components having a cycle. A *dominating cycle*  $C$  of a graph  $G$  is one such that for any edge  $e$  of  $G$ , at least one of the end vertices of  $e$  is contained in  $C$ . See [4, 8, 15, 16, 20] for other results and conjectures, and also a survey [6].

In addition to those conjectures that are equivalent to Conjecture 1, it is known that Conjecture 1 implies the following conjecture.

**Conjecture 2 (Bondy, see [9])** *There exists a constant  $c_0$  with  $0 < c_0 \leq 1$  such that every cyclically 4-edge-connected cubic graph  $H$  has a cycle of length at least  $c_0|V(H)|$ .*

**Proposition 3** *If Conjecture 1 is true, then Conjecture 2 is also true.*

For the proof of Proposition 3, see Section 3 in a survey [6]. Indeed, we can prove Proposition 3, by combining the argument on the relation between line graphs and preimage graphs (see Section 2 in this paper), the result in [9] (see Theorem 6 in this paper), and an observation that every dominating cycle in a cubic graph  $H$  has length at least  $\frac{3}{4}|V(H)|$ . (Recall that for a line graph  $G$  of a graph  $H$ ,  $H$  is called the *preimage* graph of  $G$ , or sometimes called the *root* graph of  $G$ .) Therefore if Conjecture 1 is true, then Conjecture 2 is also true with  $c_0 = \frac{3}{4}$ .

Although Proposition 3 holds, nothing about the converse has been shown. In fact, the converse of Proposition 3 was posed in [6] as an open problem. In this paper, we

focus on this problem, and show that Conjecture 2 implies the following conjecture, which is a slightly weaker version of Conjecture 1.

**Conjecture 4** *Every 4-connected line graph with minimum degree at least 5 has a Hamiltonian cycle.*

**Theorem 5** *If Conjecture 2 is true, then Conjecture 4 is also true.*

Zhan [24], and independently Jackson [13] proved that every 7-connected line graph has a Hamiltonian cycle, and several researchers [12, 23] have shown results on Hamiltonicity of 6-connected line graphs with additional conditions on the set of vertices of degree exactly 6. Recently, Kaiser and Vrána [14] improved this result by showing that every 5-connected line graph with minimum degree at least 6 has a Hamiltonian cycle. Theorem 5 suggests that Conjecture 2 is more difficult than the result by Kaiser and Vrána [14].

This paper is organized as follows. In Section 2, we give several definitions and lemmas for the proof of Theorem 5. The proof of Theorem 5 appears in Section 3, and is divided into two theorems (Theorems 10 and 11). We will prove them in Sections 4 and 5, respectively. In the last section of this paper (Section 6), we give a conclusion and open problems concerning Theorem 5.

## 2 Preliminaries

In this paper, we consider only finite graphs that may have multiple edges, but no loops. For terminology and notation not defined in this paper, we refer readers to [7].

Let  $H$  be a graph. A closed trail  $T$  in  $H$  is called a *dominating closed trail* in  $H$  if for any edge  $e$  of  $H$ , at least one of the end vertices of  $e$  is contained in  $T$ . (Note that in case that  $T$  is a cycle, we call  $T$  a *dominating cycle*.) In [11], it is shown that for a connected graph  $H$  with  $|E(H)| \geq 3$ ,  $H$  has a dominating closed trail if and only if the line graph of  $H$  has a Hamiltonian cycle. For a closed trail  $T$  in a graph  $H$ ,  $\text{dom}_H(T)$  denotes the number of edges  $e$  in  $H$  such that at least one of the end vertices of  $e$  is contained in  $T$ . Specifically,  $T$  is a dominating closed trail in  $H$  if and only if  $\text{dom}_H(T) = |E(H)|$ .

