Forbidden subgraphs generating almost the same sets

Shinya Fujita[1*] Michitaka Furuya[2†]
Kenta Ozeki[3 4‡]

[1]*Department of Integrated Design Engineering,*
Maebashi Institute of Technology,
460-1 Kamisadori, Maebashi, Gunma 371-0816, Japan
[2]*Department of Mathematical Information Science,*
Tokyo University of Science,
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan
[3]*National Institute of Informatics,*
2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan
[4]*JST, ERATO, Kawarabayashi Large Graph Project, Japan*

Abstract

Let $\mathcal{H}$ be a set of connected graphs. A graph G is said to be $\mathcal{H}$-free if G does not contain any element of $\mathcal{H}$ as an induced subgraph. Let $\mathcal{F}_k(\mathcal{H})$ be the set of k-connected $\mathcal{H}$-free graphs. When we study the relationship between forbidden subgraphs and a certain graph property, we often allow a finite exceptional set of graphs. But if the symmetric difference of $\mathcal{F}_k(\mathcal{H}_1)$ and $\mathcal{F}_k(\mathcal{H}_2)$ is finite and we allow a finite number of exceptions, no graph property can distinguish them. Motivated by this observation, we study when we obtain a finite symmetric difference. In this paper, our main aim is the following; If

[*]shinya.fujita.ph.d@gmail.com, This work was supported by JSPS KAKENHI Grant Number 23740095.

[†]michitaka.furuya@gmail.com

[‡]ozeki@nii.ac.jp

$|\mathcal{H}| \le 3$ and the symmetric difference of $\mathcal{F}_1(\{H\})$ and $\mathcal{F}_1(\mathcal{H})$ is finite, then either $H \in \mathcal{H}$ or $|\mathcal{H}| = 3$ and $H = C_3$. Furthermore, we prove that if the symmetric difference of $\mathcal{F}_k(\{H_1\})$ and $\mathcal{F}_k(\{H_2\})$ is finite, then $H_1 = H_2$.

Key words and phrases. forbidden subgraph, k-connected graphs, finite symmetric difference.
AMS 2010 Mathematics Subject Classification. 05C75.

1 Introduction

In this paper, all graphs are finite, simple, and undirected. For a set $\mathcal{H}$ of connected graphs, a graph G is said to be $\mathcal{H}$-*free* if G does not contain any element of $\mathcal{H}$ as an induced subgraph. We also say that the elements of $\mathcal{H}$ are *forbidden subgraphs*. If G is $\{H\}$-free, G is simply said to be H-free.

If we appropriately choose a set $\mathcal{H}$, $\mathcal{H}$-free graphs may satisfy a certain graph property. For example, Cockayne, Ko and Shepherd [6] proved that every connected $\{K_{1,3}, \text{Net}\}$-free graph G has domination number at most $\lceil \frac{1}{3}|V(G)| \rceil$, where Net is the unique graph having degree sequence $(3, 3, 3, 1, 1, 1)$ (Figure 1). Duffus, Gould and Jacobson [7] proved that every connected $\{K_{1,3}, \text{Net}\}$-free graph has a hamiltonian path, and that if it is 2-connected, it has a hamiltonian cycle. Forbidden subgraphs have appeared in many other topics of graph theory (see, for example, [2, 4, 10, 14]).

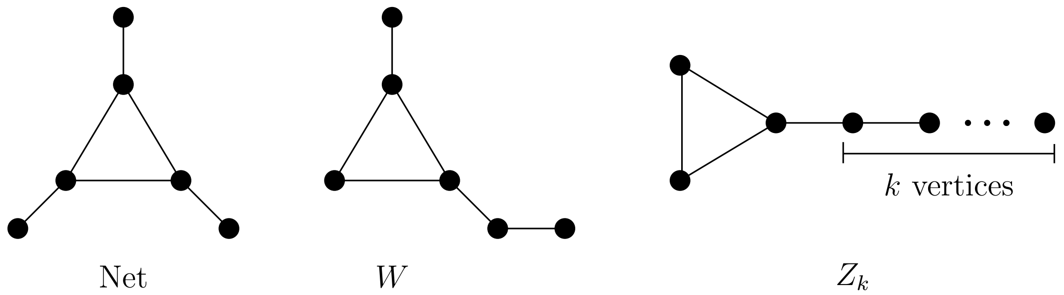

Figure 1: Net, W and Z_k

Since the result of Duffus et al. [7], several other pairs of forbidden subgraphs which imply the existence of a hamiltonian cycle were found. And finally, Bedrossian [3] characterized all such pairs. The graph W in the following theorem is the one depicted in Figure 1, and we denote the path of order k by P_k. For two sets $\mathcal{H}_1$ and $\mathcal{H}_2$ of forbidden subgraphs, we write $\mathcal{H}_1 \le \mathcal{H}_2$ if for every $H_2 \in \mathcal{H}_2$, there exists

$H_1 \in \mathcal{H}_1$ such that H_1 is an induced subgraph of H_2. It is not difficult to see that if $\mathcal{H}_1 \le \mathcal{H}_2$, then every $\mathcal{H}_1$-free graph is $\mathcal{H}_2$-free (See [13]).

Theorem A ([3]) *Let H_1 and H_2 be connected graphs of order at least three. Then every 2-connected $\{H_1, H_2\}$-free graph has a hamiltonian cycle if and only if $\{H_1, H_2\} \le \{K_{1,3}, \mathrm{Net}\}$, $\{H_1, H_2\} \le \{K_{1,3}, W\}$ or $\{H_1, H_2\} \le \{K_{1,3}, P_6\}$.*

Let Z_k be the graph obtained from K_3 and P_k by joining one vertex in K_3 with one endvertex of P_k by an edge (see Figure 1). Faudree, Gould, Ryjáček and Schiermeyer [9] proved that every 2-connected $\{K_{1,3}, Z_3\}$-free graph of order at least ten has a hamiltonian cycle. Since there exists a 2-connected $\{K_{1,3}, Z_3\}$-free non-hamiltonian graph of order nine, the assumption on the order cannot be removed. And because of this exception, the pair $\{K_{1,3}, Z_3\}$ does not appear in Theorem A.

The above observation suggests that if we allow a finite number of exceptions, or equivalently, if we confine ourselves to graphs of sufficiently large order, we may be able to enhance the set of pairs in Theorem A. Faudree and Gould [8] actually conducted this line of research, and found that even if we allow a finite number of exceptions, essentially $\{K_{1,3}, Z_3\}$ is the only pair that can be added to Bedrossian's pairs.

Theorem B ([8]) *Let H_1 and H_2 be connected graphs of order at least three. Then every 2-connected $\{H_1, H_2\}$-free graph of sufficiently large order has a hamiltonian cycle if and only if $\{H_1, H_2\} \le \{K_{1,3}, \mathrm{Net}\}$, $\{H_1, H_2\} \le \{K_{1,3}, W\}$, $\{H_1, H_2\} \le \{K_{1,3}, P_6\}$ or $\{H_1, H_2\} \le \{K_{1,3}, Z_3\}$.*

As the above example suggests, in the study of forbidden subgraphs, we often allow a finite number of exceptions in the hope of obtaining a deeper insight.

However, this approach poses a new problem. Aldred, Fujisawa and Saito [1] studied sets of forbidden subgraphs which imply the existence of a 2-factor. Let $\mathcal{H}$ be a set of connected graphs having at least two vertices, and suppose every connected $\mathcal{H}$-free graph of minimum degree at least two and sufficiently large order has a 2-factor. They proved that if $|\mathcal{H}| \le 3$, then $\mathcal{H}$ contains a star. They also proved that every connected $\{\mathrm{Chair}, \mathrm{Crown}, K_{2,3}, Z_1\}$-free graph of order at least nine and minimum degree at least two has a 2-factor, where Chair and Crown are the graphs depicted in Figure 2. By this result, they claimed that they could forbid four graphs, without using a star, to guarantee the existence of a 2-factor in a connected graph of minimum degree at least two and sufficiently large order. However, in the proof, they

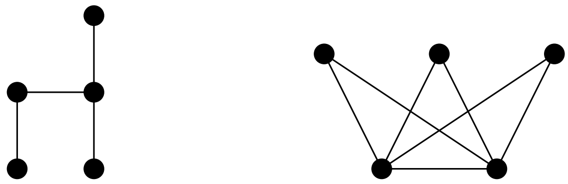

Figure 2: Chair and Crown

actually proved that every connected $\{\text{Chair}, \text{Crown}, K_{2,3}, Z_1\}$-free graph of order at least nine and minimum degree at least two is $K_{1,3}$-free. In [12], Fujisawa and Saito proved that every connected $\{K_{1,3}, Z_2\}$-free graph of minimum degree at least two and sufficiently large order has a 2-factor. This yields the result of [1] for graphs of sufficiently large order as a corollary. This phenomenon suggests that if we forbid graphs of a set $\mathcal{H}$, we may implicitly (and essentially) forbid graphs which do not belong to $\mathcal{H}$.

