Improved upper bounds for Gallai-Ramsey numbers of paths and cycles.

Martin Hall[*], Colton Magnant[*,†], Kenta Ozeki[‡,§], Masao Tsugaki[¶,‖]

December 5, 2012

Abstract

Given a graph G and a positive integer k, define the Gallai-Ramsey number to be the minimum number of vertices n such that any k-edge-coloring of K_n contains either a rainbow (all different colored) triangle or a monochromatic copy of G. In this work, we improve upon known upper bounds on the Gallai-Ramsey numbers for paths and cycles. All these upper bounds now have the best possible order of magnitude as functions of k.

1 Introduction

In this work, we consider only edge-colorings of graphs. A coloring of a graph is called *rainbow* if no two edges have the same color.

Edge-colorings of complete graphs which contain no rainbow triangle have very interesting and somewhat surprising structure. In 1967, Gallai [10] first examined this structure under the guise of transitive orientations (a translation of his paper is available in [13]). His result was restated in [12] in the terminology of graphs and can also be traced back to [4]. For the following statement, a *trivial partition* is a partition into only one part.

Theorem 1 ([4, 10, 12]) *In any coloring of a complete graph containing no rainbow triangle, there exists a non-trivial partition of the vertices (called a* Gallai partition*) such that there are at most two colors on the edges between the parts and only one color on the edges between each pair of parts.*

[*]Georgia Southern University, Statesboro, GA 30460, USA.
[†]Supported by Georgia Southern Faculty Research Committee - Research Competition Award.
[‡]National institute of Informatics, Tokyo 101-8430, Japan.
[§]JST, ERATO, Kawarabayashi Large Graph Project, Japan
[¶]Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China.
[‖]Supported by Chinese Academy of Sciences Fellowship for Young International Scientists Grant No. 2012Y1JA0004

In honor of this result, rainbow triangle-free colorings have been called *Gallai colorings*. Given a Gallai coloring of a complete graph and an associated Gallai partition, define the *reduced graph* of this partition to be the induced subgraph consisting of exactly one vertex from each part of the partition. Note that the reduced graph is a 2-colored complete graph.

When considering 2-colored complete graphs, a very natural problem to consider is the Ramsey problem of finding a monochromatic copy of some desired subgraph. Since we will be mostly considering cycles in this work, we recall a classical Ramsey result for odd cycles which will be used later in our proofs. Here, given a graph G, let $R_k(G)$ denote the k-color Ramsey number of G, namely the minimum number of vertices m such that any k coloring (using at most k colors) of K_m contains a monochromatic copy of G. The cycle of order m is denoted by C_m.

Theorem 2 ([3]) *For all integers n with $n \geq 2$,*

$$R_2(C_{2n+1}) = 4n + 1.$$

Although the reduced graph of a Gallai partition uses only two colors, the original Gallai colored complete graph could certainly use more colors. With this in mind, we consider the following generalization of the Ramsey numbers. Given two graphs G and H, the general k-colored *Gallai-Ramsey number* $\mathrm{gr}_k(G : H)$ is defined to be the minimum integer m such that every k-coloring of the complete graph on m vertices contains either a rainbow copy of G or a monochromatic copy of H. With the additional restriction of forbidding the rainbow copy of G, it is clear that $\mathrm{gr}_k(G : H) \leq R_k(H)$ for any G. The general behavior of $\mathrm{gr}_k(K_3 : H)$ was studied in [11] where the authors proved the following result.

Theorem 3 ([11]) *Let H be a fixed graph with no isolated vertices. Let k be an integer with $k \geq 1$. If H is not bipartite, then $\mathrm{gr}_k(K_3 : H)$ is exponential in k. If H is bipartite, then $\mathrm{gr}_k(K_3 : H)$ is linear in k.*

With this result in mind, the orders of magnitude in the following general bounds for paths and cycles should not be surprising. Let P_n be the path of order n.

Theorem 4 ([6]) *For all integers k and n with $k \geq 1$ and $n \geq 3$,*

$$\left\lfloor \frac{n-2}{2} \right\rfloor k + \left\lceil \frac{n}{2} \right\rceil + 1 \ \leq \ \mathrm{gr}_k(K_3 : P_n).$$

For any real number $\varepsilon > 0$ and all integers k and n sufficiently large,

$$\mathrm{gr}_k(K_3 : P_n) \ \leq \ \left(\frac{1}{2} + \varepsilon\right) nk.$$

Theorem 5 ([7]) *For all integers k and n with $k \geq 1$ and $n \geq 2$,*

$$(n-1)k + n + 1 \;\leq\; \mathrm{gr}_k(K_3 : C_{2n}) \;\leq\; (n-1)k + O(n \log n).$$

Also, for all integers k and n with $k \geq 1$ and $n \geq 2$,

$$n2^k + 1 \;\leq\; \mathrm{gr}_k(K_3 : C_{2n+1}) \;\leq\; k(n-1) + n(4n+1)3^{k-3}.$$

Note that for $\mathrm{gr}_k(K_3 : P_n)$ with $2 \leq n \leq 6$ and for $\mathrm{gr}_k(K_3 : C_n)$ with $3 \leq n \leq 6$, the exact numbers were shown in [2, 5, 6, 7, 11]. Except in the case when $C_n = K_3$, all of these exact results match the lower bounds in the above general results. We refer the reader to [8] for a survey of these and related results with an updated version available at [9].

In this work, we improve upon the general upper bounds in Theorems 4 and 5. Our main result for even cycles is the following, the proof of which can be found in Section 2. In particular, for the proof of Theorem 6, we will show a new result (Lemma 10) on "bipancyclicity" of bipartite graphs, see Section 3.

Theorem 6 *For all integers k and n with $k \geq 1$ and $n \geq 2$,*

$$(n-1)k + n + 1 \;\leq\; \mathrm{gr}_k(K_3 : C_{2n}) \;\leq\; (n-1)k + 3n.$$

In Section 4, we use Theorem 6 to produce the following improvement upon the upper bound for paths.

Theorem 7 *For all integers k and n with $k \geq 1$ and $n \geq 3$,*

$$\left\lfloor \frac{n-2}{2} \right\rfloor k + \left\lceil \frac{n}{2} \right\rceil + 1 \;\leq\; \mathrm{gr}_k(K_3 : P_n) \;\leq\; \left\lfloor \frac{n-2}{2} \right\rfloor k + 3\left\lfloor \frac{n}{2} \right\rfloor.$$

Finally, in Section 5, we improve the upper bound for odd cycles.

