Spanning trees with bounded maximum degrees of graphs on surfaces

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Abstract

For a spanning tree T of a graph G, we define the *total excess* $\mathrm{te}(T,k)$ *of T from k* as $\mathrm{te}(T,k) := \sum_{v \in V(T)} \max\{d_T(v) - k, 0\}$, where $d_T(v)$ is the degree of a vertex v in T. In this paper, we show the following; if G is a 3-connected graph on a surface with Euler characteristic $\chi < 0$, then G has a spanning $\lceil \frac{8-2\chi}{3} \rceil$-tree T with $\mathrm{te}(T,3) \leq -2\chi - 1$. We also show an application of this theorem to find "light" connected subgraphs in a 3-connected graph on a surface.

Keywords: Graphs on surfaces, Spanning k-trees, Total excess, Light subgraphs

1 Introduction

In 1956, Tutte [36] proved that every 4-connected planar graph has a Hamilton cycle, and Thomassen [34] generalized this result by showing that every 4-connected planar graph is Hamilton-connected. Note that a graph G is called *Hamilton-connected* if for every pair of two vertices, G has a Hamilton path connecting them. It is known that there exist infinite many 3-connected planar graphs without a Hamilton cycle, even a Hamilton path. Therefore, we sometimes consider a relaxation of these Hamiltonian properties in a 3-connected planar graph.

In this paper, we concentrate on a relaxation of a Hamilton path for spanning trees with bounded maximum degree. For an integer $t \geq 2$, a tree T is called a *t-tree* if the maximum degree of T is at most t. Note that a spanning 2-tree of a graph is equivalent to a Hamilton path of the graph. Barnette [2] proved the following result on a spanning 3-tree.

Theorem 1 (Barnette [2]) *Every 3-connected planar graph has a spanning 3-tree.*

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Barnette [3] improved Theorem 1 to graphs on other surfaces; every 3-connected graph on the projective plane, the torus, and the Klein bottle has a spanning 3-tree. Note that Nakamoto, Oda and Ota [24] improved these theorems as follows; every 3-connected planar graph G has a spanning 3-tree with at most $\max\left\{\frac{|V(G)|-7}{3}, 0\right\}$ vertices of degree three. They also showed the existence of spanning 3-trees with bounded number of vertices of degree three in 3-connected graphs on the projective plane, the torus, and the Klein bottle.

Recently, the existence of a spanning tree with bounded maximum degree was considered for 3-connected graphs on surfaces with higher genus. Ellingham [8] showed the following result, which was first asked by Brunet, Ellingham, Gao, Metzlar and Richter [5]; every 3-connected graph on a surface with Euler characteristic $\chi \leq 0$ has a spanning $\lceil \frac{10-2\chi}{3} \rceil$-tree. However the upper bound of the maximum degree of a spanning tree is not best possible. Sanders and Zhao [29] gave a sharp result for graphs on a surface with Euler characteristic $\chi \leq -36$. Ota and the author [26] improved it to graphs on a surface with Euler characteristic $\chi \leq 0$.

Theorem 2 (Sanders and Zhao [29], Ota and Ozeki [26]) *Let G be a 3-connected graph on a surface with Euler characteristic $\chi \leq 0$. Then G has a spanning $\lceil \frac{8-2\chi}{3} \rceil$-tree.*

Note that the complete bipartite graph $K_{3,6-2\chi}$ has no spanning $\left(\lceil \frac{8-2\chi}{3} \rceil - 1\right)$-tree, so the bound on the maximum degree of a spanning tree in Theorem 2 is best possible. See Theorem 5.1 in [29].

However, if the surface is fixed and the *representativity* is large, then the situation is different. For a graph G on a non-spherical surface $\mathbb{F}^2$, the *representativity* of G is defined as the minimum cardinality of the intersection of G with a non-contractible curve on $\mathbb{F}^2$. When a graph embedded on a surface has large representativity, it locally looks like a plane graph. In this sense, a graph on a surface with sufficiently large representativity is sometimes called a *locally planar* graph.

Thomassen [35] showed that every triangulation of an orientable surface with sufficiently large representativity has a spanning 4-tree, and that 4 cannot be replaced by 3. Yu [37] improved Thomassen's result to 3-connected graphs on a surface with sufficiently large representativity (not necessarily "triangulation"). Kawarabayashi, Nakamoto and Ota [19] further improved it as follows;

Theorem 3 (Kawarabayashi, Nakamoto and Ota [19]) *For every surface $\mathbb{F}^2$ with Euler characteristic $\chi < 0$, there exists a positive integer r such that every 3-connected graph on $\mathbb{F}^2$ with representativity at least r has a spanning 4-tree with at most $-2\chi - 1$ vertices of degree four.*

Note that the bound "$-2\chi - 1$" on the number of vertices of degree four is best possible, see Proposition 7.2 in [19].

Notice also that the integer r in Theorem 3 is enormously large. (Indeed, they used the result in [30], which is contained in the series of Graph Minor papers.)

There are infinitely many 3-connected graphs on a surface which do not satisfy the representativity assumption of Theorem 3. The main purpose of this paper is to consider spanning trees in such graphs.

As mentioned before, the bound "$\lceil \frac{8-2\chi}{3} \rceil$" in Theorem 2 is best possible because of, for example, the complete bipartite graph $K_{3,6-2\chi}$. So the maximum degree of spanning trees cannot be bounded without using the Euler characteristic of the surface where the graph is embedded.

On the other hand, we will focus on the bound "$-2\chi - 1$" of the number of vertices of degree four in Theorem 3. For a spanning 4-tree, the number of vertices of degree four can be thought as the sum of excesses of degrees from three. In this sense, we consider the *total excess* as follows. For an integer k and a spanning tree T of a graph G, we define the *total excess* $\mathrm{te}(T,k)$ *of* T *from* k as

$$\mathrm{te}(T,k) := \sum_{v \in V(T)} \max\{d_T(v) - k, 0\},$$

where $d_T(v)$ is the degree of v in T. Note that a spanning tree T satisfies $\mathrm{te}(T,k) = 0$ if and only if T is a spanning k-tree. In terms of the total excess, Theorem 3 can be restated as follows; every 3-connected locally planar graph on a surface with Euler characteristic $\chi < 0$ has a spanning 4-tree T with $\mathrm{te}(T,3) \le -2\chi - 1$.

