$\{0, 2\}$-Degree Free Spanning Forests in Graphs [*†]

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Abstract

Let G be a graph and S be a set of non-negative integers. By an S-degree free spanning forest of G we mean a spanning forest of G with no vertex degree in S. In this paper we study the existence of $\{0, 2\}$-degree free spanning forests in graphs. We show that if G is a graph with minimum degree at least 4, then there exists a $\{0, 2\}$-degree free spanning forest. Moreover, we show that every 2-connected graph with maximum degree at least 5 admits a $\{0, 2\}$-degree free spanning forest, and every 3-connected graph admits a $\{0, 2\}$-degree free spanning forest.

1 Introduction

Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The *order* of G denotes the number of vertices. We denote the minimum and maximum degree of the vertices of G by $\delta(G)$ and $\Delta(G)$, respectively. Also, the degree of v in graph G is denoted by $d_G(v)$. For every subset $U \subset V(G)$, we denote the induced subgraph on U by $\langle U \rangle$. Let $N_G(v)$ denote the neighbour set of v in G. The complete bipartite

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graph with two part sizes m and n is denoted by $K_{m,n}$. The complete graph and the cycle of order n are denoted by K_n and C_n, respectively. A *factor* in a graph G is a spanning subgraph of G. A graph is said to be *odd* if all vertices have odd degree. Let S be a set of non-negative integers. By an *S-degree free spanning forest* of G, we mean a spanning subgraph of G with no cycles and no vertex of degree in S. A tree is called a *rooted tree* if one vertex has been designated as the root. The *height* of a vertex in a rooted tree is the distance between that vertex and the root.

The main topic of this paper is motivated by the concept *HIST*. A graph is said to be *homeomorphically irreducible* if it contains no vertices of degree 2. A homeomorphically irreducible spanning tree is called an HIST, which has been much attracted, for example, see [1, 2, 3]. Note that the properties of trees can be divided into two parts; connectedness and acyclicness. This paper focuses on the latter property, and considers a *homeomorphically irreducible spanning forest* or an *HISF* for abbreviation. In this paper, we do not allow a vertex of degree 0 in an HISF, and hence an HISF is exactly a $\{0, 2\}$-degree free spanning forest.

The concept HISF is also closely related to a perfect matching and an odd factor. A perfect matching is a spanning forest in which every vertex has degree exactly one, and hence it is an HISF. On the other hand, an odd factor might have a cycle, but we can easily find an HISF in an odd factor, see Corollary 1 in Section 2.1.

The observations above suggest that the concept HISF is placed just between the concept HIST and the concept perfect matching and odd factor. Many sufficient conditions for the existence of an HIST (for example, [1, 2, 3]) and that of a perfect matching (for example, [7, 8]) have been considered, and the study in this paper follows this direction. Indeed, in this paper we give several sufficient conditions for the existence of an HISF; every graph with minimum degree at least four admits an HISF (Theorem 1), every 2-connected graph of maximum degree at least 5 admits an HISF (Theorem 2), and every 3-connected graph admits an HISF (Theorem 3).

2 Results

In this section, first we prove several lemmas, which will be needed in our proofs, and then we give our main results.

2.1 Preliminary

We start by the following interesting lemma due to [5](Lemma 16.4).

Lemma 1. *If a graph is connected then it has an odd factor if and only if the number of its vertices is even.*

Corollary 1. *Every connected graph of even order admits an odd* HISF*.*

Proof. By Lemma 1, G has an odd factor, say H. If H is a forest, then we are done. Otherwise, we may remove all edges of a cycle from H and continue this procedure until we obtain a forest. Clearly, the resultant graph is an odd HISF and the proof is complete. □

Lemma 2. *For every rooted tree T with root v, there exists a spanning forest F such that for each $u \in V(F) \setminus \{v\},\ d_F(u) \notin \{0, 2\}$.*

Proof. We construct the forest F as follows: First let $E(F) = \varnothing$. In each step we consider all vertices with maximum height whose degree are 0 or 2 in F. Then we add to $E(F)$ those edges in T, which join them to their parents. Clearly, these vertices find degree 1 or 3. We continue this procedure until the vertices of height 1. Thus, the desired forest is obtained. □