An *edge-cut* of a connected graph  $H$  is an inclusionwise minimal set of edges whose removal makes  $H$  disconnected. An *essential edge-cut*  $X$  (resp. *cyclic edge-cut*) of a connected graph  $H$  is an edge-cut such that  $H - X$  has exactly two components of orders at least 2 (resp. exactly two components having a cycle). For a positive integer  $k$ , a connected graph  $H$  is called *essentially  $k$ -edge-connected* if  $H$  has no essential edge-cut  $X$  with  $|X| \leq k - 1$ . It is known that a graph  $H$  is essentially  $k$ -edge-connected if and only if the line graph of  $H$  is  $k$ -connected or  $H$  is a complete graph, see Section 3 in [6]. Recall that a graph  $H$  is said to be *cyclically  $k$ -edge-connected* if  $H$  has no cyclic edge-cut  $X$  with  $|X| \leq k - 1$ . It is well-known that for an integer  $k$  with  $k \leq 4$ , a cubic graph  $H$  is essentially  $k$ -edge-connected if and only if  $H$  is cyclically  $k$ -edge-connected.

The *edge degree* of an edge  $e$  in a graph  $H$  is defined as the number of edges adjacent with  $e$ . Note that we count edges multiple  $e$  only once, if exist. Hence the edge degree of  $e$  in  $H$  corresponds to the degree of  $e$  in the line graph of  $H$ .

The above arguments directly imply that Conjecture 4 is equivalent to the following conjecture. (In this paper, conjectures with preimage-line graph relation have same number with prime.)

**Conjecture 4'** *Every essentially 4-edge-connected graph with minimum edge degree at least 5 has a dominating closed trail.*

Let  $H$  be a graph. An edge  $e$  of  $H$  is called a *pendant edge* if one of the end vertices of  $e$  has degree exactly 1 in  $H$ . For a vertex  $v$  in  $H$ , we denote, by  $\deg_H(v)$  and  $\text{pen}_H(v)$ , the degree of  $v$  in  $H$  and the number of pendant edges incident with  $v$  in  $H$ , respectively. For a vertex  $v$  of degree exactly 2 in a graph  $H$ , *suppressing*  $v$  is an operation to replace the path  $u_1vu_2$  in  $H$  with an edge connecting  $u_1$  and  $u_2$ , where  $u_1$  and  $u_2$  are the neighbors of  $v$ . Note that suppressing a vertex may create multiple edges. For an integer  $k$ , we denote, by  $V_k(H)$ ,  $V_{\geq k}(H)$  and  $V_{\leq k}(H)$ , the set of vertices of degree exactly  $k$ , at least  $k$  and at most  $k$  in  $H$ , respectively.

Let  $H$  be a graph and  $v \in V_{\geq 4}(H)$ , and let  $u^1, u^2, \dots, u^d$  (where  $d = \deg_H(v)$ ) be an ordering of neighbors of  $v$  (we allow repetition in case of parallel edges). Then the graph obtained from the disjoint union of  $H - v$  and the cycle  $C_v = v^1v^2 \dots v^dv^1$  by adding the edges  $u^iv^i$  for each  $1 \leq i \leq d$  is called an *inflation of  $H$  at  $v$* . If  $\delta(H) \geq 3$ , then, by successively taking an inflation at each vertex of degree greater than 3, we obtain a cubic graph  $H^I$ , called a *cubic inflation of  $H$* . An inflation of a graph at a vertex is not unique (since it depends on the ordering of neighbors of  $v$ ) and it might happen that the operation decreases the edge-connectivity of the graph. However, the following was proven in [9].

**Theorem 6 (Fleischer and Jackson [9])** *Let  $H$  be an essentially 4-edge-connected graph with  $\delta(H) \geq 3$ . Then some cubic inflation of  $H$  is cyclically 4-edge-connected.*

We also need the following lemma in Section 4. Note that a very similar lemma can be found in [17, 22].

**Lemma 7** *Let  $H$  be a cyclically 4-edge-connected cubic graph. Let  $u_1u_2$  and  $v_1v_2$  be two edges in  $H$  such that  $u_i \neq v_j$  for  $i, j = 1, 2$ . Let  $H'$  be the graph obtained from  $H$  by subdividing the edges  $u_1u_2$  and  $v_1v_2$ , and adding a new edge connecting  $w$  and  $z$ , where  $w$  and  $z$  are the vertices obtained by subdivision of the edges  $u_1u_2$  and  $v_1v_2$ , respectively. Then  $H'$  is also a cyclically 4-edge-connected cubic graph.*

