4-connected projective-planar graphs are Hamiltonian-connected

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ABSTRACT

We generalize the following two seminal results.

(1) Thomassen's result [14] in 1983, which says that every 4-connected planar graph is Hamiltonian-connected (which generalizes the old result of Tutte [15] in 1956, which says that every 4-connected planar graph is Hamiltonian).

(2) Thomas and Yu's result [11] in 1994, which says that every 4-connected projective-planar graph is Hamiltonian.

Here, *Hamiltonian-connected* means that for any two vertices u, v, there is a Hamiltonian path between u and v (and hence this generalizes the existence of Hamiltonian cycles).

Specifically, we prove the following;

Every 4-connected projective-planar graph is Hamiltonian-connected.

This proves a conjecture of Dean [3] in 1990. Our result is best possible in many senses. First, we cannot lower the connectivity 4. Secondly, we cannot generalize our result to a surface with higher genus (that is, there is a 4-connected graph on the torus or on the Klein bottle which is not Hamiltonian-connected).

Our proof is constructive in a sense that there is a polynomial time algorithm to find, given two vertices in a 4-connected projective-planar graph, a Hamiltonian path between these two vertices.

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1 Introduction

1.1 Hamiltonian cycles and Hamiltonian paths in graphs on surfaces

The study of Hamiltonian cycles was started with the connection to the famous Four Color Problem (now Theorem). It had been conjectured since the 1910's that every 3-connected cubic planar graph has

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a Hamiltonian cycle, and if true, this would imply the Four Color Problem. However, Tutte [16] in 1946 constructed a counterexample. Since then, finding a Hamiltonian cycle in planar graphs and graphs on surfaces is one of the most active topics in graph theory. As we see here, the study of Hamiltonian cycles for planar graphs is historically one of the central topics in graph theory. In the last decade, a Hamiltonian cycle in planar graphs is also studied in graph algorithm ([6], for example), because it is connected to the traveling salesmen problem.

It is now known that a 3-connected planar graph does not always have a Hamiltonian cycle, since there exist planar triangulations on n vertices whose longest cycle is of length $O(n^{\alpha})$, where $\alpha = \log 2/\log 3 \approx 0.63$ [8]. However, if one considers 4-connectivity for planar graphs, the situation dramatically changes. Whitney [17] was the first to prove that every planar triangulation with no separating triangles has a Hamiltonian cycle, and Tutte [15] proved that every 4-connected planar graph has a Hamiltonian cycle. Extending Tutte's technique, Thomassen [14] proved that every 4-connected planar graph is in fact Hamiltonian-connected, that is, there is a Hamiltonian path connecting any two prescribed vertices (a small omission in [14] was corrected by Chiba and Nishizeki [1]). Chiba and Nizhizeki [1] pointed out that Thomassen's proof implies a polynomial time algorithm to find, given a 4-connected planar graph, a Hamiltonian path between any specified vertices.

With some additional techniques and new ideas, Thomas and Yu [11] proved that every edge in a 4-connected projective-planar graph is contained in a Hamiltonian cycle, which establishes a conjecture of Grünbaum [4]. Grünbaum [4] and Nash-Williams [9] also conjectured that every 4-connected graph on the torus has a Hamiltonian cycle. While this remains open, it is shown in [12] that every 5-connected graph on the torus has a Hamiltonian cycle. Recently, it is shown in [13] that there is a Hamiltonian path in 4-connected graphs on the torus. Very recently, in [5], we prove that every 4-connected triangulation on the torus has a Hamiltonian cycle. Let us point out that there are 4-connected graphs that do not have a Hamiltonian cycle in the double torus or in the surface obtained from the sphere by adding three crosscaps. Thus this conjecture, if true, would be best possible.

On the other hand, for 4-connected graphs on the torus, certain edges may not be contained in any Hamiltonian cycle. The following example was provided by Thomassen [14]. Embed the Cartesian product of two even cycles (of length at least 4) in the torus so that every face is bounded by a cycle of length 4, and add an edge joining two non-adjacent vertices in some facial cycle. Then this new edge is not contained in any Hamiltonian cycle of the new graph. (Note that the original graph is a balanced bipartite graph and the new edge connects two vertices in the same bipartition.) Knowing such a graph, Dean [3] conjectured that every 4-connected projective-planar graph is Hamiltonian-connected. The main result of this paper is to show his conjecture, which generalizes the above mentioned Thomassen's result [14] as well as Thomas and Yu's result [11].

Theorem 1 *Every* 4*-connected graph embedded on the projective plane is Hamiltonian-connected.*

Moreover, our proof is constructive in a sense that a polynomial time algorithm to construct a Hamiltonian path between any two specified vertices can be delivered, if an input graph is 4-connected and projective-planar.

Let us observe that Theorem 1 is best possible in many senses. As we pointed out, there is a 4-connected graph embedded on the torus such that certain edges may not be contained in any Hamiltonian cycle. Similarly, we can construct 4-connected graphs embedded on the Klein bottle with the same property. So Theorem 1 would not extend to any surfaces of higher genus. Secondly, there are 3-connected planar graphs that do not have a Hamiltonian cycle. (For example, consider the face subdivision of triangulations of the projective plane.) So, 4-connectivity is definitely necessary.

Technically, we shall prove Theorem 1 by adapting the notion *Tutte subgraph*. In order to state our technical result which implies Theorem 1, we need some definitions that will be given in the next section.

1.2 A technical statement

A *block* in a graph is a maximal connected subgraph without a cut vertex. Note that any block is 2-connected, unless it is the complete graph of order at most 2. For a graph G, a pair (K_1, K_2) of subgraphs of G is called a *separation of* G if $V(G) = V(K_1) \cup V(K_2)$ and each edge of G is contained in exactly one of K_1 and K_2. A separation (K_1, K_2) of G is a *k-separation* if $|V(K_1)|, |V(K_2)| \geq k+1$ and $\bigl|V(K_1) \cap V(K_2)\bigr| = k$. Note that a graph G is k-connected if and only if G has no r-separation for each $r < k$.

For a path P and two vertices $x, y \in V(P)$, $P[x, y]$ denotes the subpath of P between x and y. For a cycle C with fixed direction, and for two vertices $x, y \in V(C)$, $C[x, y]$ denotes the subpath of C between x and y along the direction. For a cycle C of a plane graph, we usually give a direction to C in the clockwise order.

Let T be a subgraph of a graph G. A *T-bridge* of G is either (i) an edge of $G - E(T)$ with both ends on T or (ii) a subgraph of G induced by the edges in a component of $G - V(T)$ together with all edges between that component and T. A T-bridge satisfying (i) is *trivial*, while a T-bridge satisfying (ii) is *non-trivial*. For a T-bridge B of G, the vertices in $V(B) \cap V(T)$ are the *attachments* of B (on T), and let $\mathrm{Nuc}_T(B) = B - V(T)$, which is the subgraph induced by the vertices of B that are not attachments. When T is clear from the context, we simply write $\mathrm{Nuc}(B)$. We regard a vertex subset T of a graph G as a graph with vertex set T and no edges, and similarly define a T-bridge, and so on. We say that T is a *Tutte subgraph* in G if every T-bridge of G has at most three attachments on T. For a subgraph C of G, T is a *C-Tutte subgraph* in G if T is a Tutte subgraph in G and every T-bridge of G containing an edge of C has at most two attachments on T. A *Tutte path* (respectively, a *Tutte cycle*) in a graph is a path (respectively, a cycle) that is a Tutte subgraph.

Let G be a connected graph embedded on a surface $\mathbb{F}^2$. For a face R of G, the walk of G which bounds R is called the *boundary walk* of R. A *facial walk* of G is the boundary walk of some face of G. When G is a plane graph, the boundary walk of the outer face is called the *outer walk* of G. If those are cycles, then we call them a *boundary cycle*, a *facial cycle*, and the *outer cycle*, respectively.

Let G be a connected graph embedded on a non-spherical surface $\mathbb{F}^2$, let R be a face of G, and let x be a vertex of G. We define the *(x, R)-width* of G as the minimum integer k such that every essential closed curve on $\mathbb{F}^2$ passing through R at x hits G at least k times. (We here assume that any curves on $\mathbb{F}$ hit a graph G only at vertices.) The *representativity* of G is the minimum integer k such that every essential closed curve on $\mathbb{F}^2$ hits G at least k times. In other words, the representativity of G is the minimum of the (x, R)-width of G among all vertices x and all faces R of G.

Let C be a facial walk of G or a subpath of it. A *C-flap* in G is either the null graph[3] or an $\{a, b, c\}$-bridge H of G for some $a, b, c \in V(G)$ such that

(i) $a, b \in V(C) \cap V(H)$ and $a \neq b$,

(ii) H contains a subpath P of C from a to b, and

(iii) H is contained in a disk such that P and c appear in the outer walk.

When H is the null graph, then let $a = b = c$ for some vertex $a \in V(C)$ and let $\mathrm{Nuc}(H) = \emptyset$. We say that a, b, c are *attachments of H*. See Figure 1[4].

The following is the technical theorem, which implies Theorem 1.

Theorem 2 *Let G be a 2-connected graph embedded on the projective plane, let R be a face of G, and let C be the boundary walk of R. Suppose that, for every 2-separation (G_1, G_2) of G, we have*

[3]We call the graph with no vertices (and no edges) the *null graph*.

[4]In all Figures of the present paper unless we mention explicitly, rising diagonals stroke from bottom left to top right represent a C-flap, while falling diagonals stroke from top left to bottom right represent other part of a given graph. White regions represent faces.

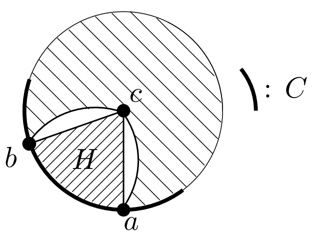

Figure 1: A C-flap H in G with attachments a, b, c.

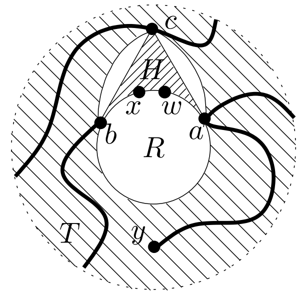

Figure 2: A C-flap H and a $\big(C - \mathrm{Nuc}(H)\big)$-Tutte path T desired in Theorem 2.

$E(C) \cap E(G_1) \neq \emptyset$ and $E(C) \cap E(G_2) \neq \emptyset$. Let $x \in V(C)$, let $y \in V(G) - \{x\}$ and let w be a neighbor of x in C. Then there exists a C-flap H in G with attachments a, b, c and there exists a $\big(C - \mathrm{Nuc}(H)\big)$-Tutte path T in $G - \mathrm{Nuc}(H)$ from b to y such that $|V(T)| \geq 3$, $a, c \in V(T)$, $x \in \big(V(H) - \{a\}\big) \cup \{b\}$, and if H is not the null graph, then $w \in V(H) - \{b\}$ and a, w, x, b appear in $C \cap H$ in this order. Furthermore, every T-bridge of G containing an edge of C is contained in a disk on the projective plane. (See Figure 2.)

Note that in Theorem 2, we allow the case where $w = a$ or $x = b$. Notice also that Theorem 2 still holds without the assumption on a 2-separation (and it is actually not hard to prove this extension from our proof of Theorem 2). However, to simplify the proof, we assume that here.

In Section 5, we will show Theorem 2. As mentioned in Section 1.1, several results on the Hamiltonicity of graphs on surfaces have been proven [1, 10, 11, 12, 13, 14, 15], actually using the notion *Tutte subgraph* as in Theorem 2. (See also Theorem 6 for an example using a Tutte subgraph.) However, our proof uses a very new idea, and in the following sense, it is different from the previously known proof method which was used to show those results in Section 1.1.

We use induction for the proof of Theorem 2, and divide the proof into four parts; The case where G has representativity at most 1 (Section 5.1), a claim on a 2-separation with a certain condition (Section 5.2), the case where G has representativity and (x, R)-width exactly 2 (Section 5.3), and the case where G has representativity at least 2 and (x, R)-width at least 3 (Section 5.4). The first three parts deal with fundamental cases (including the base case of the induction), and in fact, those are similar to the previously known proof method. However, the remaining one, which is the main part of the proof, is very different. In order to use the induction hypothesis, in all known proof methods, we delete all vertices in a certain subpath of C starting from x. In contrast, we do not need to delete so many vertices in our method; we delete just the vertex x. This is simpler than the previously known method, and hence our new idea has much advantages. (For example, it is easier to deal with the deleted part, even in the sense of algorithm.) Note that, in order to succeed that, we allow the existence of a C-flap H in G, which is in a sense "exceptional T-bridge" for a C-Tutte subgraph T.

Our proof of Theorem 2 consists of some known lemmas concerning Tutte cycles and paths, together with apparently new contributions. In the next section, we shall mention some known results, and in Section 3, we shall present our new lemmas that are needed in our proof of Theorem 2. In Sections 4, 5, and 6, we prove Theorem 1 (assuming Theorem 2), Theorem 2, and our new lemmas appeared in Section 3, respectively.

2 Known Lemmas about Tutte cycles and paths

Let us begin with some known results. The first lemma was proven by Sanders [10]. Note that he showed only the 2-connected case, but we can easily show the following, using a block decomposition. See also the paper by Thomassen [14].

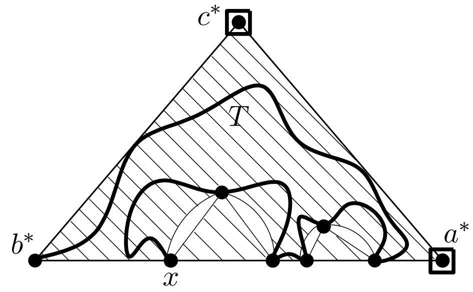

Figure 3: A $C[x, b^*]$-Tutte subgraph $T \cup \{a^*, c^*\}$ desired in Theorem 5.

Theorem 3 *Let G be a connected plane graph, let C be the outer walk of G, let $x, y \in V(G)$ with $x \neq y$, and let $e \in E(C)$. Assume that G contains a path from x to y through e. Then G has a C-Tutte path from x to y through e.*

As a corollary of Theorem 3, we obtain the following.

Theorem 4 *Let G be a connected plane graph, let C be the outer walk of G, let $x, u \in V(C)$ with $x \neq u$, and let $y \in V(G) - \{x, u\}$. Suppose that there exists a path in $G - u$ from x to y. Then G has a $C[x, u]$-Tutte subgraph consisting of u and a path T from x to y with $u \notin V(T)$.*

To show Theorem 4, indeed, consider the graph obtained from G by adding an edge connecting x and u, and use Theorem 3 to find a $\big(C[x, u] \cup \{xu\}\big)$-Tutte path from u to y through xu. Finally, deleting the added edge, we obtain the desired subgraph in Theorem 4.

The following two results are shown in [11]. By Theorem 5, we can deal with a Tutte path inside of a C-flap. Theorem 6 is a key for the proof of Thomas and Yu's result that says that every 4-connected projective-planar graph is Hamiltonian.