Lemma 3. *Let T be a rooted tree with root v and $d_T(v) = 1$. If e is the edge incident with v and there exists a spanning forest H of T such that $d_H(u) \notin \{0, 2\}$, for every $u \in V(T) \setminus \{v\}$ and $d_H(z) \geq 4$ for some vertex z, then there exists a spanning forest H' of T such that $e \in (E(H) \setminus E(H')) \cup (E(H') \setminus E(H))$ and $d_{H'}(u) \notin \{0, 2\}$ for every $u \in V(T) \setminus \{v\}$.*

Proof. Let z be a vertex of T such that $d_H(z) \geq 4$ and $d(z, v)$ has the minimum possible value. Construct the spanning forest H' as follows: Consider the path between z and v and call it by P. Let $E(H') = (E(H) \setminus E(P)) \cup (E(P) \setminus E(H))$. We claim that H' has the desired property. If $u \in V(T) \setminus V(P)$, then $d_H(u) = d_{H'}(u)$. Also if $u \in V(P) \setminus \{v, z\}$, then $d_H(u) \equiv d_{H'}(u) \pmod 2$. Since $d_H(u) \leq 3$ then $d_{H'}(u) \notin \{0, 2\}$. Moreover, $d_H(z) \geq 4$ and so $d_{H'}(z) \geq 3$. □

2.2 A minimum degree condition

The first main result deals with graphs with minimum degree at least 4. Indeed, such graphs always admit an HISF.

Theorem 1. *For every graph G with $\delta(G) \geq 4$, there exists an* HISF *for G. Moreover, this* HISF *has at most one vertex of even degree.*

Proof. Clearly, we can assume that G is a connected graph. If G has even order, then by Corollary 1, there is an HISF with the desired property. Hence assume that G has odd order. For every $I \subset V(G)$, let $R(I)$ be a connected component of $G \setminus I$ which has the maximum order. Assume that $U \subset V(G)$ and

$$|R(U)| = \max |R(X)|,$$

where the maximum takes over all subsets X of $V(G)$ such that $|X| = 5$ and $\Delta(\langle X \rangle) = 4$. Suppose that $U = \{v, u_1, u_2, u_3, u_4\}$, $C = R(U)$ and $vu_i \in E(G)$, for $i = 1, \ldots, 4$. Let $T = G \setminus (U \cup V(C))$ and $V(T) = \{t_1, \ldots, t_{|V(T)|}\}$.

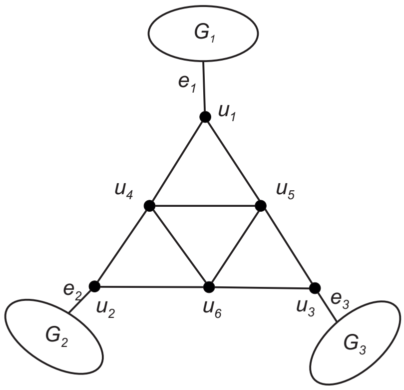
We claim that if $x \in U$ and $N(x) \cap V(C) \neq \varnothing$, then for every t_i, $1 \leq i \leq |V(T)|$, $xt_i \in E(G)$. By contradiction suppose $xt_j \notin E(G)$, for some j. Then t_j has 4 neighbors other than x, say Q. Then we have $|V(R(\{t_j\} \cup Q))| > |V(R(U))|$, which contradicts the choice of U.

Now, two cases can be considered:

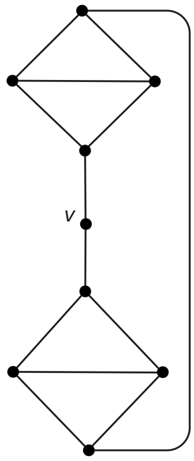
Case 1. C has even order. By Corollary 1, C has an odd HISF, say H_1. Let $x \in U$ and $N(x) \cap V(C) \neq \varnothing$. Now, define the graph H_2 with the vertex set $V(T) \cup U$ and the edge set $\{vu_1, vu_2, vu_3, vu_4\} \cup \{xt_i | 1 \leq i \leq |V(T)|\}$. Clearly, $H_1 \cup H_2$ is an HISF for G. Also since C has even order and G has odd order, $|V(T)|$ is even and if $x \neq v$, then $d_{H_1 \cup H_2}(x)$ is odd. Hence the degree of each vertex of G other than v is odd.

