Equivalence of Jackson's and Thomassen's conjectures

Roman Čada[1*†] Shuya Chiba[2‡] Kenta Ozeki[34§] Petr Vrána[1*¶]
Kiyoshi Yoshimoto[5‖]

[1]*Department of Mathematics, University of West Bohemia,*
Centre of Excellence ITI - Institute for Theoretical Computer Science, Charles University,
European Centre of Excellence NTIS - New Technologies for the Information Society,
P.O. Box 314, 306 14 Pilsen, Czech Republic

[2]*Department of Mathematics and Engineering, Kumamoto University,*
2-39-1, Kurokami, Kumamoto 860-8555, Japan

[3]*National Institute of Informatics,*
2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan

[4]*JST, ERATO, Kawarabayashi Large Graph Project, Japan*

[5]*Department of Mathematics, College of Science and Technology,*
Nihon University, Tokyo 101-8308, Japan

Abstract

A graph G is said to be claw-free if G has no induced subgraph isomorphic to $K_{1,3}$. For a cycle C in a graph G, C is called a Tutte cycle of G if C is a Hamilton cycle of G, or the order of C is at least 4 and every component of $G - C$ has at most three neighbors on C. In [On a closure concept in claw-free graphs, J. Combin. Theory Ser. B 70 (1997), 217–224], Ryjáček proved that the conjectures by Matthews and Sumner (every 4-connected claw-free graph is Hamiltonian) and by Thomassen (every 4-connected line graph is Hamiltonian) are equivalent. In this paper, we show the above conjectures are equivalent with the conjecture by Jackson in 1992 (every 2-connected claw-free graph has a Tutte cycle).

Keywords: Hamiltonian, Claw-free graphs, Line graphs, Tutte cycles
AMS Subject Classification: 05C38, 05C45

1 Introduction

In this paper, we consider only finite undirected graphs. For terminology and notation not defined in this paper, we refer the readers to [6]. Throughout this paper a graph means a *simple*

*Research partly supported by project P202/12/G061 of the Czech Science Foundation and by the European Regional Development Fund (ERDF), project NTIS - New Technologies for Information Society, European Centre of Excellence, CZ.1.05/1.1.00/02.0090

†Email address: cadar@kma.zcu.cz

‡This work was supported by JSPS KAKENHI Grant Number 26800083
Email address: schiba@kumamoto-u.ac.jp

§This work was supported by JSPS KAKENHI Grant number 25871053
Email address: ozeki@nii.ac.jp

¶Email address: vranap@kma.zcu.cz

‖This work was supported by JSPS KAKENHI Grant Number 26400190
Email address: yosimoto@math.cst.nihon-u.ac.jp

graph, i.e., without loops or multiple edges. A *multigraph* may contain multiple edges but no loops. Let G be a (multi)graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. For $X \subseteq V(G)$, we let $G[X]$ denote the subgraph induced by X in G, and let $G - X = G[V(G) - X]$. For a subgraph H of G, let $G - H = G - V(H)$. A graph G is said to be *Hamiltonian* if G has a *Hamilton cycle*, i.e., a cycle containing all the vertices of G, and *Hamilton-connected* if G has a *Hamilton path* between any pair of vertices, i.e., a path containing all the vertices of G. A graph G is said to be *claw-free* if G has no induced subgraph isomorphic to $K_{1,3}$ (here $K_{1,3}$ denotes the complete bipartite graph with partite sets of cardinalities 1 and 3, respectively). For a cycle C of G, C is said to be *maximal* if there exists no cycle C' such that $V(C) \subsetneq V(C')$.

In this paper, we will deal with many statements which are unknown to be true or not. We call two statements *equivalent* if the correctness of one statement implies that of the other and vice versa. Most of the results in this paper are motivated by the following two conjectures due to Matthews and Sumner [17] and Thomassen [23], respectively.

Conjecture A (Matthews and Sumner [17], Thomassen [23]) *The following statements are true.*

(A1) *Every* 4*-connected claw-free graph is Hamiltonian.*

(A2) *Every* 4*-connected line graph is Hamiltonian.*

Since every line graph is claw-free, statement (A2) is a special case of statement (A1). However it is known that a result on closures due to Ryjáček [18] implies that statements (A1) and (A2) are even equivalent.

Theorem B (Ryjáček [18]) *Statements* (A1) *and* (A2) *are equivalent.*

Like Theorem B, many statements that are seemingly stronger or weaker than statements (A1) and (A2) have been proven to be equivalent to it as follows (see a survey [5] for more details). Note that statements (A5) and (A6) were conjectured by Ash and Jackson [1] and Fleischner [8], respectively.

Theorem C *All of the following statements are equivalent to statements* (A1) *and* (A2).

(A3) *Every* 4*-connected claw-free graph is Hamilton-connected* [19].

(A4) *Every* 4*-connected line graph is* 1*-Hamilton-connected* (2*-edge-Hamilton-connected*) [15].

(A5) *Every essentially* 4*-edge-connected multigraph has a dominating closed trail* [9].

(A6) *Every cyclically* 4*-edge-connected cubic graph has a dominating cycle* [9].

(A7) *Every cyclically* 4*-edge-connected cubic graph that is not* 3*-edge-colorable has a dominating cycle* [12].

(A8) *Every snark has a dominating cycle* [3].

Recently, as a positive result related to Conjecture A, Kaiser and the fourth author [16] proved that every 5-connected claw-free graph with minimum degree at least 6 is Hamilton-connected.

On the other hand, it is known that considering "Tutte cycles" is an effective approach to some problems on Hamiltonicity, where a cycle C of a graph G is called a *Tutte cycle* of G if (i) C is a Hamilton cycle of G, or (ii) $|V(C)| \geq 4$ and every component of $G - C$ has at most three neighbors on C. Note that every Tutte cycle C of a 4-connected graph G is a Hamilton cycle, since otherwise the neighbors of a component of $G - C$ form a cut set of order at most three, contradicting 4-connectedness of G. One can show that every 4-connected planar graph is Hamiltonian by showing assertions on the existence of certain Tutte cycles in 2-connected planar graphs (see [22, 24]). Starting with this result, many researchers have been studying the existence of certain Tutte cycles not only in planar graphs but also in projective planar graphs or graphs on other surfaces in order to show Hamiltonicity of such graphs (for example, see [20, 21, 25]). Thus, they have succeeded to show Hamiltonicity of 4-connected planar graphs or graphs on surfaces, considering stronger concept "Tutte cycles".

Motivated by the above situation for planar graphs, we concentrate on Tutte cycles in claw-free graphs in this paper. As a possible approach to solve Conjecture A, Jackson [11] proposed the following conjecture (see also [2, Conjecture 5.37] and a survey [7, Conjecture 2a.5]).

Conjecture D (Jackson [11]) *The following statement is true.*

(A9) *Every* 2*-connected claw-free graph has a Tutte cycle.*

As mentioned above, Tutte cycles in 4-connected graphs are Hamilton cycles, and hence statement (A9) implies statement (A1). The main result of this paper is to show that the converse also holds. In fact, we prove the following theorem.

Theorem 1 *Statements* (A1) *and* (A9) *are equivalent.*

On the other hand, if a graph has a Tutte cycle, then we can expect that it is long since it can avoid only vertices in a component of the graph after deleting a cut set of order at most three. Actually, Tutte cycles in 4-connected graphs are Hamilton cycles, i.e., Tutte cycles in 4-connected graphs are longest cycles of the graphs. How about 2-connected (or 3-connected) claw-free graphs? In view of Theorem 1, it would be natural to ask that every 2-connected (or 3-connected) claw-free graph has a Tutte cycle which is a longest cycle of the graph. As an answer to this problem, in Section 7, we will give a 3-connected claw-free graph in which any Tutte cycle is not longest. Thus it is not always true that a 2-connected (or 3-connected) claw-free graph has a longest cycle being also a Tutte cycle. However, the following theorem, which is also our main theorem, implies that if every 2-connected claw-free graph has a Tutte cycle, then we can always take it so that it is maximal.

Theorem 2 *Statement* (A9) *is equivalent to the following statement.*

(A10) *Every 2-connected claw-free graph has a Tutte cycle which is a maximal cycle of the graph.*

In Section 3, we will mention a new statement and give two theorems (Theorems 3 and 4) in order to show Theorems 1 and 2. We also give the proofs of Theorems 1 and 2 by using the two theorems in this section. In Sections 4 and 5, we will prove Theorems 3 and 4 by using closure concept and other related results, some of which are also new (see also the relations between statements in Appendix).

2 Notation and terminology

In this section, we prepare notation and terminology which we use in subsequent sections.

Let G be a (multi)graph and $v \in V(G)$. The *degree* of v is the number of edges incident with v in G. We denote by $d_G(v)$ and $N_G(v)$ the degree and the neighborhood of v in G, respectively, and denote by $\delta(G)$ the minimum degree of G. For an integer l, let $V_l(G) = \{v \in V(G) : d_G(v) = l\}$, and let $V_{\geq l}(G) = \bigcup_{m \geq l} V_m(G)$ and $V_{\leq l}(G) = \bigcup_{m \leq l} V_m(G)$. For a subgraph H of G and a vertex v in $G - H$, let $N_H(v) = N_G(v) \cap V(H)$. For subgraphs H and F of G with $V(F) \cap V(H) = \emptyset$, we define $N_H(F) = \bigcup_{v \in V(F)} N_H(v)$.

We use $L(G)$ for the line graph of G. Let e be an edge of G. We denote by v_e a vertex in $L(G)$ corresponding to e. Let $V(e)$ be the set of end vertices of e, and we define $E_G(e) = \{f \in E(G) : V(f) \cap V(e) \neq \emptyset\}$. An edge e is called a *pendant edge* if one of the end vertices of e has degree exactly 1 in H, i.e., $V(e) \cap V_1(H) \neq \emptyset$. The *edge degree* of e in G is defined by the number of elements of $E_G(e) \setminus \{e\}$, i.e., the number of edges adjacent to e. Note that for a (multi)graph G, the minimum edge degree of G is d if and only if the minimum degree of $L(G)$ is d. For subsets X and Y of $V(G)$ with $X \cap Y = \emptyset$, let $E_G(X, Y)$ denote the set of edges between X and Y in G, and let $e_G(X, Y) = |E_G(X, Y)|$. We often identify a subgraph H of G with its vertex set $V(H)$. For example, we write $E_G(H, F)$ instead of $E_G(V(H), V(F))$ for two disjoint subgraphs H and F of G. For a (multi)graph H and an edge set X, $H + X$ means the graph with the vertex set $V(H) \cup \left(\bigcup_{e \in X} V(e)\right)$ and the edge set $E(H) \cup X$. For a subgraph H of G, let $E_G(H) = E(G[V(H)]) \cup E_G(H, G - H)$. A *star* is a multigraph consisting of a vertex and edges incident with the vertex (note that a star is not necessary a tree in this paper).