Theorem 5 (Theorem (2.4) in [11]) *Let G be a connected plane graph, let C be the outer walk of G, and let a^*, x, b^*, c^* be four distinct vertices in C^* such that the subpath $C[x, b^*]$ contains neither a^* nor c^*. Then there exists a $C[x, b^*]$-Tutte subgraph in G consisting of the two vertices a^* and c^* and a path T from x to b^* with $a^*, c^* \notin V(T)$. (See Figure 3.)*

Theorem 6 (Theorem (4.1) in [11]) *Let G be a 2-connected graph embedded on the projective plane, let C be a facial walk of G, and let $f \in E(C)$. Then there exists a C-Tutte cycle T in G such that $f \in E(T)$, and every T-bridge of G containing an edge of C is contained in a disk on the projective plane.*

Note that in the proof of Theorem 2, we will use Theorem 6 in order to show one case (see the first paragraph in Section 5.4). However, it is also possible to prove Theorem 2 without using Theorem 6. We use it just to simplify the proof.

3 New lemmas used in our proof.

Let us present our new lemmas. In this section, we just give the statement of the lemmas. Proof of these lemmas can be found in Section 6.

Before that we need a definition. Let G be a graph, and let u_1, u_2, v_1, v_2 be vertices of G with $u_1 \neq u_2$ and $v_1 \neq v_2$. Then two vertex-disjoint paths T_1 and T_2 in G *connect* $\{u_1, u_2\}$ *and* $\{v_1, v_2\}$ if T_1 connects u_i and v_j and T_2 connects u_{3-i} and v_{3-j} for some $i, j \in \{1, 2\}$. Note that if $u_1 = v_1$, then one of the paths T_1 and T_2 must consist of only u_1.

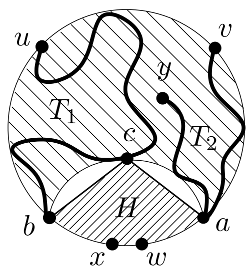

Figure 4: A C-flap H and a $\big(C - \mathrm{Nuc}(H)\big)$-Tutte subgraph $T_1 \cup T_2$ desired in Lemma 7.

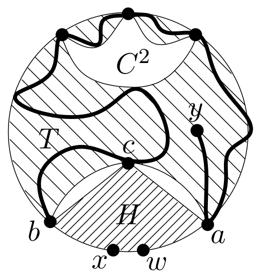

Figure 5: A C^1-flap H and a $\big(C^1 \cup C^2 - \mathrm{Nuc}(H)\big)$-Tutte path T desired in Lemma 9.

Lemma 7 *Let G be a 2-connected plane graph, let C be the outer cycle of G, let $x, u, v \in V(C)$ with $u \neq v$, let $y \in V(G) - \{x\}$, and let w be a neighbor of x in C. Suppose that $\{x, y\} \neq \{u, v\}$. Then there exists a C-flap H in G with attachments a, b, c and there exists a $\big(C - \mathrm{Nuc}(H)\big)$-Tutte subgraph in $G - \mathrm{Nuc}(H)$ consisting of two vertex-disjoint paths T_1 and T_2 such that $a, c \in V(T_1 \cup T_2)$, T_1 and T_2 connect $\{b, y\}$ and $\{u, v\}$, $x \in \big(V(H) - \{a\}\big) \cup \{b\}$, and if H is not the null graph, then $w \in V(H) - \{b\}$ and a, w, x, b appear in $C \cap H$ in this order. (See Figure 4.)*

Lemma 8 *Let G be a 2-connected plane graph, let C be the outer cycle of G, let $x, u, v \in V(C)$ with $x \neq v$ and $u \neq v$, and let $y \in V(G) - \{x\}$. Suppose that $u \in V(C[x, v]) - \{v\}$ and $\{x, y\} \neq \{u, v\}$. Then there exists a $C[x, v]$-Tutte subgraph in G consisting of two vertex-disjoint paths T_1 and T_2 such that T_1 and T_2 connect $\{x, y\}$ and $\{u, v\}$.*

Note that if G is 4-connected, then both Lemmas 7 and 8 give two vertex-disjoint paths that connect $\{x, y\}$ and $\{u, v\}$ and contain all vertices in G.

Lemma 9 *Let G be a 2-connected plane graph, let C^1 be the outer cycle of G, let C^2 be another facial cycle of G with $V(C^1) \cap V(C^2) \neq \emptyset$, let $x \in V(C^1)$, let $y \in V(G) - \{x\}$, and let w be a neighbor of x in C^1. Then there exists a C^1-flap H in G with attachments a, b, c and there exists a $\big(C^1 \cup C^2 - \mathrm{Nuc}(H)\big)$-Tutte path T from b to y in $G - \mathrm{Nuc}(H)$ such that $|V(T)| \geq 3$, $a, c \in V(T)$, $x \in \big(V(H) - \{a\}\big) \cup \{b\}$, T passes through at least one vertex in $C^1 \cap C^2$, and if H is not the null graph, then $w \in V(H) - \{b\}$ and a, w, x, b appear in $C^1 \cap H$ in this order. (See Figure 5.)*

Note that Lemma 9 holds even if we remove the assumption "$V(C^1) \cap V(C^2) \neq \emptyset$". (Indeed, we can show that using the same method as the proof of Theorem (3.2) in [11]. However, for the proof of our main theorem, it is enough to show Lemma 9 assuming it.)

Lemma 10 *Let G be a 2-connected plane graph, let C be the outer cycle of G, let u_1, v_1, u_2, v_2 be four distinct vertices in C, let $y \in V(G) - \{v_1, v_2\}$, and let w be a neighbor of v_1 in C. Suppose that u_1, w, v_1, u_2, v_2 appear in C in this clockwise order. (Possibly $w = u_1$.) Suppose also that G has no 2-separation (K_1, K_2) such that $v_1 \in V(K_1 \cap K_2)$ and $z \in V(C[u_2, v_2])$, where $\{v_1, z\} = V(K_1 \cap K_2)$. Then there exists a $C[u_1, v_1]$-flap H in G with attachments a, v_1, c and there exists a $\big(C[u_1, v_1] \cup C[u_2, v_2] - \mathrm{Nuc}(H)\big)$-Tutte subgraph in $G - \mathrm{Nuc}(H)$ consisting of the vertex v_1 and two vertex-disjoint paths T_1 and T_2 such that $v_1 \notin V(T_1 \cup T_2)$, $a, c \in V(T_1 \cup T_2) \cup \{v_1\}$, T_1 and T_2 connect $\{v_2, y\}$ and $\{u_1, u_2\}$, and if H is not the null graph, then $w \in V(H) - \{v_1\}$. (See Figure 6).*

In the following lemma, we can cut a 2-connected graph on the projective plane with representativity and (x, R)-width exactly 2.

A *chain of blocks* is a sequence $b_0, B_1, b_1, B_2, \ldots, b_{l-1}, B_l, b_l$ such that each B_i is a block, $V(B_i) \cap V(B_j) = \emptyset$ for each $1 \leq i < j \leq l$ with $j \neq i + 1$, $\{b_i\} = V(B_i) \cap V(B_{i+1})$ for each $1 \leq i \leq l - 1$, and if $|V(B_1)| \geq 2$, then $b_0 \in V(B_1) - \{b_1\}$ and $b_l \in V(B_l) - \{b_{l-1}\}$. When a chain of blocks

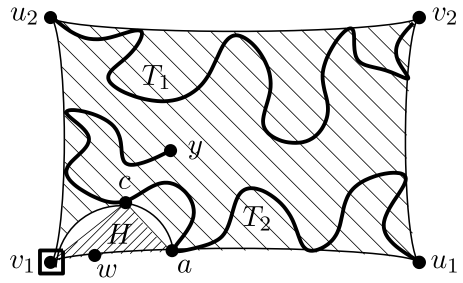

Figure 6: A $C[u_1,v_1]$-flap H and a $\big(C[u_1,v_1]\cup C[u_2,v_2]-\mathrm{Nuc}(H)\big)$-Tutte subgraph $\{v_1\}\cup T_1\cup T_2$ desired in Lemma 10.

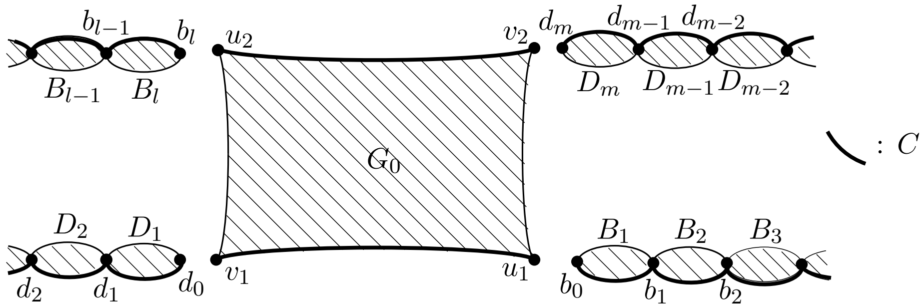

Figure 7: A 2-connected plane graph G_0 and two chains of blocks $b_0, B_1, b_1, B_2, \ldots, b_{l-1}, B_l, b_l$ and $d_0, D_1, \ldots, d_{m-1}, D_m, d_m$ desired in Lemma 11. The bold line means the cycle C.

$b_0, B_1, b_1, B_2, \ldots, b_{l-1}, B_l, b_l$ satisfies that $|V(B_1)| = 1$, then we call it *trivial*. (In the trivial one, $l = 1$ and it consists of only one vertex $b_0 = b_1$.)

Lemma 11 *Let G be a 2-connected graph embedded on the projective plane with representativity exactly 2, let R be a face of G, let C be the boundary cycle of R, and let $xw \in E(C)$. Suppose that G has (x,R)-width exactly 2. Then G can be decomposed into a 2-connected plane graph G_0 and two chains of blocks $b_0, B_1, b_1, B_2, \ldots, b_{l-1}, B_l, b_l$ and $d_0, D_1, \ldots, d_{m-1}, D_m, d_m$ satisfying the following properties (G1)–(G5). (Possibly, one of them or both may be trivial.) Let C_0 be the outer cycle of G_0, let C_{B_i} be the outer walk of B_i for each $1 \le i \le l$, and let C_{D_j} be the outer walk of D_j for each $1 \le j \le m$. (See Figure 7.)*

(G1) *G_0 has four distinct vertices u_1, v_1, u_2 and v_2 such that they appear in C_0 in this clockwise order.*

(G2) *The graph G is obtained from G_0 and the two chains of blocks $b_0, B_1, b_1, B_2, \ldots, b_{l-1}, B_l, b_l$ and $d_0, D_1, \ldots, d_{m-1}, D_m, d_m$ by identifying u_1 and b_0, u_2 and b_l, v_1 and d_0, and v_2 and d_m, respectively.*

(G3) $E(C) = E(C_0[u_1,v_1] \cup C_0[u_2,v_2]) \cup \bigcup_{i=1}^{l} E(C_{B_i}[b_{i-1},b_i]) \cup \bigcup_{j=1}^{m} E(C_{D_j}[d_{j-1},d_j])$.

(G4) *$x = d_k$ for some $0 \le k \le m-1$ and $w \in V(C_0[u_1,v_1])$ (when $k=0$) or $w \in V(C_{D_k}[d_{k-1},d_k])$ (when $k \ge 1$).*

(G5) *If $x = d_0$, then G_0 has no 2-separation (K_1,K_2) such that $v_1 \in V(K_1 \cap K_2)$ and $z \in V(C_0[u_2,v_2])$, where $\{v_1,z\} = V(K_1 \cap K_2)$.*

4 Proof of Theorem 1 assuming Theorem 2

Let G be a 4-connected graph on the projective plane, and let $x, y \in V(G)$ with $x \neq y$. It is enough to find a Hamiltonian path between x and y in G.

Let R be a face of G with $x \in V(C)$, where C is the boundary cycle of R. We can choose such a face R so that $y \notin V(C)$; Otherwise, $\{x, y\}$ would be a 2-cut of G, contradicting that G is 4-connected. Let w be a neighbor of x in C. Since G is 4-connected, G trivially satisfies the condition on a 2-separation in Theorem 2. So, by Theorem 2, there exists a C-flap H in G with attachments a, b, c and there exists a $\big(C - \mathrm{Nuc}(H)\big)$-Tutte path T in $G - \mathrm{Nuc}(H)$ from b to y such that $|V(T)| \geq 3$, $a, c \in V(T)$, $x \in \big(V(H) - \{a\}\big) \cup \{b\}$, and if H is not the null graph, then $w \in V(H) - \{b\}$ and a, w, x, b appear in $C \cap H$ in this order. Furthermore, every T-bridge of G containing an edge of C is contained in a disk on the projective plane.

Suppose first that $|V(T)| \geq 4$. If H is not the null graph, then the attachments of H form a 3-cut of G separating $\mathrm{Nuc}(H)$ from $V(T) - V(H)$, contradicting that G is 4-connected. This implies that H is the null graph, and by the same argument, we can show that there exists no non-trivial T-bridge of G. Hence T is a Hamiltonian path in G between $b = x$ and y.

Suppose next that $|V(T)| = 3$. Since $y \notin V(C)$, there exists some T-bridge B of G containing an edge of C. (Possibly $B = H$.) By the same reason as above, we have $V(B) = V(G)$. However, it follows from the condition on T that B is contained in a disk on the projective plane. Then the two edges of T could be added in order to obtain a planar embedding of G. It follows from Theorem 3 with specifying an appropriate edge that G has a Tutte path T' from x to y with $|V(T')| \geq 4$. By the same arguments as above, T' is a Hamiltonian path.

This completes the proof of Theorem 1. $\square$

5 Proof of Theorem 2

We prove Theorem 2 by induction on $|V(G)|$. If $|V(G)| \leq 3$, then we are trivially done. Hence we may assume that $|V(G)| \geq 4$. In particular, as mentioned in the last part of Section 1.2, we divide the proof into four parts; The case where G has representativity at most 1 (Section 5.1), a claim on a 2-separation with a certain condition (Section 5.2), the case where G has representativity and (x, R)-width exactly 2 (Section 5.3), and the case where G has representativity at least 2 and (x, R)-width at least 3 (Section 5.4).

5.1 The case where G has representativity at most 1

If G has representativity 0, then G can be regarded as a plane graph, where C is still a facial cycle of G. So, by Theorem 3, there exists a C-Tutte path T in G from x to y. In particular, by specifying an appropriate edge as e, we can take such a C-Tutte path so that $|V(T)| \geq 3$. Then letting H be the null graph and letting $a = b = c = x$, we are done.

Suppose next that G has representativity 1. Let γ be an essential closed curve on the projective plane that hits G exactly once. If γ does not passes through R, then G can be re-embedded into the plane so that C is the outer cycle. By Theorem 3 with specifying an appropriate edge as e, there exists a C-Tutte path T in G from x to y with $|V(T)| \geq 3$. Then letting H be the null graph and letting $a = b = c = x$, we are also done.