**Proof of Lemma 7.** Suppose not. Then there exists a cyclic edge-cut  $X'$  of  $H'$  with  $|X'| \leq 3$ . Let  $X$  be the set of edges of  $H$  obtained from  $X'$  by deleting the edge  $wz$ , replacing the edge  $wu_i$  with  $u_1u_2$  for  $i = 1, 2$ , and replacing the edge  $zv_j$  with  $v_1v_2$  for  $i = 1, 2$ , if  $X'$  contains them, respectively. Since  $X'$  is an edge-cut of  $H'$ , it follows from the construction of  $H'$  and  $X$  that  $X$  is an edge-cut of  $H$ . Since  $|X| \leq |X'| \leq 3$  and  $H$

is cyclically 4-edge-connected, at least one component  $D$  of  $H - X$  cannot contain a cycle. By the construction of  $X$ ,  $H' - X'$  has the component  $D'$  containing all vertices in  $D$ . Since  $X'$  is a cyclic edge-cut of  $H'$ ,  $D'$  has a cycle. However, since  $D$  contains no cycle, we obtain  $w, z \in V(D')$  and  $D'$  has exactly one cycle, which passes through the edge  $wz$ . In particular,  $D'$  has exactly  $|V(D')|$  edges, since  $D'$  is connected and has exactly one cycle. On the other hand, since  $u_i \neq v_j$  for  $i, j = 1, 2$ ,  $D'$  has at least four vertices, that is,  $w, z, u_i$  and  $v_j$  for some  $i, j = 1, 2$ . Hence

$$|X'| = \sum_{v \in V(D')} \deg_{H'}(v) - 2|E(D')| = 3|V(D')| - 2|V(D')| \geq 4,$$

contradicting the assumption that  $|X'| \leq 3$ . This completes the proof of Lemma 7.  $\square$

### 3 Proof of Theorem 5

The proof of Theorem 5 is divided into two parts. To do that, we need the following two “intermediate” conjectures and two theorems. Those might be interesting themselves, see Section 6.

**Conjecture 8** *There exists a constant  $c_1$  with  $0 < c_1 \leq 1$  such that every essentially 4-edge-connected graph  $H$  has a closed trail  $T$  with  $\text{dom}_H(T) \geq c_1|E(H)|$ .*

**Conjecture 9** *There exist a constant  $c_2$  with  $0 < c_2 \leq 1$  and an integer  $k$  with  $k \geq 5$  such that every essentially 4-edge-connected graph  $H$  with minimum edge degree at least  $k$  has a closed trail  $T$  with  $\text{dom}_H(T) \geq c_2|E(H)|$ .*

**Theorem 10** *If Conjecture 2 is true, then Conjecture 8 is also true.*

**Theorem 11** *If Conjecture 9 is true, then Conjecture 4 is also true.*

We can easily see that if Conjecture 8 is true, then Conjecture 9 is true. Hence Theorems 10 and 11 imply Theorem 5. We will show Theorems 10 and 11 in Sections 4 and 5, respectively.

### 4 Proof of Theorem 10

Suppose that Conjecture 2 is true. Then there exists a constant  $c_0$  with  $0 < c_0 \leq 1$  such that every cyclically 4-edge-connected cubic graph  $H$  has a cycle of length at least  $c_0|V(H)|$ . Let

$$c_1 = \frac{c_0}{12}.$$

We will show that every essentially 4-edge-connected graph  $H$  has a closed trail  $T$  with  $\text{dom}_H(T) \geq c_1|E(H)|$ . Let  $H$  be an essentially 4-edge-connected graph. We construct

the new graph  $\tilde{H}$  by suppressing all vertices of degree 2 in  $H$ . Note that  $\tilde{H}$  is an essentially 4-edge-connected graph without vertices of degree 2.

If  $\tilde{H}$  is a star, then the assumption that  $H$  is essentially 4-edge-connected implies that  $\tilde{H} = H$  and the center of  $H$  forms a closed trail  $T$  with  $\text{dom}_H(T) = |E(H)|$ . Hence we may assume that  $\tilde{H}$  is not a star. Let  $H_0$  be the graph obtained from  $\tilde{H}$  by deleting all pendant edges of  $\tilde{H}$ . Note that  $H_0$  is essentially 4-edge-connected, and  $\delta(H_0) \geq 3$  since  $\tilde{H}$  has no vertices of degree 2. Since  $\tilde{H}$  is essentially 4-edge-connected,  $\text{pen}_{\tilde{H}}(v) = 0$  for  $v \in V_3(H_0)$ ; Otherwise the set of edges in  $H_0$  incident with  $v$  is an essential edge-cut of size 3 in  $\tilde{H}$ , a contradiction. Notice also that for each  $v \in V(\tilde{H})$ ,  $\text{deg}_{\tilde{H}}(v) = \text{deg}_{H_0}(v) + \text{pen}_{\tilde{H}}(v)$ , in particular,  $\text{deg}_{\tilde{H}}(v) = \text{deg}_{H_0}(v)$  if  $\text{deg}_{\tilde{H}}(v) = 3$ . This implies that