In this paper, we show the following result, which was asked by Kawarabayashi [18] and Ota [27], independently.

Theorem 4 *Let G be a 3-connected graph on a surface with Euler characteristic $\chi < 0$. Then G has a spanning $\lceil \frac{8-2\chi}{3} \rceil$-tree T with* $\mathrm{te}(T,3) \le -2\chi - 1$.

A spanning tree obtained from Theorem 4 has the same upper bound on the maximum degree as a spanning tree obtained from Theorem 2, and the same upper bound on the total excess as a spanning tree obtained from Theorem 3. In this sense, Theorem 4 is an improvement of both Theorems 2 and 3. From Theorem 4, we obtain that the total excess "$\mathrm{te}(T,3) \le -2\chi - 1$" can be guaranteed without the representativity assumption, although the maximum degree four cannot.

As mentioned before, both bounds "$\lceil \frac{8-2\chi}{3} \rceil$" on the maximum degree and "$-2\chi - 1$" on the total excess are best possible, see Theorem 5.1 in [29] and Proposition 7.2 in [19], respectively.

Remark: One might think that we need some representativity assumption to guarantee the existence of a spanning tree with bounded maximum degree, because of graphs that Archdeacon, Hartsfield and Little [1] constructed. They proved that for each positive integer k, there exists a k-connected triangulation of an orientable surface having representativity k in which every spanning tree has a vertex of degree at least k. However, the graph that they constructed was embedded into a surface with very small Euler characteristic, compared to the integer k. So Theorems 2 and 4 do not contradict the result by Archdeacon et al. [1].

In the next section, we show an application of Theorem 4. In Sections 3 and 4, we will give two new theorems, respectively. The one in Section 3 is used in the proof of the one in Section 4, which will play an important role in the proof of Theorem 4. Using them, we will show Theorem 4 in Section 5. In Section 6, we shall consider a spanning tree in 4-connected graphs on surfaces.

For notation not defined here, we refer to the book [23].

2 An application of Theorem 4

In this section, we show an application of Theorem 4 to find "light" connected subgraphs with given number of vertices, in the sense of the degree sum.

It is well-known that every plane graph has a vertex of degree at most five. Kotzig [21] proved that every 3-connected plane graph has an edge with degree sum of its end vertices being at most 13. Starting with these results, in plane graphs or graphs on a surface, many researchers have found connected subgraphs with few degree sum of its vertices. For example, see [11, 17, 22]. See also [16] for a good survey on light subgraphs. Kawarabayashi, Nakamoto and Ota [19] showed the following theorem.

Theorem 5 (Kawarabayashi, Nakamoto and Ota [19]) *For every non-spherical surface $\mathbb{F}^2$ and every integer t, there exist positive integers r and n such that if G is a 3-connected graph on $\mathbb{F}^2$ with representativity at least r and $|V(G)| \geq n$, then G has a connected subgraph H of t vertices such that $\sum_{v \in V(H)} d_G(v) \leq 8t - 1$.*

Note that the coefficient "8" in the bound "$8t-1$" is best possible, see Proposition 7.4 in [19].

In [19], Kawarabayashi et al. proved Theorem 5 using Theorem 3, and hence they need the representativity assumption in Theorem 5. However, they do not use the maximum degree condition of the spanning tree obtained from Theorem 3. They only need the total excess condition, and after finding such a spanning tree, the representativity assumption is not used again.

Actually, in Lemma 6.4 in [19], they consider a 4-tree T with $\mathrm{te}(T, 3) = q$, but the same holds if T only satisfies that $\mathrm{te}(T, 3) = q$. In the proof of Theorem 5 (Theorem 1.3 in [19]), the representativity assumption is only used for the existence of a spanning 4-tree T with $\mathrm{te}(T, 3) = q \leq -2\chi - 1$, and the bound "4" on the maximum degree of T is only used to apply Lemma 6.4.

So the same proof as in [19] together with Theorem 4 can work to show the following theorem, which is an improvement of Theorem 5. (Note that in order to use Lemma 6.5 in [19], we need the assumption "representativity at least three".)

Theorem 6 *For every surface $\mathbb{F}^2$ with Euler characteristic $\chi < 0$ and every integer t, there exists a positive integer n such that if G is a 3-connected graph on $\mathbb{F}^2$ with representativity at least three and $|V(G)| \geq n$, then G has a connected subgraph H of t vertices such that $\sum_{v \in V(H)} d_G(v) \leq 8t - 1$.*

3 Spanning trees with bounded degrees of vertices in a specified independent set

Frank and Gyárfás showed the following theorem, which gave a necessary and sufficient condition for the existence of a spanning tree such that every vertex in a specified *independent* set has bounded degree. A vertex set S of a graph G is called an *independent* set if no edge of G connects two vertices of S. Let $\omega(G)$ be the number of components of a graph G.

Theorem 7 (Frank and Gyárfás [12]) *Let k be a positive integer. Let G be a connected graph and let $X \subset V(G)$ be an independent set. Then G has a spanning tree T such that $d_T(x) \le k$ for every $x \in X$ if and only if $\omega(G - S) \le (k-1)|S| + 1$ for every $S \subset X$.*

In this paper, we need the following result, which we prove in this section and is used in the next section. For a tree T, $X \subset V(T)$ and an integer k, let

$$\mathrm{te}(T, k; X) := \sum_{x \in X} \max\{d_T(x) - k, 0\}.$$

Note that a spanning tree T satisfies that $d_T(x) \le k$ for every $x \in X$ if and only if $\mathrm{te}(T, k; X) = 0$.

