Spanning closed walks and TSP in 3-connected planar graphs

Ken-ichi Kawarabayashi[1]

Kenta Ozeki[2]

National Institute of Informatics, 2-1-2
Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan
and
JST, ERATO, Kawarabayashi Large Graph Project, Japan

ABSTRACT

In this paper, we prove the following theorem, which is motivated by two different contexts independently, namely graph theory and combinatorial optimization.

> Given a circuit graph (which is obtained from a 3-connected planar graph by deleting one vertex) with n vertices, there is a spanning closed walk W of length at most $4(n-1)/3$ such that each edge is used by W at most twice.

Our proof is constructive (and purely combinatorial) in the sense that there is an $O(n^2)$-time algorithm to construct such a walk in a given 3-connected planar graph. We shall construct an example that shows that the upper bound $4(n-1)/3$ is essentially tight. We also point out that 2-connected planar graphs may not have such a walk, as $K_{2,n-2}$ shows.

June, 2011, revised April 28, 2014.

Keywords: A closed walk, 3-connected planar graphs, TSP, the Held-Karp relaxation, the integrality gap, and 4/3-conjecture

1 Introduction

Finding a Hamiltonian cycle is arguably one of the most popular subjects in graph theory. There are several sufficient conditions for a graph to have a Hamiltonian cycle. Note that some of the sufficient conditions are mentioned in [8]. Finding a Hamiltonian cycle is also one of the central problems in combinatorial optimization, since it is directly connected to the Traveling Salesman Problem.

A study on Hamiltonian cycles was started with the connection to the Four Color Problem (now Theorem). It had been conjectured since 1880's that every 3-connected cubic planar graph is Hamiltonian, and if true, it would imply the Four Color Problem. However, Tutte [27] in 1946 constructed a counterexample, and in [28], he proved that every 4-connected planar graph is Hamiltonian. As we see here, a study on Hamiltonian cycles for planar graphs is historically one of the central topics in graph theory. In the last decade, a Hamiltonian cycle in planar graphs has also been studied in graph algorithm ([17], for example), because of its connection to the Traveling Salesmen Problem.

[1]Research partly supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research, by C & C Foundation, by Kayamori Foundation and by Inoue Research Award.
Email address: `k_keniti@nii.ac.jp`
[2]This work was supported by JSPS KAKENHI Grant Number 25871053.
Email address: `ozeki@nii.ac.jp`

In this paper, we consider the following problem, which is motivated by two different contexts independently, namely graph theory and combinatorial optimization. A *spanning closed walk* of a graph is a walk that visits all vertices of the graph and turns back to the starting vertex. Sometimes a spanning closed walk is called a *Hamiltonian walk*. The *length* of a spanning closed walk is the total number of transits of edges. Note that a spanning closed walk can use an edge many times, and we count such an edge twice or more for the length. A Hamiltonian cycle of a graph G is equivalent to a spanning closed walk of length exactly $|G|$.

> In a 3-connected planar graph G with n vertices, is there a spanning closed walk W of length at most $4n/3$?

Indeed, we shall show that the answer of this problem is true, and in Section 5, we also show that the bound $4n/3$ is essentially tight. We now clarify why this problem is considered independently both in graph theory and combinatorial optimization.

1.1 Hamiltonian cycles and spanning closed walks in 3-connected planar graphs

In 1956, Tutte [28] proved that every 4-connected planar graph is Hamiltonian. of Tutte. For a short proof, see Thomassen [26]. Thomas and Yu [25] extended Tutte's theorem to the projective planar case. This result cannot be extended to 3-connected planar graphs since there exist planar triangulations G on n vertices such that any longest cycle in G is of length $O(n^{\alpha})$, where $\alpha = \log 2/\log 3 \approx 0.63$; see [20]. However, there are many results showing that every 3-connected planar graph has several properties close to Hamiltonicity. Here, a *k-tree* is a tree with maximum degree at most k, and a closed *k-walk* is a closed walk that visits every vertex at most k times. Note that a spanning 2-tree is exactly a Hamiltonian path, while a spanning closed 1-walk is exactly a Hamiltonian cycle. Barnette [4] proved that every 3-connected planar graph has a spanning 3-tree. Gao and Richter [10] strengthened it by showing that every 3-connected planar graph has a spanning closed 2-walk. (It was shown that if a graph has a spanning closed 2-walk, then it also has a spanning 3-tree, see [15].) Extending the result by Barnette [4] to another direction, Nakamoto, Oda and Ota [21] proved that every 3-connected planar graph with n vertices has a spanning 3-tree in which the number of vertices of degree 3 is at most $\max\{\frac{n-7}{3}, 0\}$. Since a vertex visited twice in a closed 2-walk W corresponds to a vertex of degree 3 in the 3-tree corresponding to W, it is natural to think the following problem.

Problem 1 (Nakamoto, Oda and Ota, [21]) *Does every 3-connected planar graph with n vertices have a spanning closed 2-walk in which the number of vertices visited twice is at most $\max\{\frac{n+c}{3}, 0\}$ for some constant c?*

Problem 1 is still open. It is easily shown that if a graph with n vertices has a spanning closed 2-walk W of length at most $4n/3$, then the number of vertices visited twice in W is at most $n/3$. Thus, as a step to attack Problem 1, we are interested in a spanning closed walk of short length in a 3-connected planar graph. This is our first motivation.

Furthermore, as we have seen, every 3-connected planar graph has nice properties close to Hamiltonicity. Let us observe that Tutte's theorem says that every 4-connected planar graph with n vertices has a spanning closed walk of length exactly n. So what happens if we only consider a 3-connected planar graph, and how many transits of edges are needed to cover all of the vertices by a spanning closed walk? This is also a motivation from graph theory.

1.2 TSP and Combinatorial Optimization

In combinatorial optimization, the well-known Traveling Salesman Problem (TSP) in metric graphs is one of the most fundamental NP-hard optimization problems. The TSP is the following; Given a complete undirected graph G with n vertices with non-negative edge costs, find a Hamiltonian cycle in G of minimum cost. When the costs satisfy the triangle inequality, we call the problem *metric*. A special case of the metric TSP is the so-called graph-TSP, where, given an undirected unweighted simple underlying graph G, the complete graph on $V(G)$ is formed by defining the cost between two vertices as the number of edges on the shortest path between them. The weighted complete graph obtained from G is known as the *metric completion* of G.

In spite of a vast amount of research, several important questions remain open. While the problem is known to be APX-hard and NP-hard to approximate with a ratio better than 220/219 [23], untill 2011, the best upper bound was still the 1.5-approximation algorithm obtained by Christofides [5] more than three decades ago. (Recently, the better bound was shown in [12, 18].) A promising direction to improve this approximation guarantee has long been to understand the power of a linear program, which is known as the Held-Karp relaxation [14]. So we have to study the integrality gap $\alpha(TSP)$, which is the worst-case ratio between the optimal solution to the TSP problem and the optimal solution to its linear programming relaxation (the Held-Karp relaxation). The value $\alpha(TSP)$ gives one measure of the quality of the lower bound provided by the Held-Karp relaxation for the TSP. Moreover, a polynomial-time constructive proof for value $\alpha(TSP)$ would provide an $\alpha(TSP)$-approximation algorithm for the TSP. On the other hand, the best lower bound on $\alpha(TSP)$ is 4/3 and indeed the famous 4/3-conjecture states that this lower bound would be tight.