Case 2. C has odd order. Let $x \in U$ and $N(x) \cap V(C) \neq \varnothing$. In this case, by Corollary 1, $\langle V(C) \cup \{x\} \rangle$ has an odd HISF, say H_1. Now, define the graph H_2 with the vertex set $U \cup V(T)$ and the edge set $\{vu_1, vu_2, vu_3, vu_4\} \cup \{xt_i | 1 \leq i \leq |V(T)|\}$. Obviously, $H_1 \cup H_2$ is an HISF for G. In this case, $|V(T)|$ and $d_{H_1}(x)$ are odd. Hence if $x \neq v$, then $d_{H_1 \cup H_2}(x)$ is odd and $d_{H_1 \cup H_2}(v) = d_{H_2}(v)$ is even, and hence the degree of each vertex of G in $H_1 \cup H_2$ other than v is odd. On the other hand, if $x = v$, then the degree of each vertex of G in $H_1 \cup H_2$ other than v is odd. The proof is complete. $\square$

Remark 1. There exists a connected graph G with $\delta(G) = 3$ which has no HISF. To see this, first we introduce a graph G shown in Figure 1a in which G_1, G_2 and G_3 are isomorphic to the graph shown in Figure 1b. Denote this graph by H. Three vertices u_1, u_2 and u_3 are joined to the vertex corresponding to v in G_1, G_2 and G_3, respectively. We claim that G has no HISF. By contradiction, suppose that G admits an HISF, say F. Since H has odd order and $\Delta(H) \leq 3$, all of e_1, e_2 and e_3 are contained in F. Moreover, since G has odd order, one vertex of u_4, u_5 and u_6 has degree 4 in F. Without loss of generality let $d_F(u_4) = 4$. On the other



(a) Graph G which admits no HISF.



(b) Graph H

Figure 1

hand, since $d_F(u_1) \neq 2$, we conclude $u_1u_5 \in E(F)$ and there exists a triangle in F, a contradiction. Hence G has no HISF.

2.3 A maximum degree condition for 2-connected graphs

In the next result we show that every graph with no cut vertex whose maximum degree is large enough admits an HISF.

Theorem 2. *Every 2-connected graph of maximum degree at least 5 admits an* HISF.

Proof. First one may assume that the graph is of odd order. Otherwise, by Corollary 1, there exists an HISF for G. Let $v \in V(G)$ and $d_G(v) \geq 5$. Assume that $N_G(v) = \{v_1, \ldots, v_k\}$. We first show that

(∗) if $G \setminus \{v_1, v_i\}$ is not connected for every i with $2 \leq i \leq k$, then G contains an HISF.

For $2 \leq i \leq k$, let $G_i = G \setminus \big(N_G(v) \setminus \{v_i\}\big)$, and let H_i be the subgraph induced by the vertices u in $G \setminus \{v, v_1\}$ such that there exists a path in G_i between u and v, but there exists no path in G_j between u and v for any $2 \leq j < i$. Note that $v_i \in V(H_i)$. Since $G \setminus v_1$ is connected, we see that $\{H_i |\ i = 2, \dots, k\}$ is a partition of $V(G) \setminus \{v, v_1\}$. We further claim that H_i is connected for $2 \leq i \leq k$.