Theorem 8 *Let k, k' and t be positive integers with $k \le k'$. Let G be a connected graph and let $X \subset V(G)$ be an independent set. Then G has a spanning tree T such that $\mathrm{te}(T, k; X) \le t$ and $d_T(x) \le k'$ for every $x \in X$ if and only if for every $S \subset X$,*

$$\omega(G - S) \le (k-1)|S| + 1 + t \qquad (1)$$

$$\text{and} \qquad \omega(G - S) \le (k'-1)|S| + 1. \qquad (2)$$

This is an improvement of Theorem 7. Actually, Theorem 7 is the case where $t = 0$ and $k' = +\infty$ of Theorem 8.

We will show Theorem 8 using a similar method to the one in [10]. Notice that we use Theorem 7 in the proof of Theorem 8, but it is only for simplicity of the proof. In a similar way to the proof in [28], we can prove Theorem 8 without using Theorem 7.

Before giving the proof of it, we need the following proposition.

Proposition 9 *Let T be a tree and let $S \subset V(T)$ be an independent set. Then $\omega(T - S) = \sum_{v \in S} \big(d_T(v) - 1\big) + 1$.*

Proof. We prove this proposition by induction on $|S|$. When $|S| = 1$, say $S = \{v\}$, we obtain that $\omega(T - S) = d_T(v) = \big(d_T(v) - 1\big) + 1$. So we may assume that $|S| \ge 2$. Take a vertex $u \in S$ and let $S' := S - \{u\}$. By induction hypothesis, $\omega(T - S') = \sum_{v \in S'} \big(d_T(v) - 1\big) + 1$. Let C be the component of $T - S'$ containing

u. Since S is independent, u has no neighbors in S, and hence we obtain that $d_C(u) = d_T(u)$. Since C is divided into $d_C(u)$ components when we remove u from C, we have that

$$\begin{aligned}
\omega(T-S) &= \omega(T-S') - 1 + d_C(u) \\
&= \sum_{v\in S'} \big(d_T(v)-1\big) + 1 - 1 + d_T(u) \\
&= \sum_{v\in S} \big(d_T(v)-1\big) + 1.
\end{aligned}$$

This completes the proof of Proposition 9.

For a graph T and $S \subset V(T)$, we define *an S-bridge of T* as an edge between two vertices of S or a component C of $T-S$ together with the edges connecting C and S. For a graph T and a positive integer k, let $V_{\geq k}(T) := \{v \in V(T) : d_T(v) \geq k\}$.

Proof of Theorem 8. First we show the "only if" part. Suppose that G has a spanning tree T such that $\mathrm{te}(T,k;X) \leq t$ and $d_T(x) \leq k'$ for every $x \in X$. Let $S \subset X$. By Proposition 9, we obtain that

$$\begin{aligned}
\omega(G-S) &\leq \omega(T-S) \\
&= \sum_{v\in S} \big(d_T(v)-1\big) + 1 \\
&= \sum_{v\in S} \big(k-1\big) + 1 + \sum_{v\in S} \big(d_T(v)-k\big) \\
&\leq (k-1)|S| + 1 + \sum_{v\in X} \max\{d_T(v)-k, 0\} \\
&= (k-1)|S| + 1 + \mathrm{te}(T,k;X) \\
&\leq (k-1)|S| + 1 + t.
\end{aligned}$$

Therefore, inequality (1) holds. Similarly, inequality (2) also holds.

Next we show the "if" part. Let G be a connected graph and $X \subset V(G)$ be an independent set satisfying inequalities (1) and (2) for every $S \subset X$. By condition (2) and Theorem 7, there exists a spanning tree T such that $d_T(x) \leq k'$ for every $x \in X$. We choose such a spanning tree T of G such that $\mathrm{te}(T,k;X)$ is as small as possible.

If $\mathrm{te}(T,k;X) \leq t$, there is nothing to prove. Thus, we may assume that

$$\mathrm{te}(T,k;X) > t. \tag{3}$$

For $S \subset X$, we define $\mathcal{T}(S)$ as the set of spanning trees T' of G such that $d_T(x) \leq k'$ for every $x \in X$, and for every S-bridge C' of T', there exists an S-bridge C of T such that $V(C') = V(C)$ and $\mathrm{te}\big(C',k;X\cap V(C')\big) \leq \mathrm{te}\big(C,k;X\cap V(C)\big)$. Note that $T \in \mathcal{T}(S)$ for any $S \subset X$.

Let $T' \in \mathcal{T}(S)$. Note that for each $x \in S$, the degree of x in T' is equal to the number of S-bridges of T' containing x, because T' is a tree. Since for each S-bridge C' of T' with $x \in V(C')$, there exists an S-bridge C of T such that $x \in V(C') = V(C)$, we have that $d_{T'}(x) \le d_T(x)$. This implies that

$$\begin{aligned}
\mathrm{te}(T', k; X) &= \sum_{x \in S} \max\{d_{T'}(x) - k, 0\} + \sum_{C'\colon S\text{-bridge of } T'} \mathrm{te}\big(C', k; X \cap V(C')\big) \\
&\le \sum_{x \in S} \max\{d_T(x) - k, 0\} + \sum_{C\colon S\text{-bridge of } T} \mathrm{te}\big(C, k; X \cap V(C)\big) \\
&= \mathrm{te}(T, k; X).
\end{aligned}$$

Therefore, for every $T' \in \mathcal{T}(S)$, T' satisfies that $\mathrm{te}(T', k; X) \le \mathrm{te}(T, k; X)$. Hence by the minimality of $\mathrm{te}(T, k; X)$, we have that $\mathrm{te}(T', k; X) = \mathrm{te}(T, k; X)$ for every $T' \in \mathcal{T}(S)$ and the equality holds in the above inequality.

Let $A_0 := X \cap V_{\ge k+1}(T)$. For $i \ge 1$, let

$$A_i := A_0 \cup \{v \in X : d_{T'}(v) = k \text{ for every } T' \in \mathcal{T}(A_{i-1})\}.$$

Since $\mathcal{T}(S) \subset \mathcal{T}(S')$ if $S' \subset S$, we obtain that $A_0 \subset A_1 \subset A_2 \subset \cdots \subset X \cap V_{\ge k}(T)$.