Next consider the case where γ passes through R. In this case, after re-embedding G into the plane, C is divided into two facial cycles, say C^1 and C^2. Since C is the boundary walk of R, we have $V(C^1) \cap V(C^2) \neq \emptyset$. By symmetry, we may assume that $xw \in E(C^1)$. Then it follows from Lemma 9 that there exists a C^1-flap H in G with attachments a, b, c and there exists a $\big(C^1 \cup C^2 - \mathrm{Nuc}(H)\big)$-Tutte path T from b to y in $G - \mathrm{Nuc}(H)$ such that $|V(T)| \geq 3$, $a, c \in V(T)$, $x \in \big(V(H) - \{a\}\big) \cup \{b\}$, T

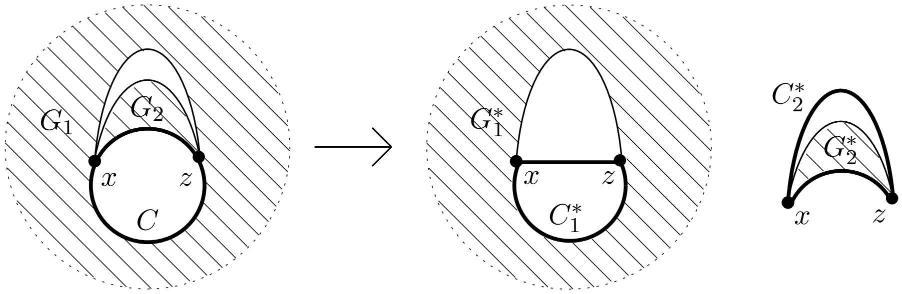

Figure 8: A 2-separation (G_1, G_2) of G and the graphs G_1^* and G_2^*.

passes through at least one vertex in $C^1 \cap C^2$, and if H is not the null graph, then $w \in V(H) - \{b\}$ and a, w, x, b appear in $C^1 \cap H$ in this order. Since T passes through at least one vertex in $C^1 \cap C^2$, every T-bridge of G containing an edge of C is contained in a disk on the projective plane. Therefore H and T are desired ones in Theorem 2.

Thus, in the remaining subsections, we may assume that the representativity of G is at least 2.

5.2 A claim on a 2-separation with a certain condition

In this section, we prove the following claim, which deals with a 2-separation with a certain condition.

Claim 1 *We may assume that there exists no 2-separation (G_1, G_2) of G such that $x \in V(G_1 \cap G_2)$.*

Proof. Suppose that G has a 2-separation (G_1, G_2) such that $x \in V(G_1 \cap G_2)$. Let $\{x, z\} = V(G_1 \cap G_2)$. By the assumption of Theorem 2, we have $z \in V(C)$. Since G is 2-connected and the representativity of G is at least 2, one of G_1 and G_2 is contained in a disk on the projective plane, say G_2 by symmetry. Let Δ be a disk that contains G_2 so that both x and z appear in the boundary. For $i = 1, 2$, let G_i^* be the graph obtained from G_i by adding an edge connecting x and z through the boundary of Δ. See Figure 8. (If G_i already has an edge connecting x and z, then we delete the original one.) Since (G_1, G_2) is a 2-separation and G_1^* is obtained from G by replacing G_2 with an edge, G_1^* is a 2-connected graph embedded on the projective plane. Similarly G_2^* is a 2-connected plane graph. We give a direction to C so that $E(C[z, x]) = E(C) \cap E(G_1)$. Let C_1^* be the facial cycle of G_1^* bounding R, and let C_2^* be the outer cycle of G_2^*. Note that $C_1^* = C[z, x] \cup \{xz\}$ and $C_2^* = C[x, z] \cup \{xz\}$. Let $y^* = y$ if $y \in V(G_1^*)$; otherwise let $y^* = z$. Let $w^* = w$ if $w \in V(G_1^*)$; otherwise let $w^* = z$.

By the induction hypothesis, there exists a C_1^*-flap H^* in G_1^* with attachments a^*, b^*, c^* and there exists a $\big(C_1^* - \mathrm{Nuc}(H^*)\big)$-Tutte path T^* in $G_1^* - \mathrm{Nuc}(H^*)$ from b^* to y^* such that $|V(T^*)| \geq 3$, $a^*, c^* \in V(T^*)$, $x \in \big(V(H^*) - \{a^*\}\big) \cup \{b^*\}$, and if H^* is not the null graph, then $w^* \in V(H^*) - \{b^*\}$ and a^*, w^*, x, b^* appear in $C_1^* \cap H^*$ in this order. Furthermore, every T^*-bridge of G_1^* containing an edge of C_1^* is contained in a disk on the projective plane. Note that the order a^*, w^*, x, b^* on $C_1^* \cap H^*$ might be different from the one we gave to C.

We divide the rest of the proof into three cases depending on whether $xz \in E(T^*)$ or not, and whether $y \in V(G_1^*)$ or not.

Case 1. $xz \in E(T^*)$.

In this case, note that $b = x$ and $y \in V(G_1^*) - V(G_2^*)$ since $|V(T^*)| \geq 3$. By Theorem 3, there exists a C_2^*-Tutte path T_2 in G_2^* from x to z. In particular, we can take such a C_2^*-Tutte path T_2 so that $xz \notin E(T_2)$ by specifying an appropriate edge as e. Let $H = H^*$, $a = a^*$, $b = b^*$, $c = c^*$, and $T = \big(T^* - \{xz\}\big) \cup T_2$. See Figures 9 and 10.[5] Note that every T-bridge B of $G - \mathrm{Nuc}(H)$ is either

[5]In Figures 9–14, the bottom curve represents a part of the cycle C, which is directed from left to right.

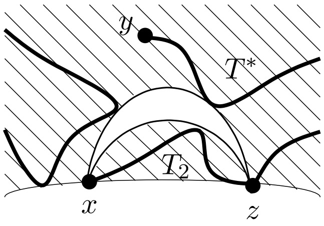

Figure 9: Case 1 with the null graph H^*.

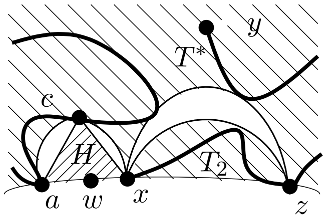

Figure 10: Case 1 with non-null graph H^*.

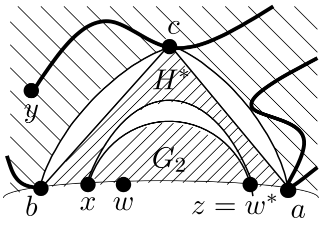

Figure 11: Case 2 with non-null graph H^* and $xz \in E(H^*)$.

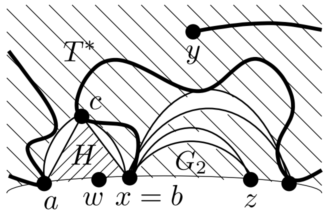

Figure 12: Case 2 with non-null graph H^* and $xz \notin E(H^*)$.

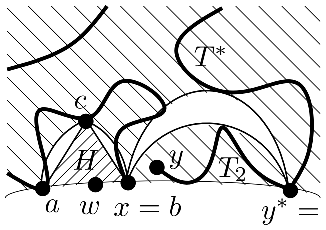

Figure 13: Case 3 with non-null graph H^* and $w \in V(G_1) - V(G_2)$.

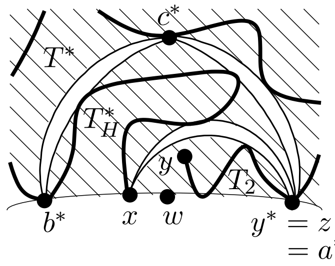

Figure 14: Case 3 with non-null graph H^* and $w \in V(G_2)$.

(i) a T^*-bridge of $G_1^* - \mathrm{Nuc}(H^*)$, or (ii) a T_2-bridge of G_2^*. Then by the condition of T^* and T_2, B has at most three attachments and at most two attachments if B contains an edge of $C_1^* \cup C_2^*$. Since $E(C) \subset E(C_1^*) \cup E(C_2^*)$, this implies that T is a $\big(C - \mathrm{Nuc}(H)\big)$-Tutte path in $G - \mathrm{Nuc}(H)$, which is the desired one in Theorem 2.

Case 2. $xz \notin E(T^*)$ and $y \in V(G_1^*)$.

Let $a = a^*$, $b = b^*$, $c = c^*$, and $T = T^*$. Suppose first that $xz \in E(H^*)$. Then let H be the subgraph of G induced by $\big(E(H^*) - \{xz\}\big) \cup E(G_2)$. See Figure 11. Note that H still has exactly three attachments a, b and c, and hence H is a C-flap in G. Then H and T are the desired ones in Theorem 2.

On the other hand, suppose that $xz \notin E(H^*)$. In this case, let $H = H^*$. See Figure 12. It follows from the condition $xz \notin E(H^*)$ that if H^* is not the null graph, then $x = b$. Furthermore, the T-bridge of G containing G_2 is the unique T-bridge that is not a T^*-bridge of G_1^*, but the attachments of it are not changed (and one of them is x). These imply that H and T are the desired ones in Theorem 2.

Case 3. $xz \notin E(T^*)$ and $y \in V(G_2^*) - V(G_1^*)$.

In this case, $y^* = z$. By Theorem 4, there exists a $C_2^*[x, z]$-Tutte subgraph in G_2^* consisting of x and a path T_2 from z to y with $x \notin V(T_2)$.

Suppose first that H^* is the null graph, or that H^* is not the null graph and $w \in V(G_1) - V(G_2)$. In the latter case, since a^*, w^*, x, b^* appear in $C_1^* \cap H^*$ in this order and $b^* \neq z = y^* \in V(T^*)$, we have $x = b^*$. Therefore in either case, $xz \notin E(H^*)$ and T is a path in G from x to y, where $T = T^* \cup T_2$. Let $H = H^*$, $a = a^*$, $b = b^*$, and $c = c^*$. See Figure 13. Note that every T-bridge B of $G - \mathrm{Nuc}(H)$ is either (i) a T^*-bridge of $G_1^* - \mathrm{Nuc}(H^*)$, or (ii) a $\big(T_2 \cup \{x\}\big)$-bridge of G_2^*. Then by the same argument as in Case 1, we see that T and H are the desired ones in Theorem 2.

Suppose next that H^* is not the null graph and $w \in V(G_2)$. Since a^*, w^*, x, b^* appear in $C_1^* \cap H^*$ in this order and $z = y^* \in V(T^*)$, we have $a^* = y^* = z$. We now give a direction to the outer walk of H^* so that it meets $C[b^*, x]$. By the choice, note that $C[b^*, x]$ contains neither a^* nor c^*. Then it follows from Theorem 5 (or trivially when $b^* = x$) that there exists a $C[b^*, x]$-Tutte subgraph in H^* consisting of the two vertices a^* and c^* and a path T_H^* from x to b^* with $a^*, c^* \notin V(T_H^*)$. Let H^* be the null graph, let $a = b = c = x$, and let $T = T_H^* \cup T^* \cup T_2$. See Figure 14. Note that every T-bridge of $G - \mathrm{Nuc}(H)$ is either (i) a T^*-bridge of $G_1^* - \mathrm{Nuc}(H^*)$, or (ii) a $\bigl(T_2 \cup \{x\}\bigr)$-bridge of G_2^*, or (iii) a $\bigl(T_H^* \cup \{a^*, c^*\}\bigr)$-bridge of H^*. Then by the same argument as in Case 1 (for (i) and (ii)) or by the conditions of T_H^* (for (iii)), we can check that H and T are the desired ones in Theorem 2.

This completes the proof of Claim 1. $\square$

5.3 The case where G has representativity and (x, R)-width exactly 2.

By Lemma 11, there exist a 2-connected plane graph G_0 and two chains of blocks $b_0, B_1, \ldots, b_{l-1}, B_l, b_l$ and $d_0, D_1, \ldots, d_{m-1}, D_m, d_m$ satisfying conditions (G1)–(G5). Let C_0 be the outer cycle of G_0, and C_{B_i} be the outer cycle of B_i for each $1 \le i \le l$.

It follows from condition (G4) that $x = d_k$ for some $0 \le k \le m-1$. If $|D_j| \ge 3$ for some $1 \le j \le m$, then we can easily find a 2-separation (G_1, G_2) with $\{x, d_{j-1}\} = V(G_1) \cap V(G_2)$ or $\{x, d_j\} = V(G_1) \cap V(G_2)$, which was already done in Claim 1. Thus, for each $1 \le j \le m$, we have $|D_j| \le 2$. By the same argument and symmetry, we only have the following four possibilities. (In the rest of this section, we deal with the vertices x and y also as vertices in G_0 and/or a vertex in $\bigcup_{i=1}^{l} B_i$. This situation is same for other vertices, for example, u_1, v_1 and so on.)

(a-i) $m = 1$, $|D_1| = 1$, $d_0 = d_1 = x$, and $w \in V(C_0[u_1, v_1])$.
(a-ii) $m = 1$, $|D_1| = 2$, $d_0 = x \ne d_1$, and $w \in V(C_0[u_1, v_1])$.
(b-i) $m = 1$, $|D_1| = 2$, $d_0 \ne d_1 = x$, and $w = d_0$.
(b-ii) $m = 2$, $|D_1| = |D_2| = 2$, $d_0 \ne d_1 = x \ne d_2$, and $w = d_0$.

We call the first two *Case a*, while the others *Case b*. Indeed, this case division depends on whether $x = d_0$ or not. Let T_D be a path consisting of only the vertex x in Cases a-i and b-i, or only the edge $d_0 d_1$ in Case a-ii, or only the edge $d_1 d_2$ in Case b-ii. Note that in either case, T_D is a path in $\bigcup_{j=1}^{m} D_j$ connecting x and d_m. Now we separately consider the case $y \in \bigcup_{i=1}^{l} V(B_i) - \{b_0, b_l\}$ (Case 1) and the case $y \in V(G_0)$ (Case 2).

Case 1. $y \in \bigcup_{i=1}^{l} V(B_i) - \{b_0, b_l\}$.

Let r be the integer with $y \in V(B_r) - \{b_r\}$. Since each B_i is a block, there exists a path in $\bigcup_{i=1}^{l} B_i$ from y to b_l through b_{r-1}. Then it follows from Theorem 3 that $\bigcup_{i=1}^{l} B_i$ has a $\bigl(\bigcup_{i=1}^{l} C_{B_i}\bigr)$-Tutte path T_B from y to b_l through b_{r-1}. If $r \ne 1$, then let $T_B' = T_B$; Otherwise, that is, if $r = 1$, then let T_B' be a path in $\bigcup_{i=1}^{l} B_i$ from y to b_l with $b_0 \notin V(T_B')$ such that $T_B' \cup \{b_0\}$ is a $\bigl(\bigcup_{i=1}^{l} C_{B_i}[b_{i-1}, b_i]\bigr)$-Tutte subgraph. (Such a path exists by Theorem 4.) Notice also that both T_B and T_B' is a path in $\bigcup_{i=1}^{l} B_i$ connecting y and b_l.

Case 1-a. $x = d_0$.

Let $G_0' = G_0 - v_1$. Suppose that there exists a 1-separation (G_1, G_2) of G_0' with $z \in V(C_0[v_2, u_1])$, where $\{z\} = V(G_1 \cap G_2)$. By symmetry, we may assume that $u_2, v_2 \in V(G_1)$ and $u_1, w \in V(G_2)$. In this case, take such a 1-separation (G_1, G_2) of G_0' so that $|V(G_1)|$ is as small as possible. On the other hand, if there exists no such a 1-separation of G_0', then let $z = u_1$. By the choice of z, there exists a path in G_0' from v_2 to u_2 through z. Let C_0' be the outer walk of G_0'. Then by Theorem 3, there exists a C_0'-Tutte path T_0 in G_0' from v_2 to u_2 through z.

Suppose first that $z \ne u_1$. In this case, let T be the path obtained from T_D, T_0 and T_B by identifying d_m and v_2, and u_2 and b_l, respectively. Then by condition (G2), T is a path in G from x to y with $|V(T)| \ge 3$.

Recall that (G_1, G_2) is a 1-separation of G_0' with $u_2, v_2 \in V(G_1)$ and $u_1, w \in V(G_2)$. Let H be the T-bridge of G containing G_2. Note that b_{r-1}, x and z are the attachments of H. Let $a = b_{r-1}$, $b = x$ and $c = z$. Note that H is a C-flap in G with attachments a, b, c such that $w \in V(H) - \{b\}$ and a, w, x, b appear in $C \cap H$ in this order. See Figure 15.