Now we formalize the problem. For a set of connected graphs $\mathcal{H}$, let $\mathcal{F}(\mathcal{H})$ denote the set of connected $\mathcal{H}$-free graphs. If $\mathcal{H}$ consists of one graph H, we write $\mathcal{F}(H)$ instead of $\mathcal{F}(\{H\})$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be sets of connected graphs. Recall that if $\mathcal{H}_1 \le \mathcal{H}_2$, then $\mathcal{F}(\mathcal{H}_1) \subseteq \mathcal{F}(\mathcal{H}_2)$ holds. However, even if $\mathcal{H}_1$ and $\mathcal{H}_2$ are not comparable with respect to the relation "$\le$", $\mathcal{F}(\mathcal{H}_1) - \mathcal{F}(\mathcal{H}_2)$ can be a finite set (see Section 2). And if $\mathcal{F}(\mathcal{H}_1) - \mathcal{F}(\mathcal{H}_2)$ is a finite set and every connected $\mathcal{H}_2$-free graph of sufficiently large order satisfies a certain graph property P, then every connected $\mathcal{H}_1$-free graph of sufficiently large order also satisfies P. If this occurs, the study of the property P of connected $\mathcal{H}_1$-free graphs only involves a finite number of graphs in $\mathcal{F}(\mathcal{H}_1) - \mathcal{F}(\mathcal{H}_2)$.

We face a more serious problem if the symmetric difference is finite. Again let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two sets of connected graphs, and suppose their symmetric difference, denoted by $\mathcal{H}_1 \triangle \mathcal{H}_2$ in this paper, is finite. Then for every graph property P, every connected $\mathcal{H}_1$-free graph of sufficiently large order satisfies P if and only if every connected $\mathcal{H}_2$-free graph of sufficiently large order satisfies P. In other words, as long as we allow a finite number of exceptions, we cannot distinguish $\mathcal{F}(\mathcal{H}_1)$ and $\mathcal{F}(\mathcal{H}_2)$, whatever graph property we choose.

Actually, it is not difficult to construct an example with infinitely many graphs. Let H be a connected graph of order k, and let $\mathcal{H}$ be the set of all connected graphs of order $k+1$ that contain H as an induced subgraph. Then $H \notin \mathcal{H}$ and $\mathcal{F}(H) \triangle \mathcal{F}(\mathcal{H}) = \{H\}$. Though these are trivial examples, there is a more complicated pair (with additional condition), see Section 7 in this paper and [1].

Motivated by the above background, we start a study on the difference and the symmetric difference of two sets of forbidden subgraphs. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two sets of connected graphs. We study the relationship between $\mathcal{H}_1$ and $\mathcal{H}_2$, assuming that $\mathcal{F}(\mathcal{H}_1)-\mathcal{F}(\mathcal{H}_2)$ or $\mathcal{F}(\mathcal{H}_1)\triangle\mathcal{F}(\mathcal{H}_2)$ is a finite set. We focus on the cases in which both $\mathcal{H}_1$ and $\mathcal{H}_2$ consist of a small number of graphs. One extreme case is that both of them are singleton sets, and even in this simple case, we observe some complications. As it has been mentioned above, we cannot judge whether $\{H_1\} \le \{H_2\}$ holds (i.e. H_2 contains H_1 as an induced subgraph) under the assumption that $\mathcal{F}(H_1)-\mathcal{F}(H_2)$ is finite. In contrast, if $\mathcal{F}(H_1)\triangle\mathcal{F}(H_2)$ is finite, then we will see $H_1 = H_2$. And this is true even if we restrict ourselves to graphs of higher connectivity. We will also investigate the case in which only one of $\mathcal{H}_1$ and $\mathcal{H}_2$ is a singleton set and a special case of $|\mathcal{H}_1| = |\mathcal{H}_2| = 2$.

The structure of the subsequent sections is as follows. In the next section, in order to demonstrate the complexity of the problem, we present an example in which H_1 and H_2 are connected graphs, neither of which is an induced subgraph of the other, but $\mathcal{F}(H_1)-\mathcal{F}(H_2)$ is finite. In Section 3, we prove several necessary conditions for $\mathcal{F}(\mathcal{H}_1)-\mathcal{F}(\mathcal{H}_2)$ to be finite. These conditions will be used in the arguments of the subsequent sections. In Sections 4–6, we study the problem of finite $\mathcal{F}(\mathcal{H}_1)\triangle\mathcal{F}(\mathcal{H}_2)$. In Section 4, we consider the case in which either $\mathcal{H}_1$ or $\mathcal{H}_2$ is a singleton set. In Section 5, we assume $|\mathcal{H}_1| = |\mathcal{H}_2| = 2$ and $\mathcal{H}_1\cap\mathcal{H}_2 \neq \emptyset$, and see what happens. And in Section 6, we consider the problem in the class of higher connectivity. We give concluding remarks in Section 7.

For terms and symbols not defined in this paper, we refer the reader to [5]. Let $\mathcal{H}$ be a set of graphs. For $k \ge 1$, let $\mathcal{F}_k(\mathcal{H}) = \{G \mid G$ is a k-connected $\mathcal{H}$-free graph$\}$. Hence $\mathcal{F}_1(\mathcal{H}) = \mathcal{F}(\mathcal{H})$. If $\mathcal{H} = \{H_1, \cdots, H_m\}$, we write $\mathcal{F}_k(H_1, \cdots, H_m)$ and $\mathcal{F}(H_1, \cdots, H_m)$ in place of $\mathcal{F}_k(\{H_1, \cdots, H_m\})$ and $\mathcal{F}(\{H_1, \cdots, H_m\})$, respectively. For graphs H_1 and H_2, we write $H_1 \prec H_2$ if H_2 contains H_1 as an induced subgraph. If $\mathcal{H}$ is a finite set, we write $|\mathcal{H}| < \infty$.

For graphs H_1 and H_2 with $V(H_1)\cap V(H_2) = \emptyset$, let H_1+H_2 be the graph obtained from $H_1 \cup H_2$ by joining every vertex of $V(H_1)$ to every vertex of $V(H_2)$. Let H be a graph. Take a set $U \subseteq V(H)$. Let $G_1^n(H;U)$ be the graph obtained from $H \cup K_n$ by joining every vertex of U to every vertex of $V(K_n)$. Let $G_2^n(H;U)$ be the graph obtained from $H \cup nK_1$ by joining every vertex of U to every vertex of $V(nK_1)$. Note that $G_1^n(H;V(H)) = H + K_n$ and $G_2^n(H;V(H)) = H + nK_1$. Take a vertex

$u \in V(H)$. Let $G_3^n(H;u)$ be the graph obtained from $H \cup P_n$ by joining u to one endvertex of P_n.

Let H be a graph. For $v \in V(H)$, we let $d_H(v)$ the degree of v in H, i.e., $d_H(v) = |N_H(v)|$. For $l \geq 0$, let $V_l(H) = \{v \in V(H) \mid d_H(v) = l\}$ and $V_{\geq l}(H) = \{v \in V(H) \mid d_H(v) \geq l\}$.

A graph H is called *special* if $\delta(H) = 1$, $\Delta(H) = |V(H)| - 1$, there exist two vertices $c_1, c_2 \in V(H)$ such that $N_H[c_1] = N_H[c_2]$ and there exist non-adjacent vertices $c_1', c_2' \in V(H)$ such that $N_H(c_1') = N_H(c_2')$. Note that every special graph has order at least five.

2 An Example of Finite $\mathcal{F}(H_1) - \mathcal{F}(H_2)$

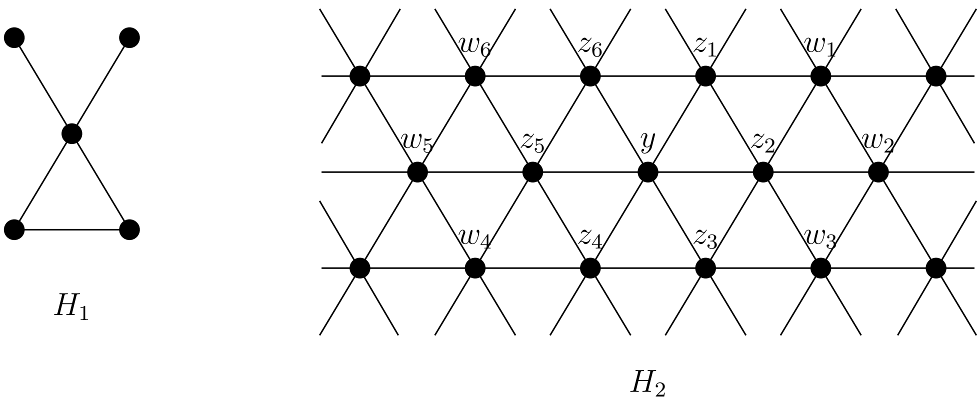

Figure 3: The graphs H_1 and H_2

In this section, in order to demonstrate the complexity of the problem, we construct an example in which neither H_1 nor H_2 is an induced subgraph of the other, but $\mathcal{F}(H_1) - \mathcal{F}(H_2)$ is finite. We will also use this example in Section 7 to show that some of the conditions we obtain in the subsequent sections are essential.

Let H_1 be the graph obtained from the triangle by attaching two pendant edges to a vertex. Let H_2 be a 6-regular triangulation of the torus. To simplify the argument, we assume that the length of the shortest non-contractible cycle of H_2 with each homotopy type is the same and large enough. See Figure 3. We show the following.

Proposition 2.1 $\mathcal{F}(H_1) - \mathcal{F}(H_2) = \{H_2\}$.

Proof. It is easy to see that $H_2 \in \mathcal{F}(H_1) - \mathcal{F}(H_2)$. We will show the converse.