A multigraph T is called a *closed trail* if T is connected and all the vertices of T have even degree in T. In this paper, we regard a K_1 as a closed trail (here K_n denotes the complete graph of order n). Let H be a multigraph, and let T be a closed trail of H. We call T a *dominating* *closed trail* of H if each edge of H is incident with a vertex in T, i.e., $H-T$ is edgeless (in case that T is a cycle, we call T a *dominating cycle*). We say that T is *edge-maximal* if there exists no closed trail T' of H such that $E_H(T) \subsetneq E_H(T')$. Note that a dominating closed trail of H is an edge-maximal closed trail of H. In [10], it is shown that for a connected multigraph H with $|E(H)| \geq 3$, H has a dominating closed trail if and only if $L(H)$ is Hamiltonian. Hence by the definition of the edge-maximal closed trail, we can easily obtain the following.

Lemma 1 *Let H be a multigraph, and let T be an edge-maximal closed trail of H and $H^* = H[V(T)] + E_H(T, H-T)$. Then $L(H^*)$ has a Hamiltonian cycle which is a maximal cycle of $L(H)$.*

Now let H be a multigraph. Here we introduce the concepts "Tutte closed trails" and "weakly Tutte closed trails", which will be used in the proofs of Theorems 1 and 2. For a closed trail T of H, T is called a *Tutte closed trail* of H if (i) $E_H(T) = E(H)$ (i.e., T is a dominating closed trail of H), or (ii) $|E_H(T)| \geq 4$ and $e_H(F,T) \leq 3$ for every component F of $H-T$. A closed trail T is called a *weakly Tutte closed trail* of H if (i) $E_H(T) = E(H)$, or (ii) $|E_H(T)| \geq 4$ and $e_H(F,T) \leq 3$ for all $F \in \mathcal{F}_H(T)$, where we let $\mathcal{F}_H(T) = \{F : F$ is a component of $H-T$ with $|V(F)| \geq 2\}$. Note that if T is a Tutte closed trail, then T is a weakly Tutte closed trail, but the converse does not hold in general. Thus the concept "Tutte closed trails" is slightly stronger. If T is a Tutte closed trail (a weakly Tutte closed trail, respectively) and an edge-maximal closed trail of H, then we call T a *Tutte* (a *weakly Tutte*, resp.) *edge-maximal closed trail* of H.

We further prepare the following notation and terminology. Let H be a connected multigraph. For an edge-cut X of H, X is called an *essential k-edge-cut* of H if $|X| = k$ and $G-X$ has exactly two components of orders at least 2. We define $\mathcal{E}_k(H) = \{X \subseteq E(H) : X$ is an essential k-edge-cut of $H\}$. For an integer $k \geq 2$, H is called *essentially k-edge-connected* if $|E(H)| \geq k+1$ and $\mathcal{E}_l(H) = \emptyset$ for all $l < k$. It is known that for a multigraph H such that $L(H)$ is not complete, H is essentially k-edge-connected if and only if $L(H)$ is k-connected. It is also known that if H is essentially 2-edge-connected and H is not a star, then $H - V_1(H)$ is 2-edge-connected.

3 Proofs of Theorems 1 and 2

We now state a new statement that plays a crucial role in the proofs of Theorems 1 and 2, and we also give two theorems.

(A11) *Every essentially 2-edge-connected multigraph has a weakly Tutte edge-maximal closed trail.*

Theorem 3 *If statement* (A1) *is true, then statement* (A11) *is also true.*

Theorem 4 *If statement* (A11) *is true, then statement* (A10) *is also true.*

Here we prove Theorems 1 and 2 assuming Theorems 3 and 4.

Proofs of Theorems 1 and 2. It is clear that statement (A10) implies statement (A9) and statement (A9) implies statement (A1). On the other hand, if statement (A1) is true, then by Theorem 3, statement (A11) is true, and by Theorem 4, statement (A10) is also true. This completes the proofs of Theorems 1 and 2. □

Thus, to prove Theorems 1 and 2, it only suffices to show Theorems 3 and 4. In the next section, we will prove Theorem 4 by using the closure concept. In Section 5, we will prove Theorem 3 by using the inflation techniques.

In the rest of this section, we also state other statements and give a theorem as follows. Note that statement (A11) implies statement (A12), and statement (A13) implies statement (A12). Note also that statement (A13) was conjectured by Jackson [11], and see also [2, Conjecture 5.38] and a survey [5, Conjecture 31]. (Actually, Jackson's original conjecture [11] is that every 2-edge-connected multigraph has a Tutte closed trail. However, it is easy to check that this statement is equivalent to statement (A13) because if H is an essentially 2-edge-connected multigraph and H is not a star, then $H - V_1(H)$ is 2-edge-connected.)

(A12) *Every essentially* 2*-edge-connected multigraph has a weakly Tutte closed trail.*

(A13) *Every essentially* 2*-edge-connected multigraph has a Tutte closed trail.*

Theorem 5 *If statement* (A12) *is true, then statement* (A13) *is also true.*

By the definition of a Tutte closed trail, it is easy to check that statement (A13) implies statement (A5) "every essentially 4-edge-connected multigraph has a dominating closed trail" (in fact, every Tutte closed trail T of an essentially 4-edge-connected multigraph H is a dominating closed trial, since otherwise the edges between T and a component of order at least 2 in $H - T$ form an essential edge-cut of order at most three, contradicting essential 4-edge-connectedness of H). Therefore, combining this with Theorems C, 3 and 5, we see that statement (A1) is also equivalent to statements (A11), (A12) and (A13). Note that it is not necessary to prove Theorem 5 for the proofs of Theorems 3 and 4, but we prove it since it may be interesting by itself (see also Appendix). We will prove Theorem 5 in Section 6.

Corollary 6 *Statements* (A1), (A11), (A12) *and* (A13) *are all equivalent.*

4 Proof of Theorem 4

In this section, we prove Theorem 4. To do that, we use the closure concept in claw-free graphs introduced by Ryjáček [18].

For a vertex v of a graph G, we call v a *locally connected vertex* of G if $G[N_G(v)]$ is connected. For a locally connected vertex v of a graph G, we call v an *eligible vertex* of G if $G[N_G(v)]$ is not compete. Let G be a claw-free graph. For an eligible vertex v of G, the operation of adding all possible edges between vertices in $N_G(v)$ is called *local completion* at v. In [18], it is shown that this operation preserves the claw-freeness of the original graph. Iterating local completions as long as possible, we obtain the graph G^* in which $G^*[N_{G^*}(v)]$ is a complete graph for every locally connected vertex v, i.e., there is no eligible vertex in G^*. We call this graph the *closure* of G, and denote it by $\mathrm{cl}(G)$. In [18], it is shown that the closure of a claw-free graph has the following property.

Theorem E (Ryjáček [18]) *Let G be a claw-free graph. Then the following hold.*

(i) $\mathrm{cl}(G)$ *is well-defined, (i.e., uniquely defined).*

(ii) *There exists a triangle-free graph H such that $L(H) = \mathrm{cl}(G)$.*

(iii) *The length of a longest cycle in G and in $\mathrm{cl}(G)$ is the same.*

To obtain Theorem E (iii), Ryjáček actually proved the following. For an eligible vertex v of a claw-free graph G, let G_v be the graph obtained from G by local completion at v.

Proposition F (Ryjáček [18]) *Let G be a claw-free graph and v be an eligible vertex of G. If C' is a longest cycle of G_v, then G has a cycle C such that $V(C) = V(C')$.*

Proposition F might not hold for a cycle C' which is not a longest cycle of G_v. However, in the proof of Proposition F, only the maximality of $|V(C')|$ is used for the fact that $N_{G_v}(v) \cup \{v\} \subseteq V(C')$ if $E(G_v[N_{G_v}(v) \cup \{v\}]) \cap E(C') \neq \emptyset$. Therefore, the same argument can work in the proof of the following proposition.

Proposition 7 *Let G be a claw-free graph and v be an eligible vertex of G. If C' is a maximal cycle of G_v, then G has a maximal cycle C such that $V(C) = V(C')$.*

As a corollary of Proposition 7, we can obtain the following, where for convenience, we call a cycle C of a graph G a *Tutte maximal cycle* of G if C is a Tutte cycle and a maximal cycle of G. Note that if C' is a Tutte cycle of G_v, then C is a Tutte cycle of G for any cycle C in G such that $V(C) = V(C')$.

Corollary 8 *Let G be a claw-free graph. If $\mathrm{cl}(G)$ has a Tutte maximal cycle, then G has a Tutte maximal cycle.*

By the definition of a weakly Tutte edge-maximal closed trail, the following holds.

Proposition 9 *Let G be a claw-free graph, and let H be a graph such that $L(H) = \mathrm{cl}(G)$. If H has a weakly Tutte edge-maximal closed trail, then $L(H)$ has a Tutte maximal cycle.*

Proof of Proposition 9. Let T be a weakly Tutte edge-maximal closed trail of H and $H^* = H[V(T)] + E_H(T, H - T)$. Then by Lemma 1, $L(H^*)$ has a Hamilton cycle C which is a maximal cycle of $L(H)$. On the other hand, by the definition of a weakly Tutte closed trail, $e_H(F, T) \leq 3$ for all $F \in \mathcal{F}_H(T)$ (recall that $\mathcal{F}_H(T) = \{F : F$ is a component of $H - T$ such that $|V(F)| \geq 2\}$). Since $E_H(F) \cap E(H^*) = E_H(F, T)$ for $F \in \mathcal{F}_H(T)$, it follows that $|N_C(L(F))| = |E_H(F) \cap E(H^*)| = e_H(F, T) \leq 3$ for $F \in \mathcal{F}_H(T)$. Moreover, again by the definition of a weakly Tutte closed trail, $V(C) = E(H^*) = E_H(T) = E(H)$ or $|V(C)| = |E(H^*)| = |E_H(T)| \geq 4$ holds. This implies that C is a Tutte cycle of $L(H)$. Thus C is a Tutte maximal cycle of $L(H)$. □

Now we are ready to prove Theorem 4.