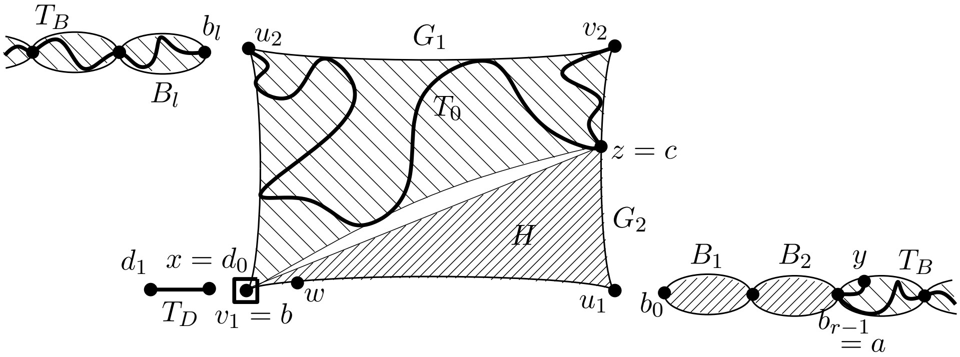

Figure 15: A C-flap H and a $\big(C - \mathrm{Nuc}(H)\big)$-Tutte path T in the case $z \neq u_1$ and $r \neq 1$ in Case 1-a-ii.

Let B be a T-bridge of $G - \mathrm{Nuc}(H)$. Then B is either (i) a $\big(T_0 \cup \{v_1\}\big)$-bridge of G_0, or (ii) a T_B-bridge of $\bigcup_{i=r}^{l} B_i$. Suppose that B satisfies (i). If v_1 is not an attachment of B, then B is also a T_0-bridge of G_0', and hence B has at most three attachments and at most two attachments if B contains an edge of C_0'. Notice that $E(C) \cap E(G_0') \subset E(C_0')$. On the other hand, if v_1 is an attachment of B, then $B - v_1$ is a T_0-bridge of G_0' containing an edge of C_0', and hence $B - v_1$ has at most two attachments. Thus, B has at most three attachments one of which is v_1. Note that by the existence of a 1-separation (G_1, G_2) of G_0' and the fact that $z \in V(T)$, B has no edge in $C_0[u_1, v_1] \cup C_0[u_2, v_2]$. On the other hand, if B satisfies (ii), then it follows from the condition of T_B that B has at most three attachments and at most two attachments if B contains an edge of $\bigcup_{i=1}^{l} C_{B_i}$. Then by condition (G3), T is a $\big(C - \mathrm{Nuc}(H)\big)$-Tutte path in $G - \mathrm{Nuc}(H)$, which is the desired one in Theorem 2.

Suppose next that $z = u_1$. Then let T be the path obtained from T_D, T_0 and T_B' by identifying d_m and v_2, and u_2 and b_l, respectively. Then by condition (G2), T is a path in G from x to y with $|V(T)| \geq 3$. (Recall that b_0 is not contained in T_B'.)

Suppose that there exists a T-bridge H of G such that H contains the edge xw and has at least three attachments. Since $x \in V(T)$, note that x is an attachment of H. Then $H - x$ is a T_0-bridge of G_0' containing an edge of C_0', and hence $H - x$ has at most two attachments on T_0 such that one of them, say a, is contained in $V(C_0'[u_1, w]) - \{w\}$, and the other, say c, is contained in $V(C_0'[w, u_2]) - \{w\}$. Let $b = x$. See Figure 16. On the other hand, if there exists no such a T-bridge H of G, then let H be the null graph, and let $a = b = c = x$. In either case, H is a C-flap in G with attachments a, b, c such that if H is not the null graph, then $w \in V(H) - \{b\}$ and a, w, x, b appear in $C \cap H$ in this order.

Note that every T-bridge B of $G - \mathrm{Nuc}(H)$ is either (i) a $\big(T_0 \cup \{v_1\}\big)$-bridge of G_0, or (ii) a $\big(T_B' \cup \{b_0\}\big)$-bridge of $\bigcup_{i=1}^{l} B_i$. By the same argument as in the case $z \neq u_1$, we see that B has at most three attachments and at most two attachments if B contains an edge in C. Then T is a C-Tutte path in $G - \mathrm{Nuc}(H)$, which is the desired one in Theorem 2.

Case 1-b. $x \neq d_0$.

Since G_0 is 2-connected, there exists a path in G_0 from v_2 to u_2 through u_1. By Theorem 3, there exists a C_0-Tutte path T_0 in G_0 from v_2 to u_2 through u_1.

Let T be the path obtained from T_D, T_0 and T_B' by identifying d_m and v_2, and u_2 and b_l, respectively. Then by condition (G2), T is a path in G from x to y with $|V(T)| \geq 3$.

Suppose that there exists a T-bridge H of G such that H contains the edge xw and has at least three attachments. Note that x is an attachment of H. Since $H - x$ is a T_0-bridge of G_0 containing an edge of C_0, $H - x$ has two attachments on T_0. In particular, one of the attachments, say a, is contained in $C_0[u_1, v_1] - \{v_1\}$, and the other, say c, is contained in $C_0[v_1, u_2] - \{v_1\}$. Let $b = x$. See Figure 17. On the other hand, if there exists no such a T-bridge H of G, then let H be the null graph, and let

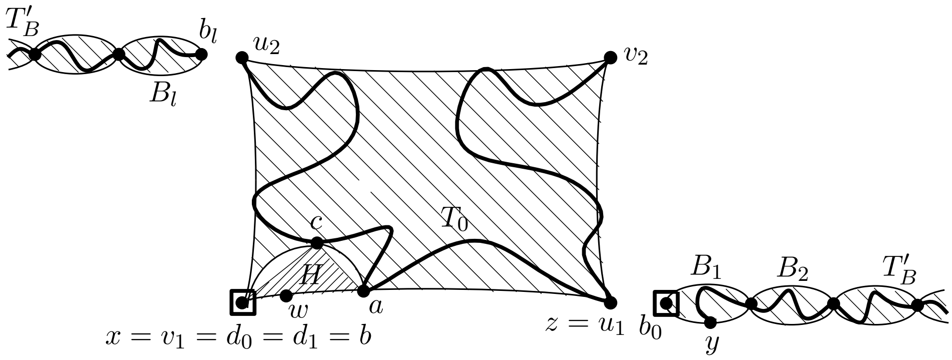

Figure 16: A C-flap H and a $\big(C - \mathrm{Nuc}(H)\big)$-Tutte path T in the case $z = u_1$ and $r = 1$ in Case 1-a-i.

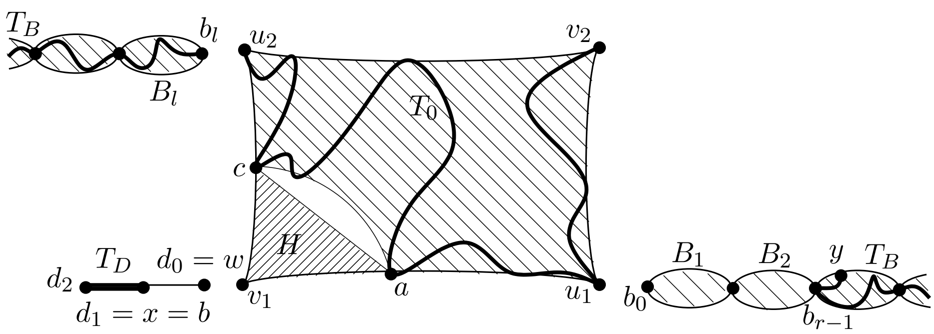

Figure 17: A C-flap H and a $\big(C - \mathrm{Nuc}(H)\big)$-Tutte path T in Case 1-b-ii.

$a = b = c = x$. In either case, H is a C-flap in G with attachments a, b, c such that if H is not the null graph, then $w \in V(H) - \{b\}$ and a, w, x, b appear in $C \cap H$ in this order. Moreover, by the same argument as in Case 1-a, we can show that H and T are the ones desired in Theorem 2.

Case 2. $y \in V(G_0)$.

In this case, it follows from Theorem 3 that $\bigcup_{i=1}^{l} B_i$ has a $\big(\bigcup_{i=1}^{l} C_{B_i}\big)$-Tutte path T_B from b_0 to b_l.

Case 2-a. $x = d_0$.

Suppose first $y = v_2$. Then, considering the reverse direction on C with $u = u_1$, $v = u_2$, it follows from Lemma 8 that there exists a $C[u_2, x]$-Tutte subgraph in G_0 consisting of two vertex-disjoint paths T_1 and T_2 such that T_1 and T_2 connect $\{x, y\}$ and $\{u_1, u_2\}$. Let T be the path obtained from T_B, T_1 and T_2 identifying u_1 and b_0, and u_2 and b_l, respectively. Then by condition (G2), T is a path in G from x to y with $|V(T)| \geq 3$. Let H be the null graph with $a = b = c = x$. By the same argument as Case 1-a, we can show that H and T are the ones desired in Theorem 2. Hence we may assume that $y \neq v_2$.

By condition (G5), G_0 has no 2-separation (K_1, K_2) such that $v_1 \in V(K_1 \cap K_2)$ and $z \in V(C_0[u_2, v_2])$, where $\{v_1, z\} = V(K_1 \cap K_2)$. Then by Lemma 10, there exists a $C_0[u_1, v_1]$-flap H in G_0 with attachments a, v_1, c and there exists a $\big(C_0[u_1, v_1] \cup C_0[u_2, v_2] - \mathrm{Nuc}(H)\big)$-Tutte subgraph in $G_0 - \mathrm{Nuc}(H)$ consisting of the vertex v_1 and two vertex-disjoint paths T_1 and T_2 such that $v_1 \notin V(T_1 \cup T_2)$, $a, c \in V(T_1 \cup T_2) \cup \{v_1\}$, T_1 and T_2 connect $\{v_2, y\}$ and $\{u_1, u_2\}$, and if H is not the null graph, then $w \in V(H) - \{v_1\}$. Let $b = x$. See Figure 18. Note that H is a C-flap in G with attachments a, b, c such that if H is not the null graph, then $w \in V(H) - \{b\}$ and a, w, x, b appear in $C \cap H$ in this order.

Let T be the path obtained from T_D, T_B, T_1 and T_2 identifying d_m and v_2, u_1 and b_0, and u_2 and b_l, respectively. Then by condition (G2), T is a path in G from x to y with $|V(T)| \geq 3$. By the same argument as Case 1-a, we can show that H and T are the ones desired in Theorem 2.

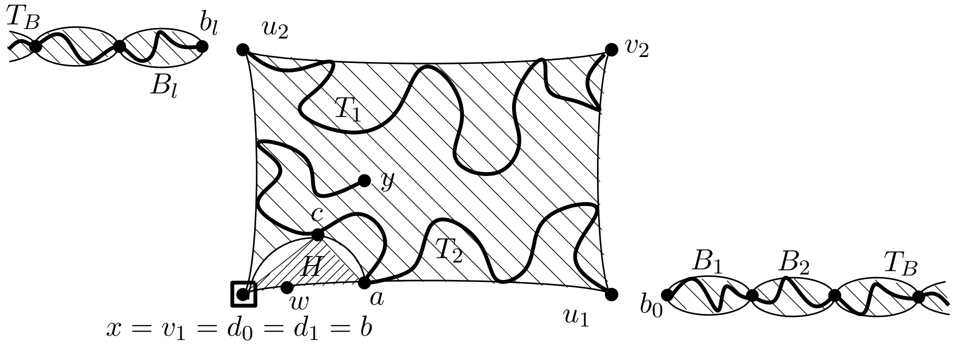

Figure 18: A C-flap H and a $\big(C - \mathrm{Nuc}(H)\big)$-Tutte path T in Case 2-a-i.

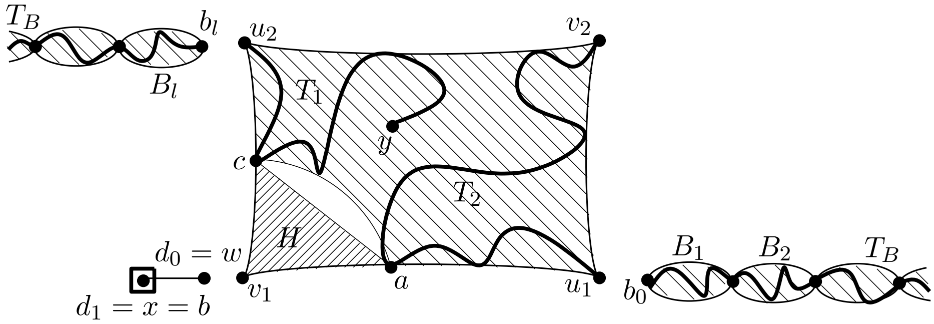

Figure 19: A C-flap H and a $\big(C - \mathrm{Nuc}(H)\big)$-Tutte path T in Case 2-b-i.

Case 2-b. $x \neq d_0$.

Note that G_0 is 2-connected and $\{v_2, y\} \neq \{u_1, u_2\}$. By Lemma 8, there exists a $C_0[u_1, v_2]$-Tutte subgraph in G_0 consisting of two vertex-disjoint paths T_1 and T_2 such that T_1 and T_2 connect $\{v_2, y\}$ and $\{u_1, u_2\}$.

Let T be the path obtained from T_D, T_1, T_2 and T_B by identifying d_m and v_2, u_1 and b_0, and u_2 and b_l, respectively. Then by condition (G2), T is a path in G from x to y with $|V(T)| \geq 3$.

Suppose that there exists a T-bridge H of G such that H contains the edge xw and has at least three attachments. Note that x is an attachment of H. Since $H - x$ is a $\big(T_1 \cup T_2\big)$-bridge of G_0 containing an edge of $C_0[u_1, v_2]$, $H - x$ has two attachments on $T_1 \cup T_2$. In particular, one of the attachments, say a, is contained in $C_0[u_1, v_1] - \{v_1\}$, and the other, say c, is contained in $V(C_0[v_1, u_2]) - \{v_1\}$. Let $b = x$. See Figure 19. On the other hand, if there exists no such a T-bridge H of G, then let H be the null graph, and let $a = b = c = x$. In either case, H is a C-flap of G with attachments a, b, c such that if H is not the null graph, then $w \in V(H) - \{b\}$ and a, w, x, b appear in $C \cap H$ in this order.

Then H and T are the ones desired in Theorem 2. This completes the case where G has representativity at least 2 and (x, R)-width exactly 2.

5.4 The case where G has representativity at least 2 and (x, R)-width at least 3.

Suppose first that $xy \in E(C)$ and let $f = xy \in E(C)$. It follows from Theorem 6 that[6] G has a C-Tutte cycle T' such that $f \in E(T')$, and every T'-bridge of G containing an edge of C is contained in a disk on the projective plane. Let T be the path obtained from T' by removing the edge f, let H be the null

[6]Here we use Theorem 6, in order to make the proof simpler. Instead of using it, we can use the same idea in the following paragraphs; Delete the vertex x, use the induction hypothesis to $G - x$, and then extend the C-flap H' in $G - x$ or the $\big(C' - \mathrm{Nuc}(H')\big)$-Tutte path T' in $\big(G - x - \mathrm{Nuc}(H')\big)$ (together with three cases).