Suppose that $H \in \mathcal{F}(H_1) - \mathcal{F}(H_2)$, and $H \neq H_2$. Note that $H_2 \prec H$, and we fix H_2 as an induced subgraph of H. Since $H \neq H_2$ and H is connected, we can find a vertex $x \in V(H) - V(H_2)$ with $N_H(x) \cap V(H_2) \neq \emptyset$. Recall that $H_1 \not\prec H$.

Claim 2.1 *Let $a \in N_H(x) \cap V(H_2)$, and let $b_1 b_2 \cdots b_6 b_1$ be the cycle of length 6 in $N_{H_2}(a)$. Then we have the following:*

(i) *For each $1 \leq i \leq 6$, at least one of b_i, b_{i+1} and b_{i+2} is a neighbor of x, where the index is taken modulo 6.*

(ii) *For some i with $1 \leq i \leq 3$, both b_i and b_{i+3} are neighbors of x, unless $\{b_j, b_{j+2}, b_{j+4}\} = N_H(x) \cap \{b_1, \ldots, b_6\}$ for some $j = 1, 2$.*

Proof. (i) Suppose not, that is, there exists an integer i such that none of b_i, b_{i+1} and b_{i+2} are neighbors of x. By symmetry, we may assume that $i = 1$. If b_5 is not a neighbor of x, then $\{a, b_1, b_2, x, b_5\}$ induces an H_1, a contradiction. Hence b_5 is a neighbor of x. However, $\{a, b_5, x, b_1, b_3\}$ induces an H_1, a contradiction again.

(ii) Suppose that for each i with $1 \leq i \leq 3$, at least one of b_i and b_{i+3} is not a neighbor of x. By (i), there exists a neighbor of x in $\{b_1, \ldots, b_6\}$, say b_1. By the assumption, $b_4 \notin N_H(x)$. Using (i) to b_3, b_4, b_5, at least one of them is a neighbor of x. Since $b_4 \notin N_H(x)$, we may assume that $b_3 \in N_H(x)$ by symmetry. Again by the assumption, $b_6 \notin N_H(x)$. Then using (i) to b_4, b_5, b_6, we have that $b_5 \in N_H(x)$. Again by the assumption, $b_2 \notin N_H(x)$. These implies that $\{b_1, b_3, b_5\} = N_H(x) \cap \{b_1, \ldots, b_6\}$. □

Now we are ready to prove Proposition 2.1. Let $y \in N_H(x) \cap V(H_2)$. Let $z_1 z_2 \ldots z_6$ be the cycle in $N_H(y) \cap V(H_2)$. By Claim 2.1 (ii) and symmetry, we have that (I) $\{z_2, z_4, z_6\} = N_H(x) \cap \{z_1, \ldots, z_6\}$, or (II) both z_2 and z_5 are neighbors of x. Let w_1, w_2, w_3 be the vertices in $(N_H(z_2) \cap V(H_2)) - \{z_1, y, z_3\}$ with $w_1 z_1, w_3 z_3 \in E(H)$. By Claim 2.1 (i), at least one of w_1, w_2, w_3 is a neighbor of x, say w_i.

Case I. $\{z_2, z_4, z_6\} = N_H(x) \cap \{z_1, \ldots, z_6\}$.

In this case, $\{x, z_2, w_i, z_4, z_6\}$ induces an H_1, a contradiction.

Case II. Both z_2 and z_5 are neighbors of x.

Let w_4, w_5, w_6 be the vertices in $(N_H(z_5) \cap V(H_2)) - \{z_4, y, z_6\}$ with $w_4 z_4, w_6 z_6 \in E(H)$. By Claim 2.1 (i), at least one of w_4, w_5, w_6 is a neighbor of x, say w_j.

Suppose first that $w_2 \notin N_H(x)$. Then by symmetry, we may assume that $i = 1$, that is, $w_1 \in N_H(x)$. By Claim 2.1 (i), at least one of w_2, w_3, z_3 is a neighbor of x, say u. Note that $u \neq w_2$. However, $\{x, z_5, w_j, w_1, u\}$ induces an H_1, a contradiction. Thus, we have that $w_2 \in N_H(x)$. By symmetry, we also have that $w_5 \in N_H(x)$.

If none of z_1, z_3, z_4 are neighbors of x, then $\{y, z_3, z_4, x, z_1\}$ induces an H_1, a contradiction. Thus, z_k is a neighbor of x for some $k = 1, 3, 4$. However, $\{x, y, z_k, w_2, w_5\}$ induces an H_1, a contradiction again. This completes the proof of Proposition 2.1. $\square$

3 $|\mathcal{F}(\mathcal{H}_1) - \mathcal{F}(\mathcal{H}_2)| < \infty$

In this section, we investigate the case in which the difference of two sets defined by forbidden subgraphs is finite. As we mentioned in Section 1, the results in this section will be used as main tools in the subsequent sections.

Lemma 3.1 *For each $1 \leq i \leq 2$, let H_i be a connected graph with $|V(H_i)| \geq 3$. Let $\mathcal{H}$ be a set of connected graphs such that $\Delta(H^*) \leq |V(H^*)| - 2$ and $\delta(H^*) \geq 2$ for every $H^* \in \mathcal{H}$. If $|\mathcal{F}(\mathcal{H} \cup \{H_1\}) - \mathcal{F}(H_2)| < \infty$ and $H^* \not\prec H_2$ for every $H^* \in \mathcal{H} \cup \{H_1\}$, then*

(i) $|V(H_1)| \geq 4$, $\Delta(H_1) = |V(H_1)| - 1$ *and* $\delta(H_1) = 1$ *and*

(ii) $|V(H_2)| \geq 2|V(H_1)| - 3$ *and* $\delta(H_2) \geq |V(H_1)| - 2$.

Proof. Suppose that $H_1 \simeq K_{1,2}$. Since $\mathcal{H}$ contains no complete graph by the assumption, $H_1 \prec H^*$ for every $H^* \in \mathcal{H}$, and so $\mathcal{F}(\mathcal{H} \cup \{H_1\}) = \mathcal{F}(H_1)$. Hence $\mathcal{F}(\mathcal{H} \cup \{H_1\}) = \{K_l \mid l \geq 1\}$. Since $H_1 \not\prec H_2$, H_2 is complete. Write $H_2 = K_\alpha$. Then $\{K_\beta \mid \beta \geq \alpha\} \subseteq \mathcal{F}(\mathcal{H} \cup \{H_1\}) - \mathcal{F}(H_2)$, which contradicts the assumption that $|\mathcal{F}(\mathcal{H} \cup \{H_1\}) - \mathcal{F}(H_2)| < \infty$. Thus

$$H_1 \not\simeq K_{1,2}. \tag{3.1}$$

Since $|\mathcal{F}(\mathcal{H} \cup \{H_1\}) - \mathcal{F}(H_2)| < \infty$ and $G_1^n(H_2; V(H_2)) \notin \mathcal{F}(H_2)$ for $n \geq 1$, $A_1 \prec G_1^{n_1}(H_2; V(H_2))$ for some $A_1 \in \mathcal{H} \cup \{H_1\}$ and some $n_1 \geq 1$. Since $A_1 \not\prec H_2$, $V(A_1) \cap V(K_{n_1}) \neq \emptyset$. Hence $A_1 = H_2' + K_{m_1}$ for a graph $H_2' \prec H_2$ and $m_1 \geq 1$. In particular, A_1 has a vertex of degree $|V(A_1)| - 1$ (and so $\Delta(A_1) = |V(A_1)| - 1$). By the assumption of lemma, $A_1 = H_1$.

Take a vertex $x \in V(H_2)$. Since $|\mathcal{F}(\mathcal{H} \cup \{H_1\}) - \mathcal{F}(H_2)| < \infty$ and $G_3^n(H_2; x) \notin \mathcal{F}(H_2)$ for $n \geq 1$, $A_2 \prec G_3^{n_2}(H_2; x)$ for some $A_2 \in \mathcal{H} \cup \{H_1\}$ and some $n_2 \geq 1$. Since $A_2 \not\prec H_2$, $V(A_2) \cap V(P_{n_2}) \neq \emptyset$. In particular, A_2 has a vertex of degree 1 (and so $\delta(A_2) = 1$). By the assumption of lemma, $A_2 = H_1$. This together with (3.1) implies that $|V(H_1)| \geq 4$. If H_1 contains two vertices of P_{n_2}, then $H_1 = K_{1,2}$ by the fact that $\Delta(H_1) = |V(H_1)| - 1$, which contradicts (3.1). Thus H_1 contains exactly one vertex of P_{n_2}. This implies that $d_{H_1}(x) = |V(H_1)| - 1$ and so $d_{H_2}(x) \geq |V(H_1)| - 2$. Hence $\delta(H_2) \geq |V(H_1)| - 2$.