Proof of Theorem 4. Suppose that statement (A11) is true. Let G be a 2-connected claw-free graph. By Theorem E (ii), there exists a triangle-free graph H such that $L(H) = \mathrm{cl}(G)$. If $L(H)$ is complete, then $L(H)$ clearly has a Hamilton cycle, and hence by Theorem E (iii), G has a Hamilton cycle, that is, G has a Tutte maximal cycle. Thus we may assume that $L(H)$ is not complete, and hence H is essentially 2-edge-connected. Since we assumed that statement (A11) is true, H has a weakly Tutte edge-maximal closed trail. Then, by Proposition 9, $L(H)$ has a Tutte maximal cycle. Hence by Corollary 8, G has a Tutte maximal cycle. Thus statement (A10) is also true and this completes the proof of Theorem 4. □

5 Proof of Theorem 3

5.1 Set up for the proof of Theorem 3

At the end of this section, we will prove Theorem 3, that is, we will prove statement (A11) assuming (A1), by induction on the number of elements of $\mathcal{E}_2(H) \cup \mathcal{E}_3(H)$, where H is a given essentially 2-edge-connected multigraph. Therefore, we need the following for the first step of the induction. Here for a multigraph H and a subset S of $E(H) \cup V(H)$, a closed trail T of H is a called an *S-closed trail* if $S \subseteq E(T) \cup V(T)$. Furthermore, if T is a dominating closed trail (a weakly Tutte closed trail, respectively) and an S-closed trail of H, we call T a *dominating* (a *weakly Tutte*, respectively) *S-closed trail* of H.

Lemma 2 *Statement* (A1) *is equivalent to the following statement.*

(A14) *Every essentially 4-edge-connected multigraph H has a dominating $V_{\geq 4}(H)$-closed trail, i.e., H has a Tutte edge-maximal closed trail.*

Proof of Lemma 2. By Theorem C, it is easy to see that statement (A14) implies statement (A1). So it suffices to show the converse. Assume that statement (A1) is true. Then by Theorem C, every essentially 4-edge-connected multigraph has a dominating closed trail. Let H be an essentially 4-edge-connected multigraph. Let H^* be the graph obtained from H by adding a pendant edge to each vertex in $V_{\geq 4}(H)$. Then H^* is also essentially 4-edge-connected and $V_{\geq 4}(H^*) = V_{\geq 4}(H)$. By the assumption, H^* has a dominating closed trail T. Since each vertex in $V_{\geq 4}(H^*)$ is incident with a pendant edge, $V_{\geq 4}(H^*) \subseteq V(T)$. Therefore by the definition of H^*, since $V_{\geq 4}(H^*) = V_{\geq 4}(H)$, it follows that T is a dominating $V_{\geq 4}(H)$-closed trail of H. $\square$

We next prepare some results to show the case of $\mathcal{E}_2(H) = \emptyset$ and $\mathcal{E}_3(H) \neq \emptyset$. To show this case, we actually consider about weakly Tutte closed trails passing through specified vertices and edges. Before mentioning the statement, we prepare the following terminology. Let H be a multigraph. For three distinct edges e_1, e_2 and e_3 in H, (e_1, e_2, e_3) is called a 3-*star* of H if there exists a vertex u of H such that $d_H(u) = 3$, $u \in V(e_1) \cap V(e_2) \cap V(e_3)$ and $V(e_3) \setminus \{u\} \subseteq V_{\geq 3}(H)$, and u is called the *center* of (e_1, e_2, e_3).

(A15) *Let H be an essentially 4-edge-connected multigraph, and let (e_1, e_2, e_3) be a 3-star of H. Then H has a dominating $\big(\{e_1, e_2\} \cup V(e_3) \cup V_{\geq 4}(H)\big)$-closed trail.*

In order to consider statement (A15), we need the concept called "$V_2(H)$-*dominated*". A multigraph H is said to be $V_2(H)$-*dominated* if for any four distinct vertices u_1, u_2, v_1 and v_2 in H with $\{u_1, u_2, v_1, v_2\} = V_2(H)$, the graph $H + \{u_1u_2, v_1v_2\}$ has a dominating $\{u_1u_2, v_1v_2\}$-closed trail. The following was proven by Kužel [14].

Theorem G (Kužel [14]) *Statement* (A1) *is equivalent to the following statement.*

(A16) *Any subgraph H of an essentially 4-edge-connected cubic graph with $\delta(H) = 2$ and $|V_2(H)| = 4$ is $V_2(H)$-dominated.*

Actually, we show the following theorem in this section.

Theorem 10 *If statement* (A16) *is true, then statement* (A15) *is also true.*

We prove Theorem 10 in the next subsection and prove Theorem 3 in Subsections 5.3 and 5.4.

At the end of this subsection, we give another theorem as follows.

Theorem 11 *If statement* (A15) *is true, then statement* (A1) *is also true.*

Combining Theorem 11 with Theorems G and 10, statement (A1) is also equivalent to statement (A15). Note that it is not necessary to prove Theorem 11 for the proof of Theorem 10, but we prove it since it may be interesting by itself (see also Appendix). We will prove Theorem 11 in Section 6.

5.2 Proof of Theorem 10

In order to show Theorem 10, we need some concepts and results.

Let $k \geq 3$ be an integer, and let H be an essentially 3-edge-connected multigraph such that $L(H)$ is not complete. Note that $V_{\leq 2}(H)$ is an independent set of H. The *core* of a graph H denoted by $\mathrm{core}(H)$, is the graph obtained by recursively deleting all the vertices of degree 1, recursively deleting the vertex z with degree 2 in H and adding the edge xy with $N_H(z) = \{x, y\}$, and recursively deleting the created loops. It is easy to see that if H is an essentially k-edge-connected multigraph such that $L(H)$ is not complete, then $\mathrm{core}(H)$ is a 3-edge-connected essentially k-edge-connected multigraph (in particular, $\delta(\mathrm{core}(H)) \geq 3$). Moreover we can obtain the following.

Lemma 3 *Let H be an essentially 4-edge-connected multigraph such that $L(H)$ is not complete, and let $H^* = \mathrm{core}(H)$. Suppose that H^* has a dominating $V_{\geq 4}(H^*)$-closed trail T^*. Then H has a dominating $V_{\geq 4}(H)$-closed trail T which satisfies the following*:

- *If $xy \in E(T^*)$, then $xy \in E(T)$ or $xz, yz \in E(T)$ for some $z \in V_2(H)$.*

Proof of Lemma 3. By the definition of the core, for each $xy \in E(H^*)$, $xy \in E(H)$ or there exists a vertex z in $V_2(H)$ such that $xz, yz \in E(H)$. Let $X = \{e \in E(H^*) : e \notin E(H)\}$. For each $e = xy \in X$, let z_e be the vertex in $V_2(H)$ such that $N_H(z_e) = \{x, y\}$. Then by replacing e with a path xz_ey for each $e = xy \in E(T^*) \cap X$, we can obtain a closed trail T of H such that $V(T) = V(T^*) \cup \{z_e : e \in E(T^*) \cap X\}$ and $E(T) = \{xz_e, yz_e : e = xy \in E(T^*) \cap X\} \cup (E(T^*) \setminus X)$. Moreover, since $V_{\geq 4}(H^*) = V_{\geq 4}(H)$ by the definition of the core and the assumption, $V_{\geq 4}(H) = V_{\geq 4}(H^*) \subseteq V(T^*) \subseteq V(T)$. Therefore, to complete the proof, we have only to show that T is a dominating closed trail of H. Note that $|E(H)| \geq 5$ because H is essentially 4-edge-connected. Let $x \in V(H - T)$. Since $V(T^*) \subseteq V(T)$, $x \notin V(T^*)$. Suppose that $N_H(x) \not\subseteq V(T)$, and let $z \in N_H(x) \setminus V(T)$. If $\{x, z\} \subseteq V_{\geq 3}(H)$, then by the definition of the core, $\{x, z\} \subseteq V(H^*)$ and $xz \in E(H^*)$. Since $x, z \notin V(T^*)$, this contradicts that T^* is a dominating closed trail of H^*. Thus $\{x, z\} \cap V_{\leq 2}(H) \neq \emptyset$. Since H is essentially 4-edge-connected and $L(H)$ is not complete, we also have $\{x, z\} \cap V_{\geq 3}(H) \neq \emptyset$. Since $x, z \in V(H - T)$ and $xz \in E(H)$, we may assume that $x \in V_{\geq 3}(H)$ and $z \in V_{\leq 2}(H)$. Note that $x \in V_3(H)$ because $V_{\geq 4}(H) \subseteq V(T)$, and hence $E_H(xz) \setminus \{xz\} \in \mathcal{E}_2(H) \cup \mathcal{E}_3(H)$, a contradiction. Thus $N_H(x) \subseteq V(T)$. Since x is an arbitrary vertex in $H - T$, this implies that T is a dominating closed trail of H. $\Box$

We also need the following operation (see [9] for more details). Let H be a multigraph and $z \in V_{\geq 4}(H)$, and let $u_1, u_2, \ldots, u_d$ ($d = d_H(z)$) be an ordering of neighbors of z (we allow repetition in case of parallel edges). Then the graph obtained from the disjoint union of $H - z$ and the cycle $C_z = z_1 z_2 \ldots z_d z_1$ by adding the edges $u_i z_i$ for each i with $1 \leq i \leq d$ is called an *inflation* of H at z. If $\delta(H) \geq 3$, then, by successively taking an inflation at each vertex of degree greater than 3, we can obtain a cubic graph H^I, called a *cubic inflation* of H. An inflation of a graph at a vertex is not unique (since it depends on the ordering of neighbors of z) and the operation may decrease the edge-connectivity. However, the following was proven in [9].

Theorem H (Fleischner and Jackson [9]) *Let H be an essentially 4-edge-connected multigraph with $\delta(H) \geq 3$. Then some cubic inflation of H is also essentially 4-edge-connected.*

Let H^I be a cubic inflation of a multigraph H and for each $z \in V(H)$, set $I(z) = V(C_z)$ if $z \in V_{\geq 4}(H)$; otherwise, set $I(z) = \{z\}$. Observing that a dominating cycle in H^I must contain at least one vertex in $I(z)$ for each $z \in V_{\geq 4}(H)$, we immediately obtain the following fact (which is implicit in [9]).

Lemma I (Fleischner and Jackson [9]) *Let H be a multigraph with $\delta(H) \geq 3$, and let H^I be a cubic inflation of H. Suppose that H^I has a dominating cycle C. Then H has a dominating $V_{\geq 4}(H)$-closed trail T which satisfies the following*:

- *If $uv \in E(C)$ with $u \in I(x)$ and $v \in I(y)$ for some $x, y \in V(H)$ $(x \neq y)$, then $xy \in E(T)$.*

Now we are ready to prove Theorem 10.