$$|E(\tilde{H})| = |E(H_0)| + \sum_{v \in V_{\geq 4}(H_0)} \text{pen}_{\tilde{H}}(v). \quad (1)$$

Now from  $H_0$ , we construct new graphs  $H_1$  and  $H_2$  as follows. First using Theorem 6 to  $H_0$ , we obtain a cubic inflation  $H_1$  of  $H_0$  such that  $H_1$  is cyclically 4-edge-connected. Clearly,  $|E(H_1)| \geq |E(H_0)|$ . Let  $v \in V_{\geq 4}(H_0)$ . Note that  $v$  corresponds to the cycle  $C_v$  in  $H_1$  of length exactly  $\text{deg}_{H_0}(v)$ . Let  $v^1, v^2, v^3, v^4$  be four consecutive vertices of  $C_v$  and let  $p = \text{pen}_{\tilde{H}}(v)$ . We subdivide the edges  $v^1v^2$  and  $v^3v^4$  exactly  $p$  times, and obtain the paths  $v^1v_1v_2 \dots v_pv^2$  and  $v^3v_{p+1}v_{p+2} \dots v_{2p}v^4$ , respectively. Then we add an edge connecting  $v_i$  and  $v_{2p+1-i}$  for  $1 \leq i \leq p$ . See Figure 1. We perform the above operation to all vertices  $v$  in  $V_{\geq 4}(H_0)$ , and let  $H_2$  be the obtained graph. Using Lemma 7 repeatedly (more precisely, using Lemma 7 exactly  $\sum_{v \in V_{\geq 4}(H_0)} \text{pen}_{\tilde{H}}(v)$  times), we see that  $H_2$  is a cyclically 4-edge-connected cubic graph.

Let  $D_v = V(C_v) \cup \{v_i : 1 \leq i \leq 2p\}$  for every vertex  $v \in V_{\geq 4}(H_0)$ . To simplify the argument, we let  $D_v = \{v\}$  for a vertex  $v$  in  $V_3(H_0)$ . Then for each vertex  $v$  in  $H_0$ , we have

$$|D_v| = \begin{cases} \text{deg}_{H_0}(v) + 2 \text{pen}_{\tilde{H}}(v) & \text{if } \text{deg}_{H_0}(v) \geq 4, \\ 1 & \text{if } \text{deg}_{H_0}(v) = 3. \end{cases} \quad (2)$$

Clearly from the construction, for each  $v \in V_{\geq 4}(H_0)$ , there are at least  $\text{pen}_{\tilde{H}}(v)$  edges inside of  $D_v$ . Hence by equality (1),

$$\begin{aligned} |E(H_2)| &\geq |E(H_1)| + \sum_{v \in V_{\geq 4}(H_0)} \text{pen}_{\tilde{H}}(v) \\ &\geq |E(H_0)| + \sum_{v \in V_{\geq 4}(H_0)} \text{pen}_{\tilde{H}}(v) = |E(\tilde{H})|. \end{aligned} \quad (3)$$

Since  $H_2$  is a cubic graph,

$$|E(H_2)| = \frac{3}{2}|V(H_2)|. \quad (4)$$

Since we assumed that Conjecture 2 is true,  $H_2$  has a cycle  $T_2$  of length at least  $c_0|V(H_2)|$ , that is,

$$|V(T_2)| \geq c_0|V(H_2)|. \quad (5)$$

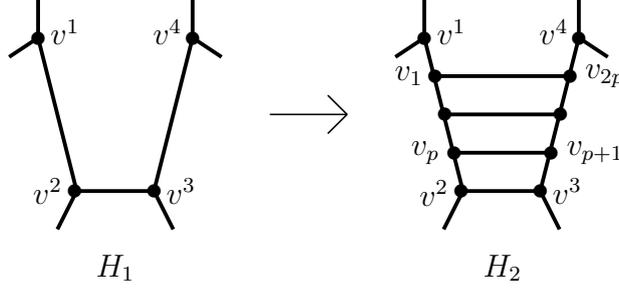


Figure 1: A cubic inflation  $H_1$  of  $H_0$  and the graph  $H_2$ .