Conjecture 1 (Goemans, [13]) *$\alpha(TSP)$ is at most 4/3.*

The ratio 4/3 is reached asymptotically by taking two disjoint triangles T_1 and T_2 and three pairwise disjoint paths of length $n/3$ joining $V(T_1)$ and $V(T_2)$. Each edge in $T_1 \cup T_2$ gets the weight 1/2, and other edges have weight 1. Then each vertex receives total weight 2. Therefore, this graph together with the assigned weight satisfies the Held-Karp relaxation. It is clear that this graph is planar. Therefore, Goemans[3] brought attention to Conjecture 1 and especially conjectured that for every plane triangulation with n vertices, we can find a solution to the Held-Karp relaxation whose value is at most $4n/3$, and therefore the 4/3-conjecture is always true for plane triangulations. This is our second motivation.

1.3 Our main results

In this paper, we answer the above problems (which are motivated independently by both graph theory and combinatorial optimization) in the affirmative. Namely;

Theorem 2 *Let G be a 3-connected planar graph with n vertices. Then G has a spanning closed walk of length at most $\frac{4n-4}{3}$. Moreover, given a 3-connected planar graph G, such a walk can be found in $O(n^2)$-time.*

In Section 5, we give an example that shows that the bound $\frac{4n-4}{3}$ is essentially tight. Theorem 2 extends an old result by Asano, Nishizeki and Watanabe [2, 3] who proved that if G is a planar triangulation with n vertices, then G has a spanning closed walk of length at most $\frac{3n}{2}$. (Also they [22] gave an algorithm to find such a walk.) In fact, they conjectured that a planar triangulation has a a spanning closed walk of length at most $\frac{4n}{3}$. Theorem 2 gives an even stronger statement. We also point out that 2-connected planar graphs may not have such a walk, for example, consider $K_{2,n-2}$.

Let us point out that although our algorithm in Theorem 2 implies an $O(n^2)$-time algorithm to give a 4/3-approximation algorithm for the TSP for 3-connected planar graphs, Klein [17] gave a much better result. He gave a linear time approximation scheme ($(1+\epsilon)$-approximation for any $\epsilon > 0$) for

[3]See "17. Maximal Planar Graphs" in http://www.math.mcgill.ca/~bshepherd/Bellairs/bellairs2007.pdf

the TSP for planar graphs. Klein's result was extended to bounded genus graphs [7] and recently to H-minor-free graphs [6]. In view of these extensions, we conjecture that Theorem 2 can be extended to the bounded genus graphs and probably to the H-minor-free graphs.

Leaving the plane to consider closed surfaces of higher genera, there are some technical difficulties. For example, the chromatic number of graphs on non-spherical closed surfaces can dramatically increase. However, for graphs which obey certain local planarity conditions, one can deduce similar properties as for planar ones. Quantitatively, we introduce the *representativity* of a graph G on a non-spherical closed surface $\mathbb{F}^2$ as the length of a shortest curve that is noncontractible in $\mathbb{F}^2$, where a *noncontractible curve* on $\mathbb{F}^2$ is a curve that does not bound a disk on $\mathbb{F}^2$. For a partial result in this direction on graphs on closed surfaces of bounded genus, we prove Theorem 3 concerning graphs with high representativity. Since this is not a main target of this paper, we put the proof of Theorem 3 in Appendix.

Theorem 3 *For every closed surface $\mathbb{F}^2$ of Euler characteristic χ with $\chi \leq 1$, there exists an integer r such that every 3-connected graph with n vertices on $\mathbb{F}^2$ with representativity at least r has a spanning closed walk of length at most $\frac{4n+2-6\chi}{3}$.*

In Section 5, we shall show that this result is also essentially tight.

Technically, we shall prove Theorem 2 by adapting the notion a *circuit graph*. In order to state our technical result which implies Theorem 2, we need some definitions that will be given in the next section.

1.4 Technical statement

For notation not defined in this paper, we refer to the book [8]. For a graph G, we write $2G$ for the multigraph obtained from G by replacing each edge of G with two multiple edges. A graph R is called *even* if each vertex of R has an even degree. It is easy to see the following:

(1) A graph G has a spanning closed walk of length at most t if and only if G has a spanning closed walk W' of length at most t such that each edge of G is used by W' at most twice.

(2) A graph G has a spanning closed walk W' such that each edge of G is used by W' at most twice and W' has the length at most t if and only if $2G$ has a spanning connected even subgraph R with $|E(R)| \leq t$.

These imply the following proposition.

Proposition 4 *A graph G has a spanning closed walk of length at most t if and only if the multigraph $2G$ has a spanning connected even subgraph R with $|E(R)| \leq t$.*

Theorem 2 concerns 3-connected planar graphs. However, for our purpose to show Theorem 2, it turns out that 3-connectivity is too strong. We have to relax it in order to apply our induction hypothesis, see Theorem 5. Therefore, we need to define "circuit graphs".

For a 2-connected plane graph G, the *outer cycle* of G is the cycle bounding the infinite face. For the outer cycle C of a 2-connected plane graph G, a pair (G, C) is called a *circuit graph* if for every vertex x in $G - V(C)$, there exist three paths in G such that they connect x and C and they are pairwise disjoint except for x. (A circuit graph is sometimes called an *internally 3-connected plane graph* or an *I3CP graph*.) In other words, there exists no vertex set that consists of at most two vertices and separates some vertices in $G - V(C)$ from C. Note that for every 3-connected plane graph G, the pair (G, C) is a circuit graph, where C is the outer cycle of G. For $u, v \in V(C)$, we denote, by $C[u, v]$, the subpath of C starting from u and ending at v in the clockwise order. A vertex v in a graph G is a 2-*vertex* if the degree of v is exactly two in G. Note that every 2-vertex of a circuit graph (G, C) is contained in C. For a positive integer i, we denote the complete graph with i vertices by K_i. Note that (K_3, C) is a circuit graph, where C is the unique cycle of K_3. For two graphs G and H, we write $G \cong H$ if G is isomorphic to H; otherwise $G \not\cong H$.

By Proposition 4, the following theorem implies Theorem 2. So in Section 3, we show Theorem 5 instead of Theorem 2.

Theorem 5 *Let (G, C) be a circuit graph with $G \not\cong K_3$. Then $2G$ has a spanning connected even subgraph R such that*

(R1) $|E(R)| \leq \frac{4|V(G)|-4}{3}$,

(R2) *every 2-vertex of G has degree two in R, and*

(R3) *for every edge e connecting two 2-vertices in G, the subgraph R contains exactly one of the two edges e_1 and e_2 of $2G$, where both e_1 and e_2 correspond to e in G.*

In Section 5, we show that the bound $\frac{4|V(G)|-4}{3}$ is best possible.

We can easily show the following.

Proposition 6 *Let $G \cong K_3$. Then $2G$ has a spanning connected even subgraph R such that $|E(R)| = 3 = \frac{4|V(G)|-3}{3}$, and R satisfies conditions (R2) and (R3).*

Note that Theorem 5 and Proposition 6 state that for every circuit graph (G, C), the multigraph $2G$ has a spanning connected even subgraph R such that (R1') $|E(R)| \leq \frac{4|V(G)|-3}{3}$ and R satisfies conditions (R2) and (R3).

In the next section, we shall give more fundamental lemmas concerning circuit graphs.

2 Some preliminary lemmas for circuit graphs

As we pointed out before, our main task is to give a proof of Theorem 5. Thus we need to give several nice properties on circuit graphs. Roughly, circuit graphs have some nice recursive structure, which allows us to apply induction.

2.1 Lemmas concerning a circuit graph

For a circuit graph, it is easy to see the following. See, for example, [10].