Proof of Theorem 10. Suppose that statement (A16) is true. Let H be an essentially 4-edge-connected multigraph, and let (e_1, e_2, e_3) be a 3-star of H (note that $V(e_3) \subseteq V_{\geq 3}(H)$ and that $V(e_1) \cup V(e_2) \subseteq V_{\geq 2}(H)$ because H is essentially 4-edge-connected). We will find a dominating $\big(\{e_1, e_2\} \cup V(e_3) \cup V_{\geq 4}(H)\big)$-closed trail of H.

If $L(H)$ is complete, then we can easily see that (i) H is a star such that $V(e_1) = V(e_2) = V(e_3)$, or (ii) H is a triangle such that e_3 is a unique simple edge in H or $V(e_i) = V(e_3)$ and $V(e_{3-i}) \neq V(e_3)$ for some i with $i \in \{1, 2\}$. In either case, clearly H has a spanning closed trail T such that $\{e_1, e_2\} \subseteq E(T)$, that is, H has a desired closed trail.

Thus we may assume that $L(H)$ is not complete. Let u be the center of (e_1, e_2, e_3). Let $H^* = \text{core}(H)$. Then H^* is an essentially 4-edge-connected multigraph with $\delta(H^*) \geq 3$. Note that $e_3 \in E(H^*)$ since $V(e_3) \subseteq V_{\geq 3}(H)$. Let e_1^* and e_2^* be two distinct edges incident with u in H^* such that $e_i^* \neq e_3$ for $i \in \{1, 2\}$, and let $e_3^* = e_3$. Let $V(e_i^*) \setminus \{u\} = \{v_i\}$ for $1 \leq i \leq 3$. Note that (e_1^*, e_2^*, e_3^*) is a 3-star with center u of H^*.

By Theorem H, there exists a cubic inflation H^I of H^* such that H^I is essentially 4-edge-connected. Note that H^I is a simple graph. Note also that by the definition of a 3-star, $I(u) = \{u\}$. For each i with $1 \leq i \leq 3$, let $v_i' \in I(v_i)$ such that $uv_i' \in E(H^I)$ (possibly $v_i' = v_i$). We claim that H^I has a dominating cycle containing uv_1', uv_2' and v_3'. Since H^I is essentially 4-edge-connected, if $v_k'v_l' \in E(H^I)$ for some k and l with $1 \leq k < l \leq 3$, then it is easy to check that $H^I \cong K_4$, and hence H^I has a desired dominating cycle. Thus we may assume that $v_k'v_l' \notin E(H^I)$ for each k, l with $1 \leq k < l \leq 3$.

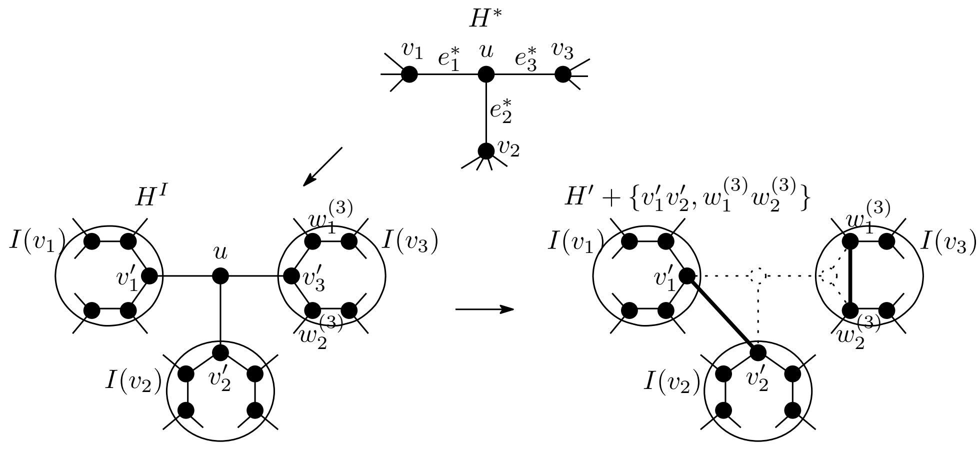

Figure 1: The subgraph H' of H^I

Let $N_{H^I}(v_3') \setminus \{u\} = \{w_1^{(3)}, w_2^{(3)}\}$. Then since $H' := H^I - \{u, v_3'\}$ is a subgraph of H^I such that $\delta(H') = 2$ and $V_2(H') = \{v_1', v_2', w_1^{(3)}, w_2^{(3)}\}$ and we assumed that statement (A16) is true, it follows that $H' + \{v_1'v_2', w_1^{(3)}w_2^{(3)}\}$ has a dominating cycle C' containing $v_1'v_2'$ and $w_1^{(3)}w_2^{(3)}$ (see Figure 1). Hence $(C' - \{v_1'v_2', w_1^{(3)}w_2^{(3)}\}) + \{uv_1', uv_2', v_3'w_1^{(3)}, v_3'w_2^{(3)}\}$ is a desired dominating cycle of H^I. Thus the assertion holds. Then by Lemma I, H^* has a dominating $\big(\{e_1^*, e_2^*\} \cup V(e_3^*) \cup V_{\geq 4}(H^*)\big)$-closed trail. Hence by Lemma 3 and the definition of e_1^*, e_2^* and e_3^*, H has a dominating $\big(\{e_1, e_2\} \cup V(e_3) \cup V_{\geq 4}(H)\big)$-closed trail. Therefore, statement (A15) is true, and this completes the proof of Theorem 10. $\square$

5.3 Preparation for the proof of Theorem 3

In this subsection, we prepare some technical lemmas to prove Theorem 3.

In the proof of Theorem 3, we will restrict maximal cycles on H to some component. To show that the resulting graph is a weakly Tutte closed trail, we use the following lemma.

Lemma 4 *Let H be a multigraph, and let T be a weakly Tutte closed trail of H. If T' is a closed trail of H such that $E_H(T') = E_H(T)$, then T' is also a weakly Tutte closed trail of H.*

Proof of Lemma 4. Let T' be a closed trail of H such that $E_H(T') = E_H(T)$, and suppose that T' is not a weakly Tutte closed trail of H. Then there exists $F' \in \mathcal{F}_H(T')$ such that $e_H(F', T') \geq 4$. (Recall that $\mathcal{F}_H(T') = \{F : F \text{ is a component of } H - T' \text{ such that } |V(F)| \geq 2\}$.) Write $E_H(F', T') = \{e_1, \ldots, e_l\}$ $(l \geq 4)$. Since $E_H(F', T') \subseteq E_H(T') = E_H(T)$, it follows that $V(T) \cap \big(\bigcup_{i=1}^{l} V(e_i)\big) \neq \emptyset$. Let $S = V(T) \cap \big(\bigcup_{i=1}^{l} V(e_i)\big)$. If $S \cap V(F') \neq \emptyset$, then $E(F') \cap E_H(T) \neq \emptyset$, and this contradicts the assumption that $E_H(T') = E_H(T)$. Thus $S \subseteq V(T')$. Since $\{e_1, \ldots, e_l\} = E_H(F', T') \subseteq E_H(T)$, this implies that $S = V(T') \cap \big(\bigcup_{i=1}^{l} V(e_i)\big)$, and hence F'

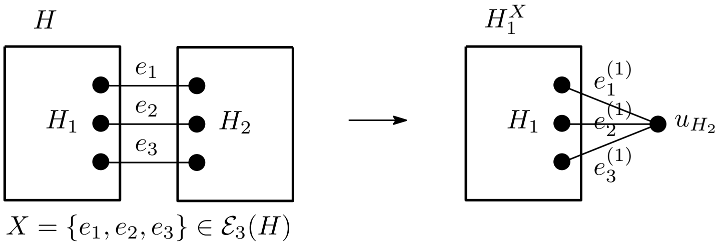

Figure 2: The graph H_1^X

is also a component of $H - T$ and $E_H(F', T') = E_H(F', T)$, which contradicts the assumption that T is a weakly Tutte closed trail of H. $\square$

In the rest of this subsection, we let k be an integer with $2 \leq k \leq 3$, and let H be an essentially k-edge-connected multigraph.

We prepare the following notation. We define $\mathcal{T}_k(H) = \{(X, H_1, H_2) : X \in \mathcal{E}_k(H)$ and, H_1 and H_2 are distinct components of $G - X\}$, and let $(X, H_1, H_2) \in \mathcal{T}_k(H)$. We further define two graphs H_1^X and H_2^X as follows. For each i with $i \in \{1, 2\}$, let H_i^X be the graph obtained from H by contracting H_{3-i} to a vertex $u_{H_{3-i}}$. Note that H_i^X is also an essentially k-edge-connected multigraph. If $X = \{e_1, \dots, e_k\}$, then for each i, j with $1 \leq i \leq 2$ and $1 \leq j \leq k$, let $e_j^{(i)}$ be the edge in H_i^X corresponding to e_j (see Figure 2).

Now we fix the following notation. Let $(X, H_1, H_2) \in \mathcal{T}_k(H)$, and write $X = \{e_1, \dots, e_k\}$.

Lemma 5 *Let $1 \leq i \leq 2$. If H_i^X has a weakly Tutte edge-maximal closed trail T_i such that $E(T_i) \cap \{e_1^{(i)}, \dots, e_k^{(i)}\} = \emptyset$, then T_i is a weakly Tutte edge-maximal closed trail of H, or H_i^X has a weakly Tutte edge-maximal closed trail R_i such that $E(R_i) \cap \{e_1^{(i)}, \dots, e_k^{(i)}\} \neq \emptyset$.*

Proof of Lemma 5. We may assume that $i = 1$. Note that T_1 is a weakly Tutte closed trail of H because $E(T_1) \cap \{e_1^{(1)}, \dots, e_k^{(1)}\} = \emptyset$. Suppose that T_1 is not a weakly Tutte edge-maximal closed trail of H. Then there exists an edge-maximal closed trail T of H such that $E_H(T_1) \subsetneq E_H(T)$. Note that $E(T) \cap X \neq \emptyset$ because T_1 is an edge-maximal closed trail of H_1^X such that $E(T_1) \cap \{e_1^{(1)}, \dots, e_k^{(1)}\} = \emptyset$. Note also that $|E(T) \cap X| = 2$ because $2 \leq k \leq 3$. We may assume that $E(T) \cap X = \{e_1, e_2\}$, and let $R_1 = (T - V(H_2)) + \{e_1^{(1)}, e_2^{(1)}\}$. Then R_1 is a closed trail of H_1^X. Since $E_H(T_1) \subseteq E_H(T)$, it follows that $E_{H_1^X}(T_1) \setminus \{e_1^{(1)}, \dots, e_k^{(1)}\} = E_H(T_1) \cap E(H_1) \subseteq E_H(T) \cap E(H_1)$. Moreover, since $u_{H_2} \in V(R_1)$, we also have $\{e_1^{(1)}, \dots, e_k^{(1)}\} \subseteq E_{H_1^X}(R_1)$, and hence $(E_H(T) \cap E(H_1)) \cup \{e_1^{(1)}, \dots, e_k^{(1)}\} = E_{H_1^X}(R_1)$. These imply that $E_{H_1^X}(T_1) \subseteq E_{H_1^X}(R_1)$. Since T_1 is an edge-maximal closed trail of H_1^X, it follows that $E_{H_1^X}(T_1) = E_{H_1^X}(R_1)$, and hence R_1 is also an edge-maximal closed trail of H_1^X. Furthermore, since T_1 is a weakly Tutte closed trail of H_1^X and $E_{H_1^X}(T_1) = E_{H_1^X}(R_1)$, it follows from Lemma 4 that R_1 is also a weakly Tutte closed trail of H_1^X. Thus R_1 is a weakly Tutte edge-maximal closed trail of H_1^X such that