Claim 1 *For every edge $xy \in E(G)$ such that x and y are contained in distinct components of $T - A_0$, we have that $x \in A_1$ or $y \in A_1$.*

Proof. Assume that there exists an edge $xy \in E(G)$ such that $x \notin A_1$, $y \notin A_1$ and x and y are contained in distinct components of $T - A_0$. Since X is an independent set in G, we may assume that $y \notin X$. Then $x \notin A_1$ implies that $x \notin X$ or G has a spanning tree $T_x \in \mathcal{T}(A_0)$ such that $d_{T_x}(x) < k$. If $x \notin X$, then let $T_x = T$.

Since x and y are contained in distinct components of $T - A_0$ and $T_x \in \mathcal{T}(A_0)$, there exists a vertex z in A_0 such that z lies on the unique path in T_x between x and y. Let zw be an edge lying on the above path. Since X is an independent set, we have that $w \notin X$. Then $T_z = (T_x - zw) \cup xy$ is a spanning tree of G such that $x \notin X$ or $d_{T_z}(x) = d_{T_x}(x) + 1 \le k \le k'$, and $d_{T_z}(z) = d_{T_x}(z) - 1 \le k' - 1$.

Note that $d_{T_z}(v) = d_{T_x}(v)$ for every $v \in X - \{x, z\}$, and hence we have that $d_{T_z}(v) \le k'$ for every $v \in X$. If $x \notin X$, then

$$\begin{aligned}
\mathrm{te}(T_x, k; X) - \mathrm{te}(T_z, k; X) &= \max\{d_{T_x}(z) - k, 0\} - \max\{d_{T_z}(z) - k, 0\} \\
&= \big(d_{T_x}(z) - k\big) - \big(d_{T_z}(z) - k\big) \\
&> 0,
\end{aligned}$$

since $k + 1 \le d_{T_x}(z) = d_{T_z}(z) + 1$. Otherwise

$$\begin{aligned}
&\mathrm{te}(T_x, k; X) - \mathrm{te}(T_z, k; X) \\
&\quad = \sum_{v \in \{x,z\}} \max\{d_{T_x}(v) - k, 0\} - \sum_{v \in \{x,z\}} \max\{d_{T_z}(v) - k, 0\} \\
&\quad = \max\{d_{T_x}(z) - k, 0\} - \max\{d_{T_z}(z) - k, 0\} \\
&\quad > 0,
\end{aligned}$$

since $\max\{d_{T_x}(x) - k, 0\} = \max\{d_{T_z}(x) - k, 0\} = 0$. In either case, we obtain that $\mathrm{te}(T_z, k; X) < \mathrm{te}(T_x, k; X) = \mathrm{te}(T, k; X)$, which contradicts the minimality of $\mathrm{te}(T, k; X)$.

Claim 2 *For every $i \geq 1$ and every edge $xy \in E(G)$ such that x and y are contained in distinct components of $T - A_i$, we have that $x \in A_{i+1}$ or $y \in A_{i+1}$.*

Proof. Assume that there exists an edge $xy \in E(G)$ such that $x \notin A_{i+1}$, $y \notin A_{i+1}$ and x and y are contained in distinct components of $T - A_i$. We take such an integer i as small as possible. Since X is an independent set in G, we may assume that $y \notin X$. Then $x \notin A_{i+1}$ implies that $x \notin X$ or G has a spanning tree $T_x \in \mathcal{T}(A_i)$ such that $d_{T_x}(x) < k$. If $x \notin X$, then let $T_x = T$.

Since x and y are contained in distinct components of $T - A_i$ and $T_x \in \mathcal{T}(A_i)$, there exists a vertex z in A_i such that z lies on the unique path in T_x between x and y. Let zw be an edge lying on the above path. Since X is an independent set, we have that $w \notin X$. By the minimality of i, $z \in A_i - A_{i-1}$, and x, y, z and w are contained in the same A_{i-1}-bridge of T. Then $T_z = (T_x - zw) \cup xy$ is a spanning tree of G such that $x \notin X$ or $d_{T_z}(x) = d_{T_x}(x) + 1 \leq k \leq k'$, and $d_{T_z}(z) = d_{T_x}(z) - 1 = k - 1$.

Note that $d_{T_z}(v) = d_{T_x}(v)$ for every $v \in X - \{x, z\}$, and hence we have that $d_{T_z}(v) \leq k'$ for every $v \in X$.

Now we show the following;

$$\begin{array}{c}\text{For every } A_{i-1}\text{-bridge } C_z \text{ of } T_z, \text{ there exists an } A_{i-1}\text{-bridge } C \text{ of } T \\ \text{with } V(C_z) = V(C) \text{ and } \mathrm{te}\big(C_z, k; X \cap V(C_z)\big) \leq \mathrm{te}\big(C, k; X \cap V(C)\big).\end{array} \quad (4)$$

Recall that $T_x \in \mathcal{T}(A_i) \subset \mathcal{T}(A_{i-1})$, and hence for every S-bridge C_x of T_x, there exists an S-bridge C of T such that $V(C_x) = V(C)$ and $\mathrm{te}\big(C_x, k; X \cap V(C_x)\big) \leq \mathrm{te}\big(C, k; X \cap V(C)\big)$. Then statement (4) is trivial for the case $x \notin X$, and for all A_{i-1}-bridge C_z of T_z with $x, y, z, w \notin V(C_z)$ since such an A_{i-1}-bridge C_z of T_z is not changed from the one in T_x. So we may assume that $x \in X$. Let C_z be an A_{i-1}-bridge of T_z with $x, y, z, w \in V(C_z)$. By the definition of T_z, there is a unique A_{i-1}-bridge C_x of T_x such that $V(C_z) = V(C_x)$. Then

$$\begin{aligned} &\mathrm{te}\big(C_x, k; X \cap V(C_x)\big) - \mathrm{te}\big(C_z, k; X \cap V(C_z)\big) \\ &= \sum_{v \in \{x,z\}} \max\{d_{C_x}(v) - k, 0\} - \sum_{v \in \{x,z\}} \max\{d_{C_z}(v) - k, 0\} \\ &= 0, \end{aligned}$$

since $d_{C_z}(x) = d_{T_z}(x) = d_{T_x}(x) + 1 = d_{C_x}(x) + 1 \leq k$ and $d_{C_z}(z) = d_{T_z}(z) = d_{T_x}(z) - 1 = k - 1$. So statement (4) holds.