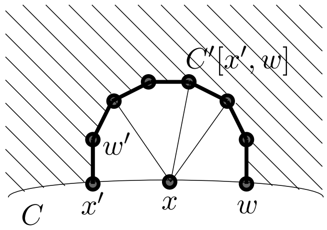

Figure 20: The vertices x' and w', and the subpath $C'[x', w]$ of C'

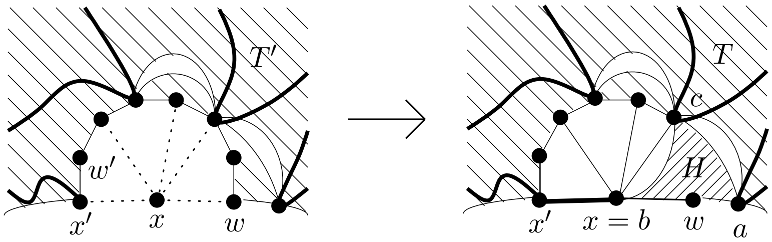

Figure 21: Case 1 with a T-bridge H with certain conditions.

graph, and let $a = b = c = x$. Since T' is a cycle, we have $|V(T)| = |V(T')| \geq 3$. Then T is a C-Tutte path in $G - \mathrm{Nuc}(H)$ from x to y, which is the desired one in Theorem 2. Thus, we may assume that $xy \notin E(C)$.

Let x' be the neighbor of x in C with $x' \neq w$. Since $xy \notin E(C)$, we have $x' \neq y$. Let $G' = G - x$. Note that G' is 2-connected by Claim 1. Let R' be the face of G' containing R, and let C' be the boundary cycle of R'. (Since G' is 2-connected and the (x, R)-width of G is at least 3, C' is actually a cycle.) Let w' be the neighbor of x' in C' such that $x'w' \notin E(C)$. We may assume that C is directed so that $E(C) = E(C'[w, x']) \cup \{xx', xw\}$. See Figure 20.[7]

By the induction hypothesis, there exists a C'-flap H' in G' with attachments a', b', c' and there exists a $\big(C' - \mathrm{Nuc}(H')\big)$-Tutte path T' in $G' - \mathrm{Nuc}(H')$ from b' to y such that $|V(T')| \geq 3$, $a', c' \in V(T')$, and if H' is not the null graph, then $w' \in V(H') - \{b'\}$ and a', w', x', b' appear in $C' \cap H'$ in this order. Furthermore, every T'-bridge of G' containing an edge of C' is contained in a disk on the projective plane.

We divide the proof into three cases, depending on the situation on H', a' and b'. Note that if H' is not the null graph, then $a' \neq x'$.

Case 1. H' is the null graph.

Let $T = T' \cup \{xx'\}$.

Suppose first that $w \notin V(T)$. Then let H be the T-bridge of G containing w. Note that x is an attachment of H, and $H - x$ is a T'-bridge of $G' - \mathrm{Nuc}(H')$ containing an edge of C'. Then $H - x$ has two attachments on T', and in particular, one of the attachments, say a, is contained in $C'[w, x'] - \{w\}$, and the other, say c, is contained in $C'[x', w] - \{w\}$. Let $b = x$.

On the other hand, when $w \in V(T)$, then let H be the null graph and let $a = b = c = x$. In either case, note that H is a C-flap in G with attachments a, b, c such that $a, c \in V(T)$, $x \in \big(V(H) - \{a\}\big) \cup \{b\}$, and if H is not the null graph, then $w \in V(H) - \{b\}$ and a, w, x, b appear in $C \cap H$ in this order. See Figure 21.

Let B be a T-bridge of $G - \mathrm{Nuc}(H)$. If x is not an attachment of B, then B is also a T'-bridge of G'. Since T' is a C'-Tutte subgraph in G', B has at most three attachments, and at most two attachments if B contains an edge of $C - \{xx', xw\} \subset C'$.

On the other hand, suppose that x is an attachment of B. Note that $B - x$ is a T'-bridge of G' containing an edge of $C'[x', w]$, and hence B has at most three attachments such that two of them are contained in $C'[x', w]$ and the other is x. Let z be an attachment of B contained in $C'[x', w]$. If B contains an edge of C, then we can find a closed curve γ on the projective plane such that γ passes through R and hits G only at x and z. If γ is essential, then this contradicts that the (x, R)-width of G is at least 3. Therefore, γ is contractible, but in this case, $\{x, z\}$ is a 2-cut of G separating x' and w, contradicting Claim 1. These arguments imply that B contains no edge of C.

Therefore, every T-bridge B of $G - \mathrm{Nuc}(H)$ has at most three attachments, and at most two attachments if B contains an edge of C. Thus, T is a $\big(C - \mathrm{Nuc}(H)\big)$-Tutte subgraph in $G - \mathrm{Nuc}(H)$,

[7] In the same manner as in Figures 9–14, the bottom curve in Figures 20–25 represents a part of C, which is directed from left to right. In Figure 20, the bold lines represent the path $C'[x', w]$ that is edge-disjoint from C.

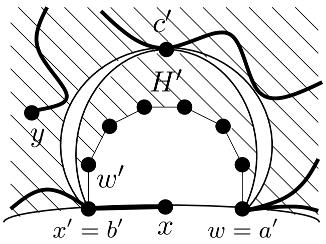

Figure 22: The T-bridge H' of G in the first case of Case 2.

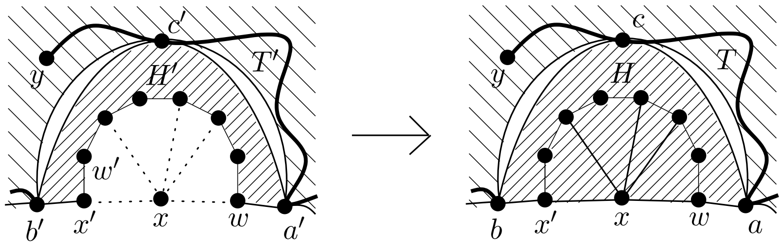

Figure 23: The C-flap H in G in the second case of Case 2.

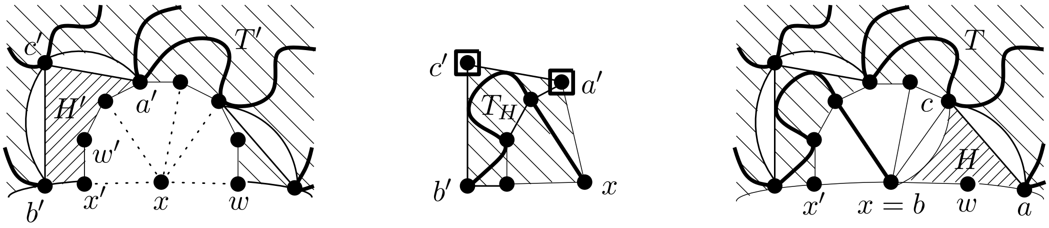

Figure 24: Case 3-(I): A C'-flap H' and a $\big(C' - \mathrm{Nuc}(H')\big)$-Tutte path T' (the left side), a $\widetilde{C}[x,b']$-Tutte subgraph $T_H \cup \{a', c'\}$ in the graph $\widetilde{H}$ (the middle), and a C-flap H and a $\big(C - \mathrm{Nuc}(H)\big)$-Tutte path T (the right side).

which is the desired one in Theorem 2.

Case 2. H' is not the null graph and a' is contained in $C'[w, x'] - \{x'\}$.

Since a', w', x', b' appear in $C' \cap H'$ in this order, note that b' is contained in $C'[a', x'] - \{a'\}$.

Suppose first that $a' = w$, $b' = x'$ and x has no neighbors in G other that x' and w. In this case, let $T = T' \cup \{xx'\}$, let H be the null graph, and let $a = b = c = x$. See Figure 22. By the conditions, H' is a T-bridge of G such that H' has exactly three attachments a', b' and c' and the edge xw is not contained in H'. Since G' is 2-connected and the (x, R)-width of G is at least 3, we have $E(C'[x', w]) \cap E(C) = \emptyset$, and hence H' contains no edge in C. Moreover, every T-bridge of G that is neither H' nor the edge xw is also a T'-bridge of G'. So, T is a $\big(C - \mathrm{Nuc}(H)\big)$-Tutte path in $G - \mathrm{Nuc}(H)$ satisfying the desired conditions.

Then we may assume that $a' \neq w$ or $b' \neq x'$ or x has a neighbor in G other than x' and w. In this case, let H be a subgraph of G induced by the edges in H' and all edges incident with x in G. Let $T = T'$, let $a = a'$, let $b = b'$, and let $c = c'$. See Figure 23. By the conditions, H is connected, and in particular, H is a C-flap in G with attachments a, b, c. Furthermore, H and T are desired ones in Theorem 2.

Case 3. H' is not the null graph and a' is contained in $C'[x', w] - \{x', w\}$.

In this case, we have two cases depending on the place of b'; (I) b' is contained in $C[w, x']$ and (II) b' is contained in $C[a', w] - \{a', w\}$. See the left side of Figures 24 and 25 for (I) and (II), respectively.

Let $\widetilde{H}$ be the graph obtained from $H' \cup \{x\}$ by adding all edges of G connecting x and a vertex in $C'[x', a']$ and a new edge connecting x and a' (if it does not exist). See the middle of Figures 24 and 25. Note that $\widetilde{H}$ is a plane graph and let $\widetilde{C}$ be the outer walk of $\widetilde{H}$. We may assume that $\widetilde{C}$ is directed so that $\widetilde{C}[x, b']$ contains neither a' nor c'. Then $\widetilde{C}[x, b'] = C'[b', x'] \cup \{xx'\}$. By Theorem 5, there exists a $\widetilde{C}[x, b']$-Tutte subgraph in $\widetilde{H}$ consisting of the two vertices a' and c' and a path T_H from x to b' with $a', c' \notin V(T_H)$. Let $T = T' \cup T_H$. See the right side of Figures 24 and 25.

Suppose that $w \notin V(T)$, and let H be the T-bridge of G containing w. Note that x is an attachment of H, and $H - x$ is either a T'-bridge of $G' - \mathrm{Nuc}(H')$ containing an edge of C' (when (I) occurs) or

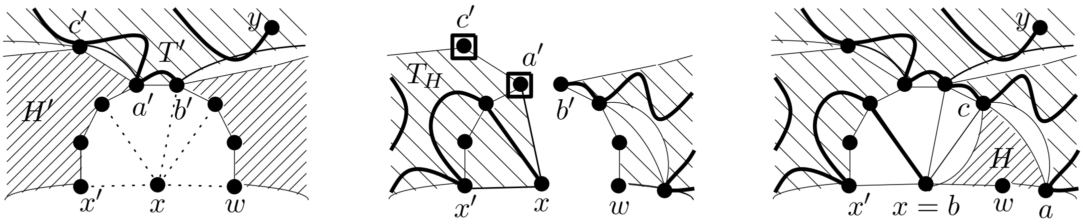

Figure 25: Case 3-(II): A C'-flap H' and a $\big(C' - \mathrm{Nuc}(H')\big)$-Tutte path T' (the left side), a $\widetilde{C}[x, b']$-Tutte subgraph $T_H \cup \{a', c'\}$ in the graph $\widetilde{H}$ (the middle), and a C-flap H and a $\big(C - \mathrm{Nuc}(H)\big)$-Tutte path T (the right side).

a $\big(T_H \cup \{a', c'\}\big)$-bridge of $\widetilde{H}$ containing an edge of $\widetilde{C}[x, b']$ (when (II) occurs). Then $H - x$ has two attachments on T' or on $T_H \cup \{a', c'\}$, and in particular, one of the attachments, say a, is contained in $C[w, x'] - \{w\}$, and the other, say c, is contained in $C'[a', w] - \{w\}$. Let $b = x$.

On the other hand, if $w \in V(T)$, then let H be the null graph and let $a = b = c = x$. In either case, note that H is a C-flap in G with attachments a, b, c such that $a, c \in V(T)$, $x \in \big(V(H) - \{a\}\big) \cup \{b\}$, and if H is not the null graph, then $w \in V(H) - \{b\}$ and a, w, x, b appear in $C \cap H$ in this order.

Let B be a T-bridge of $G - \mathrm{Nuc}(H)$. Note that either (i) B is a $\big(T' \cup \{x\}\big)$-bridge of $G - \mathrm{Nuc}(H')$ or (ii) $B \cap \widetilde{H}$ is a $(T_H \cup \{a', c'\})$-bridge of $\widetilde{H}$.

If B satisfies (i), then it follows from the same argument as in Case 1 that B has at most three attachments, and at most two attachments if B contains an edge of C.

Suppose that B satisfies (ii). If $B \cap \widetilde{H}$ has the same attachments as B, (that is, if (I) occurs, or if (II) occurs and $B \cap \widetilde{H}$ contains no vertex in $C'[b', w] - \{b', w\}$ that is a neighbor of x in G,) then B has at most three attachments and at most two attachments if B contains an edge of $\widetilde{C}[x, b']$. On the other hand, suppose that $B \neq B \cap \widetilde{H}$. Then (II) occurs, x is an attachment of B, and $B \cap \widetilde{H}$ contains a vertex in $C'[b', w] - \{b', w\}$ that is a neighbor of x in G. This implies that $B \cap \widetilde{H}$ is a $\big(T_H \cup \{a', c'\}\big)$-bridge containing an edge of $\widetilde{C}[x, b']$. Hence B has at most three attachments one of which is x. Since G is 2-connected and the (x, R)-width of G is at least three, it follows from the choice of H that B contains no edge in C.

In either case, B has at most three attachments, and at most two attachments if B contains an edge of C. Thus, T is a $\big(C - \mathrm{Nuc}(H)\big)$-Tutte subgraph in $G - \mathrm{Nuc}(H)$, satisfying the desired conditions. This completes the proof of Theorem 2. $\square$

6 Proofs of Lemmas

In this section, we shall give proofs for our new lemmas introduced in Section 3.

6.1 Proofs of Lemmas 7 and 8

We prove Lemmas 7 and 8 simultaneously by induction on $|V(G)|$. If $|V(G)| \leq 3$, then we can easily find desired two vertex-disjoint paths T_1 and T_2 (with null graph H and $a = b = c = x$), and hence Lemmas 7 and 8 holds. Therefore we may assume that $|V(G)| \geq 4$. Actually, we will show the following two statements;

(I) Lemma 8 holds for a graph G with $|V(G)| = n$ if Lemma 7 holds for all graphs G' with $|V(G')| \leq n$.

(II) Lemma 7 holds for a graph G with $|V(G)| = n$ if both Lemmas 7 and 8 hold for all graphs G' with $|V(G')| < n$.

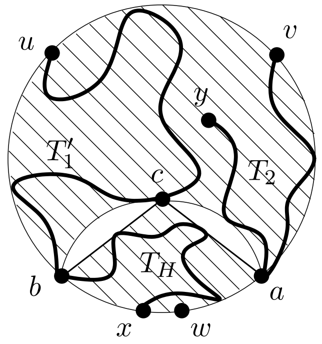

Figure 26: A $C[x,v]$-Tutte subgraph $T_1 \cup T_2$ obtained in the proof of (I) in Section 6.1.

Note that by these two statements, we can easily verify that both Lemmas 7 and 8 hold.

(I) We show Lemma 8 assuming that Lemma 7 holds for all graphs G' with $|V(G')| \leq |V(G)|$.