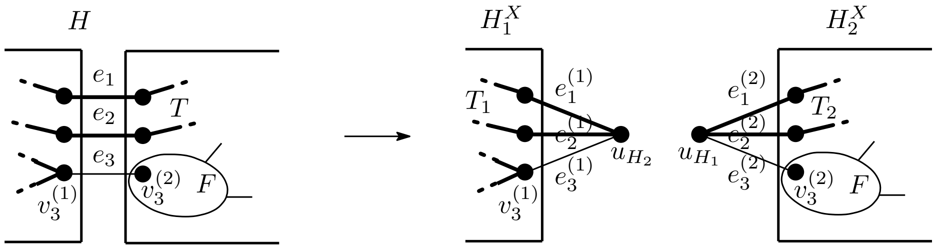

Figure 3: The component F of $H - T$

$E(R_1) \cap \{e_1^{(1)}, \ldots, e_k^{(1)}\} \neq \emptyset$. $\square$

We further fix the following notation in the following three lemmas (Lemmas 6 through 8). Let $e_i = v_i^{(1)} v_i^{(2)}$ with $v_i^{(1)} \in V(H_1)$ and $v_i^{(2)} \in V(H_2)$ for $1 \leq i \leq k$. Let l_1 and l_2 be integers with $1 \leq l_1 < l_2 \leq k$, and let T_i be a $\{e_{l_1}^{(i)}, e_{l_2}^{(i)}\}$-closed trail of H_i^X for $i \in \{1, 2\}$ and $T = \big((T_1 - u_{H_2}) \cup (T_2 - u_{H_1})\big) + \{e_{l_1}, e_{l_2}\}$.

Lemma 6 *If T_i is a weakly Tutte closed trail of H_i^X for each i with $i \in \{1, 2\}$, and $\{v_1^{(j)}, \ldots, v_k^{(j)}\} \subseteq V(T_j)$ for some j with $j \in \{1, 2\}$, then T is a weakly Tutte closed trail of H.*

Proof of Lemma 6. We may assume that $l_1 = 1$ and $l_2 = 2$, and hence $\{v_1^{(i)}, v_2^{(i)}\} \subseteq V(T_i)$ for $i \in \{1, 2\}$. By the symmetry of T_1 and T_2, we may also assume that $j = 1$, i.e., $\{v_1^{(1)}, \ldots, v_k^{(1)}\} \subseteq V(T_1)$. Let F be a component of $H - T$. Since $\{v_1^{(1)}, \ldots, v_k^{(1)}\} \subseteq V(T_1) \setminus \{u_{H_2}\} \subseteq V(T)$ and $\{e_1, e_2\} \subseteq E(T)$, we see that if $v_k^{(2)} \notin V(F)$, then F is a component of $H_i^X - T_i$ for some i with $i \in \{1, 2\}$, and hence $E_H(F, T) = E_{H_i^X}(F, T_i)$ for some i with $i \in \{1, 2\}$; if $v_k^{(2)} \in V(F)$ (note that in this case, $k = 3$), then F is a component of $H_2^X - T_2$ and $e_k^{(2)} \in E_{H_2^X}(F, T_2)$, and hence $E_H(F, T) = \big(E_{H_2^X}(F, T_2) \setminus \{e_k^{(2)}\}\big) \cup \{e_k\}$ (see Figure 3). Since T_i is a weakly Tutte closed trail of H_i^X for each i with $i \in \{1, 2\}$, this implies that T is a weakly Tutte closed trail of H. $\square$

Lemma 7 *If T_i is an edge-maximal closed trail of H_i^X for each i with $i \in \{1, 2\}$ and $\{v_1^{(j)}, \ldots, v_k^{(j)}\} \subseteq V(T_j)$ for some j with $j \in \{1, 2\}$, then T is an edge-maximal closed trail of H.*

Proof of Lemma 7. If $j = 1$, then let $A = \{v_1^{(1)}, \ldots, v_k^{(1)}\}$; otherwise, let $A = \{v_1^{(2)}, \ldots, v_k^{(2)}\}$.

Suppose that T is not an edge-maximal closed trail of H. Then there exists an edge-maximal closed trail T' of H such that $E_H(T) \subsetneq E_H(T')$. Note that $E(T') \cap X \neq \emptyset$. Let m_1 and m_2 be integers with $1 \leq m_1 < m_2 \leq k$ such that $E(T') \cap X = \{e_{m_1}, e_{m_2}\}$. For each i with $i \in \{1, 2\}$, let $R_i = (T' - V(H_{3-i})) + \{e_{m_1}^{(i)}, e_{m_2}^{(i)}\}$. Then R_i is a closed trail of H_i^X for $i \in \{1, 2\}$.

We here show that $E_{H_i^X}(T_i) \setminus \{e_1^{(i)}, \ldots, e_k^{(i)}\} = E_{H_i^X}(R_i) \setminus \{e_1^{(i)}, \ldots, e_k^{(i)}\}$ for $i \in \{1, 2\}$. Since $E_H(T) \subseteq E_H(T')$, we see that $E_{H_1^X}(T_1) \setminus \{e_1^{(1)}, \ldots, e_k^{(1)}\} = E_H(T) \cap E(H_1) \subseteq E_H(T') \cap E(H_1) = E_{H_1^X}(R_1) \setminus \{e_1^{(1)}, \ldots, e_k^{(1)}\}$. Since $u_{H_2} \in V(T_1) \cap V(R_1)$, we also have $\{e_1^{(1)}, \ldots, e_k^{(1)}\} \subseteq E_{H_1^X}(T_1) \cap$

$E_{H_1^X}(R_1)$. These imply that $E_{H_1^X}(T_1) \subseteq E_{H_1^X}(R_1)$. Since T_1 is an edge-maximal closed trail of H_1^X, it follows that $E_{H_1^X}(T_1) = E_{H_1^X}(R_1)$, i.e., $E_{H_1^X}(T_1) \setminus \{e_1^{(1)}, \dots, e_k^{(1)}\} = E_{H_1^X}(R_1) \setminus \{e_1^{(1)}, \dots, e_k^{(1)}\}$. Similarly, we have $E_{H_2^X}(T_2) \setminus \{e_1^{(2)}, \dots, e_k^{(2)}\} = E_{H_2^X}(R_2) \setminus \{e_1^{(2)}, \dots, e_k^{(2)}\}$.

Since $A \subseteq (V(T_1) \setminus \{u_{H_2}\}) \cup (V(T_2) \setminus \{u_{H_1}\}) = V(T)$, it follows that $X \subseteq E_H(T)$. This implies that $X \subseteq E_H(T')$. Thus we obtain $E_H(T) = (E_{H_1^X}(T_1) \setminus \{e_1^{(1)}, \dots, e_k^{(1)}\}) \cup (E_{H_2^X}(T_2) \setminus \{e_1^{(2)}, \dots, e_k^{(2)}\}) \cup X = (E_{H_1^X}(R_1) \setminus \{e_1^{(1)}, \dots, e_k^{(1)}\}) \cup (E_{H_2^X}(R_2) \setminus \{e_1^{(2)}, \dots, e_k^{(2)}\}) \cup X = E_H(T')$, a contradiction. $\square$

We call $(X, H_1, H_2) \in \mathcal{T}_k(H)$ a *minimal* 3*-tuple* of H if there exists no $X' \in \mathcal{E}_k(H)$ such that $H - X'$ has a component H_2' such that $V(H_2') \subsetneq V(H_2)$. Then by the definition of a minimal 3-tuple, we can obtain the following.

Lemma 8 *Suppose that $k = 3$ and (X, H_1, H_2) is a minimal 3-tuple of H. If $d_H(v_j^{(2)}) = 2$ for some j with $1 \le j \le 3$, then H_2 is isomorphic to K_2.*

Proof of Lemma 8. We may assume that $j = 3$. Since H is essentially 3-edge-connected, $X \in \mathcal{E}_3(H)$ and $d_H(v_3^{(2)}) = 2$, it follows that there exists a unique vertex v' in $N_H(v_3^{(2)}) \cap V(H_2)$. Note that $v' \in V_{\ge 3}(H)$ and $H_2 - v_3^{(2)}$ is connected. Then $X' := \{e_1, e_2, v_3^{(2)}v'\}$ is an edge-cut of H, and $H_1 + \{e_3\}$ and $H_2 - v_3^{(2)}$ are components of $H - X'$. Therefore, since (X, H_1, H_2) is a minimal 3-tuple of H, we have $|V(H_2 - v_3^{(2)})| = 1$. $\square$

5.4 Proof of Theorem 3

We finally prove Theorem 3.

Proof of Theorem 3. Assume that statement (A1) is true. Let H be an essentially 2-edge-connected multigraph. We show that H has a weakly Tutte edge-maximal closed trail by induction on $g(H) := |\mathcal{E}_2(H) \cup \mathcal{E}_3(H)|$. If $g(H) = 0$, then H is essentially 4-edge-connected. By the assumption that statement (A1) is true and Lemma 2, H has a desired closed trail, and we are done. Thus we may assume that $\alpha := g(H) \ge 1$.

By way of a contradiction, suppose that

$$H \text{ has no weakly Tutte edge-maximal closed trail.} \tag{1}$$

We define $V_{\ge 4}^*(H) = \{v \in V_{\ge 4}(H) : v \text{ is incident to a pendant edge in } H\}$, and we choose H so that $|V_{\ge 4}(H) \setminus V_{\ge 4}^*(H)|$ is as small as possible, subject to the condition $g(H) = \alpha$. Then by the choice of H, the following holds.