Therefore $T_z \in \mathcal{T}(A_{i-1})$. However, $z \in A_i$ and $d_{T_z}(z) = d_{T_x}(z) - 1 = k - 1$, which contradicts the definition of A_i.

Since $A_0 \subset A_1 \subset \cdots \subset X \cap V_{\geq k}(T)$, there exists an integer i such that $A_i = A_{i+1}$. By Claim 1 or 2, $\omega(T - A_i) = \omega(G - A_i)$. Since $X \cap V_{\geq k+1}(T) = A_0 \subset A_i \subset X \cap V_{\geq k}(T)$, we have that $\mathrm{te}(T,k;X) = \sum_{v\in A_i}\big(d_T(v)-k\big)$. It follows from Proposition 9, these equalities and inequality (3) that

$$\begin{aligned}\omega(G-A_i) &= \omega(T-A_i)\\ &= \sum_{v\in A_i}\big(d_T(v)-1\big)+1\\ &= \sum_{v\in A_i}\big(k-1\big)+1+\sum_{v\in A_i}\big(d_T(v)-k\big)\\ &= \big(k-1\big)|A_i|+1+\mathrm{te}(T,k;X)\\ &> \big(k-1\big)|A_i|+1+t,\end{aligned}$$

which contradicts assumption (1) for $S = A_i$. This completes the proof of Theorem 8. □

4 Minor minimal 3-connected graphs having no spanning tree with bounded total excess

In this section, we will show a result, which connects the existence of a spanning tree with bounded total excess to the minor relation. A graph R is called a *minor* of a graph G if R can be obtained from G by a sequence of vertex and edge deletions and edge contractions. When R is a minor of a graph G, we also say that G *contains* R *as a minor*.

Theorem 10 *Let k, k' and t be nonnegative integers with $3 \le k \le k'$, and let G be a 3-connected graph having no spanning k'-tree T with $\mathrm{te}(T,k) \le t$. Then G contains, as a minor, a 3-connected bipartite graph R with bipartition A and B such that $|B| = (k-1)|A|+2+t$ or $|B| = (k'-1)|A|+2$, where $A := \{x \in V(R) : d_R(x) \ge k+1\}$ and $B := \{x \in V(R) : d_R(x) = 3\}$.*

The method of the proof of Theorem 10 is similar to the one in [26], but in order to deal with the total excess property, we have to consider reductions of a graph more carefully. In fact, in the proof of Claim 4, we construct the spanning tree in a different way from the one used in [26].

For the proof of Theorem 10, we need some terminology and some lemmas. Let G be a 3-connected graph. For an edge $e \in E(G)$, let G/e be the graph obtained from G by contracting e. An edge e is called *contractible* if G/e is also 3-connected. An edge which is not contractible is *non-contractible*. The first lemma is a well-known result on contractible edges.

Lemma 11 (Halin [14]) *Let G be a 3-connected graph except for K_4. Then every vertex of degree three in G is incident with a contractible edge.*

A 3-connected graph G is called *minimally* 3*-connected* if the graph obtained by deleting any edge from G is not 3-connected. For a minimally 3-connected graph, Halin showed the following result. Let $V_3(G) := \{x \in V(G) : d_G(x) = 3\}$.

Lemma 12 (Halin [14, 15]) *Let G be a minimally* 3*-connected graph. Then the followings hold.*

(i) $V_3(G) \neq \emptyset$.

(ii) *Every edge connecting two vertices in $V(G) - V_3(G)$ is contractible.*

(iii) *The graph obtained by contracting any edge connecting two vertices in $V(G) - V_3(G)$ is also minimally* 3*-connected.*

(iv) *Every cycle of G contains at least two vertices of $V_3(G)$.*

As an immediate consequence of Lemma 12 (iv), we can see that $G - V_3(G)$ is a forest. Moreover, a contraction of every edge in $G - V_3(G)$ does not produce a new vertex of degree three. So, applying Lemma 12 (iii) repeatedly, we obtain the following fact.

Lemma 13 *Let G be a minimally* 3*-connected graph, and let P be any connected subgraph of $G - V_3(G)$. Then, G/P, the graph obtained from G by contracting P into a single vertex, is also a minimally* 3*-connected graph.*

In a similar way to the proofs of the "only if "part of Theorem 8 and of Proposition 9, we obtain the following proposition.

Proposition 14 *Let G be a graph having a spanning k'-tree T with $\mathrm{te}(T, k) \leq t$. Then for every $S \subset V(G)$,*

$$\omega(G - S) \leq (k-1)|S| + 1 + t$$

and

$$\omega(G - S) \leq (k'-1)|S| + 1.$$

Proof of Theorem 10. Let G be a minimal 3-connected graph having no spanning k'-tree with $\mathrm{te}(T, k) \leq t$, in the sense of the minor relation. In other words, G has no spanning tree as desired, but every minor G' of G has if G' is 3-connected and $G' \neq G$.

It is easy to show that G is not isomorphic to K_4 (K_4 has the desired spanning tree.) Note that G is minimally 3-connected, since otherwise $G - e$ has a spanning tree as desired for some edge $e \in E(G)$ by the minimality of G, but it is also a spanning tree as desired of G, contradicting the choice of G. We will show that G itself satisfies the desired conditions of R in Theorem 10. By the choice of G, this completes the proof of Theorem 10.

Let $B := V_3(G) = \{x \in V(G) : d_G(x) = 3\}$. The following claim was essentially shown by Sanders and Zhao in [29].