Let $y \in V(H_2')$ be a vertex with $d_{H_2'+K_{m_1}}(y) = 1$. Then $d_{H_2'}(y) = 0$. Hence there exist $|V(H_1)| - 2$ vertices of H_2 which are not adjacent to y in H_2. Since $d_{H_2}(y) \geq \delta(H_2) \geq |V(H_1)| - 2$, we see that $|V(H_2)| \geq 2|V(H_1)| - 3$. □

Lemma 3.2 *For each $1 \leq i \leq 2$, let H_i be a connected graph with $|V(H_i)| \geq 3$. If $|\mathcal{F}(H_1) - \mathcal{F}(H_2)| < \infty$ and $H_1 \not\prec H_2$, then*

(i) *H_1 is special and*

(ii) *$|V(H_2)| \geq 2|V(H_1)| - 3$ and $\delta(H_2) \geq |V(H_1)| - 2$.*

Proof. Let $x \in V(H_2)$. Since $|\mathcal{F}(H_1) - \mathcal{F}(H_2)| < \infty$ and $G_1^n(H_2; N_{H_2}[x]) \notin \mathcal{F}(H_2)$ for $n \geq 1$, $H_1 \prec G_1^{n_1}(H_2; N_{H_2}[x])$ for some $n_1 \geq 1$. Since $H_1 \not\prec H_2$, $|V(H_1) \cap (\{x\} \cup V(K_{n_1}))| \geq 2$. Then two vertices $c_1, c_2 \in V(H_1) \cap (\{x\} \cup V(K_{n_1}))$ satisfy $N_{H_1}[c_1] = N_{H_1}[c_2]$.

Since $|\mathcal{F}(H_1) - \mathcal{F}(H_2)| < \infty$ and $G_2^n(H_2; N_{H_2}(x)) \notin \mathcal{F}(H_2)$ for $n \geq 1$, $H_1 \prec G_2^{n_2}(H_2; N_{H_2}(x))$ for some $n_2 \geq 1$. Since $H_1 \not\prec H_2$, $|V(H_1) \cap (\{x\} \cup V(n_2K_1))| \geq 2$. Then two vertices $c_1', c_2' \in V(H_1) \cap (\{x\} \cup V(n_2K_1))$ satisfy $N_{H_1}(c_1') = N_{H_1}(c_2')$.

Applying Lemma 3.1 with $\mathcal{H} = \emptyset$, this completes the proof of Lemma 3.2. □

Note that every special graph contains K_3 as an induced subgraph. Thus we have the following corollary from Lemma 3.2.

Corollary 3.3 *For each $1 \leq i \leq 2$, let H_i be a connected graph with $|V(H_i)| \geq 3$. If $|\mathcal{F}(H_1) - \mathcal{F}(H_2)| < \infty$ and H_1 is K_3-free, then $H_1 \prec H_2$.*

4 $|\mathcal{F}(H) \triangle \mathcal{F}(\mathcal{H})| < \infty$

We now investigate the pairs of forbidden subgraphs $(\mathcal{H}_1, \mathcal{H}_2)$ such that $\mathcal{F}(\mathcal{H}_1) \triangle \mathcal{F}(\mathcal{H}_2)$ is a finite set. In this section, we discuss the case in which $\mathcal{H}_1$ is a singleton set and $\mathcal{H}_2$ contains at most three elements.

Let H be a graph. For each vertex $v \in V(H)$, let $X(H, v) = \{u \in V(H) \mid$ there exists an automorphism φ of H such that $\varphi(u) = v\}$. Let $\mathcal{X}(H) = \{X(H, v) \mid v \in V(H)\}$ and $t(H) = |\mathcal{X}(H)|$.

First, we consider what the condition $|\mathcal{F}(H) \triangle \mathcal{F}(\mathcal{H})| < \infty$ means.

Theorem 4.1 *Let H be a connected graph with $|V(H)| \geq 3$, and let $\mathcal{H}$ be a set of connected graphs such that $|V(H^*)| \geq 3$ for every $H^* \in \mathcal{H}$. If $|\mathcal{F}(H) \triangle \mathcal{F}(\mathcal{H})| < \infty$, then*

(i) *$H \in \mathcal{H}$ or*

(ii) *$H \prec H^*$ for every $H^* \in \mathcal{H}$ or*

(iii) *H is special and*
$$\left|\left\{H^* \in \mathcal{H} \mid H \prec H^*\right\}\right|$$
$$\geq \begin{cases} t(H) & (V_{|V(H)|-2}(H) = \emptyset) \\ t(H) + \min\{|V_{|V(H)|-2}(H)|, |V(H)| - 3\} - 2 & (V_{|V(H)|-2}(H) \neq \emptyset). \end{cases}$$

Proof. Let $\mathcal{H}_1 = \{H^* \in \mathcal{H} \mid H \prec H^*\}$. If $H \in \mathcal{H}$ or $\mathcal{H}_1 = \mathcal{H}$, then we have a desired result. Thus we may assume that $H \notin \mathcal{H}$ and $\mathcal{H} - \mathcal{H}_1 \neq \emptyset$.

Claim 4.1 *For every $H^* \in \mathcal{H}$,*

(i) *$|\mathcal{F}(H) - \mathcal{F}(H^*)| < \infty$ and*

(ii) *$H^* \not\prec H$.*

Proof.

(i) Since $\mathcal{F}(H) - \mathcal{F}(H^*) \subseteq \mathcal{F}(H) - \mathcal{F}(\mathcal{H}) \subseteq \mathcal{F}(H) \triangle \mathcal{F}(\mathcal{H})$ and $|\mathcal{F}(H) \triangle \mathcal{F}(\mathcal{H})| < \infty$, we have $|\mathcal{F}(H) - \mathcal{F}(H^*)| < \infty$.

(ii) Suppose that $H^* \prec H$. Since $H \neq H^*$, $H \not\prec H^*$. Hence $|V(H^*)| > |V(H)|$ by (i) and Lemma 3.2, which contradicts the assumption that $H^* \prec H$. $\square$

Claim 4.2 *The following statements hold:*

(i) *H is special.*

(ii) $|V(H^*)| \geq 2|V(H)| - 3$ *and* $\delta(H^*) \geq |V(H)| - 2 (> 1)$ *for each* $H^* \in \mathcal{H} - \mathcal{H}_1$.

Proof. Take $H^* \in \mathcal{H} - \mathcal{H}_1$. By Claim 4.1, $|\mathcal{F}(H) - \mathcal{F}(H^*)| < \infty$. Since $H \not\prec H^*$, we get the desired results by Lemma 3.2. □

Let $a \in V(H)$ be the unique vertex of degree $|V(H)| - 1$. For each $X \in \mathcal{X}(H)$, fix a vertex $w_X \in X$. Let $W = \{w_X \mid X \in \mathcal{X}(H), d_H(w_X) \leq |V(H)| - 3\}$.

Take a vertex $w \in W$. Since $|\mathcal{F}(\mathcal{H}) - \mathcal{F}(H)| < \infty$ and $G_3^n(H; w) \notin \mathcal{F}(H)$ for $n \geq 1$, $H_w \prec G_3^{n_1}(H; w)$ for some $H_w \in \mathcal{H}$ and some $n_1 \geq 1$. Since $H_w \not\prec H$ by Claim 4.1(ii), $V(H_w) \cap V(P_{n_1}) \neq \emptyset$. In particular, $\delta(H_w) = 1$. By Claim 4.2(ii), this leads to $H_w \in \mathcal{H}_1$. By the definition of W, $d_H(w) \leq |V(H)| - 3$, and hence $\Delta(H_w) \leq \Delta(H) = |V(H)| - 1$. Since $H \prec H_w$, H_w has a vertex of degree at least $|V(H)| - 1$. Since $w \in W$, this implies that a is the unique vertex of degree $|V(H)| - 1$ in H_w and so $V(H) = N_H[a] \subseteq V(H_w)$. Hence $H_w \simeq G_3^{n_w}(H; w)$ for some $n_w \geq 1$. Note that w is the unique vertex of $N_{H_w}(a)$ which is adjacent to a vertex in $V(H_w) - N_{H_w}[a]$. This together with the definition of W implies that if $w \neq w'$, then $H_w \not\simeq H_{w'}$.

Let $p = \min\{\max\{0, |V_{|V(H)|-2}(H)| - 1\}, |V(H)| - 4\}$. For each $0 \leq i \leq p$, let $U_i \subseteq V_{|V(H)|-2}(H)$ be a set with $|U_i| = i$. Since $|\mathcal{F}(\mathcal{H}) - \mathcal{F}(H)| < \infty$ and $G_2^n(H; \{a\} \cup U_i) \notin \mathcal{F}(H)$ for $n \geq 1$, $H'_{U_i} \prec G_2^{n_2}(H; \{a\} \cup U_i)$ for some $H'_{U_i} \in \mathcal{H}$ and some $n_2 \geq 1$. Since $H'_{U_i} \not\prec H$ by Claim 4.1(ii), $V(H'_{U_i}) \cap V(n_2 K_1) \neq \emptyset$. Note that $d_{H'_{U_i}}(x) \leq |U_i| + 1 \leq |V(H)| - 3$ for $x \in V(H'_{U_i}) \cap V(n_2 K_1)$. By Claim 4.2(ii), this leads to $H'_{U_i} \in \mathcal{H}_1$.

Claim 4.3 *The following statements hold:*

(i) $V_{\geq |V(H)|-2}(H) \subseteq V(H'_{U_i})$ *and* $d_{H'_{U_i}}(x) \geq |V(H)| - 2$ *for every* $x \in V_{\geq |V(H)|-2}(H)$.

(ii) $d_{H'_{U_i}}(x) = i + 1$ *for every* $x \in V(H'_{U_j}) - V(H)$.

(iii) $d_{H'_{U_i}}(x) \geq |V_{\geq |V(H)|-2}(H)|$ *or* $d_{H'_{U_i}}(x) = 1$ *for every* $x \in V(H'_{U_i}) \cap V(H)$.