Claim 1 *If uv is an edge with $d_H(u) \ge 4$ and $d_H(v) = 2$, then $u \in V_{\ge 4}^*(H)$.*

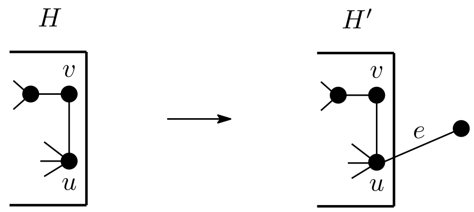

Figure 4: The graph H'

Proof. Suppose that $u \notin V^*_{\geq 4}(H)$. Let H' be the graph obtained from H by adding a pendant edge e to the vertex u (see Figure 4). By the definition of H' and since $d_H(u) \geq 3$ and $N_H(u) \subseteq V_{\geq 2}(H)$, it follows that H' is an essentially 2-edge-connected multigraph such that $g(H') = \alpha$ and $|V_{\geq 4}(H') \setminus V^*_{\geq 4}(H')| < |V_{\geq 4}(H) \setminus V^*_{\geq 4}(H)|$. Hence by the choice of H, H' has a weakly Tutte edge-maximal closed trail T'. By the definition of H' and since e is a pendant edge of H', we see that T' is also a closed trail of H. Moreover, for each component F of $H - T'$, if $u \notin V(F)$, then by the definition of H', F is also a component of $H' - T'$ and $E_{H'}(F, T') = E_H(F, T')$; otherwise, $F + e$ is a component of $H' - T'$ and $E_{H'}(F + e, T') = E_H(F, T')$. Since T' is a weakly Tutte closed trail of H', this implies that T' is also a weakly Tutte closed trail of H. Hence by (1), T' is not an edge-maximal closed trail of H. This in particular implies that

$$T' \text{ passes through the vertex } u \tag{2}$$

(otherwise, T' is an edge-maximal closed trail of H because $e \notin E_{H'}(T')$, a contradiction).

Let T be an edge-maximal closed trail of H such that $E_H(T') \subsetneq E_H(T)$. Note that T is also a closed trail of H'. Note also that

$$E_H(T) = E_{H'}(T) \setminus \{e\} \text{ and } E_H(T') = E_{H'}(T') \setminus \{e\}. \tag{3}$$

Since $uv \in E_H(T')$ by (2), we have $uv \in E_H(T)$. Since $d_H(v) = 2$, this implies that T also passes through the vertex u, and hence $e \in E_{H'}(T) \cap E_{H'}(T')$. Combining this with (3) and the fact that $E_H(T') \subsetneq E_H(T)$, we see that

$$E_{H'}(T') = E_H(T') \cup \{e\} \subsetneq E_H(T) \cup \{e\} = E_{H'}(T).$$

This contradicts that T' is an edge-maximal closed trail of H'. □

Suppose first that $\mathcal{E}_2(H) \neq \emptyset$, let $(X, H_1, H_2) \in \mathcal{T}_2(H)$ and write $X = \{e_1, e_2\}$. Then H_i^X is also essentially 2-edge-connected and $g(H_i^X) < \alpha$ for each i with $i \in \{1, 2\}$. Hence by the induction hypothesis, H_i^X has a weakly Tutte edge-maximal closed trail T_i for $i \in \{1, 2\}$. By Lemma 5 and (1), we may assume that $E(T_i) \cap \{e_1^{(i)}, e_2^{(i)}\} \neq \emptyset$ for $i \in \{1, 2\}$, and hence $\{e_1^{(i)}, e_2^{(i)}\} \subseteq E(T_i)$ for $i \in \{1, 2\}$. Then by Lemmas 6 and 7, $T := \big((T_1 - u_{H_2}) \cup (T_2 - u_{H_1})\big) + \{e_1, e_2\}$ is a weakly Tutte edge-maximal closed trail of H, which contradicts (1). Thus $\mathcal{E}_2(H) = \emptyset$.

Let (X, H_1, H_2) be a minimal 3-tuple of H in $\mathcal{T}_3(H)$. Write $X = \{e_1, e_2, e_3\}$ and $e_i = v_i^{(1)} v_i^{(2)}$ with $v_i^{(1)} \in V(H_1)$ and $v_i^{(2)} \in V(H_2)$ for each i with $1 \leq i \leq 3$. Note that H_1^X is also essentially 3-edge-connected, and $g(H_1^X) < \alpha$, and hence by the induction hypothesis, H_1^X has a weakly Tutte edge-maximal closed trail T_1. Note that by Lemma 5 and (1), T_1 passes through exactly two edges in $\{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\}$.

We divide the proof into two cases.

Case 1. $d_{H_2^X}(v_j^{(2)}) \geq 3$ for each j with $1 \leq j \leq 3$.

We may assume that $E(T_1) \cap \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\} = \{e_1^{(1)}, e_2^{(1)}\}$. By the assumption of Case 1, $(e_1^{(2)}, e_2^{(2)}, e_3^{(2)})$ is a 3-star with center u_{H_1} in H_2^X. Moreover, by the definition of a minimal 3-tuple and since $\mathcal{E}_2(H) = \emptyset$, H_2^X is essentially 4-edge-connected. Since we assumed that statement (A1) is true, it follows from Theorems G and 10 that statement (A15) is also true. Thus H_2^X has a dominating $\big(\{e_1^{(2)}, e_2^{(2)}\} \cup V(e_3^{(2)}) \cup V_{\geq 4}(H_2^X)\big)$-closed trail T_2, i.e., T_2 is a weakly Tutte edge-maximal closed trail of H_2^X, $\{e_1^{(2)}, e_2^{(2)}\} \subseteq E(T_2)$ and $\{v_1^{(2)}, v_2^{(2)}, v_3^{(2)}\} \subseteq V(T_2)$. Hence by Lemmas 6 and 7, $T := \big((T_1 - u_{H_2}) \cup (T_2 - u_{H_1})\big) + \{e_1, e_2\}$ is a weakly Tutte edge-maximal closed trail of H, which contradicts (1).

Case 2. $d_{H_2^X}(v_j^{(2)}) \leq 2$ for some j with $1 \leq j \leq 3$.

We may assume that $d_{H_2^X}(v_3^{(2)}) \leq 2$. Then by the denition of H_2^X and since $X \in \mathcal{E}_3(H)$, $d_H(v_3^{(2)}) = d_{H_2^X}(v_3^{(2)}) = 2$. Hence by Lemma 8, $H_2 \cong K_2$, i.e., $v_1^{(2)} = v_2^{(2)}$ and $v_1^{(2)} \neq v_3^{(2)}$. Note that $v_3^{(1)} \in V_{\geq 3}(H)$.

Let T_1 be a weakly Tutte edge-maximal closed trail of H_1^X and l be an integer with $1 \leq l \leq 3$ such that $e_l^{(1)} \notin E(T_1)$. If $l \in \{1, 2\}$, then let $T_2 = H_2^X - \{e_l^{(2)}\}$; otherwise, let $T_2 = H_2^X - v_3^{(2)}$. Note that T_2 is a dominating closed trail of H_2^X, i.e., T_2 is a weakly Tutte edge-maximal closed trail. We further let $T = \big((T_1 - u_{H_2}) \cup (T_2 - u_{H_1})\big) + (X \setminus \{e_l\})$ (for example, see Figure 5). We choose T_1 so that $v_3^{(1)} \in V(T_1)$ if possible.

If $l \in \{1, 2\}$, then by the definition of T_2, $\{v_1^{(2)}, v_2^{(2)}, v_3^{(2)}\} \subseteq V(T_2)$, and hence by Lemmas 6 and 7, T is a weakly Tutte edge-maximal closed trail of H, which contradicts (1). Thus $l = 3$. In particular, $\{v_1^{(1)}, v_2^{(1)}\} \subseteq V(T_1)$. Therefore, if $v_3^{(1)} \in V(T_1)$, then again by Lemmas 6 and 7, T is a weakly Tutte edge-maximal closed trail of H, which contradicts (1) again. Thus $v_3^{(1)} \notin V(T_1)$.

Since $v_3^{(1)} \notin V(T_1)$, it follows from the definition of T that

$$E_H(T) = (E_{H_1^X}(T_1) \setminus \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\}) \cup \{e_1, e_2, v_1^{(2)} v_3^{(2)}\} \text{ (see the bottom part of Figure 5).} \tag{4}$$

Since $v_1^{(2)} \in V(T)$ and $d_H(v_3^{(2)}) = 2$, we further see that for a component F of $H - T$,

the following (i)–(iii) are equivalent (see the bottom part of Figure 5):
(i) $v_3^{(1)} \in V(F)$, (ii) $v_3^{(2)} \in V(F)$, and (iii) $v_1^{(2)} v_3^{(2)} \in E_H(F, T)$. (5)

By the definition of T and (5), we can also obtain the following.