Theorem 8 *For all integers k and n with $k \geq 1$ and $n \geq 2$,*

$$n2^k + 1 \;\leq\; \mathrm{gr}_k(K_3 : C_{2n+1}) \;\leq\; \left(2^{k+3} - 3\right) n \log n.$$

By our results, all Gallai Ramsey numbers for paths and cycles are now known asymptotically as functions of k. Note that all lower bounds were previously known so we prove only the new upper bounds. Also note that the lower bound is known to be sharp for sufficiently large odd cycles by a recent result of Allen, Brightwell and Skokan [1].

Throughout this work, $|G|$ denotes the order (number of vertices) of the graph G. Also the subgraph of G induced on the edges of a particular color i is to include only the edges of color i and the vertices of G which are incident to edges of color i.

2 Gallai-Ramsey number for even cycles

For the proof of Theorem 6, we need two lemmas (Lemmas 9 and 10).

Lemma 9 *Let G be a Gallai colored complete graph. Then G has a Gallai partition such that for any color i on the edges between the parts, the subgraph of the reduced graph induced by i is connected. (Note that we have only one or two possible choices for the color i.)*

Proof. Let G be a Gallai colored complete graph. By Theorem 1, G has a Gallai partition, and let $\mathscr{V} := \{V_1, V_2, \cdots, V_t\}$ be the parts of a Gallai partition of G. We choose $\mathscr{V}$ so that t is as small as possible. Suppose that there exists a color i such that the subgraph of the reduced graph induced by i, say H, is not connected. For each component D of H, let $V_D := \bigcup_{V_j \in D} V_j$. Then

$$\{V_D : D \text{ is a component of } H\}$$

is also a Gallai partition of G, since H has at least two components, which contradicts the minimality of t. $\square$

For two vertex sets A and B, let $A + B$ denote the join of A and B. In other words, $A + B$ is the complete bipartite graph with bipartitions A and B. Furthermore, let $A_1 + A_2 + \cdots + A_t$ denote the graph $A_1 \cup A_2 \cup \cdots \cup A_t$ with the addition of all edges between A_i and A_{i+1} for all $1 \leq i \leq t-1$. Let $G_0 := B_1 + A_1 + B_2 + A_2 + B_3$ with $A_1, A_2 \neq \emptyset$, $|B_1| = |B_3|$ and $|B_2| = 1$. See Figure 1, where the heavy edges represent all possible edges between the sets.

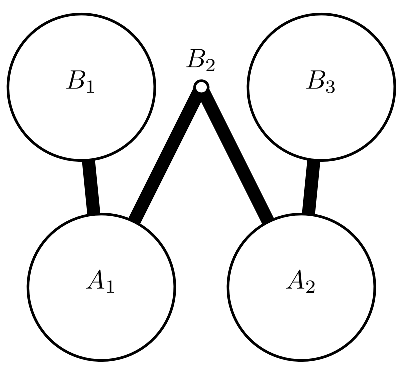

Figure 1: Construction of G_0.

Lemma 10 *Let $G = A \cup B$ be a bipartite graph with $|A| \geq 2$, $|B| \geq 4$ and minimum degree of vertices within A satisfying $\delta(A) \geq (|B|+1)/2$. Let l be an integer with $2 \leq l \leq \min\left\{|A|, \frac{|B|-1}{2}\right\}$. Then G has a C_{2l} or G is isomorphic to G_0.*

The proof of Lemma 10 is deferred to Section 3. Instead of Theorem 6, we will show the following lemma, which directly implies Theorem 6.

Lemma 11 *Let n be an integer with $n \geq 2$. Let G be a Gallai colored complete graph. Let $k'(G)$ be the number of colors that induce a graph having a component of order at least n in G. If $|G| \geq (n-1)k'(G) + 3n$, then G has a monochromatic C_{2n}.*

Proof. Let $k' := k'(G)$. If $k' = 0$, this contradicts Lemma 9, so we may assume that $k' \geq 1$. We consider the following two cases, depending on k'.

Case 1 $k' = 1$.

Then $|G| \geq 4n - 1$. Since $k' = 1$, all but one color (say red) induces a graph which is not connected. Then it follows from Lemma 9 that there exists a Gallai partition $\mathscr{V}$ of G using only red edges between the parts. Choose such a partition $\mathscr{V}$ so that $|\mathscr{V}|$ is as large as possible.

We shall prove that each set of $\mathscr{V}$ has an order at most $n-1$. Let $V \in \mathscr{V}$. Suppose that $|V| \geq n$. It follows from Lemma 9 that V also has a Gallai partition such that for any color i on the edges between the parts, the subgraph of the reduced graph induced by i is connected. Since $k' = 1$ and $|V| \geq n$, the Gallai partition of V has only one color which is red. But, adding the partition of V into $\mathscr{V}$ instead of V, this contradicts the maximality of $|\mathscr{V}|$, and hence $|V| \leq n-1$. Therefore, we can see that each set of $\mathscr{V}$ has order at most $n-1$.

Choose a bipartition of the vertices of G into A and B so that each of A and B is made up of sets of $\mathscr{V}$ and the orders of A and B are as balanced as possible. Since all the sets have order at most $n-1$, we know that $\big||A| - |B|\big| \leq n-1$. Since $|G| \geq 4n-1$, this implies that $|A|, |B| \geq \frac{3n}{2} \geq n$. Thus, it is easy to find a red C_{2n} using only edges between A and B, completing the proof in this case.

Case 2 $k' \geq 2$.

We prove this case by induction on k'. Let $D_1, D_2, \ldots, D_{k'} \subseteq V(G)$ (say $D := \bigcup_{i=1}^{k'} D_i$) be disjoint sets of vertices such that each vertex $x \in D_i$ has only the color i on the edges from x to $G - D$, and $|G - D| \geq n$. We take such sets $D_1, D_2, \ldots, D_{k'}$ so that $|D|$ is as large as possible.

Claim 1 *$|V| \leq n-1$ for each $V \in \{D_1, D_2, \cdots, D_{k'}\}$.*

Proof. Suppose that $|D_i| \geq n$ for some $1 \leq i \leq k'$. Since $|G-D| \geq n$, the graph induced on $D_i \cup (G - D)$ contains a monochromatic $K_{n,n}$, which contains a monochromatic C_{2n} colored by i. Thus, $|D_i| \leq n-1$ for each color i. □

Now consider a Gallai partition of $G - D$ as in Lemma 9, and let $V_1, V_2, \ldots, V_t$ be the parts of $G - D$ under the Gallai partition. Then any color i on the edges between the parts in this partition induces a connected graph in $G - D$. Let

$$\mathscr{V} := \{D_1, D_2, \cdots, D_{k'}\} \cup \{V_1, V_2, \cdots, V_t\}.$$

Claim 2 *Let e be an edge with color i between V and V' for some $V, V' \in \mathscr{V}$ with $V \neq V'$. Then $i \in \{1, 2, \ldots, k'\}$. In particular, if e is an edge between D_j and $D_{j'}$ for some $1 \leq j < j' \leq k'$, then $i = j$ or $i = j'$.*

Proof. Let e be an edge with color i between V and V' for some $V, V' \in \mathscr{V}$ with $V \neq V'$.