Suppose that H_i is not connected for some $2 \leq i \leq k$, that is, suppose that there exists a vertex u in H_i such that u and v_i belong to distinct components of H_i. By the condition of H_i, there exists a path P_u in G_i between u and v in G_i. Since u and v_i belong to distinct components of H_i and both are contained in P_u, there exists an edge $u'w$ in P_u such that $u' \in V(H_i)$ and $w \notin V(H_i)$. Let j be an integer with $2 \leq j \leq k$ such that $w \in V(H_j)$. Note that $j \neq i$. Then there exists a path P_w in G_j between w and v, but there is no path in $G_{j'}$ between w and v for any $2 \leq j' < j$. If $j > i$, then the subpath of P_u connecting w and v is a path in G_i, which leads to a contradiction between the second condition with $j' = i$. On the other hand, if $j < i$, then the path obtained from P_w by adding the edge $u'w$ is a path in G_j, which contradicts $u' \in V(H_i)$. Therefore in either case, we have a contradiction, and hence H_i is connected for $2 \leq i \leq k$.

Note that similar arguments imply that there is no edge between $V(H_i) \setminus \{v_i\}$ and $V(H_j) \setminus \{v_j\}$ for $j \neq i$. In fact, if there exists such an edge and $i < j$, then $V(H_j) \setminus \{v_j\}$ should be contained in H_i.

Let T_i be a spanning rooted tree of H_i with root v_i, for $i = 2, \dots, k$. Let T be a spanning tree of $G \setminus \{v_1\}$ with the edge set $\bigcup_{i=2}^{k} E(T_i) \cup \{vv_2, \dots, vv_k\}$. By Lemma 2, T has a spanning forest F in which the degree of each vertex $x \in V(T) \setminus \{v\}$ is neither 0 nor 2.

First we claim that there exists exactly one integer p with $2 \leq p \leq k$ such that H_p has odd order. Since G has odd order, H_p has odd order for at least one $p \in \{2, \ldots, k\}$. Furthermore, if there exist two indexes i and j, such that H_i and H_j have odd order, then it follows from applying Lemma 3 to T_i and T_j that there exists a spanning forest F' of T such that for every $u \in V(T) \setminus \{v\}$, $d_{F'}(u) \notin \{0, 2\}$ and $vv_j, vv_i \in E(F')$. Hence by adding vv_1 to F' we obtain the desired HISF for G and the claim is proved.

Now, we claim that for $2 \leq i \leq k$, there exists a vertex $w_i \in V(H_i)$ with $w_i \neq v_i$ such that $H_i \setminus \{w_i\}$ is connected and $w_i \in N_G(\{v_1, \ldots, v_k\} \setminus \{v_i\})$. Note that, since $G \setminus v_i$ is connected and there is no edge between $V(H_i) \setminus \{v_i\}$ and $V(H_j) \setminus \{v_j\}$ for $j \neq i$, we have $S_i := N_G(\{v_1, \ldots, v_k\} \setminus \{v_i\}) \cap V(H_i) \neq \varnothing$. Now, let $x \in S_i$. If $H_i \setminus \{x\}$ is connected, then we are done. Otherwise, let C be a component of $H_i \setminus \{x\}$ not containing v_i. Since $G \setminus \{x\}$ is connected, $S_i \cap V(C) \neq \varnothing$. Continue this procedure with one vertex of $S_i \cap V(C)$, say x'. Similarly, if $H_i \setminus \{x'\}$ is not connected, then let C' be a component of $H_i \setminus \{x'\}$ not containing v_i. Clearly, $|V(C')| < |V(C)|$ and $S_i \cap V(C') \neq \varnothing$. By continuing this procedure, we obtain a desired vertex w_i.

If there are two indexes i and j such that H_i and H_j have even order and $v_j w_i \in E(G)$, then we claim that there exists an HISF for G. To see this, let T' be a spanning tree of $G \setminus \{v_1, w_i\}$ and F' be the spanning forest of T' obtained from Lemma 2. If $vv_i \notin E(F')$ or $vv_j \in E(F')$, since $|V(H_i) \setminus \{w_i\}|$ is odd and $|V(H_j)|$ is even, then we can use Lemma 3. Hence there exists a spanning forest F'' of T' such that $vv_i \in E(F''), vv_j \notin E(F'')$ and $d_{F''}(u) \notin \{0, 2\}$, for every $u \in V(T') \setminus \{v\}$. Now, if we add $v_j w_i, vv_j$ and vv_1 to F'', then the desired HISF is obtained and the claim is proved.