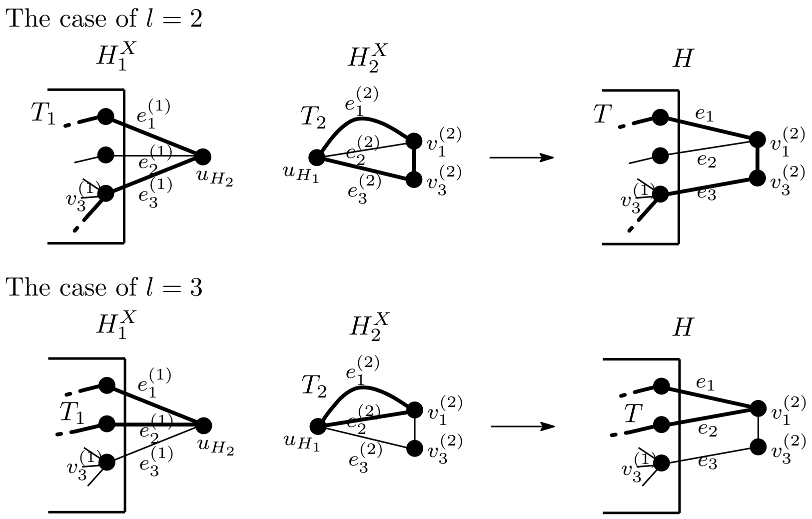

Figure 5: The closed trails T_1, T_2 and T

Claim 2 *T is a weakly Tutte closed trail of H.*

Proof. We show that $e_H(F,T) \leq 3$ for $F \in \mathcal{F}_H(T)$ (recall that $\mathcal{F}_H(T) = \{F : F$ is a component of $H - T$ such that $|V(F)| \geq 2\}$). Let $F \in \mathcal{F}_H(T)$. Assume for the moment that $v_3^{(1)} \notin V(F)$. Then by the definition of T and (5), F is also a component of $H_1^X - T_1$ such that $F \in \mathcal{F}_{H_1^X}(T_1)$ and $E_H(F,T) = E_{H_1^X}(F,T_1)$. Since T_1 is a weakly Tutte closed trail of H_1^X, we have $e_H(F,T) = e_{H_1^X}(F,T_1) \leq 3$. Thus we may assume that $v_3^{(1)} \in V(F)$. Then again by the definition of T and (5), $F - v_3^{(2)}$ is a component of $H_1^X - T_1$ and $E_{H_1^X}(F - v_3^{(2)}, T_1) = (E_H(F,T) \setminus \{v_1^{(2)}v_3^{(2)}\}) \cup \{e_3^{(1)}\}$; in particular, $e_{H_1^X}(F - v_3^{(2)}, T_1) = e_H(F,T)$. Thus we may assume that $|V(F - v_3^{(2)})| = 1$, i.e., $V(F - v_3^{(2)}) = \{v_3^{(1)}\}$ (otherwise, $F - v_3^{(2)} \in \mathcal{F}_{H_1^X}(T_1)$, and hence $e_H(F,T) = e_{H_1^X}(F - v_3^{(2)}, T_1) \leq 3$). Then $v_3^{(1)}$ is not incident to a pendant edge. Since $v_3^{(1)}v_3^{(2)} \in E(H)$ and $d_H(v_3^{(2)}) = 2$, this together with Claim 1 implies that $d_H(v_3^{(1)}) = 3$. Thus $e_H(F,T) = e_{H_1^X}(F - v_3^{(2)}, T_1) = d_{H_1^X}(v_3^{(1)}) = d_H(v_3^{(1)}) = 3$. Therefore T is a weakly Tutte closed trail of H. $\square$

By Claim 2 and (1), T is not an edge-maximal closed trail of H. Let T' be an edge-maximal closed trail of H such that $E_H(T) \subsetneq E_H(T')$. Note that $|E(T') \cap X| = 2$ because $v_1^{(2)}v_3^{(2)} \in E_H(T) \subseteq E_H(T')$. Let l_1 and l_2 be integers with $1 \leq l_1 < l_2 \leq 3$ such that $E_H(T') \cap X = \{e_{l_1}, e_{l_2}\}$. Let $R_1 = (T' - V(H_2)) + \{e_{l_1}^{(1)}, e_{l_2}^{(1)}\}$. Then R_1 is a closed trail of H_1^X. Since $E_H(T) \subseteq E_H(T')$, it follows that $E_{H_1^X}(T_1) \setminus \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\} = E_H(T) \cap E(H_1) \subseteq E_H(T') \cap E(H_1) = E_{H_1^X}(R_1) \setminus \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\}$. Since $u_{H_2} \in V(T_1) \cap V(R_1)$, we also have $\{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\} \subseteq E_{H_1^X}(T_1) \cap E_{H_1^X}(R_1)$. These imply that $E_{H_1^X}(T_1) \subseteq E_{H_1^X}(R_1)$. Since T_1 is an edge-maximal closed trail of H_1^X, we have $E_{H_1^X}(T_1) = E_{H_1^X}(R_1)$. Hence by Lemma 4 and the fact that T_1 is a weakly Tutte closed trail of H_1^X, we see that R_1 is a weakly Tutte closed trail of H_1^X.

Thus R_1 is a weakly Tutte edge-maximal closed trail of H_1^X, and it follows from the choice of T_1 that $\{l_1, l_2\} = \{1, 2\}$ and $v_3^{(1)} \notin V(R_1)$. Combining this with the definition of R_1, we get $E(T') \cap X = \{e_1, e_2\}$ and $v_3^{(1)} \notin V(T')$. Therefore we have

$$E_H(T') = (E_{H_1^X}(R_1) \setminus \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\}) \cup \{e_1, e_2, v_1^{(2)} v_3^{(2)}\}. \tag{6}$$

Since $E_{H_1^X}(T_1) \setminus \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\} = E_{H_1^X}(R_1) \setminus \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\}$, it follows from (4) and (6) that $E_H(T) = E_H(T')$, which contradicts the fact that $E_H(T) \subsetneq E_H(T')$.

This completes the proof of Theorem 3. $\square$

6 Proofs of Theorems 5 and 11

As mentioned in the paragraph following Theorem 5 and the paragraph following Theorem 11 in Section 3 and Subsection 5.1, respectively, we prove Theorems 5 and 11 in this section.

Proof of Theorem 5. Assume that statement (A12) is true. Let H be an essentially 2-edge-connected multigraph. Let H^* be the graph obtained from H by adding a pendant edge to each vertex in $V_{\geq 4}(H)$. Then H^* is also essentially 2-edge-connected and $V_{\geq 4}(H^*) = V_{\geq 4}(H)$. Since we assumed that statement (A12) is true, H^* has a weakly Tutte closed trail T. Then by the definition of H^*, T is also a weakly Tutte closed trail of H. We show that T is a Tutte closed trail of H. Suppose that T is not a Tutte closed trail of H. Since T is a weakly Tutte closed trail of H, there exists a component F of $H - T$ such that $|V(F)| = 1$, say $V(F) = \{x\}$, and $x \in V_{\geq 4}(H)$. Then by the definition of H^*, there exists a vertex y in $N_{H^*}(x) \cap V_1(H^*)$. Since $x \notin V(T)$ and $V(H - T) \subseteq V(H^* - T)$, we see that xy is a graph in $\mathcal{F}_{H^*}(T)$ such that $e_{H^*}(\{x, y\}, T) = d_H(x) \geq 4$, which contradicts that T is a weakly Tutte closed trail of H^*. Thus T is a Tutte closed trail of H^*. Hence statement (A13) is also true, and this completes the proof of Theorem 5. $\square$

Proof of Theorem 11. By Lemma 2, it is enough to show that statement (A15) implies statement (A14). Assume that statement (A15) is true. Let H be an essentially 4-edge-connected multigraph. We will find a dominating $V_{\geq 4}(H)$-closed trail. If $L(H)$ is complete, then H is a star or a triangle, and hence we can easily see that H has a desired dominating closed trail. Thus, we may assume that $L(H)$ is not complete.

Then $H^* := \mathrm{core}(H)$ is an essentially 4-edge-connected graph with $\delta(H^*) \geq 3$. By Theorem H, there exists a cubic inflation H^I of H^* such that H^I is essentially 4-edge-connected. Since we assumed that statement (A15) is true, taking any vertex in H^I as the center of a 3-star, we can find a dominating cycle of H^I. By Lemma I, H^* has a dominating $V_{\geq 4}(H^*)$-closed trail. By Lemma 3, H also has a dominating $V_{\geq 4}(H)$-closed trail. Hence statement (A14) is also true, and this completes the proof of Theorem 11. $\square$

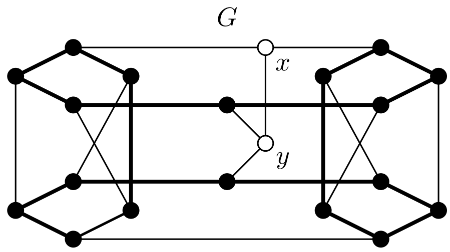

Figure 6: The cubic graph to construct the example

7 Concluding remarks

In 1992, Jackson posed a possible approach to the well-known conjecture on the existence of a Hamilton cycle in 4-connected claw-free graphs (Conjecture A), using a Tutte cycle. Indeed, he conjectured that statement (A9) "every 2-connected claw-free graph has a Tutte cycle" is true (Conjecture D), which directly implies Conjecture A. In this paper, we have concentrated on Tutte cycles in claw-free graphs and seen that many statements (A1)–(A16) are equivalent, see Theorems B, C, G, 1–4, 10, 11, Corollary 6, Lemma 2 and also Appendix. (In order to show the equivalence of statements (A1) and (A13) "Every essentially 2-edge-connected multigraph has a Tutte closed trail" (Corollary 6), we applied Theorem 3, that is, we considered the concept of "weakly Tutte edge-maximal closed trails", see the paragraph following Theorem 5 in Section 3. However, we can also show this equivalence by considering "Tutte closed trails". In fact, we can replace "weakly Tutte edge-maximal closed trail" to "Tutte closed trail" in Lemmas 5 and 6 in Subsection 5.3, and hence by combining this with Theorems G, 10, Lemma 2 and a similar argument as in the proof of Theorem 3, we can show the equivalence of statements (A1) and (A13) slightly simpler.)

By the above fact, it follows that statement (A10) "every 2-connected claw-free graph has a Tutte maximal cycle" is seemingly stronger than statement (A9), that is, if (A9) is true, then we can always take a Tutte cycle so that it is maximal. However, as mentioned in Section 1, it is not always true that a 3-connected claw-free graph has a Tutte cycle which is longest even if statement (A9) is true. The following is the 3-connected claw-free graph showing this. Let G be the graph illustrated in Figure 6. Then it is easy to check that G is an essentially 3-edge-connected (3-connected) cubic graph which is not Hamiltonian. Moreover, the edges depicted in Figure 6 by bold lines induce a cycle C such that $V(C) = V(G) \setminus \{x, y\}$ and C is a maximal cycle of G. Let $d \geq 3$ be an integer. Let G^* be the graph obtained from G by adding $d-2$ pendant edges to each vertex in $\{x, y\}$ and at least $2d-2$ pendant edges to each vertex in $V(G) \setminus \{x, y\}$, and let X be the set of pendant edges which are incident with $\{x, y\}$ in G^*. Note that $|X \cup \{xy\}| = 2d-3$. Then by the definition of G^* and since G is essentially 3-edge-connected, it follows that G^* is also essentially 3-edge-connected and the minimum edge