Suppose first that $V, V' \in \{V_1, V_2, \ldots, V_t\}$ and $i \notin \{1, 2, \ldots, k'\}$. Since any color on the edges between the parts in the partition of $G - D$ induces a connected graph, i is also connected in the reduced graph of the partition. Since $i \notin \{1, 2, \ldots, k'\}$, $G - D$ has at most $n - 1$ vertices. Then, by Claim 1, $(n-1)k' + 3n \leq |V(G)| \leq (n-1)k' + (n-1)$, a contradiction. So, if $V, V' \in \{V_1, V_2, \ldots, V_t\}$, then $i \in \{1, 2, \ldots, k'\}$.

Suppose next that $V \in \{D_1, D_2, \cdots, D_{k'}\}$ and $V' \in \{V_1, V_2, \cdots, V_t\}$. In this case, all edges between V and V' is colored by j, where $V = D_j$. Then $i = j \in \{1, 2, \cdots, k'\}$.

Suppose finally that $V = D_j$ and $V' = D_{j'}$ for some $1 \leq j < j' \leq k'$ and $i \neq j, j'$. Let $e := x_j x_{j'}$ with $x_j \in D_j$ and $x_{j'} \in D_{j'}$. Taking a vertex $y \in V(G) - D$, the edges e, $x_i y$ and $x_j y$ are colored by i, j, and j', respectively, the triangle $x_i x_j y$ is rainbow, a contradiction. $\square$

Claim 3 $|V| \leq n - 1$ *for each* $V \in \{V_1, V_2, \cdots, V_t\}$.

Proof. If $|V_i| \geq n$ for some $1 \leq i \leq t$, then, by Claim 2, we can add $(G - D) - V_i$ into D, contradicting the maximality of $|D|$. $\square$

Subcase 1 $k' = 2$.

In this case, note that $|V(G)| \geq 5n - 2$. We take a subset $B' \subseteq V(G)$ so that B' consists of the union of sets in $\mathscr{V}$ with the additional restriction that $|B'| \geq 2n + 1$. Choose such a set B' so that $|B'|$ is as small as possible. By Claims 1 and 3, $|V| \leq n - 1$ for each $V \in \mathscr{V}$, and hence the choice of a smallest set B' implies that $|B'| \leq 3n - 1$. Hence, we have

$$|V(G) - B'| \geq 5n - 2 - (3n - 1) = 2n - 1.$$

When $|B'|$ is odd, then let $B := B'$; otherwise let $B := B' - \{u\}$ for some u. We construct B so that $|B|$ is odd in either case.

For $i = 1, 2$, let

$$A^i \quad := \quad \{x \in V(G) - B' : x \text{ has at least } \frac{|B| + 1}{2} \text{ edges to } B \text{ with color } i.\}.$$

By Claim 2, all edges between B' and $V(G) - B'$ are colored by either color 1 or color 2. Hence, we have that $V(G) - B' = A^1 \cup A^2$. We may assume that $|A^1| \geq \left\lceil \frac{|V(G)-B'|}{2} \right\rceil \geq n$.

Let H be the bipartite subgraph of G on $A^1 \cup B$ induced by the edges colored by color 1. Then $H = A^1 \cup B$ is a bipartite graph with $\delta_B(A^1) \geq \frac{|B|+1}{2}$, and $|B| \geq 2n+1 \geq 5$. Hence, by Lemma 10, either H contains a C_{2l} for each $2 \leq l \leq \min\left\{|A^1|, \frac{|B|-1}{2}\right\}$ or H is isomorphic to G_0. If the former holds, then in particular, H contains a C_{2n} since $\min\left\{|A^1|, \frac{|B|-1}{2}\right\} \geq n \geq 2$. Thus, $C_{2n} \subseteq H \subseteq G$ so G contains the desired cycle colored by 1. Therefore, we may assume that H is isomorphic to G_0.

Let $H := B_1 + A_1 + B_2 + A_2 + B_3$ with $A_1, A_2 \neq \emptyset$, $|B_1| = |B_3|$ and $|B_2| = 1$. Note that $|B_1| = |B_3| = \frac{|B|-1}{2} \geq n$.

$$\begin{aligned} \text{Let} \quad \widetilde{A_1} &:= \{x \in A^1 \cup A^2 : \text{all edges between } x \text{ and } B_3 \text{ have color } 2\}, \\ \text{and} \quad \widetilde{A_2} &:= V(G) - B' - \widetilde{A_1}. \end{aligned}$$

Since H has no edges between A_1 and B_3, we have that $A_1 \subseteq \widetilde{A_1}$. Note that for all $x \in A_2$, we have that $x \in \widetilde{A_2}$ and all edges between x and B_1 are colored by 2. If there exist two edges xy_1 and xy_3 colored by 1 such that $x \in A^2$, $y_1 \in B_1$ and $y_3 \in B_3$, then, together with x and the two edges xy_1 and xy_3, we can easily find a monochromatic C_{2n} colored by 1 in H. Thus, we may assume that for all $x \in A^2$, all edges between x and B_j are colored by 2 for some $j = 1, 3$. Hence, we also have that for each $x \in \widetilde{A_2}$, all edges between x and B_1 have color 2.

Suppose first that $|\widetilde{A_1}| \geq n$. Then with $\widetilde{A_1}$ and B_3 forming a complete bipartite graph with all edges colored in color 2 and $|\widetilde{A_1}|, |B_3| \geq n$, we can easily find a monochromatic C_{2n} colored by 2. Thus, we may assume that $|\widetilde{A_1}| < n$. Since $\widetilde{A_1} \cup \widetilde{A_2} = V(G) - B'$ has at least $2n-1$ vertices, we have that $|\widetilde{A_2}| \geq n$. In this case, with $\widetilde{A_2}$ and B_1 forming a complete bipartite graph with all edges having color 2 and $|\widetilde{A_2}|, |B_1| \geq n$, we are also done, completing the proof in this case.

Subcase 2 $k' \geq 3$.

Since there are at most two colors between the parts of $G - D$, there is an integer i with $1 \leq i \leq k'$, by symmetry say $i = k'$, such that the color k' does not appear on the edges between different parts in $G - D$. Let $G' := G - D_{k'}$. By Claim 1,

$$\begin{aligned} |G'| &\geq |G| - |D_{k'}| \\ &\geq (n-1)(k'-1) + 3n. \end{aligned}$$

By Claim 2, for every $1 \leq p < q \leq k' - 1$, every edge between D_p and D_q is not colored by k'. This and the choice of $i = k'$ implies that every component of the subgraph induced on color k' in G' is contained in some $V \in \mathscr{V}$. Since $|V| \leq n-1$, the color k' has no component in G' of order at least n. This implies that $k'(G') \leq k'(G) - 1$, and by the induction hypothesis, G' has a monochromatic C_{2n}. Since $G' \subseteq G$, G has a monochromatic C_{2n}. This completes the proof of Lemma 11. $\square$

3 Proof of Lemma 10

For a graph G and an integer l, let $\mathcal{C}_l$ denote the set of cycles in G of length exactly l.