Lemma 7 *Let (G, C) be a circuit graph and let C' be a cycle in G. Let G' be the graph induced by C' together with its interior edges. Then (G', C') is a circuit graph.*

Lemma 8 *Let (G, C) be a circuit graph, and let u, v be two vertices of C such that $G - \{u, v\}$ is not connected. Let D be a component of $G - \{u, v\}$ and $\overline{D} = G - \{u, v\} - D$. (Note that both D and $\overline{D}$ contains a vertex of C, by the definition of a circuit graph.) Let G' be the plane graph obtained from G by deleting $\overline{D}$ and adding a path P from u to v so that the outer cycle C' of G' contains P and $E(C) \cap E(G - \overline{D})$. Then (G', C') is also a circuit graph.*

A circuit graph (G, C) is *edge-minimal* if for every edge e of G, the pair (G', C') is not a circuit graph, where $G' = G - e$ and C' is the outer cycle of G'. It follows from Lemma 7 that for an edge e in C, the pair (G', C') is not a circuit graph if and only if G' is not 2-connected.

A *block* in a graph H (possibly H might not be connected, or H is 2-connected) is a maximal subgraph of H that has no cut vertex. Note that every block of any graph is 2-connected or isomorphic to K_2. A block B is an *end block* of a graph H if B contains at most one cut vertex of H.

A *chain of blocks* in a graph H is a sequence $B_0B_1 \cdots B_m$ such that each B_i is a block of H, $B_i \cap B_j = \emptyset$ for $0 \leq i < j \leq m$ with $j \neq i+1$, and $|V(B_i \cap B_{i+1})| = 1$ for $0 \leq i \leq m-1$. So the reader can think of this chain of blocks as a block decomposition such that the abstract tree is a path. (For the definition of a block decomposition and the abstract tree, see, for example, [29].) For a block B of a graph H, let $I_H(B)$ be the set of vertices x in B such that x is not a cut vertex in H.

The following lemma is also obvious. (See [21].)

Lemma 9 *Let (G, C) be an edge-minimal circuit graph, and let e be an edge in C. Then the following holds;*

(i) *The graph G' is a chain of blocks $B_0B_1 \cdots B_m$ with $m \geq 1$, where $G' = G - e$.*

(ii) *One end vertex of e, say x_0, is contained in $I_{G'}(B_0)$, and the other, say x_{m+1}, is contained in $I_{G'}(B_m)$.*

(iii) *Let G_0 be the graph obtained from G by contracting $I_{G'}(B_m)$ into one vertex, and let C_0 be the outer cycle of G_0. Then (G_0, C_0) is also a circuit graph. Moreover, if $d_G(x_0) \geq 3$, then $G_0 \not\cong K_3$.*

For a circuit graph (G, C), the following lemma guarantees the existence of the end block of $G - V(C)$ with an "end" property. This can be easily shown by the planarity, so we omit the proof.

Lemma 10 *Let (G, C) be a circuit graph with $V(G) - V(C) \neq \emptyset$. Let $H = G - V(C)$. Then there exist an end block B_0 of H and two neighbors u_0 and v_0 of $I_H(B_0)$ in C such that*

(i) *$C[u_0, v_0]$ contains all neighbors of $I_H(B_0)$ in C, and*

(ii) *$C[u_0, v_0] - \{u_0, v_0\}$ contains no neighbors of $H - I_H(B_0)$ in C.*

2.2 An extended chain of blocks

In this subsection, we shall define an *extended chain of blocks* and its maximality, both of which plays a key role in our proof of Theorem 5. This concept is somewhat technical, but roughly speaking, we want to contract a certain part of a circuit graph (G, C) and apply the induction hypothesis. It turns out that some special property of blocks in $G - V(C)$ (which is exactly "an extended chain of blocks") is suitable for this purpose.

Let (G, C) be a circuit graph, and let $H = G - V(C)$. Let B_0, u_0, and v_0 be an end block of H and two neighbors of $I_H(B_0)$ in C as in Lemma 10, respectively.

An *extended chain of blocks* in H for C from B_0 is either (i) a chain $B_0B_1 \cdots B_m$ of blocks or (ii) two chains $B_0B_1 \cdots B_k$ and $B_{k+1} \cdots B_m$ of blocks of H such that they satisfy the following conditions (B1)–(B7). See the left side of Figure 13 for (i), and the right side for (ii). Note that the outer ellipse represents the outer cycle C. Let $\{x_{i+1}\} = V(B_i \cap B_{i+1})$ for $0 \leq i \leq m-1$ with $i \neq k$ if (ii) occurs.

(B1) If (ii) occurs, then both B_k and B_{k+1} are end blocks of H, and $B_k \cap B = \emptyset$ for each block B of H with $B \neq B_k, B_{k-1}$.

(B2) For $0 \leq i \leq m-1$, the cut vertex x_{i+1} is contained in exactly two blocks B_i and B_{i+1} of H, unless (ii) occurs and $i = k$.

(B3) For $0 \leq i \leq m$, the block B_i contains at most two cut vertices of H.

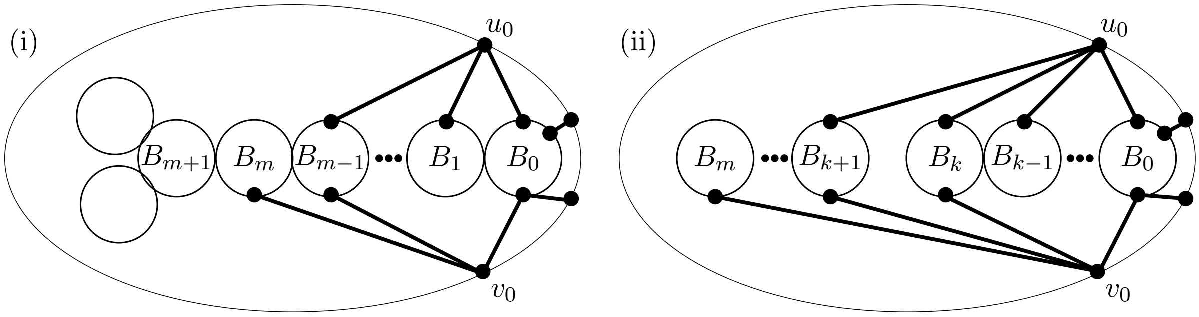

Figure 1: An extended chain of blocks of H for C from B_0.

(B4) For $0 \leq i \leq m$, at least one of u_0 and v_0 has a neighbor in $B_i - x_i$.

(B5) If (ii) occurs, then both u_0 and v_0 has a neighbor in $I_H(B_{k+1})$.

(B6) If there exists a block B_{m+1} such that $B_{m+1} \neq B_{m-1}$ and $|V(B_m \cap B_{m+1})| = 1$, then $B_m - x_{m+1}$ has no neighbors in $C[v_0, u_0] - \{u_0, v_0\}$, where $\{x_{m+1}\} = V(B_m \cap B_{m+1})$.

(B7) If (ii) occurs and $k = m - 1$, then B_m itself is a component of H, or $I_H(B_m)$ has no neighbors in $C[v_0, u_0] - \{u_0, v_0\}$.

Note that the sequence consisting of only B_0 satisfies conditions (B1)–(B7). Notice also that by conditions (B1)–(B3), an extended chain of blocks in H for C from B_0 consists of at most two chains of blocks whose abstract trees are paths in H. Moreover, if (ii) occurs, then the chain of blocks containing B_0 is a component in H.

An extended chain $B_0 B_1 \cdots B_m$ of blocks in H for C from B_0 is called *maximal* if $B_0 B_1 \cdots B_m B_{m+1}$ is not an extended chain of blocks in H for every block B_{m+1} of H.