Claim 3 *$B \cup \{x \in V(G) : 4 \le d_G(x) \le k\}$ is independent.*

Proof. Assume that there exist two vertices $y_1, y_2 \in B \cup \{x \in V(G) : 4 \le d_G(x) \le k\}$ such that $y_1y_2 \in E(G)$.

Suppose first that y_1 is incident with a contractible edge, say y_1u (possibly $u = y_2$). By the minimality of G, $G/(y_1u)$ has a spanning k'-tree T' with $\mathrm{te}(T', k) \le t$. Let $\widetilde{T}$ be the subgraph of G which has the same edge set as T'. When there exist more than one edge corresponding to one edge of T', we choose one of them arbitrarily. Let v_{y_1u} be the vertex of $G/(y_1u)$ corresponding to the edge y_1u of G. Notice that $d_{\widetilde{T}}(y_1) + d_{\widetilde{T}}(u) = d_{T'}(v_{y_1u})$ and $d_{\widetilde{T}}(y_1) \le d_G(y_1) - 1 \le k - 1$.

Suppose first that $d_{\widetilde{T}}(y_1) \ge 1$. Let $T := \widetilde{T} \cup y_1u$. Then

$$d_T(u) = d_{T'}(v_{y_1u}) - d_{\widetilde{T}}(y_1) + 1 \le d_{T'}(v_{y_1u}) \le k',$$

and hence T is a spanning k'-tree of G. Moreover,

$$\begin{aligned} \mathrm{te}(T, k) &= \mathrm{te}(T', k) - \max\{d_{T'}(v_{y_1u}) - k, 0\} \\ &\qquad + \max\{d_T(u) - k, 0\} + \max\{d_T(y_1) - k, 0\} \\ &\le \mathrm{te}(T', k) \\ &\le t, \end{aligned}$$

since $d_T(u) \le d_{T'}(v_{y_1u})$ and $d_T(y_1) \le k$. Thus, T is a spanning k'-tree of G with $\mathrm{te}(T, k) \le t$, which contradicts the choice of G. Hence we may assume that $d_{\widetilde{T}}(y_1) = 0$. This implies that $\widetilde{T}$ is a spanning tree of $G - y_1$. Then $T := \widetilde{T} \cup y_1y_2$ is a spanning k'-tree of G with $\mathrm{te}(T, k) = \mathrm{te}(T', k) \le t$, because $d_T(y_2) \le d_G(y_2) \le k$, a contradiction again.

Therefore, we may assume that every edge incident with y_1 is not contractible. By symmetry, every edge incident with y_2 is not contractible. In particular, y_1y_2 is not contractible, and hence $y_1 \in V_3$ or $y_2 \in V_3$ by Lemma 12 (ii), which contradicts Lemma 11.

Claim 4 *There exist no edge in $G - B$.*

Proof. Suppose that there exists an edge in $G - B$. Let $P := x_0x_1 \dots x_l$ ($l \ge 1$) be a maximal path in $G - B$. Since $G - B$ is a forest by Lemma 12 (iv), x_0 and x_l are leaves of $G - B$, and $x_ix_j \notin E(G)$ for every $0 \le i < j \le l$ with $j \ne i + 1$. By Lemma 13, G/P is 3-connected. Let v_P be the vertex in G/P obtained by the contraction of P.

By the minimality of G, there is a spanning k'-tree T' of G/P with $\mathrm{te}(T', k) \le t$. We consider the subgraph $\widetilde{T}$ of G which has the same edge set as T'. Similarly to the proof of Claim 3, when there exist more than one edge corresponding to one edge of T', we choose one of them arbitrarily. Note that $\widetilde{T}$ is a spanning forest of G consisting of $l + 1$ components each of which contains one of $x_0, x_1, \dots, x_l$.

Since both x_0 and x_l have degree one in $G-B$, we can take vertices $u_1 \in N_G(x_0) \cap B$ and $u_2 \in N_G(x_l) \cap B$. (Possibly $u_1 = u_2$.) Let $X := \{x \in V(P) : d_{\widetilde{T}}(x) \geq 1\}$. Note that

$$k' \geq d_{T'}(v_P) = \sum_{x \in X} d_{\widetilde{T}}(x). \tag{5}$$

Now we consider three cases depending on the cardinality of X, and define a spanning tree T of G as follows.

Case 1. $|X| = 1$.

Let $x_i \in X$, and let T be the subgraph of G obtained from $\widetilde{T}$ by adding the paths $u_1 x_0 P x_{i-1}$ (if $i \neq 0$) and $x_{i+1} P x_l u_2$ (if $i \neq l$). By equality (5), note that $d_T(x_i) = d_{\widetilde{T}}(x_i) = d_{T'}(v_P) \leq k'$, and $d_T(x_j) \leq 2$ for every $0 \leq j \leq l$ with $j \neq i$.

Case 2. $|X| = 2$.

Let $x_i, x_j \in X$ with $i < j$, and let T be the subgraph of G obtained from $\widetilde{T}$ by adding the paths $x_i P x_j$, $u_1 x_0 P x_{i-1}$ (if $i \neq 0$) and $x_{j+1} P x_l u_2$ (if $j \neq l$). By equality (5), note that $d_T(x_i) = d_{\widetilde{T}}(x_i) + 1 \leq d_{T'}(v_P) \leq k'$, $d_T(x_j) = d_{\widetilde{T}}(x_j) + 1 \leq d_{T'}(v_P) \leq k'$. Notice also that $d_T(x_r) \leq 2$ for every $0 \leq r \leq l$ with $r \neq i, j$.

Case 3. $|X| \geq 3$.

Let $T := \widetilde{T} \cup P$. By equality (5), $d_T(x_i) \leq d_{\widetilde{T}}(x_i) + 2 \leq d_{T'}(v_P) \leq k'$ for every $0 \leq i \leq l$, because there exist at least two vertices $x \in V(P) - \{x_i\}$ with $d_{\widetilde{T}}(x) \geq 1$.