We divide the proof of Lemma 10 into three cases depending on the connectivity of G. In particular, we show the following three lemmas. After that, we will prove Lemma 10 using these three lemmas.

Lemma 12 *Let $G = A \cup B$ be a connected bipartite graph having a cut vertex in B. If $|A| \geq 2$ and $\delta(A) \geq (|B|+1)/2$, then $\delta(A) = (|B|+1)/2$ and G is isomorphic to G_0.*

Lemma 13 *Let $G = A \cup B$ be a connected bipartite graph having no cut vertex in B. If $|A| \geq 2$, $|B| \geq 4$ and $\delta(A) \geq (|B|+1)/2$, then $\mathcal{C}_{2l} \neq \emptyset$ for any $2 \leq l \leq \min\{|A|, \delta(A)-1\}$.*

Lemma 14 *Let $G = A \cup B$ be a disconnected bipartite graph. If $|A| \geq 2$, $|B| \geq 4$ and $\delta(A) \geq (|B|+1)/2$, then $\mathcal{C}_{2l} \neq \emptyset$ for any $2 \leq l \leq \min\{|A|, \delta(A)-1\}$.*

The proof of Lemma 12 is easy so we begin with the proof of Lemma 13. Before it, we need to define some terminology used in this section. For a vertex x in a graph G, we denote, by $d_G(x)$ and $N_G(x)$, the degree and the neighborhood of a vertex x in G, respectively. For a subgraph H of G and a vertex $x \in V(G) - V(H)$, we denote $N_H(x) := N_G(x) \cap V(H)$. We write a cycle C with a given cyclic orientation by $\overrightarrow{C}$. Let C be a cycle or a path. For $x, y \in V(C)$, we denote by $x\overrightarrow{C}y$ a path from x to y on $\overrightarrow{C}$. The reverse sequence of $x\overrightarrow{C}y$ is denoted by $y\overleftarrow{C}x$. For $x \in V(C)$, we denote the successor and the predecessor of x on $\overrightarrow{C}$ by x^+ and x^-, respectively. For $X \subseteq V(C)$, we define $X^+ := \{x^+ : x \in X\}$ and $X^- := \{x^- : x \in X\}$. For an integer $i \geq 2$, we inductively define $x^{+i} := (x^{+(i-1)})^+$.

Proof of Lemma 13. We prove Lemma 13 by induction on l.

Claim 4 $\mathcal{C}_4 \neq \emptyset$.

Proof. Suppose that $|A| = 2$, that is $A := \{a_1, a_2\}$. Since G has no cut vertex in B, $|N_G(a_1) \cap N_G(a_2)| \geq 2$. Then we can easily see $\mathcal{C}_4 \neq \emptyset$. Hence, we may assume that $|A| \geq 3$. Suppose that $\mathcal{C}_4 = \emptyset$. Then $|N_G(a_1) \cap N_G(a_2)| \leq 1$ for any $a_1, a_2 \in A$ with $a_1 \neq a_2$. Let $a_1, a_2, a_3 \in A$ be three distinct vertices. Then $|B| \geq |N_G(a_1) \cup N_G(a_2) \cup N_G(a_3)| \geq 3((|B|+1)/2) - 3$, that is $|B| \leq 3$, a contradiction. Hence, $\mathcal{C}_4 \neq \emptyset$. $\square$

Claim 5 *For any $3 \leq l \leq \min\{|A|, \delta(A)-1\}$, if $\mathcal{C}_{2l-2} \neq \emptyset$, then $\mathcal{C}_{2l} \neq \emptyset$.*

Proof. Suppose, for a contradiction, that $\mathcal{C}_{2l-2} \neq \emptyset$ and $\mathcal{C}_{2l} = \emptyset$ for some $3 \leq l \leq \min\{|A|, \delta(A) - 1\}$. For $C \in \mathcal{C}_{2l-2}$ and $x \in V(G) - V(C)$, let $H_C := G - C$ and $H_C^x := N_G(x) \cap H_C$. If $A \subseteq V(C)$ for some $C \in \mathcal{C}_{2l-2}$, then $l \leq |A| = |C|/2 = l - 1$, a contradiction. Hence, $A \cap H_C \neq \emptyset$ for any $C \in \mathcal{C}_{2l-2}$. Suppose that $|H_C^x| \leq 1$ for some $C \in \mathcal{C}_{2l-2}$ and $x \in H_C \cap A$. Then $\delta(A) \leq d_G(x) \leq |C \cap B| + 1 = l \leq \delta(A) - 1$, a contradiction. Hence, $|H_C^x| \geq 2$ for any $C \in \mathcal{C}_{2l-2}$ and $x \in H_C \cap A$.

Suppose that there exist no cycle $C \in \mathcal{C}_{2l-2}$ and $x \in H_C \cap A$ such that $N_C(x) \neq \emptyset$. Let $C \in \mathcal{C}_{2l-2}$. Since G is connected, there exist $x \in H_C \cap A$ and $y \in H_C^x$ such that $N_C(y) \neq \emptyset$. By assumption, $N_C(x) = \emptyset$. For convenience, we abbreviate H_C and H_C^x by H and H^x, respectively. Let $u \in N_C(y)$. Suppose that $y \in N_H(u^{+2})$. Then $C' := u^{+2}\overrightarrow{C}uyu^{+2}$ is a cycle such that $C' \in \mathcal{C}_{2l-2}$, $x \in H_{C'} \cap A$ and $N_{C'}(x) \neq \emptyset$, a contradiction. Hence, we get $y \notin N_H(u^{+2})$. Since $\mathcal{C}_{2l} = \emptyset$, $N_H(u^{+2}) \cap N_H(x) - \{y\} = \emptyset$. Then $|B| = |C \cap B| + |H \cap B| \geq |N_C(u^{+2})| + |N_H(u^{+2})| + |N_H(x)| \geq 2((|B|+1)/2) = |B|+1$, a contradiction. Hence, there exist $C \in \mathcal{C}_{2l-2}$ and $x \in H_C \cap A$ such that $N_C(x) \neq \emptyset$.