This maximal extended chain plays a key role in our proof, because what we are trying to do is to contract the subgraphs in the maximal extended chain, together with some vertices on the outer cycle. Let us observe that each block in the maximal extended chain is a circuit graph and therefore, we can use the induction hypothesis to each block. We want to show that the resulting graph after the contraction satisfies the assumptions of Theorem 5. This is not quite true, but Lemma 11 (see the end of this section) tells us that this is *almost* true. Assuming Lemma 11, we will try to glue a spanning closed walk in the extended maximal chain and a spanning closed walk in the resulting graph together, and then obtain a spanning closed walk of length at most $4(n-1)/3$ in the original graph. As we see here, Lemma 11 below is a key, but in order to mention this lemma, we need to analyze the structure of a maximal extended chain more carefully.

Let $B_0 B_1 \cdots B_m$ be a maximal extended chain of blocks in H for C from B_0. By the maximality of m, we can divide the properties of the sequence $B_0 B_1 \cdots B_m$ into the following types.

Type 1. There exists a block B_{m+1} such that $B_{m+1} \neq B_{m-1}$ and $|V(B_m \cap B_{m+1})| = 1$.

This condition implies that the sequence $B_0 \cdots B_m B_{m+1}$ satisfies conditions (B1), (B5) and (B7). Let $\{x_{m+1}\} = V(B_m \cap B_{m+1})$. By the maximality of m, we can also divide Type 1 into the following four types. Note that the sequence $B_0 \cdots B_{m+1}$ of Types 1.1, 1.2, 1.3 and 1.4 does not satisfy conditions (B2), (B3), (B4) and (B6), respectively.

Type 1.1. The vertex x_{m+1} is contained in at least three blocks of H.
Type 1.2. The block B_{m+1} contains at least three cut vertices of H.

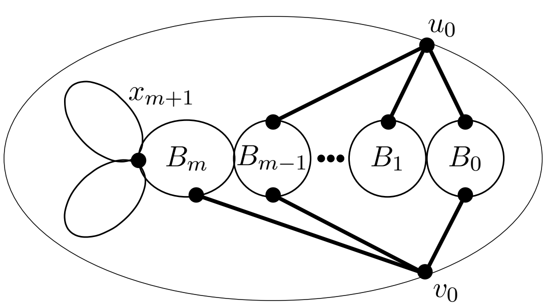

Figure 2: Type 1.1

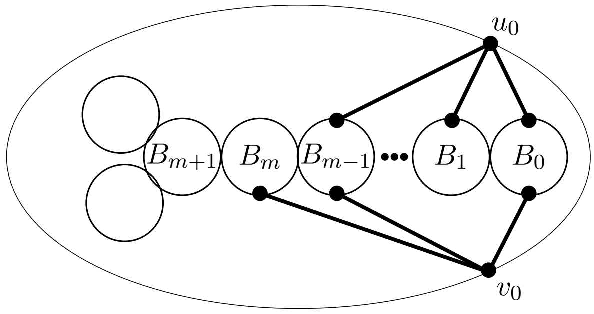

Figure 3: Type 1.2

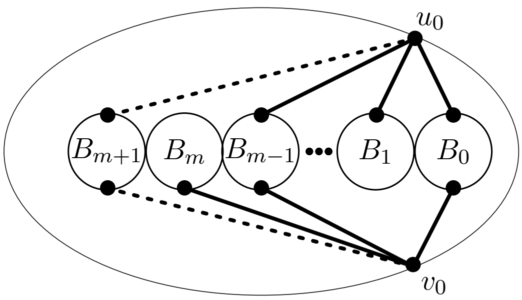

Figure 4: Type 1.3

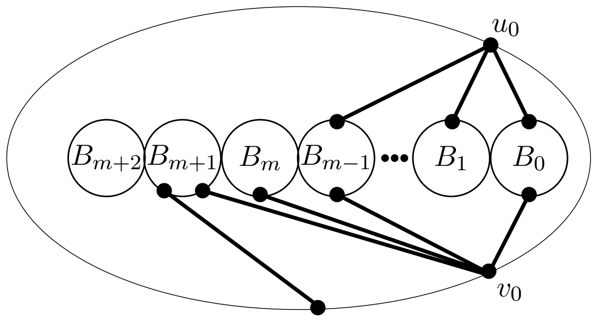

Figure 5: Type 1.4

Type 1.3. Neither u_0 nor v_0 is a neighbor of $B_{m+1} - x_{m+1}$.
Type 1.4. There exists a block B_{m+2} such that $B_{m+2} \neq B_m$, $|V(B_{m+1} \cap B_{m+2})| = 1$, and $B_{m+1} - x_{m+2}$ has a neighbor in $C[v_0, u_0] - \{u_0, v_0\}$, where $\{x_{m+2}\} = V(B_{m+1} \cap B_{m+2})$.

Type 2. There exists no block B such that $B \neq B_{m-1}$ and $|V(B \cap B_m)| = 1$.

This condition implies that B_m is an end block of H. So, for all end blocks B of H with $B \neq B_i$ for $0 \leq i \leq m$, the sequence $B_0 \cdots B_m B$ satisfies conditions (B1) and (B2). If there exists a block B_{m+1} of $H - \bigcup_{i=0}^{m} B_i$ such that B_{m+1} itself is a component of H and both u_0 and v_0 is a neighbor of B_{m+1}, then $B_0 \cdots B_m B_{m+1}$ also satisfies conditions (B3)–(B7), contradicting the maximality of m. Thus, Type 2 can be also divided into the following types.

Type 2.1. There exists a component D' of $H - \bigcup_{i=0}^{m} B_i$ such that D' has at least two blocks and both u_0 and v_0 is a neighbor of D'.
Type 2.2. There exists no component D' of $H - \bigcup_{i=0}^{m} B_i$ such that both u_0 and v_0 is a neighbor of D'.

We have the following lemma, which states that the contraction of a maximal extended chain of blocks keeps the property "being a circuit graph", with a few exceptions. (See Figure 6. In Figure 6, the chain of blocks $B_0 B_1 \ldots B_m$ together with four blocks inside of C represents D.) This plays an important role in the proof of Theorem 5.

Lemma 11 *Let (G, C) be a circuit graph, and let $H = G - V(C)$. Let B_0, u_0 and v_0 be an end block of H and two neighbors of $I_H(B_0)$ in C as in Lemma 10, respectively. Let $B_0 B_1 \ldots B_m$ be a maximal extended chain of blocks of H for C from B_0. Let G_0 be the graph obtained from G by contracting $\bigcup_{i=0}^{m-1} B_i \cup I_H(B_m) \cup C[u_0, v_0]$ into one vertex. Let C_0 be the outer cycle of G_0. If $|V(G_0)| \geq 3$, then*

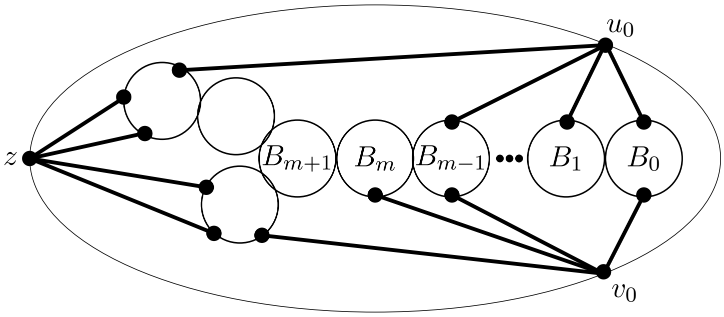

Figure 6: An exception to Lemma 11.