Let  $U_{T_2}(H_0)$  be the set of vertices  $v$  of  $H_0$  with  $D_v \cap V(T_2) \neq \emptyset$ . Note that  $V(T_2) \subset \bigcup_{v \in U_{T_2}(H_0)} D_v$ , and hence  $|V(T_2)| \leq \sum_{v \in U_{T_2}(H_0)} |D_v|$ . Let  $\tilde{T}$  be the subgraph of  $\tilde{H}$  obtained from  $T_2$  by contracting all vertices in  $D_v$  into one vertex for each  $v \in U_{T_2}(H_0) \cap V_{\geq 4}(H_0)$  and deleting all loops (but we keep multiple edges if exist). Note that  $\tilde{T}$  is a closed trail of  $\tilde{H}$  and  $V(\tilde{T}) = U_{T_2}(H_0)$ . Since each edge is dominated by  $\tilde{T}$  from at most two end vertices of it, we have  $2 \cdot \text{dom}_{\tilde{H}}(\tilde{T}) \geq \sum_{v \in V(\tilde{T})} \text{deg}_{\tilde{H}}(v)$ . Then it follows from the above arguments and (in)equalities (2)–(5) that

$$\begin{aligned}
\text{dom}_{\tilde{H}}(\tilde{T}) &\geq \frac{1}{2} \sum_{v \in U_{T_2}(H_0)} \text{deg}_{\tilde{H}}(v) \\
&= \frac{1}{2} \sum_{v \in U_{T_2}(H_0) \cap V_{\geq 4}(H_0)} \left( \text{deg}_{H_0}(v) + \text{pen}_{\tilde{H}}(v) \right) + \frac{1}{2} \sum_{v \in U_{T_2}(H_0) \cap V_3(H_0)} \text{deg}_{H_0}(v) \\
&\geq \frac{1}{4} \sum_{v \in U_{T_2}(H_0) \cap V_{\geq 4}(H_0)} \left( \text{deg}_{H_0}(v) + 2 \text{pen}_{\tilde{H}}(v) \right) + \frac{3}{2} \sum_{v \in U_{T_2}(H_0) \cap V_3(H_0)} 1 \\
&= \frac{1}{4} \sum_{v \in U_{T_2}(H_0) \cap V_{\geq 4}(H_0)} |D_v| + \frac{3}{2} \sum_{v \in U_{T_2}(H_0) \cap V_3(H_0)} |D_v| \\
&\geq \frac{1}{4} \sum_{v \in U_{T_2}(H_0)} |D_v| \geq \frac{1}{4} |V(T_2)| \\
&\geq \frac{c_0}{4} |V(H_2)| = \frac{c_0}{6} |E(H_2)| \geq \frac{c_0}{6} |E(\tilde{H})|.
\end{aligned}$$

Then  $\tilde{T}$  is a closed trail of  $\tilde{H}$  with  $\text{dom}_{\tilde{H}}(\tilde{T}) \geq \frac{c_0}{6} |E(\tilde{H})|$ .

We obtain the closed trail  $T$  of  $H$  by subdividing all suppressed edges in  $\tilde{T}$ . Note that  $\text{dom}_H(T) \geq \text{dom}_{\tilde{H}}(\tilde{T})$ . On the other hand, since  $H$  is essentially 4-edge-connected, there are no two consecutive vertices of degree 2 in  $H$ . Hence each edge of  $\tilde{H}$  is obtained by suppressing a vertex of degree 2 in  $H$  at most once, and hence  $|E(\tilde{H})| \geq \frac{1}{2} |E(H)|$ . These imply that

$$\text{dom}_H(T) \geq \text{dom}_{\tilde{H}}(\tilde{T}) \geq \frac{c_0}{6} |E(\tilde{H})| \geq c_1 |E(H)|.$$