For convenience, we again abbreviate H_C and H_C^x by H and H^x, respectively. Let $v \in N_C(x)$ and $Z := (N_C(v^+)^- \cap N_C(x)^+) - \{v^+\}$. For $z \in Z$, $C' := v^+z^+\overrightarrow{C}vxz^-\overleftarrow{C}v^+$ is a cycle such that $C' \in \mathcal{C}_{2l-2}$ and $z \in H_{C'} \cap A$. Hence, for each $z \in Z$, we can take a set $O_z \subseteq H_{C'}^z \subseteq H_C$ with $|O_z| = 2$. Note that $O_z \cap O_{z'} = \emptyset$ for all $z, z' \in Z$ with $z \neq z'$, since otherwise $v^+z^+\overrightarrow{C}z'^-xv\overleftarrow{C}z'x'z\overleftarrow{C}v^+$ is a cycle in $\mathcal{C}_{2l}$, where $x' \in O_z \cap O_{z'}$ and z appears in $v^+\overrightarrow{C}z'$. Let $O^* := \bigcup_{z \in Z} O_z$. Then $|O^*| = 2|Z|$. Since $\mathcal{C}_{2l} = \emptyset$, $(N_H(x) \cup N_H(v^+)) \cap O^* = \emptyset$ and $N_H(x) \cap N_H(v^+) = \emptyset$. Then $|C \cap B| = |C \cap A| \geq |N_C(v^+)^- \cup N_C(x)^+| = |N_C(v^+)| + |N_C(x)| - |Z| - 1$ and $|H \cap B| \geq |N_H(v^+)| + |N_H(x)| + |O^*|$. Then

$$
\begin{aligned}
|B| &\geq \big(|N_C(v^+)| + |N_C(x)| - |Z| - 1\big) + \big(|N_H(v^+)| + |N_H(x)| + 2|O^*|\big) \\
&\geq d_G(v^+) + d_G(x) - 1 + |Z| \\
&\geq 2\big((|B| + 1)/2\big) - 1 + |Z| \geq |B|,
\end{aligned}
$$

so equality holds in the above inequalities. In particular, $Z = \emptyset$ and $C \cap A = N_C(v^+)^- \cup N_C(x)^+$. Let $M := N_H(v^+)$. Then $M \cap H^x = \emptyset$ and $H \cap B = M \cup H^x$.

If $M = \emptyset$, then $N_G(v^+) \subseteq V(C) \cap B$, and $2l - 2 = |C| \geq 2|N_G(v^+)| \geq 2\delta(A) \geq 2l + 2$, a contradiction. Hence, $M \neq \emptyset$.

Suppose that $N_C(x) - \{v\} \neq \emptyset$, say $v' \in N_C(x) - \{v\}$. By the symmetry of v and v', $N_H(v'^+) = H \cap B - H^x = M$. Let $w \in M$. Then $wv^+\overrightarrow{C}v'xv\overleftarrow{C}v'^+w$ is a cycle in $\mathcal{C}_{2l}$, a contradiction. Hence, $N_C(x) = \{v\}$ and this implies that $C \cap B = N_C(v^+)$.

Therefore, $N_C(y) = \emptyset$ for any $y \in H^x$, since otherwise, $v^+\overleftarrow{C}uyxv\overrightarrow{C}u^+v^+$ is a cycle in $\mathcal{C}_{2l}$, where $u \in N_C(y)$, a contradiction. Since $v \in B$, v is not a cut vertex of G, and hence there exist $y \in H^x$ and $x' \in N_H(y) - \{x\}$ such that $N_C(x') - \{v\} \neq \emptyset$ or $N_H(x') \cap M \neq \emptyset$. If $N_C(x') - \{v\} \neq \emptyset$, then, for $v' \in N_C(x') - \{v\}$, let $C' := xyx'v'\overleftarrow{C}v^+v'^{+2}\overrightarrow{C}vx$. If $N_H(x') \cap M \neq \emptyset$, then, for $w \in N_H(x') \cap M$, let $C' := vxyx'wv^+v^{+4}\overrightarrow{C}v$. Then $C' \in \mathcal{C}_{2l}$, a contradiction. □

Proof of Lemma 14. Since $\delta(A) \geq (|B|+1)/2$, there exists a component G' of G such that $A \subseteq V(G')$. Let $G' := A \cup B'$, where $B' \subseteq B$. Since $d_{G'}(a) \geq \delta(A) \geq (|B|+1)/2 > (|B'|+1)/2$ for any $a \in V(G') \cap A$, it follows from Lemma 12 that G' has no cut vertex in B'. Suppose that $|B'| \geq 4$. Then, by Lemma 13, $\mathcal{C}_{2l} \neq \emptyset$ for any $2 \leq l \leq \min\{|A|, \delta_{G'}(A)-1\} = \min\{|A|, \delta(A)-1\}$. Hence, we may assume that $|B'| \leq 3$. Then $5/2 \leq (|B|+1)/2 \leq \delta(A) \leq |B'| \leq 3$, that is $\delta(A) = |B'| = 3$. This implies that G' is a complete bipartite graph, and $\min\{|A|, \delta(A)-1\} = 2$ since $|A| \geq 2$. Therefore, $\mathcal{C}_{2l} \neq \emptyset$ for any $2 \leq l \leq \min\{|A|, \delta(A)-1\}$. □

Proof of Lemma 10. Let $G := A \cup B$ be a bipartite graph with $|A| \geq 2$, $|B| \geq 4$ and $\delta(A) \geq (|B|+1)/2$. Let l be an integer with $2 \leq l \leq \min\{|A|, \frac{|B|-1}{2}\}$. If G is disconnected, or if G is connected and has no cut vertex in B, then it follows from Lemmas 13 and 14 and the fact $l \leq \frac{|B|-1}{2} \leq \delta(A) - 1$ that G has a C_{2l}. If G is connected and has a cut vertex in B, then by Lemma 12, G is isomorphic to G_0. □

4 Gallai-Ramsey number for paths

In light of Theorem 6, the proof of our improved upper bound on Gallai-Ramsey numbers for paths, becomes relatively easy.