(iv) *For each* $l \neq l'$, $H'_{U_l} \not\simeq H'_{U_{l'}}$.

Proof.

(i) Since $|V_{\geq |V(H)|-2}(H'_{U_i})| \leq |V_{\geq |V(H)|-2}(H)|$ and $H \prec H'_{U_i}$, we see that $V_{\geq |V(H)|-2}(H) \subseteq V(H'_{U_i})$ and $d_{H'_{U_i}}(x) \geq |V(H)| - 2$ for each $x \in V_{\geq |V(H)|-2}(H)$.

(ii) By (i), we get the desired result.

(iii) Take a vertex $y \in V(H'_{U_i}) \cap (V(H) - (V_{\geq |V(H)|-2}(H) \cup V_1(H)))$. Then y is adjacent to every vertex of $V_{\geq |V(H)|-2}(H)$ in H'_{U_i}. By (i), this implies that $d_{H'_{U_i}}(y) \geq |V_{\geq |V(H)|-2}(H)|$. Take a vertex $y' \in V_{\geq |V(H)|-2}(H)$. By (i), $d_{H'_{U_i}}(y') \geq |V(H)|-2$. Since H is special, we see that $|V(H)|-2 \geq |V_{\geq |V(H)|-2}(H)|$. Hence $d_{H'_{U_i}}(y') \geq |V_{\geq |V(H)|-2}(H)|$. Consequently, we get the desired result.

(iv) By the definition of p, $2 \leq i+1 \leq |V_{|V(H)|-2}(H)| = |V_{\geq |V(H)|-2}(H)| - 1$ for each $1 \leq i \leq p$. Hence, for each $1 \leq j \leq p$, H'_{U_j} has a vertex of degree $i+1$ if and only if $j = i$ by (ii) and (iii). This implies $H'_{U_l} \not\simeq H'_{U_{l'}}$ for each $l \neq l'$. □

For $w \in W$ and $0 \leq i \leq p$, the radius of H_w is 2 and the radius of H'_{U_i} is 1, and hence $H_w \not\simeq H'_{U_i}$. Let $\mathcal{H}' = \{H_w \mid w \in W\}$ and $\mathcal{H}'' = \{H'_{U_i} \mid 0 \leq i \leq p\}$. Then $|\mathcal{H}_1| \geq |\mathcal{H}'| + |\mathcal{H}''|$. If $V_{|V(H)|-2}(H) = \emptyset$, then $|\mathcal{H}'| = t(H) - 1$ and $|\mathcal{H}''| = 1$, as desired. If $V_{|V(H)|-2}(H) \neq \emptyset$, then $|\mathcal{H}'| = t(H) - 2$ and $|\mathcal{H}''| = \min\{|V_{|V(H)|-2}(H)| - 1, |V(H)| - 4\} + 1$, as desired.

This completes the proof of Theorem 4.1. □

Let H be a special graph with $|V_{|V(H)|-2}(H)| = 1$. Then $|V_1(H)| = |V_{|V(H)|-2}(H)| = |V_{|V(H)|-1}(H)| = 1$. Since $|V(H)| \geq 5$, this implies that $t(H) \geq 4$. Therefore Theorem 4.1 leads to the following corollary.

Corollary 4.2 *Let H be a connected graph with $|V(H)| \geq 3$, and let $\mathcal{H}$ be a set of connected graphs such that $|V(H^*)| \geq 3$ for every $H^* \in \mathcal{H}$. If $|\mathcal{F}(H) \triangle \mathcal{F}(\mathcal{H})| < \infty$, then $H \in \mathcal{H}$ or $|\{H^* \in \mathcal{H} \mid H \prec H^*\}| \geq \min\{|\mathcal{H}|, 3\}$.*

For $v \in V(H)$, we define $\mathrm{ecc}(v) = \max\{d(v,u) \mid u \in V(H)\}$. Let $s(H) = \max\{\mathrm{ecc}(v) \mid v \in V_1(H)\}$; if $V_1(H) = \emptyset$, we set $s(H) = 0$. Then the following lemma clearly holds.

Lemma 4.3 *Let H be a connected graph with $V_1(H) \neq \emptyset$. Let $x \in V_1(H)$ be a vertex of H with $\mathrm{ecc}(x) = s(H)$, and let $y \in V(H) - \{x\}$ be a vertex such that $H - y$ is connected. Then $s(H - y) \geq s(H) - 1$.*

Next, we restrict Corollary 4.2 to the case $|\mathcal{H}| \leq 3$.

Theorem 4.4 *Let H be a connected graph with $|V(H)| \geq 3$, and let $\mathcal{H}$ be a set of connected graphs with $|\mathcal{H}| \leq 3$ such that $|V(H^*)| \geq 3$ for every $H^* \in \mathcal{H}$. If $|\mathcal{F}(H) \triangle \mathcal{F}(\mathcal{H})| < \infty$ and $H \notin \mathcal{H}$, then $|\mathcal{H}| = 3$ and $H \simeq C_3$.*

Proof. Set $k = |\mathcal{H}| \leq 3$, and write $\mathcal{H} = \{H_1, \cdots, H_k\}$. Suppose that $H \notin \mathcal{H}$. It suffices to show that $k = 3$ and $H \simeq C_3$. By Corollary 4.2, $H \prec H_i$ for every i. In particular, $|V(H)| < |V(H_i)|$ for every i. For each i, let H'_i be a connected graph with $|V(H'_i)| = |V(H)|+1$ and $H \prec H'_i \prec H_i$ (so H'_i may be H_i). Let $\mathcal{H}' = \{H'_1, \cdots, H'_k\}$. For each i, since $H \prec H'_i$ and $|V(H'_i)| = |V(H)|+1$, there exists a vertex $u_i \in V(H'_i)$ such that $H'_i - u_i \simeq H$.

Claim 4.4 $|\mathcal{F}(\mathcal{H}') - \mathcal{F}(H)| < \infty$.

Proof. Since $\mathcal{F}(\mathcal{H}') - \mathcal{F}(H) \subseteq \mathcal{F}(\mathcal{H}) - \mathcal{F}(H)$ and $|\mathcal{F}(\mathcal{H}) - \mathcal{F}(H)| < \infty$, $|\mathcal{F}(\mathcal{H}') - \mathcal{F}(H)| < \infty$. □

Take a set $U \subseteq V(H)$. Since $|\mathcal{F}(\mathcal{H}') - \mathcal{F}(H)| < \infty$ and $G_1^n(H;U) \notin \mathcal{F}(H)$ for $n \geq 1$, $H'_{i_U} \prec G_1^{n_U}(H;U)$ for some $1 \leq i_U \leq 3$ and some $n_U \geq 1$. Choose (H'_{i_U}, n_U) so that n_U is as small as possible. Since $|V(H)| < |V(H'_{i_U})|$, $V(H'_{i_U}) \cap V(K_{n_U}) \neq \emptyset$. By the choice of (H'_{i_U}, n_U), we have $V(K_{n_U}) \subseteq V(H'_{i_U})$. Since $|V(H)| < |V(H'_{i_U})|$ again, we have $n_U \geq |U - V(H'_{i_U})| + 1$. Hence every vertex of $V(K_{n_U})$ has degree $|U \cap V(H'_{i_U})| + n_U - 1 \geq |U|$ in H'_{i_U}.

We may assume that $i_{V(H)} = 1$. Since $|V(H'_1)| = |V(H)| + 1$, we see that $\Delta(H'_1) = |V(H)|(= |V(H'_1)| - 1)$ (and so $V_{|V(H)|}(H'_1) \neq \emptyset$).

Claim 4.5 *H'_1 has no cutvertex. In particular, $\delta(H'_1) \geq 2$.*

Proof. Since $\Delta(H'_1) = |V(H)| = |V(H'_1)| - 1$, every vertex of $V(H'_1) - V_{|V(H)|}(H'_1)$ is not a cutvertex. Let $u \in V_{|V(H)|}(H'_1)$. It suffices to show that $H'_1 - u$ is connected. If $|V_{|V(H)|}(H'_1)| \geq 2$, then $H'_1 - u$ has a vertex of degree $|V(H'_1)| - 2$, and so $H'_1 - u$ is connected, as desired. Thus we may assume that $|V_{|V(H)|}(H'_1)| = 1$. By the definition of $n_{V(H)}$, this implies that $n_{V(H)} = 1$, and hence $H'_1 = G_1^1(H; V(H))$. Then we have $H'_1 - u = H$. Since H is connected, $H'_1 - u$ is connected. □

Claim 4.6 *If $H'_1 - v \simeq H$, then $v \in V_{|V(H)|}(H'_1)$.*

Proof. By the construction of $G_1^{n_{V(H)}}(H; V(H))$, $n_{V(H)} \geq |V_{|V(H)|-1}(H) - V(H'_1)| + 1$. Hence $|V_{|V(H)|}(H'_1)| \geq n_{V(H)} + |V_{|V(H)|-1}(H) \cap V(H'_1)| \geq |V_{|V(H)|-1}(H)| + 1$. If $v \notin V_{|V(H)|}(H'_1)$, then $|V_{|V(H)|-1}(H)| = |V_{|V(H)|-1}(H'_1 - v)| \geq |V_{|V(H)|}(H'_1)| \geq |V_{|V(H)|-1}(H)| + 1$, a contradiction. Thus $v \in V_{|V(H)|}(H'_1)$. □

Take a set $U \subseteq V(H)$. Since $|\mathcal{F}(\mathcal{H}') - \mathcal{F}(H)| < \infty$ and $G_2^n(H;U) \notin \mathcal{F}(H)$ for $n \geq 1$, $H'_{j_U} \prec G_2^{m_U}(H;U)$ for some $1 \leq j_U \leq 3$ and some $m_U \geq 1$. Choose (H'_{j_U}, m_U)

so that m_U is as small as possible. Since $|V(H)| < |V(H'_{j_U})|$, $V(H'_{j_U}) \cap V(m_U K_1) \neq \emptyset$. By the choice of (H'_{j_U}, m_U), we have $V(m_U K_1) \subseteq V(H'_{j_U})$. If $U = \{u\}$, we write $G_2^n(H; u)$, j_u and m_u instead of $G_2^n(H; U)$, j_U and m_U, respectively. For each $u \in V(H)$, since $V(H'_{j_u}) \cap V(m_u K_1) \neq \emptyset$, $\delta(H'_{j_u}) = 1$ and so $j_u \neq 1$ by Claim 4.5.