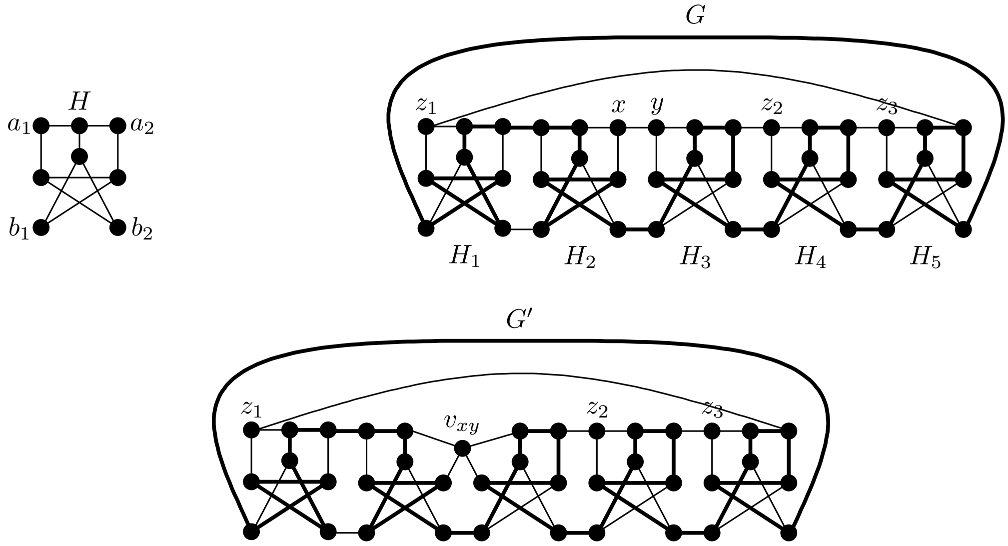

Figure 7: The cubic graph introduced by Kochol [13]

degree of G^* is just d. Furthermore, since G is not Hamiltonian and C is a maximal cycle consisting of $V(G) \setminus \{x, y\}$, it follows that $|E_{G^*}(T)| < |E_{G^*}(C)|$ for every closed trail (cycle) T of G^* with $V(T) \neq V(C)$. These imply that $L(G^*)$ is a 3-connected claw-free graph with $\delta(L(G^*)) = d$, and $V(D) = E_{G^*}(C) = E(G^*) \setminus (X \cup \{xy\})$ for any longest cycle D of $L(G^*)$. Since $|E_{G^*}(C) \cap E_{G^*}(xy)| = e_{G^*}(\{x, y\}, V(G^*) \setminus \{x, y\}) = 4$, every cycle D of $L(G^*)$ with $V(D) = E_{G^*}(C)$ is not a Tutte cycle of $L(G^*)$. Thus any Tutte cycle of $L(G^*)$ is not longest.

In addition, if statement (A9) is true, then we can also take Tutte closed trails (weakly Tutte closed trails, weakly Tutte edge-maximal closed trails) in essentially 2-edge-connected graphs (see statements (A11)–(A13)). Moreover, it is also true that every essentially 4-edge-connected graph has a Tutte edge-maximal closed trail if statement (A9) is true (see statement (A14)). However, it is not always true that an essentially 3-edge-connected graph has a Tutte edge-maximal closed trail. We finally give the graph showing this. We use the methods of Kochol [13] for constructions of snarks with a maximal cycle that is not a dominating cycle. (Note that by using this method, we can also construct a 3-connected claw-free graph, other than the above graph, in which any Tutte cycle is not longest.) Let G be the graph in the right side of Figure 7. It arises from five copies of the graph H $(H_1, H_2, H_3, H_4, H_5)$ illustrated in the left side of Figure 7 after joining the vertices a_i and b_i of degree 2 as in depicted in the figure. Then G is an essentially 3-edge-connected (3-connected) cubic graph and the cycle C depicted by bold lines is a maximal cycle of G such that $V(C) = V(G) \setminus \{x, y, z_1, z_2, z_3\}$. Let G' be the graph obtained from G by contracting xy to a vertex v_{xy} (see Figure 7), and let G^* be the graph obtained from G' by adding a pendant edge to each vertex in $V(G') \setminus \{v_{xy}, z_1, z_2, z_3\}$. Then G^* is also essentially 3-edge-connected and C is a dominating closed trail of G^*, i.e., C is an edge-maximal closed trail of G^*. Since each vertex in $V(G') \setminus \{v_{xy}, z_1, z_2, z_3\}$ is incident with a pendant edge in G^* and $E_{G^*}(C) = E(G^*)$, it follows that every edge-maximal closed trail of G^* contains $V(G') \setminus \{v_{xy}, z_1, z_2, z_3\}$. On the other hand, since C is a maximal cycle of G consisting of $V(G) \setminus \{x, y, z_1, z_2, z_3\}$ and by the definition of H, G, G' and G^*, we can see that for every closed trail T of G^* with $v_{xy} \in V(T)$, $V(G) \setminus \{x, y, z_1, z_2, z_3\} = V(G') \setminus \{v_{xy}, z_1, z_2, z_3\} \nsubseteq V(T)$ holds (note that there exists no Hamilton path in H from a_1 to $\{a_2, b_2\}$ and H has no two disjoint paths covering $V(H)$ from a_1 to $\{a_2, b_2\}$ and from b_1 to $\{a_2, b_2\}$, respectively, see [13] for more details). Thus C is a unique edge-maximal closed trail of G^*. But since C is not a Tutte closed trail of G^*, any Tutte closed trail of G^* is not an edge-maximal closed trail of G^*.

Acknowledgments

The authors would like to thank the referee for valuable suggestions and comments.

References

[1] P. Ash and B. Jackson, Dominating cycles in bipartite graphs, In: Progress in Graph Theory, (J.A. Bondy and U.S.R. Murty, Eds.), Academic Press, New York, (1984), pp. 81–87.

[2] J.A. Bondy, Basic graph theory: paths and circuits, in Handbook of combinatorics, Vol. 1, 2, Elsevier, Amsterdam (1995), 3–110.

[3] H.J. Broersma, G. Fijavž, T. Kaiser, R. Kužel, Z. Ryjáček and P. Vrána, Contractible subgraphs, Thomassen's conjecture and the dominating cycle conjecture for snarks, Discrete Math. **308** (2008), 6064–6077.

[4] H.J. Broersma, M. Kriesell and Z. Ryjáček, On factors of 4-connected claw-free graphs, J. Graph Theory **37** (2001), 125–136.

[5] H.J. Broersma, Z. Ryjáček and P. Vrána, How many conjectures can you stand? A survey, Graphs Combin. **28** (2012), 57–75.

[6] R. Diestel, Graph Theory. Second edition. Graduate Texts in Mathematics, 173, Springer-Verlag, New York, (2000).

[7] R.J. Faudree, E. Flandrin, Z. Ryjáček, Claw-free graphs - a survey, Discrete Math. **164** (1997), 87-147.

[8] H. Fleischner, Cycle decompositions, 2-coverings, removable cycles and the four-color disease, In: Progress in Graph Theory, (J.A. Bondy and U.S.R. Murty, Eds.), Academic Press, New York, (1984), pp. 233–246.

[9] H. Fleischner and B. Jackson, A note concerning some conjectures on cyclically 4-edge-connected 3-regular graphs, In: Graph Theory in Memory of G. A. Dirac, Annals of Discrete Mathematics, 41 (L. D. Andersen, I. T. Jakobsen, C. Thomassen, B. Toft, and P. D. Vestergaard, Eds.), North-Holland, Amsterdam, (1989), pp. 171–177.

[10] F. Harary and C.St.J.A. Nash-Williams, On Eulerian and Hamiltonian graphs and line graphs, Can. Math. Bull. **8** (1965), 701–710.

[11] B. Jackson, Concerning the circumference of certain families of graphs, Memorandum 1076, Univ. of Twente, Enschede (1992), 87–94.

[12] M. Kochol, Equivalence of Fleischner's and Thomassen's conjectures, J. Combin. Theory Ser. B **78** (2000), 277–279.

[13] M. Kochol, Stable dominating circuits in snarks Discrete Math. **233** (2001), 247–256.

[14] R. Kužel, A note on the dominating circuit conjecture and subgraphs of essentially 4-edge-connected cubic graphs, Discrete Math. **308** (2008), 5801–5804.

[15] R. Kužel, Z. Ryjáček and P. Vrána, Thomassen's conjecture implies polynomiality of 1-Hamilton-connectedness in line graphs, J. Graph Theory **69** (2012), 241–250.

[16] T. Kaiser and P. Vrána, Hamilton cycles in 5-connected line graphs, European J. Combin. **33** (2012), 924–947.

[17] M.M. Matthews and D.P. Sumner, Hamiltonian results in $K_{1,3}$-free graphs, J. Graph Theory **8** (1984), 139–146.

[18] Z. Ryjáček, On a closure concept in claw-free graphs, J. Combin. Theory Ser. B **70** (1997), 217–224.

[19] Z. Ryjáček and P. Vrána, Line graphs of multigraphs and hamilton-connectedness of claw-free graphs, J. Graph Theory **66** (2011), 152–173.

[20] R. Thomas and X. Yu, 4-connected projective-planar graphs are Hamiltonian, J. Combin. Theory Ser. B **62** (1994), 114–132.

[21] R. Thomas, X. Yu and W. Zang, Hamilton paths in toroidal graphs, J. Combin. Theory Ser. B **94** (2005), 214–236.

[22] C. Thomassen, A theorem on paths in planar graphs, J. Graph Theory **7** (1983), 169–176.

[23] C. Thomassen, Reflections on graph theory, J. Graph Theory **10** (1986), 309–324.

[24] W.T. Tutte, A theorem on planar graphs, Trans. Amer. Math. Soc. **82** (1956), 99–116.

[25] X. Yu, Disjoint paths, planarizing cycles, and spanning walks, Trans. Amer. Math. Soc. **349** (1997), 1333–1358.

Appendix: A list of conjectures and their relations in this paper

(A1) Every 4-connected claw-free graph is Hamiltonian. [17]

(A2) Every 4-connected line graph is Hamiltonian. [23]

(A3) Every 4-connected claw-free graph is Hamilton-connected [19].

(A4) Every 4-connected line graph is 1-Hamilton-connected (2-edge-Hamilton-connected) [15].

(A5) Every essentially 4-edge-connected multigraph has a dominating closed trail [9].

(A6) Every cyclically 4-edge-connected cubic graph has a dominating cycle [9].

(A7) Every cyclically 4-edge-connected cubic graph that is not 3-edge-colorable has a dominating cycle [12].

(A8) Every snark has a dominating cycle [3].

(A9) Every 2-connected claw-free graph has a Tutte cycle. [11]

(A10) Every 2-connected claw-free graph has a Tutte cycle which is a maximal cycle of the graph.

(A11) Every essentially 2-edge-connected multigraph has a weakly Tutte edge-maximal closed trail.

(A12) Every essentially 2-edge-connected multigraph has a weakly Tutte closed trail.

(A13) Every essentially 2-edge-connected multigraph has a Tutte closed trail.

(A14) Every essentially 4-edge-connected multigraph H has a dominating $V_{\geq 4}(H)$-closed trail.

(A15) Let H be an essentially 4-edge-connected multigraph, and let (e_1, e_2, e_3) be a 3-star of H. Then H has a dominating $\big(\{e_1, e_2\} \cup V(e_3) \cup V_{\geq 4}(H)\big)$-closed trail

(A16) Any subgraph H of an essentially 4-edge-connected cubic graph with $\delta(H) = 2$ and $|V_2(H)| = 4$ is $V_2(H)$-dominated. [14]