Hence there exist two indexes i and j such that H_i and H_j have even order and either $v_1 w_i, v_1 w_j \in E(G)$ or $v_p w_i, v_p w_j \in E(G)$. Let T' be the spanning tree of $G \setminus \{v_1, w_i, w_j\}$ and F' be the spanning forest of T' obtained from Lemma 2.

If $vv_i \notin E(F')$ or $vv_j \notin E(F')$, since $|V(H_i) \setminus \{w_i\}|$ and $|V(H_j) \setminus \{w_j\}|$ are odd, then we can use Lemma 3. Hence there exists a spanning forest F'' of T' such that $vv_i, vv_j \in E(F'')$ and $d_{F''}(u) \notin \{0, 2\}$, for every $u \in V(T') \setminus \{v\}$. In the first case that $v_1w_i, v_1w_j \in E(G)$, by adding v_1w_j, v_1w_i and vv_1 to F'' we obtain an HISF for G. In the other case, by adding v_pw_i, v_pw_j and vv_1 to F'' the desired HISF is obtained and this completes the proof of $(*)$.

Therefore one may assume that $G \setminus \{v_1, v_2\}$ is connected. We next claim that if $G \setminus \{v_1, v_2, v_i\}$ is connected for some $i = 3, \ldots, k$, then G contains an HISF. To see this, since $G \setminus \{v_1, v_2, v_i\}$ has an even order, by Corollary 1, it has an HISF F. Now, by adding vv_1, vv_2 and vv_i to F, we obtain the desired HISF for G, and we are done. Hence we may assume that $G \setminus \{v_1, v_2, v_i\}$ is not connected, for $i = 3, \ldots, k$. Now, define H'_i as follows:

Let H'_i be the subgraph induced by the vertices u in $G \setminus \{v, v_1, v_2\}$ such that there exists a path in G_i between u and v, but there exists no path in G_j between u and v for $3 \leq j < i$. Note that $v_i \in V(H_i)$ for $i = 3, \ldots, k$. Since $G \setminus \{v_1, v_2\}$ is connected, we see that $\{H'_i |\ i = 3, \ldots, k\}$ is a partition of $V(G) \setminus \{v, v_1, v_2\}$. By the same arguments as for H_i, we see that H'_i is connected for $3 \leq i \leq k$ and there is no edge between $V(H'_i) \setminus \{v_i\}$ and $V(H'_j) \setminus \{v_j\}$ for $j \neq i$.

Let T'_i be a spanning rooted tree of H'_i with root v_i, for $i = 3, \ldots, k$. Let T' be a spanning tree of $G \setminus \{v_1, v_2\}$ with the edge set $\bigcup_{i=3}^{k} E(T'_i) \cup \{vv_3, \ldots, vv_k\}$. By Lemma 2, T' has a spanning forest F_1 in which the degree of each vertex $x \in V(T') \setminus \{v\}$ is neither 0 nor 2. Now, noting to F_1, two cases may be considered:

(i) $d_{F_1}(v) \neq 0$. In this case add vv_1 and vv_2 to F_1 to obtain an HISF for G.

(ii) $d_{F_1}(v) = 0$. First we claim that one can assume that H'_i has even order, for $i = 3, \ldots, k$. If not, then there is some index j, such that H'_j has odd order. Thus, either $vv_j \in E(F_1)$ which is Case (i) or there is $z \in V(H'_j)$ such that $d_{F_1}(z) \geq 4$.

Now, by Lemma 3, there exists a spanning forest F'_1 of T' such that for every $u \in V(T') \setminus \{v\}$, $d_{F'_1}(u) \notin \{0, 2\}$ and $vv_j \in E(F'_1)$. Hence Case (i) occurs and we are done.