Let w be the neighbor of x in C such that $w \in V(C[v,x]) - \{x\}$. Since Lemma 7 holds for G, there exists a C-flap H in G with attachments a, b, c and there exists a $\bigl(C - \mathrm{Nuc}(H)\bigr)$-Tutte subgraph in $G - \mathrm{Nuc}(H)$ consisting of two vertex-disjoint paths T_1' and T_2 such that $a, c \in V(T_1' \cup T_2)$, T_1' and T_2 connect $\{b, y\}$ and $\{u, v\}$, $x \in \bigl(V(H) - \{a\}\bigr) \cup \{b\}$, and if H is not the null graph, then $w \in V(H) - \{b\}$ and a, w, x, b appear in $C \cap H$ in this order. By symmetry, we may assume that b is an end vertex of T_1'.

If $b \neq x$, then by Theorem 5, there exists a $C[x,b]$-Tutte subgraph in H consisting of the two vertices a and c and a path T_H in H from x to b with $a, c \notin V(T_H)$. On the other hand, if $b = x$, then let T_H be the path consisting of only x. Let $T_1 = T_1' \cup T_H$. Note that T_1 and T_2 connect $\{x, y\}$ and $\{u, v\}$. See Figure 26.

Note that each $\bigl(T_1 \cup T_2\bigr)$-bridge of G is either (i) a $\bigl(T_1' \cup T_2\bigr)$-bridge of $G - \mathrm{Nuc}(H)$, or (ii) a $\bigl(T_H \cup \{a, c\}\bigr)$-bridge of H. Note also that $E\bigl(C[x,v]\bigr) \cap E\bigl(G - \mathrm{Nuc}(H)\bigr) = E\bigl(C[b,v]\bigr)$ and $E\bigl(C[x,v]\bigr) \cap E(H) = E\bigl(C[x,b]\bigr)$. Since $T_1' \cup T_2$ is a $\bigl(C - \mathrm{Nuc}(H)\bigr)$-Tutte subgraph in $G - \mathrm{Nuc}(H)$ and $T_H \cup \{a, c\}$ is a $C[x,b]$-Tutte subgraph in H, we have that $T_1 \cup T_2$ is a $C[x,v]$-Tutte subgraph of G. This completes the proof of the statement (I). $\square$

(II) We will show Lemma 7 assuming that both Lemmas 7 and 8 hold for all graphs G' with $|V(G')| < |V(G)|$.

By the symmetry between u and v, we may assume that $x \neq v$ and w, x, u, v appear in C in this clockwise order. (Possibly $x = u$ and/or $w = v$.) We first show the following two claims.

Claim 2 *We may assume that there exists no 2-separation (G_1, G_2) of G such that $x \in V(G_1 \cap G_2)$.*

Proof. Suppose that there exists a 2-separation (G_1, G_2) of G such that $x \in V(G_1 \cap G_2)$. Let $\{x, z\} = V(G_1 \cap G_2)$.

First, suppose that $u, v \in V(G_1) - \{z\}$. Note that $z \in V(C[v,x] \cup C[x,u]) - \{x\}$ or $E(G_2) \cap E(C) = \emptyset$. For $i = 1, 2$, let Δ_i be a disk containing G_i, and let G_i^* be the graph obtained from G_i by adding an edge between x and z so that it appears on the boundary of Δ_i. (If G_i already has an edge connecting x and z, then we delete the original one.) Note that G_i^* is a 2-connected plane graph with $|V(G_i^*)| < |V(G)|$. Let C_i^* be the outer cycle of G_i^*, and let $y^* = y$ if $y \in V(G_1^*)$; otherwise let $y^* = z$. Since we assumed that Lemma 7 holds for all graphs G' with $|V(G')| < |V(G)|$, there exists a C_1^*-flap H^* in G_1^* with attachments a^*, b^*, c^* and there exists a $\bigl(C_1^* - \mathrm{Nuc}(H^*)\bigr)$-Tutte subgraph in $G_1^* - \mathrm{Nuc}(H^*)$ consisting of two vertex-disjoint paths T_1^* and T_2^* such that $a^*, c^* \in V(T_1^* \cup T_2^*)$, T_1^* and T_2^* connect $\{b^*, y^*\}$ and $\{u, v\}$, $x \in \bigl(V(H^*) - \{a^*\}\bigr) \cup \{b^*\}$, and if H^* is not the null graph, then $w \in V(H^*) - \{b^*\}$ and a^*, w, x, b^* appear in $C_1^* \cap H^*$ in this order. Then by the same arguments as in the proof of Claim 1 in Section

5.2, we can extend the C_1^*-flap H^* or one of the two paths T_1^* and T_2^*, and obtain a C-flap H in G with attachments a, b, c and a $\big(C - \mathrm{Nuc}(H)\big)$-Tutte subgraph in $G - \mathrm{Nuc}(H)$, which are desired in Lemma 7.

Then we may assume that $z \in V(C[u, v])$. In this case, we will find a desired C-flap H in G and a desired $\big(C - \mathrm{Nuc}(H)\big)$-Tutte subgraph $T_1 \cup T_2$ with null graph H and $a = b = c = x$. So we do not care the vertex w, and hence we can use the symmetry between u and v and also the symmetry between G_1 and G_2. Then we may assume that $E(C) \cap E(G_1) = E(C[x, z])$, and $E(C) \cap E(G_2) = E(C[z, x])$. Note that $u \in V(G_1) - \{z\}$ and $v \in V(G_2) - \{x, z\}$.

Suppose that $z = y$. By Theorem 4 (or trivially when $x = u$ and/or $z = y = v$), there exists a $C[x, z]$-Tutte subgraph in G_1 consisting of z and a path T_1 from x to u with $z \notin V(T_1)$, and there exists a $C[z, x]$-Tutte subgraph in G_2 consisting of x and a path T_2 from y to v with $x \notin V(T_2)$. Since every $(T_1 \cup T_2)$-bridge of G is either (i) a $(T_1 \cup \{z\})$-bridge of G_1 or (ii) a $(T_2 \cup \{x\})$-bridge of G_2, T_1 and T_2 are desired paths. Then we may assume that $z \neq y$. Again by the symmetry between G_1 and G_2, we may assume that $y \in V(G_1) - V(G_2)$.

For $i = 1, 2$, let Δ_i be a disk containing G_i, and let G_i^* be the graph obtained from G_i by adding an edge between x and z so that it appears on the boundary of Δ_i. (If G_i already has an edge connecting x and z, then we delete the original one.) Note that G_i^* is a 2-connected plane graph with $|V(G_i^*)| < |V(G)|$. Let C_i^* be the outer cycle of G_i^*. Since we assumed that Lemma 8 holds for all graphs G' with $|V(G')| < |V(G)|$, G_1^* has a $C_1^*[x, z]$-Tutte subgraph in G consisting of two vertex-disjoint paths T_1 and T_2' such that T_1 and T_2' connect $\{x, y\}$ and $\{u, z\}$. By symmetry, we may assume that z is an end vertex of T_2'. On the other hand, it follows from Theorem 4 that G_2^* has a $C_2^*[z, x]$-Tutte subgraph consisting of x and a path T_2^* from z to v with $x \notin V(T_2^*)$. Let $T_2 = T_2' \cup T_2^*$, let H be the null graph, and let $a = b = c = x$. Then by the same arguments as in (I), we can show that T_1, T_2 and H are desired ones in Lemma 7. This completes the proof of Claim 2. $\square$

Claim 3 *We may assume that $x \neq u$.*

Proof. Suppose not, that is, suppose $x = u$. Let T_1 be the path in G consisting of only $x = u$, and let $G^* = G - x$. By Claim 2, G^* is a 2-connected plane graph. Let C^* be the outer cycle of G^*, and let z be the neighbor of x in C with $z \neq w$. Since G^* is 2-connected, there exists a path in G^* from y to v through z. Then by Theorem 3, G^* has a C^*-Tutte path T_2 from y to v through z. Note that T_1 and T_2 are two vertex-disjoint paths in G connecting $\{x, y\}$ and $\{u, v\}$.

Suppose that $w \notin V(T_2)$, and let H be the $\big(T_1 \cup T_2\big)$-bridge of G containing w. See the left side of Figure 27. Note that x is an attachment of H, and $H - x$ is a T_2-bridge of G^*. Since $H - x$ contains an edge of C^*, $H - x$ has exactly two attachments on T_2. In particular, one of the attachments, say a, is contained in $V(C^*[v, w]) - \{w\}$, and the other, say c, is contained in $V(C^*[w, v]) - \{w\}$. Since $z \in V(T_2)$, indeed, we have $c \in V(C^*[w, z]) - \{w\}$. Let $b = x$.

On the other hand, if $w \in V(T_2)$, then let H be the null graph and let $a = b = c = x$. See the right side in Figure 27. In either case, note that H is a C-flap in G such that $a, c \in V(T_1 \cup T_2)$, $x \in \big(V(H) - \{a\}\big) \cup \{b\}$, and if H is not the null graph, then $w \in V(H) - \{b\}$ and a, w, x, b appear in $C \cap H$ in this clockwise order.

So it suffices to show that $T_1 \cup T_2$ is a $\big(C - \mathrm{Nuc}(H)\big)$-Tutte subgraph in $G - \mathrm{Nuc}(H)$. Let B be a $\big(T_1 \cup T_2\big)$-bridge of $G - \mathrm{Nuc}(H)$. If x is not an attachment of B, then B is also a T_2-bridge of G^*. Hence B has at most three attachments and at most two attachments if B contains an edge of C^*. Note that $E(C) \cap E(G^*) \subset E(C^*)$. On the other hand, suppose that x is an attachment of B. Then $B - x$ is a T_2-bridge of G^* containing an edge of C^*, and hence B has at most three attachments on $T_1 \cup T_2$, one of which is x. By the choice of H, we have $E(B) \cap E(C^*) \subset E(C^*[c, z])$. Since G^* is 2-connected, $E(C^*[c, z]) \cap E(C) = \emptyset$, and hence B contains no edge of C. These imply that $T_1 \cup T_2$ is a $\big(C - \mathrm{Nuc}(H)\big)$-Tutte subgraph in $G - \mathrm{Nuc}(H)$, which are desired in Lemma 7.

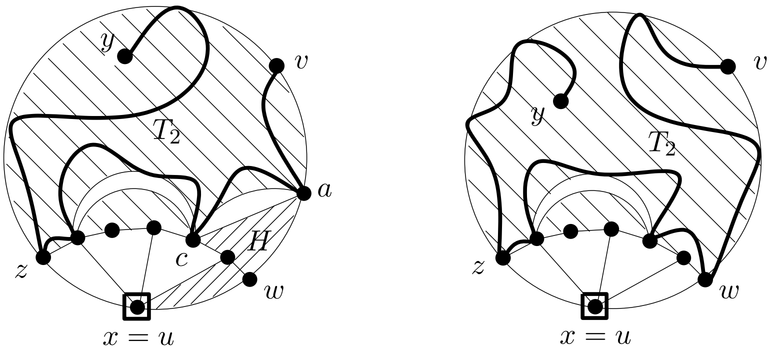

Figure 27: A C-flap H and a $\big(C - \mathrm{Nuc}(H)\big)$-Tutte subgraph $T_1 \cup T_2$ in Claim 3. (The left side and the right side correspond to the case $w \in V(T_2)$ and the case $w \notin V(T_2)$, respectively.)

This completes the proof of Claim 3. $\Box$

Now we are ready to complete the proof of statement (II). Let x' be the neighbor of x in C with $x' \neq w$. Suppose first $x' = y$. Let $e = xy \in E(C)$. By Theorem 3, G has a C-Tutte path T' from u to v through e. Let T_1 and T_2 be the two paths obtained from T' by removing the edge e. Note that $T_1 \cup T_2$ is a C-Tutte subgraph in G, and T_1 and T_2 connect $\{x, y\}$ and $\{u, v\}$. Hence letting H be the null graph and letting $a = b = c = x$, H, T_1 and T_2 are desired ones in Lemma 7. So, we may assume $x' \neq y$.

By Claim 3, we have $x \neq u, v$. Let $G' = G - x$. It follows from Claim 2 that G' is 2-connected. Let C' be the outer cycle of G', and let w' be the neighbor of x' in C' with $x'w' \notin E(C)$. Since we assumed that Lemma 7 holds for all graphs G'' with $|V(G'')| < |V(G)|$, there exists a C'-flap H' in G' with attachments a', b', c' and there exists a $\big(C' - \mathrm{Nuc}(H')\big)$-Tutte subgraph in $G' - \mathrm{Nuc}(H')$ consisting of two vertex-disjoint paths T_1' and T_2 such that $a', c' \in V(T_1' \cup T_2)$, T_1' and T_2 connect $\{b', y\}$ and $\{u, v\}$, and if H' is not the null graph, then $w' \in V(H') - \{b'\}$ and a', w', x', b' appear in $C' \cap H'$ in this order. By symmetry, we may assume that x' is an end vertex of T_1'. By the same arguments[8] as in Section 5.4, we see that G has desired ones in Lemma 7.

This completes the proof of the statement (II) and the proofs of Lemmas 7 and 8. $\Box$

6.2 Proof of Lemma 9

Let $l := |V(C^1) \cap V(C^2)|$. Since $V(C^1) \cap V(C^2) \neq \emptyset$, G can be decomposed into l graphs $G_1, G_2, \dots, G_l$ by the l vertices in $V(C^1) \cap V(C^2)$ such that each G_i is 2-connected or an edge, and satisfying the following properties;

(i) When $l = 1$, then G is obtained from G_1 by identifying two distinct vertices of G_1, say u_1 and v_1.

(ii) When $l \geq 2$, then each G_i has two distinct vertices u_i and v_i and G is obtained from $G_1, G_2, \dots, G_l$ by identifying u_i and v_{i+1} for each $1 \leq i \leq l$, where $v_{l+1} = v_1$.

For each $1 \leq i \leq l$, let C_i be the outer walk of G_i. We may assume that $C_i[v_i, u_i] = C^1 \cap G_i$ and $C_i[u_i, v_i] = C^2 \cap G_i$. We may also assume that $xw \in E(C_1)$ and v_1, w, x, u_1 appear in C_1 in this clockwise order (possibly $v_1 = w$ and/or $x = u_1$). In particular, $x \neq v_1$.

[8]Actually the only difference is that (i) here we consider the planar case, while the projective-planar case was considered in Section 5.4, and (ii) here we consider two vertex-disjoint paths T_1' and T_2, while one path T' was considered in Section 5.4. In the sense of (i), this case is easier than the one in Section 5.4. In the sense of (ii), since we only have to care the end vertex b' in T_1' or T_2 instead of the end vertex b' in T' in Section 5.4, completely same arguments can work.

Claim 4 *We may assume that $x \neq u_1$.*

Proof. Suppose contrary $x = u_1$. We show the existence of T and H desired in Lemma 9 with the null graph H and $a = b = c = x$. So throughout the proof of Claim 4, let H be the null graph and let $a = b = c = x$. Since H is the null graph, we can ignore the vertex w.

Suppose first $y \in V(G_1) - \{u_1, v_1\}$. Then by Theorem 3, there exists a C_1-Tutte path T in G_1 from x to y through v_1. Since $y \neq v_1$, we have $|V(T)| \geq 3$. Note that each T-bridge B of G satisfies that either (i) B is a T-bridge of G_1, or (ii) $B = \bigcup_{i=2}^{l} G_i$. Since T is a C_1-Tutte subgraph in G_1 and $\bigcup_{i=2}^{l} G_i$ has exactly two attachments u_1 and v_1, in either case, B has at most three attachments and at most two attachments if B contains an edge in $C^1 \cup C^2$. Hence T is a $\left(C^1 \cup C^2\right)$-Tutte path in G from x to y. Therefore, T satisfies the conditions in Lemma 9.