This holds for every essentially 4-edge-connected graph  $H$ , and hence Conjecture 8 is also true. This completes the proof of Theorem 10.  $\square$

## 5 Proof of Theorem 11

By the argument in Section 2, it is enough to show that if Conjecture 4' is false, then Conjecture 9 is also false. Suppose that Conjecture 4' is false. Then there exists an essentially 4-edge-connected graph  $H$  with minimum edge degree at least 5 such that  $H$  has no dominating closed trail. If there exists an edge  $e$  of  $H$  connecting two vertices in  $V_{\leq 3}(H)$ , then the edge degree of  $e$  is at most 4, contradicting the minimum edge degree condition on  $H$ . Hence there exists no such edge  $e$  of  $H$ . Therefore if  $H$  has a closed trail  $T$  that passes through all vertices in  $V_{\geq 4}(H)$ , then  $T$  is a dominating closed trail in  $H$ , contradicting the choice of  $H$ . Hence we have the following claim.

**Claim 1** *For any closed trail  $T$  in  $H$ , there exists a vertex  $v$  in  $V_{\geq 4}(H)$  such that  $v$  is not visited by  $T$ .*

We construct an infinite sequence of graphs  $H_0, H_1, \dots$  as follows; Let  $H_0 = H$ , and take any vertex  $v$  in  $V_{\geq 4}(H)$ . For  $i \geq 1$ , the graph  $H_i$  is obtained from  $H_{i-1}$  and  $|V_{\geq 4}(H_{i-1})|$  copies of  $H$  by identifying each vertex in  $V_{\geq 4}(H_{i-1})$  and the vertex  $v$  in a copy of  $H$ . Since  $\deg_H(v) \geq 4$ , for  $i \geq 0$ ,  $H_i$  is an essentially 4-edge-connected graph with minimum edge degree at least 5. Notice also that  $|V_{\geq 4}(H_i)| = h \cdot |V_{\geq 4}(H_{i-1})|$ , where  $h := |V_{\geq 4}(H)|$ . Since  $|V_{\geq 4}(H_0)| = h$ , we have

$$|V_{\geq 4}(H_i)| = h^{i+1}. \quad (6)$$

For a graph  $H'$ , let  $f_{\geq 4}(H')$  be the maximum number of vertices  $v$  in  $V_{\geq 4}(H')$  such that  $v$  is visited by a closed trail  $T'$ , where  $T'$  is taken over all closed trails in  $H'$ . The following claim plays a crucial role in the proof of Theorem 11.

**Claim 2**  $f_{\geq 4}(H_i) \leq (h-1)^{i+1}$ .

**Proof.** First we show that  $f_{\geq 4}(H_i) \leq (h-1) \cdot f_{\geq 4}(H_{i-1})$ . Let  $T_i$  be any closed trail in  $H_i$ . Let  $T_{i-1}$  be the closed trail in  $H_{i-1}$  such that  $T_{i-1}$  is the restriction of  $T_i$  on  $H_{i-1}$ . By the definition of  $f_{\geq 4}(H_{i-1})$ ,  $T_{i-1}$  visits at most  $f_{\geq 4}(H_{i-1})$  vertices in  $V_{\geq 4}(H_{i-1})$ . Let  $u$  be a vertex in  $V_{\geq 4}(H_{i-1})$  that is visited by  $T_{i-1}$ . By the above argument, we have at most  $f_{\geq 4}(H_{i-1})$  choices for such a vertex  $u$ . Let  $H_u$  be the copy of  $H$  that is added to  $u$  when we construct  $H_i$  from  $H_{i-1}$ , and let  $T_u$  be the closed trail in  $H_u$  such that  $T_u$  is the restriction of  $T_i$  on  $H_u$ . By Claim 1, at most  $h-1$  vertices in  $V_{\geq 4}(H_u)$  can be visited by  $T_u$ . Hence  $T_i$  can visit at most  $(h-1) \cdot f_{\geq 4}(H_{i-1})$  vertices in  $V_{\geq 4}(H_i)$ . This implies that  $f_{\geq 4}(H_i) \leq (h-1) \cdot f_{\geq 4}(H_{i-1})$ .