In any case, notice that $d_T(u_1) \leq d_G(u_1) \leq k \leq k'$ and $d_T(u_2) \leq d_G(u_2) \leq k \leq k'$. Then T is a spanning k'-tree of G. We will show that $\mathrm{te}(T, k) \leq t$. This contradicts that G has no spanning tree as desired and hence this completes the proof of Claim 4.

Let $X' := \{x \in V(P) : d_T(x) \geq k\}$. Note that $X' \subset X$ and

$$\begin{aligned} \mathrm{te}(T, k) &= \mathrm{te}(T', k) - \max\{d_{T'}(v_P) - k, 0\} + \sum_{x \in V(P)} \max\{d_T(x) - k, 0\} \\ &\leq t - \max\{d_{T'}(v_P) - k, 0\} + \sum_{x \in X'} \big(d_T(x) - k\big). \end{aligned}$$

Thus, if $X' = \emptyset$, then

$$\begin{aligned} \mathrm{te}(T, k) &\leq t - \max\{d_{T'}(v_P) - k, 0\} \\ &\leq t, \end{aligned}$$

and if $|X'| = 1$, say $x \in X'$, then

$$\begin{aligned} \mathrm{te}(T, k) &\leq t - \max\{d_{T'}(v_P) - k, 0\} + \big(d_T(x) - k\big) \\ &= t - d_{T'}(v_P) + d_T(x) \\ &\leq t, \end{aligned}$$

since $d_T(x) \leq d_{T'}(v_P)$ in either case. Therefore, we may assume that $|X'| \geq 2$, in particular, Case 1 does not occur.

By equality (5) and construction of T, note that

$$\begin{aligned}\sum_{x \in X'} d_T(x) &\leq \sum_{x \in X'} \big(d_{\widetilde{T}}(x) + 2\big) \\ &\leq d_{T'}(v_P) + 2|X'|,\end{aligned}$$

and equality holds only in Case 3 (for the first inequality) and the case $X' = X$ (for the second one). This implies that

$$\begin{aligned}\mathrm{te}(T,k) &\leq t - \max\{d_{T'}(v_P) - k, 0\} + \sum_{x \in X'} \big(d_T(x) - k\big) \\ &\leq t - \big(d_{T'}(v_P) - k\big) + \sum_{x \in X'} d_T(x) - k|X'| \\ &\leq t + k + 2|X'| - k|X'| \\ &= t - (k-2)(|X'| - 1) + 2.\end{aligned}$$

If $|X'| \geq 3$ or if $\sum_{x \in X'} d_T(x) < d_{T'}(v_P) + 2|X'|$, then we obtain that $\mathrm{te}(T,k) \leq t$ because $k \geq 3$. Thus, we may assume that $|X'| \leq 2$ and $\sum_{x \in X'} d_T(x) = d_{T'}(v_P) + 2|X'|$. This implies that $X' = X$ and Case 3 occurs, that is $|X'| = |X| \geq 3$, but this is impossible. This completes the proof of Claim 4.

By Claim 4, there exists no edge in $G - B$, and hence $A := V(G) - B$ is independent. Then Claim 3 implies that G is a bipartite graph with bipartition A and B and $d_G(x) \geq k + 1$ for all $x \in A$.

If G has a spanning tree T such that $\mathrm{te}(T,k;A) \leq t$ and $d_T(x) \leq k'$ for every $x \in A$, then T is a spanning tree as desired of G, because $d_T(y) \leq d_G(y) = 3 \leq k$ for all $y \in B$. This contradicts the minimality of G. Hence there exists no such a spanning tree in G, and it follows from Theorem 8 that there exists $\widetilde{A} \subset A$ such that $\omega(G - \widetilde{A}) \geq (k-1)|\widetilde{A}| + 2 + t$ or $\omega(G - \widetilde{A}) \geq (k'-1)|\widetilde{A}| + 2$.

Suppose that $A - \widetilde{A} \neq \emptyset$ or $\omega(G - \widetilde{A}) > (k-1)|\widetilde{A}| + 2 + t$ or $\omega(G - \widetilde{A}) > (k'-1)|\widetilde{A}| + 2$. In the first case, we can choose $y \in B$ such that y is not an isolated vertex in $G - \widetilde{A}$. In the remaining two cases, we choose $y \in B$ arbitrarily. By Lemma 11, there exists a contractible edge incident with y, say yx and let $G' := G/(yx)$. We name the new vertex of G' as v_{yx}. Let $\widetilde{A}' := \big(\widetilde{A} - \{x\}\big) \cup \{v_{yx}\}$ if $x \in \widetilde{A}$; otherwise let $\widetilde{A}' := \widetilde{A}$. By the choice of y, we obtain that $\omega(G' - \widetilde{A}') \geq \omega(G - \widetilde{A})$ in the first case, and $\omega(G' - \widetilde{A}') \geq \omega(G - \widetilde{A}) - 1$, in the remaining two cases. In either case, $\omega(G' - \widetilde{A}') \geq (k-1)|\widetilde{A}'| + 2 + t$ or $\omega(G - \widetilde{A}') \geq (k'-1)|\widetilde{A}'| + 2$. However, by Proposition 14, G' does not have a spanning k'-tree T' with $\mathrm{te}(T',k) \leq t$, which contradicts the minimality of G.

Thus, we have that $\widetilde{A} = A$, $\omega(G - A) = \omega(G - \widetilde{A}) \leq (k-1)|\widetilde{A}| + 2 + t$ and $\omega(G - A) \leq (k'-1)|\widetilde{A}| + 2$. Since $|B| = \omega(G - A) \geq (k-1)|\widetilde{A}| + 2 + t$ or $|B| = \omega(G - A) \geq (k'-1)|\widetilde{A}| + 2$, we have that $|B| = (k-1)|A| + 2 + t$ or $|B| = (k'-1)|A| + 2$. This completes the proof of Theorem 10. $\square$

5 Proof of Theorem 4

In order to prove Theorem 4, we will use the following lemma. Since it directly follows from Euler's formula, we omit the proof.