Proof of Theorem 7. Let G be a Gallai colored complete graph of order at least $\lfloor \frac{n-2}{2} \rfloor k + 3 \lfloor \frac{n}{2} \rfloor$ and suppose, for a contradiction, that G contains no monochromatic P_n. By Theorem 6, there exists a monochromatic C_m with $m = 2\lfloor \frac{n}{2} \rfloor$. If n is even, we can easily obtain a monochromatic P_n. Hence, we may assume that n is odd. For convenience, let t be chosen so that $2t+1 = n$ and note that G contains no monochromatic P_{2t+1}. Let k be the color of the edges of C_m. Suppose that there exists no monochromatic P_{2t+1}. This choice implies that any edge between a vertex in $V(G) - V(C)$ and a vertex in C cannot have the color k. If there exists a vertex $x \in V(G) - V(C)$ such that x has at least two colors on the edges from x to $V(C)$, then two of the colors together with k induces a rainbow triangle, a contradiction. Thus, each vertex $x \in V(G) - V(C)$ has only one color on the edges from x to $V(C)$. Therefore, $V(G) - V(C)$ can be partitioned into $D_1, D_2, \ldots, D_{k-1}$ so that each vertex $x \in D_i$ has only one color i on the edges from x to $V(C)$. Since there exists no monochromatic P_{2t+1} and $|C| = 2t$, we see that $|D_i| \leq t-1$ for any $1 \leq i \leq k-1$. Hence, $(t-1)k + 3t \leq |V(G)| \leq |V(C)| + \sum_{1 \leq i \leq k-1} |D_i| \leq 2t + (t-1)(k-1)$, a contradiction. □

5 Gallai-Ramsey number for odd cycles

Before proving our main result for odd cycles, we first state a general counting lemma which will be used in its proof.

Lemma 15 *For $1 \leq t \leq n$, any Gallai colored complete graph having a Gallai partition with at least $4\left\lceil \frac{n}{t} \right\rceil + 1$ parts each of order at least t contains a monochromatic C_{2n+1}.*

Proof. Suppose $1 \leq t \leq n$, G is a Gallai colored complete graph and there exists a Gallai partition of G with at least $4\left\lceil \frac{n}{t} \right\rceil + 1$ parts of order at least t. Let H be the subgraph of the reduced graph of this partition which is induced on the vertices corresponding to the parts of order at least t. Then H has at least $4\left\lceil \frac{n}{t} \right\rceil + 1$ vertices, and hence by Theorem 2, there exists a monochromatic C_m with $m = 2\left\lceil \frac{n}{t} \right\rceil + 1$ in H.

Let $H_1, H_2, \dots, H_m$ be the parts of G corresponding to these vertices of H used in the cycle (in order) and let $x_{i,1}, x_{i,2}, \dots, x_{i,t}$ be t vertices of each part H_i. Construct a monochromatic cycle in G using these vertices as follows. Start by using t vertices of H_1 and H_2 by taking the path $x_{1,1}x_{2,1}x_{1,2}x_{2,2}x_{1,3}x_{2,3}\dots x_{1,t}x_{2,t}$ and jumping to $x_{3,1}$. Then repeat this process on the pair H_3 and H_4 and so on until H_{m-1} is used. From $x_{m-1,t}$, we take the path $x_{m-1,t}x_{m,1}x_{1,1}$ to close a cycle. Note that by construction, this cycle contains at least $2t\left\lceil \frac{n}{t} \right\rceil + 1 \geq 2n + 1$ vertices and has odd length. By bypassing vertices in the construction, for example using the edge $x_{1,1}x_{2,2}$ in place of the path $x_{1,1}x_{2,1}x_{1,2}x_{2,2}$, we can easily remove an even number of vertices if necessary to produce the desired monochromatic C_{2n+1}. $\square$

We may now prove our main result for this section, a general upper bound on the Gallai-Ramsey number for odd cycles. Recall Theorem 8 says that for all integers k and n with $k \geq 1$ and $n \geq 2$,

$$\operatorname{gr}_k(K_3 : C_{2n+1}) \leq \left(2^{k+3} - 3\right) n \log n.$$

For integers $n, k, t_1, t_2, \dots, t_k$, with $n \geq 2$ and $k \geq 1$. let

$$f_n(k; t_1, t_2, \dots, t_k) = \left(2^{k+3} - 3\right) n \log n - \sum_{i=1}^{k} t_i.$$

We show the following lemma, which directly implies Theorem 8 by setting $T_1 = T_2 = \dots = T_k = \emptyset$.

Lemma 16 *Let k and n be integers with $k \geq 1$ and $n \geq 2$. Let G be a Gallai colored complete graph with at most k colors, and let $T_1, T_2, \dots, T_k$ be disjoint sets of vertices with $T_i \cap V(G) = \emptyset$ for all $1 \leq i \leq k$. Let $t_i := |T_i|$ for $1 \leq i \leq k$. Suppose that $0 \leq t_i \leq n - 1$ and all edges between T_i and $V(G)$ are colored by the color i for all $1 \leq i \leq k$. If $|G| \geq f_n(k; t_1, t_2, \dots, t_k)$, then $G \cup \bigcup_{i=1}^{k} T_i$ has a monochromatic C_{2n+1}.*

Proof. We prove Lemma 16 by induction on k. If $k = 1$, then a complete graph with

$$f_n(1; t_1) = 13n \log n - t_1 > 2n + 1$$

vertices colored entirely in one color certainly contains a monochromatic C_{2n+1}. Thus, we may assume that $k \geq 2$, and suppose that G does not have a monochromatic C_{2n+1}.

Let $D_1, D_2, \ldots, D_k$ be the sets of vertices of G such that $\sum_{j=1}^{k} |D_j| \leq (k+2)n$ and for each $1 \leq i \leq k$, D_i has all edges of the color i to $G - \bigcup_{j=1}^{k} D_j$. (Note that some D_i's might be empty). Choose such $D_1, D_2, \ldots, D_k$ so that $\sum_{j=1}^{k} |D_j|$ is as large as possible under these restrictions. Let $H := G - \bigcup_{j=1}^{k} D_j$. For each $1 \leq i \leq k$, let $T_i' := T_i \cup D_i$ and $t_i' := |T_i'| = t_i + |D_i|$. We have the following claim.

Claim 6 *For all $1 \leq i \leq k$, we have that $t_i' \leq n - 1$.*

Proof. Suppose, for a contradiction, that $t_i' \geq n$ for some $1 \leq i \leq k$. By symmetry, we may assume that $i = k$. It follows from the conditions $t_k \leq n-1$ and $\sum_{j=1}^{k} |D_j| \leq (k+2)n$ that

$$\begin{aligned}
|H| = |G| - \sum_{j=1}^{k} |D_j| \quad &\geq \quad \left(2^{k+3} - 3\right) n \log n - \sum_{j=1}^{k-1} t_j - t_k - (k+2)n \\
&\geq \quad \left(2^{k+3} - 3\right) n \log n - (k+3)n + 1 - \sum_{j=1}^{k-1} t_j \\
&\geq \quad \left(2^{k+2} - 3\right) n \log n - \sum_{j=1}^{k-1} t_j \\
&= \quad f_n(k-1; t_1, \ldots, t_{k-1}).
\end{aligned}$$

(Recall that $2^{k+2} n \log n \geq (k+3)n$ since $k \geq 2$ and $n \geq 2$.) In particular, $|H| \geq n+1$. If H has an edge x_1x_2 colored by k, then together with such an edge, n vertices in T_k' and $n+1$ vertices in H including x_1 and x_2 form a C_{2n+1} colored entirely by color k, a contradiction. Thus, H has no edge colored by k. It follows from the induction hypothesis that $H \cup \bigcup_{i=1}^{k-1} T_i$ has a monochromatic C_{2n+1}, a contradiction. $\square$

Now consider a Gallai partition of H. Let V be a part of this partition with maximum order.