(G_0, C_0) is a circuit graph, unless there exists a subgraph D of H such that D satisfies the following properties;

(D1) *The subgraph D consists of some components of H.*

(D2) *For $0 \leq i \leq m$, the block B_i is contained in D.*

(D3) *The subgraph D has exactly one neighbor in $C[v_0, u_0] - \{u_0, v_0\}$, say z.*

(D4) *The subgraph D' is a chain of blocks with at least two end blocks, where $D' = D - \bigcup_{i=0}^{m-1} B_i - I_H(B_m)$, and z is a neighbor of both end blocks.*

(D5) *The pair (G'_0, C_0) is a circuit graph with $G'_0 \not\cong K_3$, where $G'_0 = G_0 - D'$.*

We will prove Lemma 11 in Section 4.

3 Proof of Theorem 5

Assume that Theorem 5 does not hold and let (G, C) be a minimum counter example, that is, $2G$ has no spanning connected even subgraph with desired conditions, but $2G'$ has for every circuit graph (G', C') with $|V(G')| < |V(G)|$, or $|V(G')| = |V(G)|$ and $|E(G')| < |E(G)|$.

Suppose that (G, C) is not an edge-minimal circuit graph, that is, there exists an edge e in G such that (G', C') is also a circuit graph, where $G' = G - e$ and C' is the outer cycle of G'. Then by the minimality of (G, C), the multigraph $2G'$ has a spanning connected even subgraph with desired conditions. However it is also a spanning connected even subgraph with desired conditions in $2G$, a contradiction. Hence the following holds:

Fact 1 *The circut graph (G, C) is edge-minimal.*

Now we show the following claim.

Claim 2 *Every edge in C is incident with a 2-vertex of G.*

Proof of Claim 2. Suppose not, and let e be an edge of C which is incident with no 2-vertex. Let $G' = G - e$. By Lemma 9 (i)–(ii), the graph G' is a chain of blocks $B_0 B_1 \cdots B_m$ with $m \geq 1$ such that one end vertex of e, say x_0, is contained in $I_{G'}(B_0)$, and the other, say x_{m+1}, is contained in $I_{G'}(B_m)$. By the choice of e, we have $d_G(x_0) \geq 3$ and $d_G(x_{m+1}) \geq 3$. If $m = 1$ and $|B_0| = |B_m| = 3$, then we can

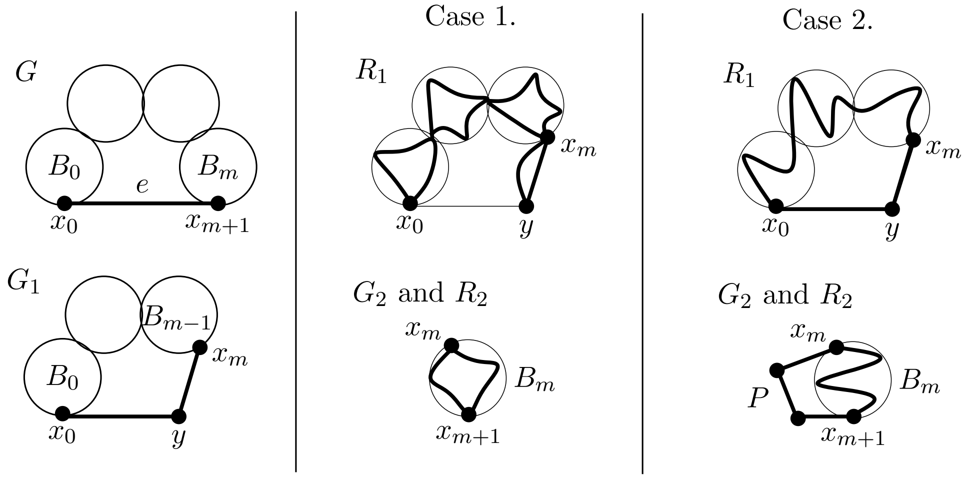

Figure 7: The graphs G_1, G_2, R_1 and R_2 in the proof of Claim 2.

easily find a spanning connected even subgraph with desired conditions in $2G$. Hence by symmetry, we may assume that $m \geq 2$ or $|B_0| \geq 4$.

Let G_1 be the graph obtained from G by contracting $I_{G'}(B_m)$ into one vertex, say y, and let C_1 be the outer cycle of G_1. See the left side of Figure 7. Note that y is a 2-vertex in G_1.

Since $d_G(x_{m+1}) \geq 3$, we have $|B_m| \geq 3$, and hence $|V(G_1)| = |V(G)| - |I_{G'}(B_m)| + 1 < |V(G)|$. By Lemma 9 (iii), the pair (G_1, C_1) is also a circuit graph with $G_1 \ncong K_3$. Then it follows from the minimality of (G, C) that $2G_1$ has a spanning connected even subgraph R_1 satisfying conditions (R1)–(R3). Let $\widetilde{R_1} = R_1 - y$. Note that possibly except for x_0 and x_m, each vertex of $\widetilde{R_1}$ has an even degree, where $\{x_m\} = V(B_{m-1} \cap B_m)$. Since y is a 2-vertex in G_1, it follows from condition (R2) that $d_{R_1}(y) = 2$. Hence by condition (R1), we have

$$|E(\widetilde{R_1})| \leq \frac{4|V(G_1)| - 4}{3} - 2. \tag{1}$$

We divide the proof into 2 cases depending on parities of the degrees of x_0 and x_m: the case where both x_0 and x_m have even degrees in $\widetilde{R_1}$ (Case 1), and the case where both have odd degrees (Case 2). Before considering two cases separately, we show a common property in both cases. In Case 1, note that the two edges of R_1 incident with y come from one edge of G. In this case, clearly $\widetilde{R_1}$ is connected. On the other hand, in Case 2, the two edges of R_1 that are incident with y come from two distinct edges of G. In this case, if $\widetilde{R_1}$ is not connected, then each component of $\widetilde{R_1}$ contains exactly one of x_0 and x_m, and hence it does not satisfy the handshake lemma, a contradiction. Therefore, in either case, $\widetilde{R_1}$ is connected.

Case 1. Both x_0 and x_m have even degrees in $\widetilde{R_1}$.

In this case, see the middle of Figure 7. Let $G_2 = B_m$ and let C_2 be the outer cycle of G_2. By Lemma 7 (and since $|B_m| \geq 3$), the pair (G_2, C_2) is also a circuit graph. By the minimality of (G, C) or by Proposition 6, the multigraph $2G_2$ has a spanning connected even subgraph R_2 satisfying conditions (R1'), (R2) and (R3).

Let $R = \widetilde{R_1} \cup R_2$. Since each vertex of $\widetilde{R_1}$ and R_2 has an even degree and both $\widetilde{R_1}$ and R_2 are connected, the graph R is a spanning connected even subgraph of $2G$. Since both $\widetilde{R_1}$ and R_2 satisfy conditions (R2) and (R3), the subgraph R of $2G$ also satisfies conditions (R2) and (R3), (Note that each 2-vertex of G is also a 2-vertex in one of the graphs G_1 and G_2) Notice also that

$$|V(G_2)| = |B_m| = |V(G)| - |V(G_1)| + 2.$$

Then by inequality (1) and condition (R1') for R_2, we have

$$\begin{aligned}
|E(R)| &= |E(\widetilde{R_1})| + |E(R_2)| \\
&\leq \frac{4|V(G_1)| - 4}{3} - 2 + \frac{4|V(G_2)| - 3}{3} \\
&< \frac{4\big(|V(G_1)| + |V(G_2)| - 2\big) - 4}{3} \\
&= \frac{4|V(G)| - 4}{3}.
\end{aligned}$$

Hence R also satisfies condition (R1), contradicting the assumption that (G, C) is a minimum counter example. □

Case 2. Both x_0 and x_m have odd degrees in $\widetilde{R_1}$.