Claim 4.7 *For each $u \in V_{\geq 2}(H)$,*

(i) *if $H'_{j_u} - v \simeq H$, then $v \in V_1(H'_{j_u})$ and*

(ii) *H'_{j_u} is isomorphic to a graph obtained from $G_2^{m_u}(H; u)$ by deleting $m_u - 1$ vertices of $V_1(G_2^{m_u}(H; u))$.*

Proof. Since $d_H(u) \geq 2$, note that $|V_1(G_2^{m_u}(H; u))| = |V_1(H)| + m_u$. Since $H \prec H'_{j_u} \prec G_2^{m_u}(H; u)$, $|V(H'_{j_u})| = |V(H)| + 1$ and $|V(G_2^{m_u}(H; u))| = |V(H'_{j_u})| + (m_u - 1)$, we see that $|V_1(H'_{j_u})| \leq |V_1(H)| + 1$ and $|V_1(G_2^{m_u}(H; u))| \leq |V_1(H'_{j_u})| + (m_u - 1)$. This together with $|V_1(G_2^{m_u}(H; u))| = |V_1(H)| + m_u$ forces $|V_1(H'_{j_u})| = |V_1(H)| + 1$ and $|V_1(G_2^{m_u}(H; u))| = |V_1(H'_{j_u})| + (m_u - 1)$. Since $|V(H'_{j_u})| = |V(H)| + 1$ and $|V_1(H'_{j_u})| = |V_1(H)| + 1$, if $H'_{j_u} - v \simeq H$, then $v \in V_1(H'_{j_u})$ and so (i) holds. Since $|V(H'_{j_u})| = |V(G_2^{m_u}(H; u))| - (m_u - 1)$ and $|V_1(H'_{j_u})| = |V_1(G_2^{m_u}(H; u))| - (m_u - 1)$, there exists a set $L \subseteq V_1(G_2^{m_u}(H; u))$ with $|L| = m_u - 1$ such that $G_2^{m_u}(H; u) - L \simeq H'_{j_u}$. $\square$

Take a vertex $u \in V(H)$. Since $|\mathcal{F}(\mathcal{H}') - \mathcal{F}(H)| < \infty$ and $G_3^n(H; u) \notin \mathcal{F}(H)$ for $n \geq 1$, $H'_{h_u} \prec G_3^{l_u}(H; u)$ for some $1 \leq h_u \leq 3$ and some $l_u \geq 1$. Choose (H'_{h_U}, l_U) so that l_U is as small as possible. Since $|V(H)| < |V(H'_{h_u})|$, $V(H'_{h_u}) \cap V(P_{l_u}) \neq \emptyset$. By the choice of (H'_{h_U}, l_U), we have $V(P_{l_u}) \subseteq V(H'_{h_u})$. Since $V(H'_{h_u}) \cap V(P_{l_u}) \neq \emptyset$, $\delta(H'_{h_u}) = 1$ and so $h_u \neq 1$ by Claim 4.5.

Claim 4.8 *For each $u \in V(H)$, if $H'_{h_u} - v \simeq H$, then $v \in V_1(H'_{h_u})$.*

Proof. Note that $|E(G_3^{l_u}(H; u))| = |E(H)| + l_u$. Since $H \prec H'_{h_u} \prec G_3^{l_u}(H; u)$, $|V(H'_{h_u})| = |V(H)| + 1$ and $|V(G_3^{l_u}(H; u))| = |V(H'_{h_u})| + (l_u - 1)$, we see that $|E(H'_{h_u})| \geq |E(H)| + 1$ and $|E(G_3^{l_u}(H; u))| \geq |E(H'_{h_u})| + (l_u - 1)$. This together with $|E(G_3^{l_u}(H; u))| = |E(H)| + l_u$ forces $|E(H'_{h_u})| = |E(H)| + 1$. Since $H \prec H'_{h_u}$ and $|V(H'_{h_u})| = |V(H)| + 1$, we have $v \in V_1(H'_{h_u})$. $\square$

Claim 4.9 $\delta(H) \geq 2$.

Proof. Suppose that $\delta(H)=1$. Let $a\in V_1(H)$ be a vertex with $\mathrm{ecc}(a)=s(H)$. We consider $G_3^{l_a}(H;a)$ and H'_{h_a}. Recall that $h_a\neq 1$. Without loss of generality, we may assume that $h_a=2$. Note that $s(G_3^{l_a}(H;a))=s(H)+l_a$. Since $|V(H'_2)|=|V(G_3^{l_a}(H;a))|-(l_a-1)$ and $H'_2\prec G_3^{l_a}(H;a)$, there exists a set $L_1\subseteq V(G_3^{l_a}(H;a))$ with $|L_1|=l_a-1$ such that $H'_2=G_3^{l_a}(H;a)-L_1$. Then by Lemma 4.3, we can check that $s(H'_2)=s(G_3^{l_a}(H;a)-L_1)\geq s(G_3^{l_a}(H;a))-(l_a-1)$. Hence $s(H'_2)\geq s(H)+1$.

Write $N_H(a)=\{b\}$. We consider $G_2^{m_b}(H;b)$ and H'_{j_b}. Note that $s(G_2^{m_b}(H;b))=s(H)$. By Claim 4.7(ii), H'_{j_b} is isomorphic to a graph obtained from $G_2^{m_b}(H;b)$ by deleting m_b-1 vertices of $V_1(G_2^{m_b}(H;b))$. Recall that $V_1(H'_{j_b})\neq\emptyset$. Hence we can check that $s(H'_{j_b})\leq s(G_2^{m_b}(H;b))$. Thus $s(H'_{j_b})\leq s(H)$, and so $j_b\neq 2$. Recall that $j_b\neq 1$. Therefore $j_b=3$. In particular, $\delta(H'_2)=\delta(H'_3)=1$.

Let $A=N_H[a](=\{a,b\})$. We consider $G_1^{n_A}(H;A)$ and H'_{i_A}. Note that b is a cutvertex of $G_1^{n_A}(H;A)$. Recall that $V(G_1^{n_A}(H;A))-V(H)\subseteq V(H'_{i_A})$ and every vertex of $V(G_1^{n_A}(H;A))-V(H)$ has degree at least $|A|(=2)$ in H'_{i_A}. Since $\delta(H)=1$ and $H\prec H'_{i_A}$, H'_{i_A} is not complete. Since $G_1^{n_A}(H;A)-(V(H)-A)$ is complete, $b\in V(H'_{i_A})$ and $V(H'_{i_A})\cap(V(H)-A)\neq\emptyset$. Hence b is a cutvertex of H_{i_A} and H_{i_A} has an endblock which is complete and has order at least three. Then by Claim 4.5, $i_A\neq 1$, and so $i_A\in\{2,3\}$. Recall that $h_a=2$ and $j_b=3$. By Claims 4.7(i) and 4.8, there exists a vertex $v\in V_1(H'_{i_A})$ such that $H'_{i_A}-v\simeq H$. In particular, H has an endblock which is complete and has order at least three.

Let C' be a maximum complete endblock of H, and let b' be the unique cutvertex of H in C'. Let $D=V(C')$. We consider $G_1^{n_D}(H;D)$ and H'_{i_D}. Note that b' is a cutvertex of $G_1^{n_D}(H;D)$. Recall that $V(G_1^{n_D}(H;D))-V(H)\subseteq V(H'_{i_D})$ and every vertex of $V(G_1^{n_D}(H;D))-V(H)$ has degree at least $|D|$ in H'_{i_D}. Since $\delta(H)=1$ and $H\prec H'_{i_D}$, H'_{i_D} is not complete. Since $G_1^{n_D}(H;D)-(V(H)-D)$ is complete, $b'\in V(H'_{i_D})$ and $V(H'_{i_D})\cap(V(H)-D)\neq\emptyset$. Hence b' is a cutvertex of H'_{i_D} and H'_{i_D} has an endblock which is complete and has order at least $|D|+1$. Then by Claim 4.5, $i_D\neq 1$, and so $i_D\in\{2,3\}$. Recall that $h_a=2$ and $j_b=3$. By Claims 4.7(i) and 4.8, there exists a vertex $v\in V_1(H'_{i_D})$ such that $H'_{i_D}-v\simeq H$. In particular, H has an endblock which is complete and has order at least $|D|+1(=|V(C')|+1)$, which contradicts the maximality of C'. $\square$

Take an integer i' with $V_1(H'_{i'})\neq\emptyset$. Since $H\prec H'_{i'}$ and $|V(H'_{i'})|=|V(H)|+1$, we see that $H'_{i'}-u\simeq H$ if and only if $u\in V_1(H'_{i'})$ by Claim 4.9. This forces $|V_1(H'_{i'})|=1$. Thus we see that $|V_1(H'_i)|\leq 1$ for every $1\leq i\leq 3$. Fix a vertex $x\in V(H)$. We may assume that $j_x = 2$. By the construction of $G_2^{m_x}(H;x)$ and H'_2, we can check that $H'_2 = G_2^1(H;x)$.