Claim 7 $|V| \geq n + 1$.

Proof. Suppose that $|V| \leq n$. For $1 \leq i \leq n$, let h_i be the number of parts of H under the Gallai partition whose order is at least i. Then, by Lemma 15, $h_i \leq 4 \left\lceil \frac{n}{i} \right\rceil$ for $1 \leq i \leq n$. Since $|V| \leq n$, we have $|H| = \sum_{i=1}^{n} h_i \leq \sum_{i=1}^{n} 4 \left\lceil \frac{n}{i} \right\rceil$. Hence, by Claim 6 and the fact that $k \geq 2$ and $n \geq 2$,

$$
\begin{aligned}
|G| &\leq \sum_{j=1}^{k} |D_j| + \sum_{i=1}^{n} 4 \left\lceil \frac{n}{i} \right\rceil \\
&< \sum_{j=1}^{k} (t'_j - t_j) + \sum_{i=1}^{n} \Big(\frac{4n}{i} + 4 \Big) \\
&\leq \sum_{j=1}^{k} (t'_j - t_j) + 4n\Big(1 + \int_1^n \frac{1}{x}\ dx\Big) + 4n \\
&\leq k(n-1) - \sum_{j=1}^{k} t_j + 4n \log n + 8n \\
&= \Big(\frac{k+8}{\log n} + 7\Big) n \log n - 3n \log n - k - \sum_{j=1}^{k} t_j \\
&< \big(2^{k+3} - 3\big) n \log n - \sum_{j=1}^{k} t_j \\
&= f_n(k; t_1, t_2, \ldots, t_k),
\end{aligned}
$$

a contradiction. □

The following is the most important claim in this section. It forces a bottleneck of the function $f_n(k; t_1, t_2, \ldots, t_k)$.

Claim 8 *For any sets $V_1, V_2 \subseteq V(H)$ satisfying $V_1 \cap V_2 = \emptyset$ and $|V(H) - V_1 - V_2| \leq n$, the set $G[V_1] \cup G[V_2]$ contains an edge of every color i.*

Proof. Suppose that there exist $V_1, V_2 \subseteq V(H)$ and a color i such that $V_1 \cap V_2 = \emptyset$, $|V(H) - V_1 - V_2| \leq n$, and neither $G[V_1]$ nor $G[V_2]$ contains an edge of color i. We may assume that $i = k$. If $|V_1| \geq f_n(k-1; t'_1, t'_2 \ldots, t'_{k-1})$, then by Claim 6, we can use the induction hypothesis for $G[V_1]$ together with $T'_1, T'_2, \ldots, T'_{k-1}$. Then $G[V_1] \cup \bigcup_{i=1}^{k-1} T'_i$ has a monochromatic C_{2n+1}, which is also contained in $G \cup \bigcup_{i=1}^{k} T_i$, a contradiction. Hence, $|V_1| < f_n(k-1; t'_1, t'_2 \ldots, t'_{k-1})$. By symmetry, $|V_2| < f_n(k-1; t'_1, t'_2 \ldots, t'_{k-1})$. Then by

Claim 6,

$$\begin{aligned}
|G| &= |V_1| + |V_2| + |V(H) - V_1 - V_2| + \sum_{i=1}^{k} |D_i| \\
&\le 2f_n(k-1; t'_1, t'_2, \ldots, t'_{k-1}) - 2 + n + \sum_{i=1}^{k} |D_i| \\
&= 2\Big(\big(2^{k+2} - 3\big)n \log n - \sum_{i=1}^{k-1} t'_i\Big) + n + \sum_{i=1}^{k} |D_i| - 2 \\
&= \big(2^{k+3} - 3\big)n \log n - 3n \log n - \sum_{i=1}^{k-1}(t'_i - |D_i|) - \sum_{i=1}^{k-1} t'_i + |D_k| + n - 2 \\
&\le \big(2^{k+3} - 3\big)n \log n - 3n \log n - \sum_{i=1}^{k} t_i + t_k + |D_k| + n - 2 \\
&\le f_n(k; t_1, t_2, \ldots, t_k) - 3n \log n + 2n - 3 \\
&< f_n(k; t_1, t_2, \ldots, t_k),
\end{aligned}$$

a contradiction. □

Claim 9 $|H - V| \ge 2(n+1)$.

Proof. Suppose that $|H - V| < 2(n+1)$. Since V is a part of the Gallai partition of H, each of the vertices of $H - V$ have all edges of a single color to V, and hence each vertex of $H - V$ can be added into D_i for some i. Let D'_i be the sets we obtained from D_i by the above operation. By Claim 6,

$$\begin{aligned}
\sum_{j=1}^{k} |D'_j| &= \sum_{j=1}^{k} |D_i| + |H - V| \\
&< k(n-1) + 2(n+1) \\
&\le (k+2)n.
\end{aligned}$$

This contradicts the choice of the sets D_i. □

Here we call the colors on the edges between parts of the Gallai partition of H red and blue. (Possibly blue might not be used in the Gallai partition.) Let

$$V_R := \{u \in V(H) - V : \ uv \text{ is colored by red for any } v \in V\}$$

and

$$V_B := \{u \in V(H) - V : \ uv \text{ is colored by blue for any } v \in V\}.$$

Since V is a part of the Gallai partition, $V(H) - V = V_R \cup V_B$. Let $\mathcal{R}$ and $\mathcal{B}$ be the sets of parts of the Gallai partition contained in V_R and V_B, respectively. Since there is no monochromatic C_{2n+1}, the following fact holds by Claims 7.

Fact 10 **(i)** *If* $|V_R| \geq n+1$, *then neither* $G[V]$ *nor* $G[V_R]$ *have a red edge.*

(ii) *If* $|V_B| \geq n+1$, *then neither* $G[V]$ *nor* $G[V_B]$ *have a blue edge.*

Claim 11 $|V_R| \geq n+1$ *and* $|V_B| \geq n+1$.

Proof. Suppose not. By Claim 9 and symmetry, we may assume that $|V_R| \geq n+2$ and $|V_B| \leq n$. By Fact 10 (i), there is no red edge in $G[V]$ and $G[V_R]$. However, this contradicts Claim 8 with $V_1 = V$ and $V_2 = V_R$. $\square$

Since there exists no monochromatic C_{2n+1}, Fact 10 and Claim 11 imply the following claim.