In this case, see the middle of Figure 7. Note that R_1 contains an edge connecting x_0 and y and an edge connecting y and x_m.

Let G_2 be the graph obtained from B_m by adding the path P of length three from x_m to x_{m+1} (with two new vertices) so that the outer cycle C_2 of G_2 contains P and $E(C) \cap E(B_m)$. Since $m \geq 2$ or $|B_0| \geq 4$, we have $|V(G_1)| \geq 5$. Hence

$$|V(G_2)| = |B_m| + 2 = |V(G)| - |V(G_1)| + 4 < |V(G)|.$$

Note that $G_2 \ncong K_3$. By Lemma 8 for $D = B_m - \{x_m, x_{m+1}\}$, the pair (G_2, C_2) is also a circuit graph. It follows from the minimality of (G, C) that $2G_2$ has a spanning connected even subgraph R_2 satisfying conditions (R1)–(R3). Let $\widetilde{R_2}$ be the graph obtained from R_2 by removing two internal vertices of P. By conditions (R1) and (R2), we have $|E(\widetilde{R_2})| \leq \frac{4|V(G_2)|-4}{3} - 3$. By condition (R3), both x_m and x_{m+1} have odd degree in $\widetilde{R_2}$, but all other vertices have even degree.

Let $R = \widetilde{R_1} \cup \widetilde{R_2} \cup \{x_0 x_{m+1}\}$. By the construction, the graph R is a spanning connected even subgraph of $2G$ such that R satisfies conditions (R2) and (R3). Moreover, by inequality (1) and condition (R1) for R_2, we have

$$\begin{aligned}
|E(R)| &= |E(\widetilde{R_1})| + |E(\widetilde{R_2})| + 1 \\
&\leq \frac{4|V(G_1)| - 4}{3} - 2 + \frac{4|V(G_2)| - 4}{3} - 3 + 1 \\
&= \frac{4\big(|V(G_1)| + |V(G_2)| - 4\big) - 4}{3} \\
&= \frac{4|V(G)| - 4}{3},
\end{aligned}$$

and hence R also satisfies condition (R1), a contradiction again. This discussion completes the proof of Claim 2. □

If $G = C$, then C itself is a spanning connected even subgraph of $2G$ such that C satisfies conditions (R1)–(R3). Hence we may assume that $V(G) - V(C) \neq \emptyset$. Then it follows from Claim 2 that $|V(C)| \geq 6$. Let $H = G - V(C)$. Let B_0, u_0, and v_0 be an end block of H and two neighbors of $I_H(B_0)$ in C as in Lemma 10, respectively. Let $B_0 B_1 \ldots B_m$ be a maximal extended chain of blocks of H for C from B_0. Let G_0 be the graph obtained from G by contracting $\bigcup_{i=0}^{m-1} B_i \cup I_H(B_m) \cup C[u_0, v_0]$ to one vertex. See Figure 8. We next prove the following claim.

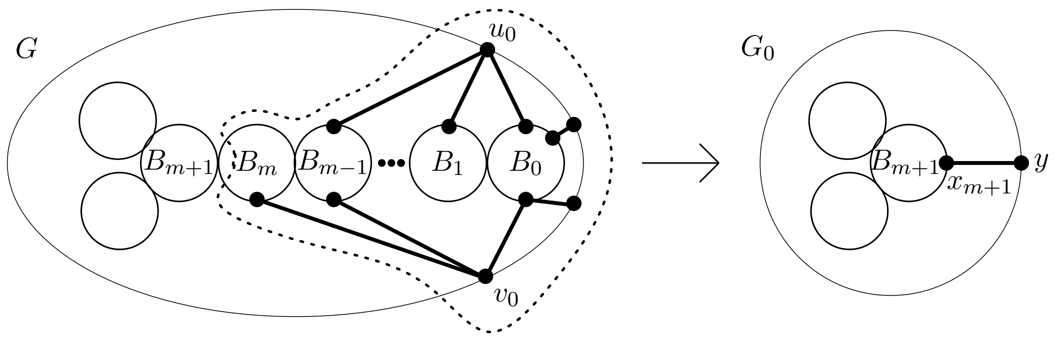

Figure 8: The graph G_0 for an extended chain of Type 1.

Claim 3 $|V(G_0)| \geq 4$.

Proof. Assume that $|V(G_0)| \leq 3$. Note that $|C[v_0, u_0]| \leq 4$. Hence by Claim 2, H has no neighbors in $C[v_0, u_0] - \{u_0, v_0\}$.

Suppose first $H \neq \bigcup_{i=0}^{m} B_i$. Then there exists a vertex x in $H - \bigcup_{i=0}^{m} V(B_i)$. It follows from Claim 2 that $C[v_0, u_0] - \{u_0, v_0\} \neq \emptyset$, and hence we have $|V(G_0) - V(H)| \geq 2$. This inequality implies that $V(H) = \{x\} \cup \bigcup_{i=0}^{m} B_i$. Since x has no neighbors in $C[v_0, u_0] - \{u_0, v_0\}$, both u_0 and v_0 are neighbors of x. However, letting B_{m+1} be the block of H containing x, (which consists of only two vertices,) the sequence $B_0 \cdots B_m B_{m+1}$ is also an extended chain of blocks of H, contradicting the maximality of m. Hence we have

$$H = \bigcup_{i=0}^{m} B_i.$$

Suppose next $m = 0$. If $|B_0| \leq 3$, then we can easily show that $2G$ has a spanning connected even subgraph satisfying conditions (R1)–(R3). So we may assume that $|B_0| \geq 4$. Let $G_1 = B_0$ and C_1 be the outer cycle of G_1. Then G_1 is 2-connected, and hence by Lemma 7, the pair (G_1, C_1) is a circuit graph with $4 \leq |V(G_1)| = |V(G)| - |V(C)|$. By the minimality of (G, C), the multigraph $2G_1$ has a spanning connected even subgraph R_1 satisfying conditions (R1)–(R3). Let e and e' be the edges in $2G$ such that both of e and e' correspond to an edge connecting B_0 and C in G. Let $R = R_1 \cup C \cup \{e, e'\}$. Then R is a spanning connected even subgraph of $2G$ such that R satisfies conditions (R2) and (R3). Moreover, since $|V(G_1)| = |V(G)| - |V(C)|$ and $|V(C)| \geq 6$ (by Claim 2), we have

$$\begin{aligned} |E(R)| &= |E(R_1)| + |E(C)| + 2 \\ &\leq \frac{4|V(G_1)| - 4}{3} + |V(C)| + 2 \\ &= \frac{4\big(|V(G_1)| + |V(C)|\big) - 4}{3} - \frac{|V(C)| - 6}{3} \\ &\leq \frac{4|V(G)| - 4}{3}. \end{aligned}$$

Then R also satisfies condition (R1).

Now we suppose $m \geq 1$. This inequality implies that $|V(H)| \geq 3$.

If H is not connected, then a component of H not containing B_0 can have only two neighbors u_0 and v_0, contradicting that (G, C) is a circuit graph. This discussion implies that H is connected, and hence $|V(B_i \cap B_{i+1})| = 1$ for $0 \leq i \leq m - 1$. Since the pair (G, C) is a circuit graph and H has no neighbors in $C[v_0, u_0] - \{u_0, v_0\}$, the vertex u_0 is a neighbor of $I_H(B_m)$. Let G_2 be the subgraph of G induced by $V(H) \cup \{u_0\}$. See Figure 9. By condition (B4) and the assumption that "H is connected and $m \geq 1$", the graph G_2 is 2-connected. Hence (G_2, C_2) is a circuit graph, where C_2 is the outer cycle of G_2. Since $m \geq 1$, we have $|V(G_2)| \geq 4$. It follows from the minimality of (G, C) that $2G_2$ has a spanning connected even subgraph R_2 satisfying conditions (R1)–(R3).