Let Y be a maximum subset of $V(H)$ so that $N_H(u) = N_H(v)$ for every $u, v \in Y$, and let $Y' = N_H(Y)$. We consider $G_2^{m_{Y'}}(H;Y')$ and $H'_{j_{Y'}}$. Let $Y^* = (Y \cap V(H'_{j_{Y'}})) \cup (V(G_2^{m_{Y'}}(H;Y')) - V(H))$. Note that $N_{H'_{j_{Y'}}}(u) = N_{H'_{j_{Y'}}}(v)$ for every $u, v \in Y^*$. By the construction of $G_2^{m_{Y'}}(H;Y')$, $m_{Y'} \geq |Y - V(H'_{j_{Y'}})| + 1$. Hence $|Y^*| = |Y \cap V(H'_{j_{Y'}})| + m_{Y'} \geq |Y| + 1$. Therefore, $H'_{j_{Y'}} - u \simeq H$ implies that $u \in Y^*$ by the maximality of Y. Since $|Y^*| \geq 2$, every vertex of Y^* has degree at most $|V(H)| - 1$ in $H'_{j_{Y'}}$. By Claim 4.6, this implies that $j_{Y'} \neq 1$. Since $|V_1(H'_{j_{Y'}})| \leq 1$, every vertex of Y^* has degree at least two. Recall that $j_x = 2$. By Claim 4.7(i), this implies that $j_{Y'} \neq 2$. Hence $j_{Y'} = 3$. Also we see that $\delta(H'_3) \geq 2$.

Take a vertex $y \in V(H)$. Since $\delta(H'_{j_y}) = 1$, $j_y = 2$. By the construction of $G_2^{m_y}(H;y)$ and H'_2, we can check that $H'_2 \simeq G_2^1(H;y)$. This implies that $t(H) = 1$, and hence H is regular. Set $m = \delta(H)$. By Claim 4.6, $\delta(H'_1) = m + 1$. Recall that $|Y^*| \geq 2$. Since, for every $u \in Y^*$, $H'_3 - u \simeq H$ and $d_{H'_3 - u}(v) = d_{H'_3}(v)$ for every $v \in Y^* - \{u\}$, $\delta(H'_3) = m$.

Suppose that $m \geq 3$. Take a vertex $z \in V(H)$. Let Z be a subset of $N_H(z)$ with $|Z| = m - 1$. We consider $G_2^{m_Z}(H;Z)$ and H'_{j_Z}. Note that every vertex of $V(G_2^{m_Z}(H;Z)) - V(H)$ has degree $m - 1$. Since $V(G_2^{m_Z}(H;Z)) - V(H) \subseteq V(H'_{j_Z})$, $\delta(H'_{j_Z}) \leq m - 1$, and hence $j_Z = 2$. Since every vertex of H'_2 of degree at most $m - 1$ belongs to $V_1(H'_2)$, $Z - V(H'_2) \neq \emptyset$. By the definition of $G_2^{m_Z}(H;Z)$, $m_Z \geq |Z - V(H'_2)| + 1 \geq 2$. Hence there exist two vertices of degree one in H'_2, a contradiction. Thus $m \leq 2$. This implies that H is a cycle.

Suppose that $|V(H)| \geq 4$. Let $e = w_1w_2$ be an edge of H, and let $W = \{w_1, w_2\}$. We consider $G_1^{n_W}(H;W)$ and H'_{i_W}. Since $H \prec H'_{i_W}$, H'_{i_W} has an induced cycle of order $|V(H)|$. This together with $|V(H'_{i_W})| = |V(H)| + 1$ implies $H'_{i_W} \simeq G_1^1(H;W)$. Since $V_1(H'_{i_W}) = V_{|V(H)|}(H'_{i_W}) = \emptyset$, $i_W \neq 1, 2$. Hence $i_W = 3$. However, there exist no vertices $u, v \in V(H'_3)$ with $u \neq v$ such that $N_{H'_3}(u) = N_{H'_3}(v)$, a contradiction. Thus $|V(H)| = 3$, and so $H \simeq C_3$.

This completes the proof of Theorem 4.4. $\square$

Theorem 4.4 leads to the following results.

Corollary 4.5 *For each $1 \leq i \leq 3$, let H_i be a connected graph with $|V(H_i)| \geq 3$. If $|\mathcal{F}(H_1) \triangle \mathcal{F}(H_2, H_3)| < \infty$, then $H_1 \in \{H_2, H_3\}$.*

Corollary 4.6 *For each $1 \le i \le 3$, let H_i be a connected graph with $|V(H_i)| \ge 3$. If $|\mathcal{F}(H_1) \triangle \mathcal{F}(H_2, H_3)| < \infty$ and H_1 is not special, then there exists an integer $2 \le i \le 3$ such that $H_1 = H_i \prec H_{3-i}$.*

Proof. By Corollary 4.5, $H_1 \in \{H_2, H_3\}$. We may assume that $H_1 = H_2$. Then $\mathcal{F}(H_1) \triangle \mathcal{F}(H_2, H_3) = \mathcal{F}(H_1) \triangle \mathcal{F}(H_1, H_3) = \mathcal{F}(H_1) - \mathcal{F}(H_3)$. Hence $|\mathcal{F}(H_1) - \mathcal{F}(H_3)| < \infty$. Since H_1 is not special, $H_1 \prec H_3$ by Lemma 3.2, as desired. $\square$

Corollary 4.7 *For each $1 \le i \le 2$, let H_i be a connected graph with $|V(H_i)| \ge 3$. If $|\mathcal{F}(H_1) \triangle \mathcal{F}(H_2)| < \infty$, then $H_1 = H_2$.*

5 $|\mathcal{H}_1| = |\mathcal{H}_2| = 2$ and $\mathcal{H}_1 \cap \mathcal{H}_2 \neq \emptyset$

In this section, we focus on the case in which $\mathcal{F}(\{H_1, H_2\}) \triangle \mathcal{F}(\{H_1, H_3\})$ is finite.

Theorem 5.1 *For each $1 \le i \le 3$, let H_i be a connected graph with $|V(H_i)| \ge 3$. If $|\mathcal{F}(H_1, H_2) \triangle \mathcal{F}(H_1, H_3)| < \infty$, $\Delta(H_1) \le |V(H_1)| - 2$ and $\delta(H_1) \ge 2$, then either $H_1 \prec H_2$ and $H_1 \prec H_3$, or $H_2 = H_3$.*

Proof. Suppose that $H_1 \prec H_2$ or $H_1 \prec H_3$. We may assume that $H_1 \prec H_2$. Then $\mathcal{F}(H_1, H_2) \triangle \mathcal{F}(H_1, H_3) = \mathcal{F}(H_1) \triangle \mathcal{F}(H_1, H_3) = \mathcal{F}(H_1) - \mathcal{F}(H_3)$. Hence $|\mathcal{F}(H_1) - \mathcal{F}(H_3)| < \infty$. Since H_1 is not special, $H_1 \prec H_3$ by Lemma 3.2, as desired. Thus we may assume that $H_1 \not\prec H_2$ and $H_1 \not\prec H_3$.

Suppose that $H_2 \neq H_3$. We may assume that $H_2 \not\prec H_3$. Then by Lemma 3.1, $|V(H_3)| \ge 2|V(H_2)| - 3$ and $|V(H_2)| \ge 4$. If $|V(H_2)| \ge |V(H_3)|$, then we see that $|V(H_2)| \le 3$, a contradiction. Thus $|V(H_2)| < |V(H_3)|$. In particular, $H_3 \not\prec H_2$. Then by Lemma 3.1, $|V(H_2)| \ge 2|V(H_3)| - 3$. This together with $|V(H_2)| < |V(H_3)|$ implies that $|V(H_3)| \le 2$, a contradiction. Therefore $H_2 = H_3$. $\square$

6 k-Connected graphs

In this section, we extend Corollary 4.7 to k-connected graphs.

In our proof, we use the Cartesian product of two graphs. The *Cartesian product* $G_1 \square G_2$ of two graphs G_1 and G_2 has vertex set $V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) are joined by an edge if and only if $u_i v_i \in E(G_i)$ and $u_{3-i} = v_{3-i}$

for some $1 \leq i \leq 2$. Xu and Yang [15] proved the following results concerning the connectivity of the Cartesian product of two graphs.