Claim 12 *Suppose that there exists a matching with two edges* e *and* f *between* V_R *and* V_B *such that* e *and* f *have the same color. Then the following holds.*

(i) *If both* e *and* f *are colored by red, then there exists* $B \in \mathcal{B}$ *such that both end vertices of* e *and* f *in* V_B *are contained in* B, *and moreover,* $G[B]$ *contains no red edges.*

(ii) *If both* e *and* f *are colored by blue, then there exists* $R \in \mathcal{R}$ *such that both end vertices of* e *and* f *in* V_R *are contained in* R, *and moreover,* $G[R]$ *contains no blue edges.*

Proof. By symmetry, it is enough to show only (i). Suppose that there exists a matching with two red edges e and f between V_R and V_B. Let $e := x_1y_1$ and $f := x_2y_2$ with $x_1, x_2 \in V_R$ and $y_1, y_2 \in V_B$. If the color of y_1y_2 is red, then y_1, y_2, n vertices in V_R including x_1 and x_2, and $n-1$ vertices in V form a red C_{2n+1}, a contradiction. Hence, the color of y_1y_2 is not red. In particular, by Fact 10 (ii) and Claim 11, y_1 and y_2 are contained in a same part in $\mathcal{B}$, say $B \in \mathcal{B}$. Furthermore, if $G[B]$ contains a red edge $y_1'y_2'$, then we can find a matching with two red edges x_1y_1' and x_2y_2' between V_R and V_B, which can lead to the same contradiction as above. $\square$

Claim 13 $G[V_R]$ *contains a blue edge and* $G[V_B]$ *contains a red edge.*

Proof. Suppose that $G[V_R]$ contains no blue edge. Then, by Facts 10 (ii) and Claim 11, neither $G[V \cup V_R]$ nor $G[V_B]$ contains a blue edge. However, this contradicts Claim 8 with $V_1 = V \cup V_R$ and $V_2 = V_B$. So $G[V_R]$ contains a blue edge. By symmetry, $G[V_B]$ contains a red edge. $\square$

Now we divide the rest of the proof depending on the sizes of $\mathcal{R}$ and $\mathcal{B}$. By symmetry, it is enough to consider the following five cases.

Case 1. $|\mathcal{R}| = 1$ and $|\mathcal{B}| = 1$.

By the symmetry of red and blue, we may assume that all the edges between V_R and V_B are colored by red. Then, by Claim 12 (i), $G[V_B]$ does not contain a red edge. This contradicts Claim 13.

Case 2. $|\mathcal{R}| = 1$ and $|\mathcal{B}| \geq 2$.

Let $\mathcal{R} := \{R_1\}$ and $\mathcal{B} := \{B_1, B_2, \ldots\}$. If there exists no blue edge between V_R and V_B, then we can find a matching with two red edges e and f such that e is between R_1 and B_1 and f is between R_1 and B_2. However, this contradicts Claim 12 (i). On the other hand, if there exists a matching with two blue edges e and f between V_R and V_B, then by Claim 12 (ii), R_1 contains no blue edge, but this contradicts Claim 13. Thus, there exists a blue edge between V_R and V_B, but no matching with two blue edges between V_R and V_B.

These and the symmetry among B_i's imply that we may assume that $|B_1| = 1$, all the edges between R_1 and B_1 are colored by blue, and all the edges between R_1 and $V_B - B_1$ are colored by red. By Claim 12 (i), $\mathcal{B} = \{B_1, B_2\}$ and there exists no red edge in $G[B_2]$. Then neither $G\big[V \cup B_2\big]$ nor $G[R_1]$ contains a red edge. However, this contradicts Claim 8 with $V_1 = V \cup B_2$ and $V_2 = R_1$.

Case 3. $|\mathcal{R}| = 2$ and $|\mathcal{B}| = 2$.

Let $\mathcal{R} := \{R_1, R_2\}$ and $\mathcal{B} := \{B_1, B_2\}$. By the symmetry of V_R and V_B, we may assume that all the edges between R_1 and B_1 are colored by red. By Claim 12 (i), this implies that all the edges between R_2 and B_2 are colored by blue. If the color of edges between R_1 and B_2 is red (resp. blue), then by Claim 12 (i) (resp. (ii)), we have that $|R_1| = 1$ (resp. $|B_2| = 1$). By the symmetry of V_R and V_B, we may assume that $|R_1| = 1$.

First, suppose that $G[R_2]$ contains a blue edge. Then, by Claim 12 (ii), $|B_2| = 1$ and all the edges between R_2 and B_1 are colored by red. Then, by Claim 11, $|B_1| \geq n \geq 2$, and hence it follows from Claim 12 (i) that B_1 does not contain a red edge. Then neither $G[V \cup B_1]$ nor $G[R_2]$ contains a red edge. However, this contradicts Claim 8 with $V_1 = V \cup B_1$ and $V_2 = R_2$.

Next, suppose that $G[R_2]$ does not contain a blue edge. Then neither $G[V \cup R_2]$ nor $G[V_B]$ contains a blue edge by Fact 10 (ii) and Claim 11, and hence this contradicts Claim 8 again with $V_1 = V \cup R_2$ and $V_2 = V_B$.

Case 4. $|\mathcal{R}| = 2$ and $|\mathcal{B}| \geq 3$.

Let $\mathcal{R} := \{R_1, R_2\}$ and $\mathcal{B} := \{B_1, B_2, \ldots\}$. By Claim 11, we have $|V_R| \geq n+1$, and so by symmetry, we may assume that $|R_1| \geq 2$. By Claim 12 (i), we may also assume that all edges between R_1 and B_i are colored by blue for all $i \neq 1$, and by Claim 12 (ii), $G[R_1]$ has no blue edge. If $|R_2| \geq 2$, then by the same argument as above, all edges between R_1 and B_i are colored by blue for all but at most one i, however, we can find a matching with two blue edges one of which is incident with a vertex in R_1 and the other is incident with a vertex in R_2, contradicting Claim 12 (ii). Thus, we also have that $|R_2| = 1$. Then by Fact 10 (ii) and Claim 11, neither $G[V \cup R_1]$ nor $G[V_B]$ have a blue edge. However, this contradicts Claim 8 with $V_1 = V \cup R_1$ and $V_2 = V_B$.

Case 5. $|\mathcal{R}| \geq 3$ and $|\mathcal{B}| \geq 3$.

Take three vertices x_1, x_2 and x_3 from distinct parts in $\mathcal{R}$, and three vertices y_1, y_2 and y_3 from distinct parts in $\mathcal{B}$. Consider a matching with three edges x_1y_1, x_2y_2 and x_3y_3. By the symmetry, we may assume that x_1y_1 and x_2y_2 are colored by red. This contradicts Claim 12 (i).

This completes the proof of Lemma 16. $\square$

Acknowledgments

The authors would like to thank Professor Katsuhiro Ota for his valuable suggestions.

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