Moreover, we claim that there exists a vertex $z_i \in V(H'_i)$ with $z_i \neq v_i$ such that $H'_i \setminus \{z_i\}$ is connected and $z_i \in N_G(\{v_1, \ldots, v_k\} \setminus \{v_i\})$. Note that, since $G \setminus v_i$ is connected and there is no edge between $V(H'_i) \setminus \{v_i\}$ and $V(H'_j) \setminus \{v_j\}$ for $j \neq i$, we have $S'_i := N_G(\{v_1, \ldots, v_k\} \setminus \{v_i\}) \cap (V(H'_i) \setminus \{v_i\}) \neq \varnothing$. Now, let $x \in S'_i$. If $H'_i \setminus \{x\}$ is connected, then we are done. Otherwise, let C be a component of $H'_i \setminus \{x\}$ not containing v_i. Since $G \setminus \{x\}$ is connected, $S'_i \cap V(C) \neq \varnothing$. Continue this procedure with one vertex of $S'_i \cap V(C)$, say x'. Similarly, if $H'_i \setminus \{x'\}$ is not connected, then let C' be a component of $H'_i \setminus \{x'\}$ not containing v_i. Clearly, $|V(C')| < |V(C)|$ and $S'_i \cap V(C') \neq \varnothing$. By continuing this procedure we obtain z_i.
If one z_i is adjacent to a vertex, say v_p in $\{v_3, \ldots, v_k\} \setminus \{v_i\}$, then we construct an HISF as follows:

Consider the spanning tree T''_1 of $G \setminus \{v_1, v_2, z_i\}$ with root v. Also, let F_2 be the spanning forest of T''_1 obtained from Lemma 2. If $vv_i \notin E(F_2)$ or $vv_p \in E(F_2)$, since $|V(H'_i) \setminus \{z_i\}|$ is odd and $|V(H'_p)|$ is even, then we can use Lemma 3. Hence, there exists a spanning forest F'_2 of T''_1 such that $vv_i \in E(F'_2)$, $vv_p \notin E(F'_2)$ and $d_{F'_2}(u) \notin \{0, 2\}$, for every $u \in V(T''_1) \setminus \{v\}$. Now, by adding vv_1, vv_2, vv_p and $z_i v_p$ to F'_2 we obtain an HISF for G.
So, assume that for every i, $N_G(z_i) \cap \{v_1, v_2\} \neq \varnothing$. Thus, without loss of generality suppose that v_1 is adjacent to z_3 and z_4. (Recall that $k \geq 5$.) Now, consider the spanning tree T''_2 of $G \setminus \{v_1, v_2, z_3, z_4\}$, and the spanning forest F_3 of T''_2. If vv_3 or vv_4 is not contained in $E(F_3)$, since $|V(H'_3) \setminus \{z_3\}|$ and $|V(H'_4) \setminus \{z_4\}|$ are odd, we can use Lemma 3. Hence, there is a spanning forest F'_3 of T''_2 such that $vv_3, vv_4 \in E(F'_3)$ and $d_{F'_3}(u) \notin \{0, 2\}$, for every $u \in V(T''_2) \setminus \{v\}$. Now, by adding $vv_1, vv_2, v_1 z_3$ and $v_1 z_4$ to F'_3 we obtain an HISF for G. The proof is complete. □

Remark 2. Every connected graph G of order n with maximum degree at least $\lceil \frac{n}{2} \rceil + 1$ admits an HISF. To see this, let v be a vertex of maximum degree in G. Let T be a spanning rooted tree of G with root v such that $d_T(v) \geq \lceil \frac{n}{2} \rceil + 1$. Let S be the set of pendant edges incident with v. By Lemma 2, T admits a spanning forest F such that for each $u \in V(F) \setminus \{v\}$, $d_F(u) \notin \{0, 2\}$. Since $d_T(v) \geq \lceil \frac{n}{2} \rceil + 1$, we have $|S| \geq 3$. Since all edges in S have to be used in F, we have $d_F(v) \geq |S| \geq 3$. Hence F is an HISF. Note that this lower bound is sharp. To show this, let T' be a tree with one vertex of degree $\lceil \frac{n}{2} \rceil$ adjacent to two vertices of degree 1 and $\lceil \frac{n}{2} \rceil - 2$ vertices of degree 2. Clearly, T' has no HISF.