Suppose next $y = v_1$. Note that at least one of G_1 and $\bigcup_{i=2}^{l} G_i$ has at least three vertices. If G_1 has at least three vertices, then by Lemma 3 with specifying an appropriate edge as e, we can take a C_1-Tutte path T in G_1 from x to y with $|V(T)| \geq 3$; Otherwise, again by Lemma 3 with specifying an appropriate edge as e, we can take a $\bigcup_{i=2}^{l} C_i$-Tutte path T in $\bigcup_{i=2}^{l} G_i$ from x to y with $|V(T)| \geq 3$. In either case, we can check that T satisfies the desired conditions in Lemma 9.

Thus, we may assume that $y \notin V(G_1)$. Since $x = u_1$ and we ignore the vertex w, here we can use the symmetry between G_1 and G_2. Hence we may also assume that $y \notin V(G_2)$. Therefore, $y \in V(G_r) - \{v_r\}$ for some r with $3 \leq r \leq l$. By Theorem 3, $\bigcup_{i=2}^{r} G_i$ has a $\left(\bigcup_{i=2}^{r} C_i\right)$-Tutte path T from x to y through u_r. Then T satisfies the conditions in Lemma 9.

This completes the proof of Claim 4. $\square$

By Claim 4, we may assume that $x \neq u_1$. We divide the rest of the proof of Lemma 9 into three cases, depending on the place of y.

Case 1. $y \in V(G_1) - \{v_1\}$.

By Lemma 7, there exists a C_1-flap H in G_1 with attachments a, b, c and there exists a $\left(C_1 - \mathrm{Nuc}(H)\right)$-Tutte subgraph in $G_1 - \mathrm{Nuc}(H)$ consisting of two vertex-disjoint paths T_1 and T_{l+1} such that $a, c \in V(T_1 \cup T_{l+1})$, T_1 and T_{l+1} connect $\{b, y\}$ and $\{u_1, v_1\}$, and if H is not the null graph, then $w \in V(H) - \{b\}$ and a, w, x, b appear in $C_1 \cap H$ in this order. By symmetry, we may assume that y is an end vertex of T_{l+1}. By Theorem 3, G_i has a C_i-Tutte path T_i from v_i to u_i for each $2 \leq i \leq l$. Let $T = \bigcup_{i=1}^{l+1} T_i$. See the left side of Figure 28. Note that T is a path in G from b to y. Notice also that every T-bridge of $G - \mathrm{Nuc}(H)$ is either (i) a $\left(T_1 \cup T_{l+1}\right)$-bridge of $G_1 - \mathrm{Nuc}(H)$, or (ii) a T_i-bridge of G_i for some i with $2 \leq i \leq l$. Therefore, T is a $\left(C^1 \cup C^2 - \mathrm{Nuc}(H)\right)$-Tutte path in $G - \mathrm{Nuc}(H)$ from x to y.

Now we check that $|V(T)| \geq 3$. Suppose that $|V(T)| \leq 2$. This easily implies that H is the null graph (otherwise T has to pass through three vertices a, b and c), $l = 1$ (otherwise $|V(T)| \geq |V(T_1)| + |V(T_{l-1})| \geq 3$. Recall that $x \neq v_1$ and $y \neq v_1$), and $|V(T_1)| + |V(T_{l+1})| = 3$. The last equality further implies that $y = u_1$ because of Claim 4. Since $|V(T_1)| + |V(T_{l+1})| = 3$ and $y = u_1$, we see that $T_1 \cup T_{l+1}$ consists of only three vertices u_1, v_1 and x. Note that the fact $l = 1$ implies that $E(C^1) = E(C^1) \cap E(C_1) = E\big(C_1[v_1, u_1]\big)$ and $E(C^2) = E(C^2) \cap E(C_1) = E\big(C_1[u_1, v_1]\big)$, and hence $|V(C_1[v_1, u_1])|, |V(C_1[u_1, v_1])| \geq 4$. Thus, $V(C_1[v_1, u_1]) - V(T) \neq \emptyset$, and let B_1 be a T-bridge of G_1 containing a vertex in $V(C_1[v_1, u_1]) - V(T)$. Since B_1 contains an edge in C_1, B_1 has exactly two attachments. If x is not an attachment of B_1, then $\{u_1, v_1\}$ separates B_1 from x, but this contradicts that G is 2-connected. (Note that G is obtained from G_1 by identifying u_1 and v_1.) Hence x is an attachment of B_1, and in particular, x is also contained in both C^1 and C_1. Since $V(C_1[u_1, v_1]) - V(T) \neq \emptyset$, it follows from the same argument for C^2 that x is contained in both C^2 and C_1. Then x appears in both $C_1[v_1, u_1] - \{u_1, v_1\}$ and $C_1[u_1, v_1] - \{u_1, v_1\}$, and hence x is a cut vertex of G_1 separating u_1 and v_1, which contradicts that G_1 is 2-connected. Therefore, we have $|V(T)| \geq 3$.

Then T and H satisfy the conditions in Lemma 9.

Case 2. $y \in V(G_r) - \{v_r\}$ for some $2 \leq r \leq l - 1$.

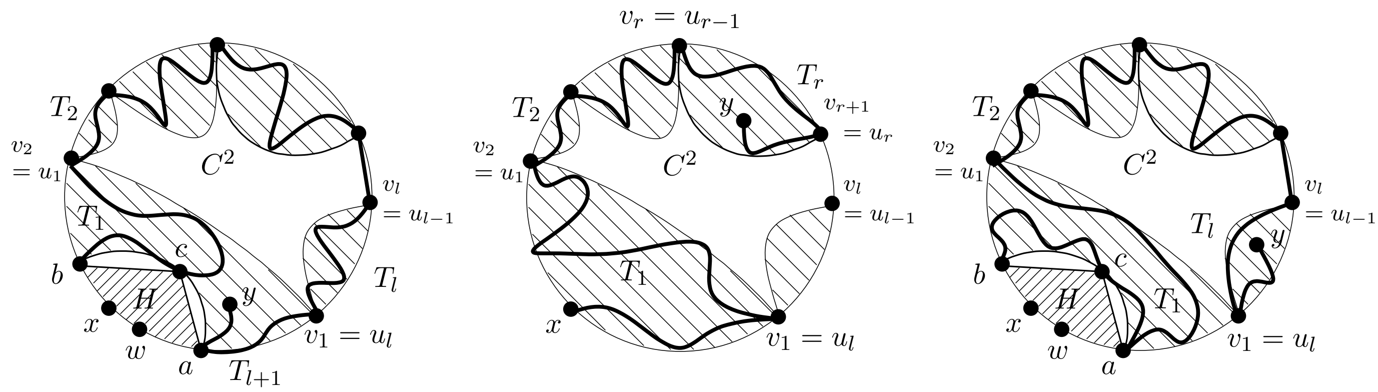

Figure 28: A C^1-flap H and a $\big(C^1 - \mathrm{Nuc}(H)\big)$-Tutte subgraph $T_1 \cup T_2$ in the proof of Lemma 9. (The left side, the middle and the right side correspond to Cases 1, 2 and 3, respectively.)

By Theorem 3, G_1 has a C_1-Tutte path T_1 from x to u_1 through v_1, G_i has a C_i-Tutte path T_i from v_i to u_i for each $2 \le i \le r-1$, and G_r has a C_r-Tutte path T_r from v_r to y through u_r. Let $T = \bigcup_{i=1}^{r} T_i$, let H be the null graph, and let $a = b = c = x$. See the middle of Figure 28. Clearly $|V(T)| \ge 3$. Notice that for every T-bridge B of $G - \mathrm{Nuc}(H)$, either (i) B is a T_i-bridge of G_i for some i with $1 \le i \le r$, or (ii) $B = \bigcup_{i=r+1}^{l} G_i$, which is a $\{v_{r+1}, u_l\}$-bridge of G. Then T satisfies the conditions in Lemma 9.

Case 3. $y \in V(G_l) - \{v_l\}$.

By Lemma 7, Then there exists a C_1-flap H in G_1 with attachments a, b, c and there exists a $\big(C_1 - \mathrm{Nuc}(H)\big)$-Tutte subgraph in $G_1 - \mathrm{Nuc}(H)$ consisting of two vertex-disjoint paths T_1 and T_{l+1} such that $a, c \in V(T_1 \cup T_{l+1})$, T_1 and T_{l+1} connect $\{b, v_1\}$ and $\{u_1, v_1\}$, $x \in \big(V(H) - \{a\}\big) \cup \{b\}$, and if H is not the null graph, then $w \in V(H) - \{b\}$ and a, w, x, b appear in $C_1 \cap H$ in this order. By symmetry, we may assume that T_{l+1} consists of only the vertex v_1. By Theorem 3, G_i has a C_i-Tutte path T_i from v_i to u_i for each $2 \le i \le l-1$, and G_l has a C_l-Tutte path T_l from v_l to y through u_l. Let $T = \bigcup_{i=1}^{l} T_i$. See the right side of Figure 28. Clearly $|V(T)| \ge 3$. Notice that every T-bridge of $G - \mathrm{Nuc}(H)$ is either (i) a $\big(T_1 \cup T_{l+1}\big)$-bridge of $G_1 - \mathrm{Nuc}(H)$, or (ii) a T_i-bridge of G_i for some i with $2 \le i \le l$. Then T satisfies the conditions in Lemma 9.

This completes the proof of Lemma 9. $\square$

6.3 Proof of Lemma 10

We prove Lemma 10 by induction on $|V(G)|$. If $|V(G)| \le 4$, we can easily find desired H, T_1 and T_2 with null graph H and $a = c = v_1$. So, we may assume that $|V(G)| \ge 5$. We first show the following claim.

Claim 5 *We may assume that there exists no 2-separation (G_1, G_2) of G such that $v_1 \in V(G_1 \cap G_2)$.*

Proof. Suppose that there exists a 2-separation (G_1, G_2) of G such that $v_1 \in V(G_1 \cap G_2)$. Let $\{v_1, z\} = V(G_1 \cap G_2)$.

Case 1. $z \in V(C[u_1, v_1] \cup C[v_1, u_2]) - \{v_1\}$ or $E(G_i) \cap E(C) = \emptyset$ for some $i \in \{1, 2\}$.

In this case, we may assume that $u_1, v_1, u_2, v_2 \in V(G_1)$. For $i = 1, 2$, let Δ_i be a disk containing G_i, and let G_i^* be the graph obtained from G_i by adding an edge connecting v_1 and z so that it appears on the boundary of Δ_i. (If G_i already has the edge connecting v_1 and z, then we delete the original one.) Note that G_i^* is a 2-connected plane graph with $|V(G_i^*)| < |V(G)|$. Let C_i^* be the outer cycle of G_i^*. Let $y^* = y$ if $y \in V(G_1)$; otherwise let $y^* = z$. Let $w^* = z$ if $z \in V(C[u_1, v_1]) - \{v_1\}$; otherwise let

$w^* = w$. By the induction hypothesis, there exists a $C_1^*[u_1, v_1]$-flap H^* in G_1^* with attachments a, v_1, c and there exists a $\big(C_1^*[u_1, v_1] \cup C_1^*[u_2, v_2] - \mathrm{Nuc}(H^*)\big)$-Tutte subgraph in $G_1^* - \mathrm{Nuc}(H^*)$ consisting of the vertex v_1 and two vertex-disjoint paths T_1^* and T_2^* such that $a, c \in V(T_1^* \cup T_2^*) \cup \{v_1\}$, T_1^* and T_2^* connect $\{v_2, y^*\}$ and $\{u_1, u_2\}$, and if H^* is not the null graph, then $w^* \in V(H^*) - \{v_1\}$. Then by the same arguments as in the proof of Claim 1 in Section 5.2, we can extend the $C_1^*[u_1, v_1]$-flap H^* or one of the two paths T_1^* and T_2^*, and obtain a $C[u_1, v_1]$-flap H in G with attachments a, b, c and a $\big(C[u_1, v_1] \cup C[u_2, v_2] - \mathrm{Nuc}(H)\big)$-Tutte subgraph in $G - \mathrm{Nuc}(H)$, which are desired in Lemma 10.

Case 2. $z \in V(C[u_2, v_2])$.

This case does not occur, since otherwise there exists a 2-separation (G_1, G_2) of G with $\{v_1, z\} = V(G_1 \cap G_2)$ and $z \in V(C[u_2, v_2])$, which contradicts the assumption of Lemma 10.

Case 3. $z \in V(C[v_2, u_1]) - \{u_1, v_2\}$.

In this case, we may assume that $v_2, u_2 \in V(G_1)$ and $w, u_1 \in V(G_2)$. We take a 2-separation (G_1, G_2) so that $C[v_1, z]$ is as short as possible. By this choice, there exists no 2-separation (K_1, K_2) of G such that $v_1 \in V(K_1 \cap K_2)$ and $z' \in V(C[u_2, v_2] \cup C[v_2, z]) - \{z\}$, where $\{v_1, z'\} = V(G_1 \cap G_2)$. Then $G_1' = G_1 - v_1$ is 2-connected, and let C_1' be the outer cycle of G_1'. We give a direction to C_1' so that $C_1'[u_2, z] = C[u_2, z]$.

Case 3.1. $y \in V(G_1)$.

By Lemma 8, there exists a $C_1'[z, v_2]$-Tutte subgraph in G_1' consisting of two vertex-disjoint paths T_1' and T_2 such that T_1' and T_2 connect $\{v_2, y\}$ and $\{z, u_2\}$. By symmetry, we may assume that z is an end vertex of T_1' and u_2 is an end vertex of T_2. Let C_2 be the outer walk of G_2 with a direction such that $C_2[z, v_1] = C[z, v_1]$. By Theorem 4, there exists a $C_2[z, v_1]$-Tutte subgraph in G_2 consisting of v_1 and a path T' from z to u_1 with $v_1 \notin V(T')$. Let $T_1 = T_1' \cup T'$. Note that T_1 and T_2 connect $\{v_2, y\}$ and $\{u_1, u_2\}$. Let H be the null graph and let $a = c = v_1$.

Let B be a $(\{v_1\} \cup T_1 \cup T_2)$-bridge of G. Note that B is either (i) a $(\{v_1\} \cup T_1' \cup T_2)$-bridge of G_1, or (ii) a $(\{v_1\} \cup T')$-bridge of G_2.

Suppose first that B satisfies (i). If v_1 is not an attachment of B, then B is also a $(T_1' \cup T_2)$-bridge of G_1'. So B has at most three attachments and at most two attachments if B contains an edge of $C_1'[z, v_2]$. Note that $E\big(C[u_1, v_1] \cup C[u_2, v_2]\big) \cap E(G_1') = E(C[u_2, v_2]) \subset E(C_1'[z, v_2])$. On the other hand, if v_1 is an attachment of B, then $B - v_1$ is a $\big(T_1' \cup T_2\big)$-bridge of G_1' containing an edge of $C_1'[z, v_2]$. Hence $B - v_1$ has at most two attachments on $T_1' \cup T_2$, so B has at most three attachments on $\{v_1\} \cup T_1' \cup T_2$ one of which is v_1. Note that B contains no edge in $C[u_1, v_1] \cup C[u_2, v_2]$.

Suppose next that B satisfies (ii). Then B has at most three attachments and at most two attachments if B contains an edge of $C_2[z, v_1]$. Note that $E\big(C[u_1, v_1] \cup C[u_2, v_2]\big) \cap E(G_2) = E\big(C[u_1, v_1]\big) \subset E\big(C_2[z, v_1]\big)$.