Since  $f_{\geq 4}(H_0) \leq h-1$  by Claim 1, we obtain  $f_{\geq 4}(H_i) \leq (h-1)^{i+1}$ . This completes the proof of Claim 2.  $\square$

Now we are ready to show that Conjecture 9 does not hold. Let  $c_2$  be any constant with  $0 < c_2 \leq 1$  and  $k$  be any integer with  $k \geq 5$ . Since

$$\lim_{i \rightarrow \infty} \frac{1 + (h-1)^{i+1}}{1 + h^{i+1}} = \lim_{i \rightarrow \infty} \frac{\frac{1}{h^{i+1}} + \left(1 - \frac{1}{h}\right)^{i+1}}{\frac{1}{h^{i+1}} + 1} = 0,$$

there exists an integer  $i$  such that

$$\frac{1 + (h - 1)^{i+1}}{1 + h^{i+1}} < c_2. \quad (7)$$

For a non-negative integer  $t$ , let  $H_i(t)$  be the graph obtained from  $H_i$  by adding  $t$  pendant edges to all vertices in  $V_{\geq 4}(H_i)$ . Since we added pendant edges only to vertices in  $V_{\geq 4}(H_i)$ ,  $H_i(t)$  is still essentially 4-edge-connected, and moreover, the minimum edge degree of  $H_i(t)$  is at least  $t + 3$ . Note that  $|E(H_i(t))| = |E(H_i)| + t \cdot |V_{\geq 4}(H_i)|$ .

Let  $T$  be any closed trail in  $H_i(t)$ . By the definition of  $f_{\geq 4}(H_i)$ ,  $T$  can pass through at most  $f_{\geq 4}(H_i)$  vertices in  $V_{\geq 4}(H_i)$ . Hence  $T$  can dominate at most  $t \cdot f_{\geq 4}(H_i)$  of added pendant edges. This implies that for any closed trail  $T$  in  $H_i(t)$ , we have  $\text{dom}_{H_i(t)}(T) \leq |E(H_i)| + t \cdot f_{\geq 4}(H_i)$ .

Let  $H' = H_i(m)$ , where  $m = \max\{|E(H_i)|, k - 3\}$ . Note that  $H'$  is essentially 4-edge-connected and the minimum edge degree of  $H_i(m)$  is at least  $k$ . Then by equality (6), Claim 2, and inequality (7), for each closed trail  $T'$  in  $H'$ ,

$$\begin{aligned} \frac{\text{dom}_{H'}(T')}{|E(H')|} &\leq \frac{|E(H_i)| + m \cdot f_{\geq 4}(H_i)}{|E(H_i)| + m \cdot |V_{\geq 4}(H_i)|} \\ &\leq \frac{1 + (h - 1)^{i+1}}{1 + h^{i+1}} < c_2. \end{aligned}$$

This means that for each constant  $c_2$  with  $0 < c_2 \leq 1$  and each integer  $k$  with  $k \geq 5$ , there exists an essentially 4-edge-connected graph  $H'$  with minimum edge degree at least  $k$  such that any closed trail  $T'$  in  $H'$  satisfies  $\text{dom}_{H'}(T') < c_2 |E(H')|$ . So Conjecture 9 does not hold. This completes the proof of Theorem 11.  $\square$

**Remark:** In the proof of Theorem 11, assuming that Conjecture 4' is not true, we construct, for each constant  $c_2$  with  $0 < c_2 \leq 1$ , the graph  $H'$  with  $\frac{\text{dom}_{H'}(T')}{|E(H')|} < c_2$  for any dominating closed trail  $T'$  in  $H'$ . We point out here that if Conjecture 4' is not true, then the correct magnitude of  $\max_{T'} \text{dom}_{H'}(T')$  is at most  $|E(H')|^\alpha$ , where  $\alpha = \log_h(h - 1)$ . Indeed, the proof of Theorem 11 also shows that if Conjecture 4' is not true, then for any integer  $k$  with  $k \geq 5$ , there exist infinitely many essentially 4-edge-connected graphs  $H'$  with minimum edge degree at least  $k$  such that for any closed trail  $T'$  in  $H'$ ,  $\text{dom}_{H'}(T') = O(|E(H')|^\alpha)$ . Recall that  $h = |V_{\geq 4}(H)|$  and  $H$  is a counterexample of Conjecture 4'.