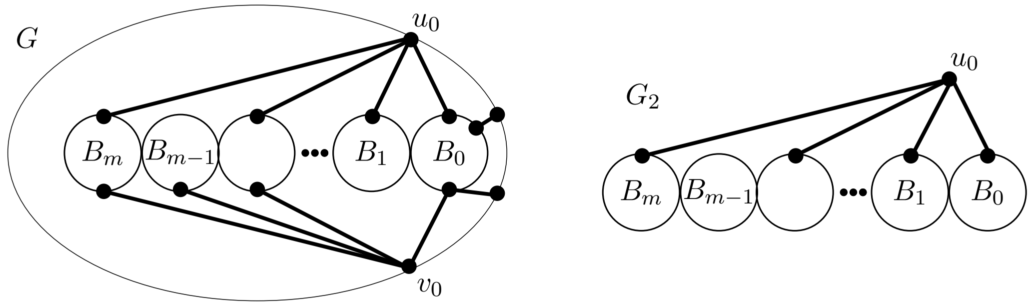

Figure 9: The graphs G_2 and R_2 in the proof of Claim 3.

Let $R = R_2 \cup C$. Then R is a spanning connected even subgraph of $2G$ such that R satisfies conditions (R2) and (R3). Moreover, since $|V(G_2)| = |V(G)| - |V(C)| + 1$ and $|V(C)| \geq 6$ (by Claim 2), we have

$$\begin{aligned} |E(R)| &= |E(R_2)| + |E(C)| \\ &\leq \frac{4|V(G_2)| - 4}{3} + |V(C)| \\ &= \frac{4\big(|V(G_2)| + |V(C)| - 1\big) - 4}{3} - \frac{|V(C)| - 4}{3} \\ &\leq \frac{4|V(G)| - 4}{3}. \end{aligned}$$

This discussion completes the proof of Claim 3. $\square$

Let y be the vertex of G_0 obtained by contracting $\bigcup_{i=0}^{m-1} B_i \cup I_H(B_m) \cup C[u_0, v_0]$. Now we prove the following claim. Let C_0 be the outer cycle of G_0.

Claim 4 *The multigraph $2G_0$ has a spanning connected even subgraph satisfying conditions (R1)–(R3).*

Proof. If (G_0, C_0) is a circuit graph, then it follows from Claim 3 and the minimality of (G, C) that the statement is obvious. Then we may assume that (G_0, C_0) is not a circuit graph. By Lemma 11 and Claim 3, there exists a subgraph D of H such that D satisfies conditions (D1)–(D5). By condition (D5), the pair (G_0', C_0) is a circuit graph, where $G_0' = G_0 - D'$. Let G_0'' be the subgraph of G_0 induced by $V(D') \cup \{z\}$, where z is the unique neighbor of D in $C[v_0, u_0]$ with $z \neq u_0$ and $z \neq v_0$. By conditions (D3) and (D4), the graph G_0'' is 2-connected, and hence it follows from Lemma 7 that (G_0'', C_0'') is a circuit graph, where C_0'' is the outer cycle of G_0''. Note that $|V(G_0')| < |V(G)|$, $|V(G_0'')| < |V(G)|$, $G_0' \not\cong K_3$, and $G_0'' \not\cong K_3$. Then it follows from the minimality of (G, C) that both $2G_0'$ and $2G_0''$ have spanning connected even subgraphs R_0' and R_0'' satisfying conditions (R1)–(R3), respectively.

Let $R_0 = R_0' \cup R_0''$. Then R_0 is a spanning connected even subgraph of $2G_0$ such that R_0 satisfies conditions (R2) and (R3). Moreover, since $|V(G_0)| = |V(G_0')| + |V(G_0'')| - 1$, we obtain

$$\begin{aligned} |E(R_0)| &= |E(R_0')| + |E(R_0'')| \\ &\le \frac{4|V(G_0')| - 4}{3} + \frac{4|V(G_0'')| - 4}{3} \\ &= \frac{4\big(|V(G_0')| + |V(G_0')| - 1\big) - 4}{3} \\ &= \frac{4|V(G_0)| - 4}{3}, \end{aligned}$$

and hence R_0 also satisfies condition (R1). This discussion completes the proof of Claim 4. $\square$

By Claim 4, the multigraph $2G_0$ has a spanning connected even subgraph R_0 satisfying conditions (R1)–(R3). In the remaining parts of the proof, we deal with two types (Types 1 and 2) for $B_0 \cdots B_m$ at that same time. To do that we first set some terminology.

Suppose first that the sequence $B_0 \cdots B_m$ is of Type 1. In this case, the vertex x_{m+1} exists, and $2G_0$ has the two edges, say e and e', such that e and e' connect x_{m+1} and y. Let $\widetilde{R_0}$ be the subgraph of $2G$ such that $\widetilde{R_0}$ is induced by the all edges of R_0 except for the edges e and e' if R_0 contains them. Then each vertex has even degree in $\widetilde{R_0}$, possibly except for u_0, v_0 and x_{m+1}.

Suppose next that the sequence $B_0 \cdots B_m$ is of Type 2. In this case, the vertex x_{m+1} does not exist. Let $\widetilde{R_0}$ be the subgraph of $2G$ such that $\widetilde{R_0}$ is induced by the edges of R_0. Then each vertex has even degree in $\widetilde{R_0}$, possibly except for u_0 and v_0. If $B_0 \cdots B_m$ is of Type 2, then ignore the vertex x_{m+1} and the edges e and e'.

Depending on parities of the degrees of x_{m+1}, u_0 and v_0 in $\widetilde{R_0}$, we divide the proof into two cases.

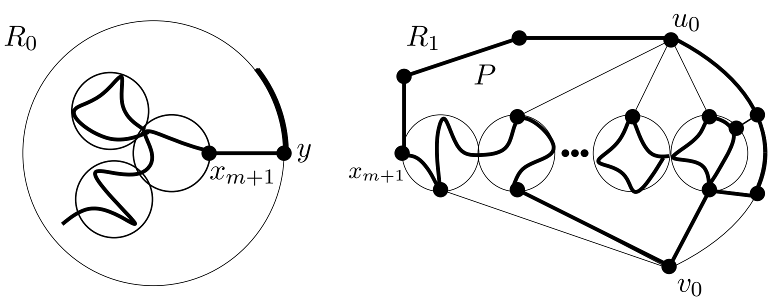

Figure 10: The graphs R_0 and R_1 for Case 1.

Case 1. The sequence $B_0 \cdots B_m$ is of Type 1 and exactly one of e and e' is contained in R_0.