Lemma 6.1 (Xu and Yang [15]) *For each $i = 1, 2$, let G_i be a connected graph. Then $\kappa(G_1 \Box G_2) \geq \min\{\kappa(G_1) + \delta(G_2), \kappa(G_2) + \delta(G_1)\}$.*

Lemma 6.2 *Let k be a positive integer. For each $1 \leq i \leq 2$, let H_i be a connected graph with $|V(H_i)| \geq 3$. If $|\mathcal{F}_k(H_1) - \mathcal{F}_k(H_2)| < \infty$ and $H_1 \not\prec H_2$, then either $|V(H_1)| < |V(H_2)|$, or $|V(H_1)| = |V(H_2)|$ and $|E(H_1)| > |E(H_2)|$.*

Proof. Assume that $|V(H_1)| \geq |V(H_2)|$. It suffices to show that $|V(H_1)| = |V(H_2)|$ and $|E(H_1)| > |E(H_2)|$.

Note that $G_1^n(H_2; V(H_2))$ is k-connected for $n \geq k-1$. Since $|\mathcal{F}_k(H_1) - \mathcal{F}_k(H_2)| < \infty$ and $G_1^n(H_2; V(H_2)) \notin \mathcal{F}_2(H_2)$, $H_1 \prec G_1^{n_1}(H_2; V(H_2))$ for some $n_1 \geq k-1$. Since $H_1 \not\prec H_2$, $V(H_1) \cap V(K_{n_1}) \neq \emptyset$. Hence $H_1 = H_2' + K_{m_1}$ for a graph $H_2' \prec H_2$ and $m_1 \geq 1$. In particular, H_1 has a vertex of degree $|V(H_1)| - 1$ (and so $\Delta(H_1) = |V(H_1)| - 1$).

For $n \geq k$, let $G_4^n = H_2 \Box K_{n,n}$. Since $\kappa(K_{n,n}) = \delta(K_{n,n}) = n \geq k$, G_4^n is k-connected by Lemma 6.1. Since $|\mathcal{F}_k(H_1) - \mathcal{F}_k(H_2)| < \infty$ and $G_4^n \notin \mathcal{F}_2(H_2)$, $H_1 \prec G_4^{n_2}$ for some $n_2 \geq k$. Recall that $\Delta(H_1) = |V(H_1)| - 1$. We may assume that (u_1, u_2) has degree $|V(H_1)| - 1$ in $H_1 (\prec G_4^{n_2})$. Since $H_1 \not\prec H_2$, $(u_1, v_2) \in V(H_1) (\prec G_4^{n_2})$ for some $v_2 \in V(K_{n,n}) - \{u_2\}$. Since (u_1, v_2) is adjacent to (u_1, u_2), (u_1, v_2) has degree 1 in $H_1 (\prec G_4^{n_2})$ by the definition of $G_4^{n_2}$. In particular, H_1 has a vertex of degree 1.

Recall that $H_1 = H_2' + K_{m_1}$. If $m_1 \geq 2$ or $H_2' = H_2$, then H_1 has no vertex of degree 1, a contradiction. Thus $m_1 = 1$ and $H_2' \neq H_2$. This together with $|V(H_1)| \geq |V(H_2)|$ implies that $|V(H_1)| = |V(H_2)| = |V(H_2')| + 1$. Write $V(H_2) - V(H_2') = \{x\}$. If $d_{H_2}(x) = |V(H_2)| - 1$, then we have $H_2 \simeq H_2' + K_1$, which contradicts the fact that $H_1 \not\simeq H_2$. Thus $d_{H_2}(x) \leq |V(H_2)| - 2$. Then $|E(H_1)| = |E(H_2' + K_1)| = |E(H_2')| + (|V(H_2)| - 1) > |E(H_2')| + d_{H_2}(x) = |E(H_2)|$. Therefore we have the desired result. $\square$

Theorem 6.3 *Let k be a positive integer. For each $1 \leq i \leq 2$, let H_i be a connected graph with $|V(H_i)| \geq 3$. If $|\mathcal{F}_k(H_1) \triangle \mathcal{F}_k(H_2)| < \infty$, then $H_1 = H_2$.*

Proof. Suppose that $H_1 \neq H_2$. We may assume that $H_1 \not\prec H_2$. Since $|\mathcal{F}_k(H_1) - \mathcal{F}_k(H_2)| < \infty$, either $|V(H_1)| < |V(H_2)|$, or $|V(H_1)| = |V(H_2)|$ and $|E(H_1)| > |E(H_2)|$ by Lemma 6.2. Suppose that $|V(H_1)| < |V(H_2)|$. Then $H_2 \not\prec H_1$. Since

$|\mathcal{F}_k(H_2) - \mathcal{F}_k(H_1)| < \infty$, $|V(H_2)| \le |V(H_1)|$ by Lemma 6.2, a contradiction. Thus $|V(H_1)| = |V(H_2)|$ and $|E(H_1)| > |E(H_2)|$. Then $H_2 \not< H_1$. Since $|\mathcal{F}_k(H_2) - \mathcal{F}_k(H_1)| < \infty$ and $|V(H_1)| = |V(H_2)|$, $|E(H_2)| > |E(H_1)|$ by Lemma 6.2, a contradiction.

Therefore $H_1 = H_2$. $\square$

7 Concluding Remarks

In this paper, we have studied when the difference and the symmetric difference of sets of graphs defined by forbidden subgraphs become finite.

As in Section 2, let H_1 be the graph obtained from the triangle by attaching two pendant edges to a vertex and let H_2 be a 6-regular triangulation of the torus. Then we have seen that H_1 is not an induced subgraph of H_2, but $\mathcal{F}(H_1) - \mathcal{F}(H_2)$ is finite.

Let $\mathcal{H} = \{G_1^1(H_1; U) \mid U \subseteq V(H_1), U \neq \emptyset\} \cup \{H_2\}$. Then $(H_1, \mathcal{H})$ is a pair that satisfies the assumption of Theorem 4.1 with $\mathcal{F}(H_1) \triangle \mathcal{F}(\mathcal{H}) = \{H_1, H_2\}$, but it does not satisfy the conclusions (i) and (ii) of Theorem 4.1. Therefore the condition (iii) of Theorem 4.1 is necessary. Let H_3 be a 6-regular triangulation of the torus which is different from H_2. Then the pair $(\{H_1, H_2\}, \{H_1, H_3\})$ satisfies the assumption of Theorem 5.1 with $\mathcal{F}(H_1, H_2) \triangle \mathcal{F}(H_1, H_3) = \{H_2, H_3\}$, except for the degree condition, but it does not satisfy the conclusion of Theorem 5.1. Therefore the degree condition on H_1 of Theorem 5.1 is necessary.

Acknowledgment

The authors would like to thank Professor Akira Saito for his helpful suggestion and assistance in correction of this paper.

References

[1] R.E.L. Aldred, J. Fujisawa and A. Saito, Forbidden subgraphs and the existence of 2-factor, J. Graph Theory **64** (2010) 250–266.

[2] M.D. Barrus, M. Kumbhat and S.G. Hartke, Graph classes characterized both by forbidden subgraphs and degree sequences, J. Graph Theory **57** (2008) 131–148.

[3] P. Bedrossian, *Forbidden subgraphs and minimum degree conditions for hamiltonicity*, Ph.D. Thesis, University of Memphis, 1991.

[4] H. Broersma, R.J. Faudree, A. Huck, H. Trommel and H.J. Veldman, Forbidden subgraphs that imply Hamiltonian-connectedness, J. Graph Theory **40** (2002) 104–119.

[5] G. Chartrand and L. Lesniak, Graphs & Digraphs (4th ed.), Chapman and Hall/CRC, Boca Raton, Florida, U.S.A. (2005).

[6] E.J. Cockayne, C.W. Ko and F.B. Shepherd, Inequalities concerning dominating sets in graphs, Technical Report DM-370-IR, Dept. Math., Univ. Victoria, 1985.

[7] D. Duffus, R.J. Gould and M.S. Jacobson, Forbidden subgraphs and the hamiltonian theme, in: Chartrand, Alavi, Goldsmith, Lesniak and Lick, eds, The Theory and Applications of Graphs (1981) 297–316.

[8] R.J. Faudree and R.J. Gould, Characterizing forbidden pairs for Hamiltonian properties, Discrete Math. **173** (1997), 45–60.

[9] R.J. Faudree, R.J. Gould, Z. Ryjáček and I. Schiermeyer, Forbidden subgraphs and pancyclicity, Congress Numer. **109** (1995) 13–32.

[10] J. Fujisawa, S. Fujita, M.D. Plummer, A. Saito and I. Schiermeyer, A pair of forbidden subgraphs and perfect matchings in graphs of high connectivity, Combinatorica, to appear.

[11] J. Fujisawa, K. Ota, K. Ozeki and G. Sueiro, Forbidden induced subgraphs for star-free graphs, Discrete Math. **311** (2011) 2475–2484.

[12] J. Fujisawa and A. Saito, A pair of forbidden subgraphs and 2-factors, Combin. Probab. Comput. to appear.

[13] S. Fujita, K. Kawarabayashi, C. L. Lucchesi, K. Ota, M. Plummer and A. Saito, A pair of forbidden subgraphs and perfect matchings, J. Combin. Theory Ser. B **96** (2006) 315–324.

[14] B. Randerath and I. Schiermeyer, Vertex colouring and forbidden subgraphs–a survey, Graphs Combin. **20** (2004) 1–40.

[15] J.M. Xu and C. Yang, Connectivity of Cartesian product graphs, Discrete Math. **306** (2006) 159–165.