In either case, B has at most three attachments and at most two attachments if B contains an edge of $C[u_1, v_1] \cup C[u_2, v_2]$. Then H, T_1 and T_2 are desired ones in Lemma 10.

Case 3.2. $y \in V(G_2) - V(G_1)$.

Since G_1' is 2-connected, G_1' has a path from u_2 to v_2 through z. Then by Theorem 3, there exists a C_1'-Tutte path T_1 in G_1' from u_2 to v_2 through z.

Let $G_2' = G_2 - v_1$ and let C_2' be the outer walk of G_2'. We give a direction to C_2' so that $C_2'[z, w] = C[z, w]$. If $G_2' - z$ is not connected, then we can find a 2-separation (G_1', G_2') of G such that $E(G_i') \cap E(C) = \emptyset$ for some $i \in \{1, 2\}$, contradicting the choice of (G_1, G_2). Hence $G_2' - z$ is connected, and in particular, $G_2' - z$ has a path from y to u_1. Then by Theorem 4, there exists a $C_2'[u_1, z]$-Tutte subgraph in G_2' consisting of z and a path T_2 from y to u_1 with $z \notin V(T_2)$.

We show that T_1 and T_2 are the desired paths in Lemma 10.

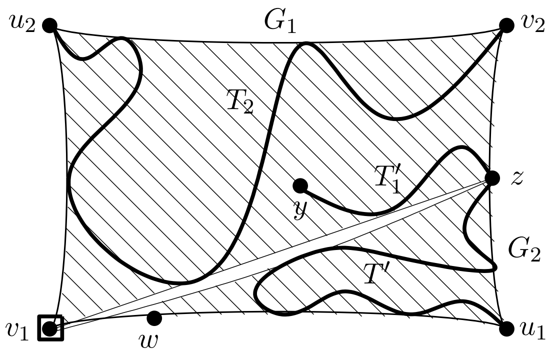

Figure 29: A $\big(C[u_1,v_1]\cup C[u_2,v_2]-\mathrm{Nuc}(H)\big)$-Tutte subgraph $\{v_1\}\cup T_1\cup T_2$ in Case 3.1 in the proof of Lemma 10.

Suppose that $w\notin V(T_1\cup T_2)$, and let H be the $(\{v_1\}\cup T_1\cup T_2)$-bridge of G containing w. Note that v_1 is an attachment of H in G, and $H-v_1$ is a T_2-bridge of G_2' containing an edge of $C_2'[u_1,z]$. Hence it follows from the conditions of T_2 that $H-v_1$ has two attachments on T_2 such that one of them is contained in $C_2'[u_1,w]-\{w\}$, say a, and the other is contained in $C_2'[w,z]-\{w\}$, say c.

On the other hand, if $w\in V(T_1\cup T_2)$, then let H be the null graph and let $a=c=v_1$. Then in either case, H is a $C[u_1,v_1]$-flap in G with attachments a,v_1,c. See Figure 30.

Let B be a $(\{v_1\}\cup T_1\cup T_2)$-bridge of $G-\mathrm{Nuc}(H)$. Note that B is either (i) a $(\{v_1\}\cup T_1)$-bridge of G_1, or (ii) a $(\{v_1,z\}\cup T_2)$-bridge of G_2.

Suppose first that B satisfies (i). If v_1 is not an attachment of B, then B is also a T_1-bridge of G_1'. So B has at most three attachments, and at most two attachments if B contains an edge of C_1'. Note that $E\big(C[u_1,v_1]\cup C[u_2,v_2]\big)\cap E(G_1')=E(C[u_2,v_2])\subset E(C_1')$. On the other hand, if v_1 is an attachment of B, then $B-v_1$ is a T_1-bridge of G_1' containing an edge of C_1'. Hence $B-v_1$ has at most two attachments on T_1, so B has at most three attachments on $\{v_1\}\cup T_1$ one of which is v_1. Note that B contains no edge in $C[u_1,v_1]\cup C[u_2,v_2]$.

Suppose next that B satisfies (ii). If v_1 is not an attachment of B, then by the same argument as above, B has at most three attachments and at most two attachments if B contains an edge in $C_2'[u_1,z]$. Note that $E\big(C[u_1,v_1]\cup C[u_2,v_2]\big)\cap E(G_2')=E(C[u_1,w])\subset E(C_2'[u_1,z])$. On the other hand, suppose that v_1 is an attachment of B. In this case, again by the same argument as above, B has at most three attachments on $\{v_1,z\}\cup T_2$ such that one of them is v_1, and B contains no edge in $C[u_1,v_1]\cup C[u_2,v_2]$.

In either case, B has at most three attachments and at most two attachments if B contains an edge of $C[u_1,v_1]\cup C[u_2,v_2]$. Then H,T_1 and T_2 are desired ones in Lemma 10. This completes the proof of Claim 5. $\square$

Now we are ready to prove Lemma 10. Let $G'=G-v_1$. By Claim 5, G' is 2-connected. Let C' be the outer cycle of G', and we give a direction to C' so that $C'[u_2,v_2]=C[u_2,v_2]$. By the choice, we have $E\big(C[u_1,v_1]\cup C[u_2,v_2]\big)\cap E(G')=E\big(C[u_1,w]\cup C[u_2,v_2]\big)\subset E(C'[u_1,v_2])$. By Lemma 8, there exists a $C'[u_1,v_2]$-Tutte subgraph in G' consisting of two vertex-disjoint paths T_1 and T_2 such that T_1 and T_2 connect $\{v_2,y\}$ and $\{u_1,u_2\}$.

Suppose that $w\notin V(T_1\cup T_2)$, and let H be a $\big(\{v_1\}\cup T_1\cup T_2\big)$-bridge of G containing w. Note that v_1 is an attachment of H in G, and $H-v_1$ is a $(T_1\cup T_2)$-bridge of G' containing an edge of $C'[u_1,v_2]$. Therefore, $H-v_1$ has two attachments on $T_1\cup T_2$ such that one of them is contained in $C'[u_1,w]\cap C$, say a, and the other is contained in $C'[w,u_2]$, say c. On the other hand, if $w\in V(T_1\cup T_2)$, then let H be the null graph and $a=c=v_1$. In either case, note that H is a $C[u_1,v_1]$-flap with attachments a,v_1,c.

Let B be a $(\{v_1\}\cup T_1\cup T_2)$-bridge of $G-\mathrm{Nuc}(H)$. If v_1 is not an attachment of B, then B has at most three attachments and at most two attachments if B contains an edge of $C'[u_1,v_2]$. If v_1 is

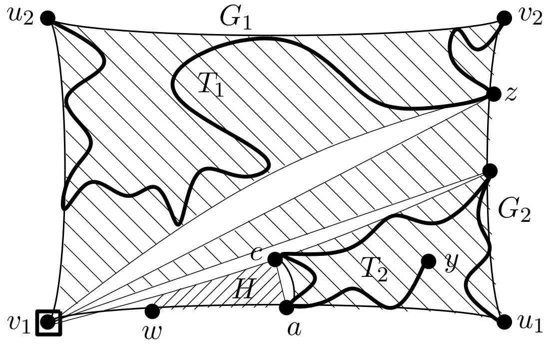

Figure 30: A $C[u_1, v_1]$-flap H and a $\big(C[u_1, v_1] \cup C[u_2, v_2] - \mathrm{Nuc}(H)\big)$-Tutte subgraph $\{v_1\} \cup T_1 \cup T_2$ in Case 3.2 in the proof of Lemma 10.

an attachment of B, then $B - v_1$ is a $\big(T_1 \cup T_2\big)$-bridge of G' containing an edge of $C'[u_1, v_2]$. Hence $B - v_1$ has at most two attachments on $T_1 \cup T_2$, so B has at most three attachments on $\{v_1\} \cup T_1 \cup T_2$ one of which is v_1. Note that by the choice of H and by the planarity of G, B contains no edge in $C[u_1, v_1] \cup C[u_2, v_2]$.

In either case, B has at most three attachments and at most two attachments if B contains an edge of $C[u_1, v_1] \cup C[u_2, v_2]$. Then H, T_1 and T_2 are desired ones in Lemma 10. This completes the proof of Lemma 10. $\square$

6.4 Proof of Lemma 11

Let G be a 2-connected graph on the projective plane with representativity exactly 2, let R be a face of G, let C be the boundary cycle of R, and let $xw \in E(C)$. Suppose that G has (x, R)-width exactly 2.

Since the (x, R)-width of G is exactly 2, there exists an essential closed curve γ on the projective plane such that γ hits G at exactly two vertices, which are x and a vertex in C, say u. By cutting open G along γ, we obtain a connected plane graph G' with four distinct vertices u'_1, x'_1, u'_2, x'_2 on the outer walk, say C', with the following properties; They appear in C' in that order, and G is obtained from G' by identifying u'_1 and u'_2 into u, and x'_1 and x'_2 into x, respectively. Moreover, we may assume that $C'[u'_1, x'_1] \cup C'[u'_2, x'_2]$ becomes C after these identifications.

Suppose that G' has a 1-separation (K_1, K_2). Note that $\mathrm{Nuc}(K_1) = K_1 - V(K_2)$. If $|V(K_i) \cap \{u'_1, x'_1, u'_2, x'_2\}| \geq 2$ for each $i = 1, 2$, then there exists an essential curve on the projective plane which passes through G only at $V(K_1) \cap V(K_2)$, contradicting that G has representativity at least 2. Thus, by symmetry between K_1 and K_2, we may assume that $|V(K_1) \cap \{u'_1, x'_1, u'_2, x'_2\}| \leq 1$. On the other hand, if $V\big(\mathrm{Nuc}(K_1)\big) \cap \{u'_1, x'_1, u'_2, x'_2\} = \emptyset$, then $(K_1, \overline{K_1})$ is a 1-separation of G, a contradiction, where $\overline{K_1} = G - \mathrm{Nuc}(K_1)$. Therefore, $|V\big(\mathrm{Nuc}(K_1)\big) \cap \{u'_1, x'_1, u'_2, x'_2\}| = 1$. Then for each 1-separation (K_1, K_2) of G', we may assume that $|V\big(\mathrm{Nuc}(K_1)\big) \cap \{u'_1, x'_1, u'_2, x'_2\}| = 1$. If G' does not have any 1-separation, then let $G_0 = G'$; otherwise let

$$G_0 = \bigcap_{(K_1, K_2):\text{1-separation of } G'} K_2.$$

Note that G_0 is 2-connected. Let C_0 be the outer cycle of G_0.

If there exists a 1-separation (K_1, K_2) of G' with $\{u'_1\} = V\big(\mathrm{Nuc}(K_1)\big) \cap \{u'_1, x'_1, u'_2, x'_2\}$, then let $u_1 \in V(G_0)$ such that u_1 separates u'_1 from G_0 in G'; otherwise let $u_1 = u'_1$. If there exists a 1-separation (K_1, K_2) of G' with $\{x'_1\} = V\big(\mathrm{Nuc}(K_1)\big) \cap \{u'_1, x'_1, u'_2, x'_2\}$, then let $v_1 \in V(G_0)$ such that v_1 separates x'_1 from G_0 in G'; otherwise let $v_1 = x'_1$. Similarly, we define u_2 and v_2. Note that u_1, v_1, u_2, v_2 appear in C_0 in this order. So condition (G1) holds. Notice also that G_0 is a $\{u_1, x_1, u_2, x_2\}$-bridge of G'.

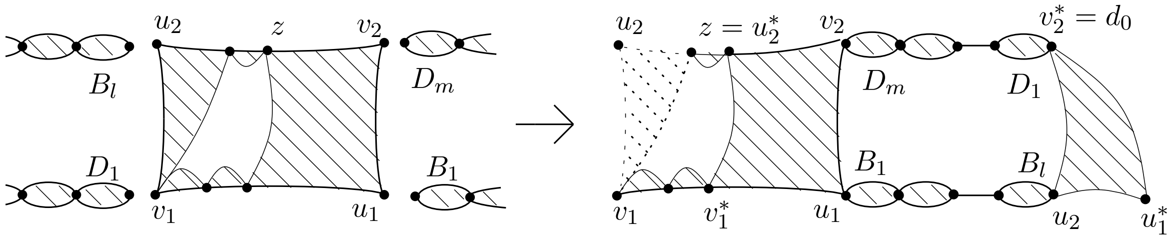

Figure 31: The graph G_0^* in the proof of Lemma 11.

Note that $G' - \mathrm{Nuc}(G_0)$ has four components each of which contains a vertex in $\{u_1', x_1', u_2', x_2'\}$, and hence each of them is a chain of blocks. Then identifying u_1' and u_2', and x_1' and x_2', respectively, $G - \mathrm{Nuc}(G_0)$ consists of two chains of blocks, say $b_0, B_1, b_1, B_2, \ldots, b_{l-1}, B_l, b_l$ and $d_0, D_1, \ldots, d_{m-1}, D_m, d_m$, where $b_0 = u_1$, $b_l = u_2$, $d_0 = v_1$ and $d_m = v_2$. Note that G is obtained from G_0 by identifying u_1 and b_0, u_2 and b_l, v_1 and d_0, and v_2 and d_m, respectively. Then condition (G2) also holds.

Moreover, since $C'[u_1', x_1'] \cup C'[u_2', x_2']$ becomes C after the identification, we have that

$$E(C) = E(C_0[u_1, v_1] \cup C_0[u_2, v_2]) \cup \bigcup_{i=1}^{l} E(C_{B_i}[b_{i-1}, b_i]) \cup \bigcup_{j=1}^{m} E(C_{D_j}[d_{j-1}, d_j]),$$

where C_{B_i} is the outer walk of B_i for each $1 \le i \le l$, and C_{D_j} is the outer walk of D_j for each $1 \le j \le m$. Hence condition (G3) holds.

Since $d_0, D_1, \ldots, d_{m-1}, D_m, d_m$ is obtained from two components of $G' - \mathrm{Nuc}(G_0)$ by identifying x_1' and x_2' into x, we have $x = d_k$ for some $0 \le k \le m$. By the symmetry between v_1 and v_2, we may assume that $x = d_k$ for some $0 \le k \le m-1$ and $w \in V(C_0[u_1, v_1])$ (when $k = 0$) or $w \in V(C_{D_k}[d_{k-1}, d_k])$ (when $k \ge 1$). So condition (G4) holds.

Suppose that $x = d_0$ and that there exists a 2-separation (K_1, K_2) of G_0 such that $x \in V(K_1 \cap K_2)$ and $z \in V(C_0[u_2, v_2])$, where $\{x, z\} = V(K_1 \cap K_2)$. Take a 2-separation (G_1, G_2) of G_0 so that $C_0[z, v_2]$ is as short as possible. Then we can find an essential closed curve γ^* on the projective plane that hits G only at x and z. Let $(G')^*$ be the graph obtained from G by cutting along with γ^* such that $(G')^*$ corresponds to G'. Similarly, we define C_0^*, $(x_1')^*$, $\ldots$ that correspond to C_0, x_1', $\ldots$, respectively. Then either $x \ne d_0^*$, or G_0^* has no 2-separation (K_1^*, K_2^*) such that $v_1^* \in V(K_1^* \cap K_2^*)$ and $z^* \in V(C_0^*[u_2^*, v_2^*])$, where $\{v_1^*, z^*\} = V(K_1^* \cap K_2^*)$. See Figure 31. Then condition (G5) also holds.

This completes the proof of Lemma 11. $\square$

Acknowledgments

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