It should be mentioned here that Blinski, Jackson, Ma, and Yu [3] recently showed that every essentially 3-edge-connected graph  $H'$  has a closed trail  $T'$  with  $\text{dom}_{H'}(T') \geq \left(\frac{|E(H')|}{12}\right)^\beta + 2$ , where  $\beta \approx 0.753$ , consider the preimage version of Theorem 1.2 in [3]. So, if Conjecture 4' is not true, then the gap of bounds on  $\max_{T'} \left(\frac{\log \text{dom}_{H'}(T')}{|E(H')|}\right)$  between the essentially 4-edge-connected case and the essentially 3-edge-connected case would be only the difference between  $\alpha$  and  $\beta$ .

## 6 Conclusion and open problems

In this paper, we have shown that Conjecture 2 implies Conjecture 4, which is a weaker version of Conjecture 1. Together with Proposition 3, we see the following situation;

$$\text{Conjecture 1} \xrightarrow{\text{Proposition 3}} \text{Conjecture 2} \xrightarrow{\text{Theorem 5}} \text{Conjecture 4}$$

However, we do not know about the converse of these two implications. Indeed, as mentioned in Section 1, the converse of Proposition 3 appeared in [6] as an open problem. In addition to that, we left an open problem on the converse of Theorem 5.

**Problem 12** *Is Conjecture 4 equivalent to Conjecture 2, or moreover to Conjecture 1?*

On the other hand, now we point out that Theorem 11 gives a corollary concerning Conjecture 4. It shows the equivalence of Conjecture 4 and the following conjecture, which is the line graph version of Conjecture 9, see Section 2. This corollary might be interesting itself.

**Conjecture 9'** *There exists a constant  $c_3$  with  $0 < c_3 \leq 1$  and an integer  $k$  with  $k \geq 5$  such that every 4-connected line graph  $G$  with  $\delta(G) \geq k$  has a cycle of length at least  $c_3|V(G)|$ .*

**Corollary 13** *Conjecture 4 is equivalent to Conjecture 9'.*

**Proof.** It is easy to see that if Conjecture 4 is true, then Conjecture 9' is also true with  $c_3 = 1$  and  $k = 5$ . On the other hand, suppose that Conjecture 9' is true. Since Conjecture 9 is the preimage version of Conjecture 9', Conjecture 9 is also true with  $c_2 = c_3$ . By Theorem 11, Conjecture 4 is also true.  $\square$

Thus, by Corollary 13, in order to solve Conjecture 4 instead of a Hamiltonian cycle, it is enough to find a cycle of length  $c_3$  times the order of a given graph, even for  $c_3 = 1/1000000000$ , assuming high minimum degree. We hope that Corollary 13 gives a step to solve Conjecture 4.

Note that the similar situation can be found for Barnette's conjecture [2]; every 3-connected cubic plane bipartite graph has a Hamiltonian cycle. Recently, Harant [10] proved that Barnette's conjecture is equivalent to the following statement; there exists a constant  $c'$  with  $0 < c' \leq 1$  such that every 3-connected cubic plane bipartite graph  $G$  has a cycle of length at least  $c'|V(G)|$ .

Now we consider the above situation for Conjecture 1. It is shown in [5] that Conjecture 1 is equivalent to the following statement; there exists a function  $f$  such that  $\lim_{n \rightarrow \infty} \frac{f(n)}{n} = 0$ , and every 4-connected line graph of order  $n$  has a cycle of length at least  $n - f(n)$ . So, in order to solve Conjecture 1, it is enough to find a cycle of length at least  $n - f(n)$  in 4-connected line graphs. However, we do not know if it is enough to find a cycle of length linear on the order of a graph. Indeed, we can consider the following conjecture, which is analogous to Conjecture 9' and seemingly

weaker than Conjecture 1. Considering Corollary 13, we expect that Conjecture 8' is equivalent to Conjecture 1, and leave it as an open problem.

**Conjecture 8'** *There exists a constant  $c_4$  with  $0 < c_4 \leq 1$  such that every 4-connected line graph  $G$  has a cycle of length at least  $c_4|V(G)|$ .*

**Problem 14** *Is Conjecture 8' equivalent to Conjecture 1?*

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