Lemma 15 *Let G be a bipartite graph of order at least 3 on a surface with Euler characteristic χ. Then $|E(G)| \leq 2|V(G)| - 2\chi$.*

Proof of Theorem 4. Let $k := 3$, $t := -2\chi - 1$ and $k' := \lceil \frac{8-2\chi}{3} \rceil$. Suppose that G is a 3-connected graph on a surface with Euler characteristic $\chi < 0$ having no spanning k'-tree T with $\mathrm{te}(T, k) \leq t$. Since $\chi < 0$, we have that $k \leq k'$ and $t \geq 0$. By Theorem 10, G contains, as a minor, a 3-connected bipartite graph R with bipartition A and B such that $B = V_3(R)$, and $|B| = (k-1)|A| + 2 + t$ or $|B| = (k'-1)|A| + 2$. Note that R can be embedded in the surface where G is embedded. Notice also that $|A| \geq 3$ since G is 3-connected. Then it follows from Lemma 15 and the fact "$d_R(y) = 3$ for all $y \in B$" that

$$3|B| = |E(R)| \leq 2|V(R)| - 2\chi = 2|A| + 2|B| - 2\chi,$$

and hence

$$2|A| - 2\chi \geq |B|.$$

If $|B| = (k-1)|A| + 2 + t$, then

$$\begin{array}{rrcl} & 2|A| - 2\chi & \geq & (k-1)|A| + 2 + t, \\ \text{or} & -2\chi - 2 & \geq & (k-3)|A| + t \\ & & = & t \\ & & = & -2\chi - 1. \end{array}$$

On the other hand, if $|B| = (k'-1)|A| + 2$, then

$$\begin{array}{rrcl} & 2|A| - 2\chi & \geq & (k'-1)|A| + 2, \\ \text{or} & -2\chi - 2 & \geq & (k'-3)|A| \\ & & \geq & \left(\left\lceil \frac{8-2\chi}{3} \right\rceil - 3\right) \cdot 3 \\ & & \geq & -2\chi - 1. \end{array}$$

In either case, we obtain a contradiction, and this completes the proof of Theorem 4. $\square$

6 4-connected graphs on surfaces

In this paper, we have considered a spanning tree with bounded maximum degree and bounded total excess in 3-connected graphs on surfaces. However, not only spanning trees of 3-connected graphs, but also that of 4-connected graphs on surfaces have been extensively studied.

As mentioned in Section 1, Tutte [36] proved that every 4-connected plane graph has a Hamilton cycle. Later, Thomas and Yu [31] improved Tutte's result; every 4-connected graph on the projective plane has a Hamilton cycle. Recently, Kawarabayashi and the author [20] further improved it; every 4-connected graph on the projective plane is Hamilton-connected. For graphs on the torus, Grünbaum [13] and independently Nash-Williams [25] conjectured that every 4-connected graph on the torus has a Hamilton cycle. Although this conjecture is still open, there are some partial solutions to it. Brunet and Richter [6] proved that every 5-connected triangulation of the torus has a Hamilton cycle, and Thomas and Yu [32] improved this to 5-connected graphs (not necessarily "triangulation") on the torus. Dean and Ota [7], and Thomas, Yu and Zang [33] showed that every 4-connected graph on the torus has a 2-factor and a Hamilton path, respectively.

It is well-known that there exist infinitely many 4-connected graphs on a surface with Euler characteristic $\chi < 0$ which has no Hamilton cycle. (For example, consider the face subdivision of quadrangulations of a surface. Such graphs have no Hamilton cycle since they are not 1-tough, which is a trivial necessary condition for the existence of a Hamilton cycle.) So, instead of a Hamilton cycle, we consider a spanning tree with bounded maximum degree in such graphs. Actually, Ellingham [8] showed that every 4-connected graph on a surface with Euler characteristic $\chi \leq 0$ has a spanning $\lceil \frac{10-\chi}{4} \rceil$-tree. However, we do not know whether the upper bound on the maximum degree is best possible. Considering the bipartite graph $K_{4,-\chi+4}$, (which can be embedded into a surface with Euler characteristic χ,) the best possible upper bound seems the following;

Conjecture 1 *Let G be a 4-connected graph on a surface with Euler characteristic $\chi \leq 0$. Then G has a spanning $\lceil \frac{7-\chi}{4} \rceil$-tree.*

Ellingham and Gao [9] showed that every 4-connected triangulation of an orientable surface has a spanning 3-tree if the representativity is sufficiently large. Yu [37] improved this result to 4-connected graphs (not necessarily "triangulation") on a surface (not necessarily "orientable"). Mohar (see Page 181 in [23]) conjectured the existence of a spanning 3-tree with few total excess from 2 in 4-connected locally planar graphs.

Conjecture 2 (Mohar) *For every surface $\mathbb{F}^2$ with Euler characteristic $\chi < 0$, there exists a positive integer r such that every 4-connected graph on $\mathbb{F}^2$ with representativity at least r has a spanning 3-tree T with $\mathrm{te}(T, 2) = O(-\chi)$.*

Note that Böhme, Mohar and Thomassen [4] showed that for every $\varepsilon > 0$ and every surface $\mathbb{F}^2$ with Euler characteristic $\chi \leq 0$, there exists a positive integer r such that every 4-connected graph on $\mathbb{F}^2$ with representativity at least r has a spanning

3-tree T with $\mathrm{te}(T,2) < \varepsilon|V(G)|$. As considered Theorem 4 from Theorem 3, we conjecture the following;

Conjecture 3 *Every 4-connected graph on a surface with Euler characteristic $\chi < 0$ has a spanning $\lceil \frac{7-\chi}{4} \rceil$-tree T with $\mathrm{te}(T,2) = O(-\chi)$.*

Note that the face subdivision of a quadrangulation of a surface with Euler characteristic $\chi < 0$ has no spanning tree T with $\mathrm{te}(T,2) \leq -\chi - 2$. So if Conjecture 3 (also Conjecture 2) is true, the upper bound on $\mathrm{te}(T,2)$ is at least $-\chi - 1$. This upper bound might be best possible.

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