2.4 3-connected graphs

In the following result, we consider the 3-connected case.

Theorem 3. *Every* 3*-connected graph admits an* HISF*.*

Proof. Let G be a 3-connected graph. First one may assume that the graph is of odd order, otherwise, by Corollary 1, there exists an HISF for G. If all vertices in G have odd degree, then G is of even order, a contradiction. Hence there exists a vertex $v \in V(G)$ such that $d_G(v) \geq 4$. Assume that $N_G(v) = \{v_1, \ldots, v_k\}$ with $k = d_G(v) \geq 4$. Since G is 3-connected, $G \setminus \{v_1, v_2\}$ is connected. So $G \setminus \{v, v_1, v_2\}$ can be partitioned into $G_3, G_4, \ldots, G_k$ such that for each i with $3 \leq i \leq k$, G_i is connected and contains v_i. For simplifying the argument, for $i = 1, 2$, let G_i be the graph consisting of only v_i. Note that v has no neighbors in $G_i \setminus \{v_i\}$ for each $1 \leq i \leq k$. By symmetry, we may assume that $|V(G_1)|, \ldots, |V(G_p)|$ are odd and $|V(G_{p+1})|, \ldots, |V(G_k)|$ are even for some p, $1 \leq p \leq k$. Since $|V(G_1)| = |V(G_2)| = 1$, we have $p \geq 2$. Note that $\sum_{i=3}^{k} |V(G_i)| = |V(G)| - 3$ is even, and hence p is even. We take such a partition $G_3, \ldots, G_k$ so that

$$p \text{ is as large as possible.} \tag{1}$$

Suppose first $p \geq 4$. Then $|V(G_i) \cup \{v\}|$ is even for each i, $1 \leq i \leq p$, and $|V(G_i)|$ is even for each i, $p+1 \leq i \leq k$. Hence by Corollary 1, $\langle V(G_i) \cup \{v\} \rangle$ has an odd HISF T_i for each i, $1 \leq i \leq p$, and G_i has an odd HISF T_i for each i, $p+1 \leq i \leq k$. Let T be a spanning subgraph of G with the edge set $\bigcup_{i=1}^{k} T_i$. Note that each vertex u in T with $u \neq v$ has odd degree, and $d_T(v) = p \geq 4$. Hence T is an HISF of G.

Hence we may assume $p = 2$, that is, $|V(G_i)|$ is even for all i, $3 \leq i \leq k$. Let $G_3^1, \ldots, G_3^t$ be the components of $G_3 \setminus \{v_3\}$. Since $\sum_{j=1}^{t} |V(G_3^j)| = |V(G_3)| - 1$ is odd, there exists an index j such that $|V(G_3^j)|$ is odd. Say $j = 1$ by symmetry. Note that $G_3 \setminus V(G_3^1)$ is connected. Since $G \setminus \{v_3\}$ is 2-connected, there are at least two vertices u_3^1, u_3^2 in $G \setminus (V(G_3^1) \cup \{v_3\})$ such that $N_G(u_3^1) \cap V(G_3^1) \neq \varnothing$ and $N_G(u_3^2) \cap V(G_3^1) \neq \varnothing$. By the choice of $G_3, G_4, \ldots, G_k$, we have $u_3^1, u_3^2 \neq v$, and by the choice of $G_3^1, \ldots G_3^t$, we have $u_3^1, u_3^2 \notin \bigcup_{j=2}^{t} V(G_3^j)$. Suppose $u_3^1 \in V(G_i)$ for some i with $4 \leq i \leq k$, say $i = 4$ by symmetry. Let $G_3' = G_3 \setminus V(G_3^1)$ and $G_4' = \langle V(G_4) \cup V(G_3^1) \rangle$. Then both G_3' and G_4' are connected, and both $|V(G_3')|$ and $|V(G_4')|$ are odd, which contradict condition (1) for the choice of $G_3, G_4, \ldots, G_k$. Hence $u_3^1, u_3^2 \notin V(G_i)$ for any i, $4 \leq i \leq k$. This (and the symmetry) implies $u_3^1 = v_1$ and $u_3^2 = v_2$.

By the same argument as above for G_4, there exists a component G_4^1 of $G_4 \setminus \{v_4\}$ such that $|V(G_4^1)|$ is odd, and G_4^1 has two neighbors u_4^1 and u_4^2 with $u_4^1 = v_1$ and $u_4^2 = v_2$.

Now, let $G_1' = \langle \{v_1\} \cup V(G_3^1) \cup V(G_4^1) \rangle$, $G_2' = G_2$, $G_3' = G_3 \setminus V(G_3^1)$, $G_4' = G_4 \setminus V(G_4^1)$ and $G_i' = G_i$ for each i, $5 \leq i \leq k$. Then G_i' is connected for each i with $1 \leq i \leq k$, all of $|V(G_1')|, |V(G_2')|, |V(G_3')|$ and $|V(G_4')|$ are odd, and $|V(G_i')|$ is even for each i, $5 \leq i \leq k$. Hence by Corollary 1, $\langle V(G_i') \cup \{v\} \rangle$ has an odd HISF T_i' for each i with $1 \leq i \leq 4$, and G_i' has an odd HISF T_i' for each i, $5 \leq i \leq k$. Let T' be a spanning subgraph of G with the edge set $\bigcup_{i=1}^{k} T_i'$. Then T' is an HISF of G. This completes the proof. □

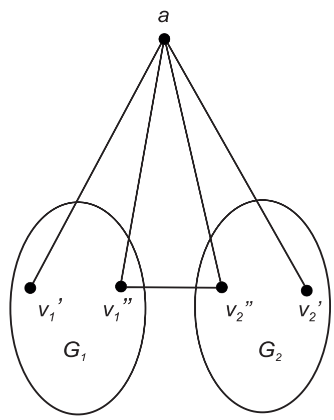


Figure 2: A 2-connected graph G of minimum degree 3 with no HISF

Remark 3. There exists a family of 2-connected graphs of minimum degree 3 with no HISF. To see this, first we construct a 2-connected 3-regular graph H, and from H, we construct the graph H' by splitting a vertex, say v, into two vertices v' and v'', such that $d_{H'}(v') = 2$ and $d_{H'}(v'') = 1$. Now, we introduce the graph G shown in Figure 2 in which G_1 and G_2 are isomorphic to H' and the vertex a is joined to v'_1, v''_1, v'_2, v''_2 which are corresponding vertices to v' and v'' in G_1 and G_2. The graph G is clearly 2-connected and its minimum degree is 3.

We claim that G has no HISF. By contradiction, let F be an HISF of G. Since G has odd number of vertices and $d_G(u) = 3$ for every $u \in V(G) \setminus \{a\}$, we have $d_F(a) = 4$. In particular, we have $v''_1 v''_2 \notin E(F)$. On the other hand, $G'_i \setminus \{v'_i, v''_i\}$ has an odd order, for $i = 1, 2$. Therefore, among the edges between $\{v'_i, v''_i\}$ and $V(G'_i) \setminus \{v'_i, v''_i\}$ for $i = 1, 2$, the HISF F must use exactly 1 or 3 edges. However, since $v''_1 v''_2 \notin E(F)$, these imply that either v'_i or v''_i is of degree 2 in F, a contradiction. So, G has no HISF.

Also, the graph shown in Figure 2 shows that the condition $\Delta(G) \geq 5$ is not superfluous in the statement of Theorem 2.

3 Open Problems

As mentioned in Section 1, the concept HISF is placed just between the concept HIST and the concept perfect matching and odd factor. For the existence of an HIST, it was shown in [1] that the problem of deciding if a given graph has an HIST or not is NP-complete. On the other hand, the problem for $\{0, 2\}$-degree free spanning subgraph can be solved in Polynomial-time [4, 6]. Then the following problem seems interesting.

Problem 1. *Is the problem of deciding if a given graph has an HISF NP-complete, or polynomial-time solvable?*

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