In this case, see the left side of Figure 10. Note that x_{m+1} and exactly one of u_0 and v_0 have odd degrees in $\widetilde{R_0}$. By symmetry, we may assume that u_0 has an odd degree in $\widetilde{R_0}$. It follows from condition (R1) for R_0 that

$$|E(\widetilde{R_0})| \le \frac{4|V(G_0)| - 4}{3} - 1.$$

Let G_1 be the graph obtained from the subgraph of G induced by $\bigcup_{i=0}^{m} V(B_i) \cup C[u_0, v_0]$ by adding the path P of length three from x_{m+1} to u_0 (with two new vertices) so that the outer cycle C_1 of G_1 contains P and $C[u_0, v_0]$. See the right side of Figure 10. Note that

$$|V(G_1)| = \left| \bigcup_{i=0}^{m} V(B_i) \cup C[u_0, v_0] \right| + 2 = |V(G)| - |V(G_0)| + 4.$$

Note that $G_1 \not\cong K_3$. By Lemmas 7 and 8 and by condition (B6), the pair (G_1, C_1) is also a circuit graph. It follows from the minimality of (G, C) that $2G_1$ has a spanning connected even subgraph R_1 satisfying conditions (R1)–(R3). Let $\widetilde{R_1}$ be the graph obtained from R_1 by removing two internal vertices of P. By conditions (R1) and (R3) for R_1, note that

$$|E(\widetilde{R_1})| \leq \frac{4|V(G_1)| - 4}{3} - 3.$$

Let $R = \widetilde{R_0} \cup \widetilde{R_1}$. It follows from the construction that R is a spanning connected even subgraph of $2G$ such that R satisfies conditions (R2) and (R3). Moreover, we have

$$\begin{aligned}
|E(R)| &= |E(\widetilde{R_0})| + |E(\widetilde{R_1})| \\
&\leq \frac{4|V(G_0)| - 4}{3} - 1 + \frac{4|V(G_1)| - 4}{3} - 3 \\
&= \frac{4\big(|V(G_0)| + |V(G_1)| - 4\big) - 4}{3} \\
&= \frac{4|V(G)| - 4}{3},
\end{aligned}$$

and hence R also satisfies condition (R1), a contradiction to the minimality of (G, C), again. □

Case 2. Otherwise.

We will deal with the remaining case at that same time, that is, the sequence $B_0 \cdots B_m$ is of Type 2, or of Type 1 and both e and e' or neither e nor e' is contained in R_0. Note that in this case, even if x_{m+1} exists (Type 1), then it has even degree in $\widetilde{R_0}$. We divide the rest of the proof into two subcases; the one where both u_0 and v_0 have even degree in $\widetilde{R_0}$ (Case 2.1) and the one where both have odd degree in $\widetilde{R_0}$ (Case 2.2). To do that, we first need the following settings.

Let

$$t_e = \begin{cases} 2 & \text{if the sequence } B_0 \cdots B_m \text{ is of Type 1 and } e, e' \in E(R_0), \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$|E(\widetilde{R_0})| \leq \frac{4|V(G_0)| - 4}{3} - t_e.$$

Define the integer k as follows:

$$k = \begin{cases} \max\{i : 0 \leq i \leq m-1 \text{ and } B_i \text{ is } K_2 \text{ or } K_3, \text{ or } B_i \cap B_{i+1} = \emptyset\ \} & \text{if such an integer } i \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

For $1 \leq i \leq m-k$, let $G_i = B_{m-i+1}$ and let C_i be the outer cycle of G_i. Note that for $1 \leq i \leq m-k$, the graph G_i is isomorphic to neither K_2 nor K_3 by the choice of k, and hence by Lemma 7, the pair (G_i, C_i) is a circuit graph with $G_i \not\cong K_3$. By the minimality of (G, C), for $1 \leq i \leq m-k$, the multigraph $2G_i$ has a spanning connected even subgraph R_i satisfying conditions (R1)–(R3).

Remark: Here we remark the reason why we need to define the integer k as above. First, even if we try to treat the extended chain of block as a whole, then it is impossible, since the whole might not be 2-connected. Hence at least we have to deal with each block (or some groups of them) separately.

Next, without defining the integer k, it follows from the minimality of (G, C) and Proposition 6 that for each block B_i with $1 \leq i \leq m$, we can find a spanning connected even subgraph R_i in $2B_i$, and obtain a spanning even subgraph R of $2G$ by combining them. However, the issues are connectedness and condition (R1) for R. If the extended chain of blocks $B_0 \cdots B_m$ consists of two chains of blocks, then just combining R_0 and R_i for $1 \leq i \leq m$ cannot produce a connected subgraph. Hence in this case, we have to distinguish them, and hence we need to find an integer i with $B_i \cap B_{i+1} = \emptyset$. The second issue is condition (R1). Note that the upper bound on the number of edges in the spanning connected even subgraph obtained by Proposition 6 is slightly more than the one obtained by the minimality of (G, C). Hence by only combining R_i for $1 \leq i \leq m$, we cannot show that R satisfies condition (R1). To avoid such a situation, we define the integer k and use the minimality of (G, C) to only the blocks that are isomorphic to neither K_2 nor K_3, that is, the blocks B_i with $k+1 \geq i \geq m$. The remaining blocks will be dealt separately.

Let $\widehat{B} = \bigcup_{i=0}^{k-1} B_i \cup I_H(B_k)$. In the remaining arguments, we will construct a spanning connected even subgraph of $2G$ such that it satisfies conditions (R1)–(R3), combining $\widetilde{R_0}$ and R_i for $1 \leq i \leq m-k$ together with the connected even subgraph R_{m-k+1} obtained by the minimality of (G, C) to the certain graph containing $\widehat{B}$. (In Case 2.2, we also combine $C[u_0, v_0]$.) However, that is not enough and we need extra edges f and f', because of the following reason. Suppose that $t_e = 2$. In this case, through the process to obtain $\widetilde{R_0}$, the spanning connected even subgraph R_0 might be separated into three components; the one containing u_0, the one containing v_0, and the one containing x_{m+1}. The first two can be automatically connected through the process of combining $\widetilde{R_0}$ and R_i for $1 \leq i \leq m-k+1$, but the last one cannot. Thus, in the last case, we need to take two edges f and f' in $2G$ such that f and f' connect B_{k+1} and $B_k \cup \{u_0, v_0\}$, and add the edges f and f' instead of e and e'. Note that in the case where $t_e = 0$, the subgraph $\widetilde{R_0}$ of $2G$ has only at most two components such that one of them contains u_0 and the other contains v_0, and hence we do not need to add two such edges f and f'.

By the choice of k, we have four possibilities on B_k;
(I) $B_k \cap B_{k+1} = \emptyset$,
(II) $B_k \cap B_{k+1} \neq \emptyset$ and B_k is K_2,
(III) $B_k \cap B_{k+1} \neq \emptyset$ and B_k is K_3, and
(IV) $B_k \cap B_{k+1} \neq \emptyset$ and $B_k = B_0$.

In Case (I), the sequence $B_0 \cdots B_m$ is of Type 2. Then $I_H(B_k) = V(B_k) - \{x_k\}$, and hence $\widehat{B} = \bigcup_{i=0}^{k} B_i$. In this case, it follows from condition (B5) that both u_0 and v_0 has a neighbor in $I(B_{k+1})$. Let f and f' be the edges of $2G$ such that both f and f' correspond to an edge connecting $I(B_{k+1})$ and u_0 in G.

In Case (II), B_k consists of only one edge, and let f and f' be the edges of $2G$ such that both f and f' correspond to the unique edge in B_k.

In Case (III), B_k consists of only three vertices, which are x_k, x_{k+1} and the other one, say x'_k. Since (G, C) is a circuit graph, x'_k has to have a neighbor in C. Note that by condition (B4) and the planarity, exactly one of u_0 and v_0 is a neighbor of x'_k. Let f and f' be the edges of $2G$ such that both f and f' correspond to the edge $x_{k+1}x'_k$ of G.

In Case (IV), we have $\widehat{B} = I_H(B_0)$. By condition (B4), at least one of u_0 and v_0 has a neighbor in $B_1 - x_1$. Let f and f' be the edges of $2G$ such that both f and f' correspond to an edge connecting $B_1 - x_1$ and u_0 or v_0 in G.

We are now ready to show each of the two subcases.

Case 2.1. Both u_0 and v_0 have even degree in $\widetilde{R_0}$.

Let G_{m-k+1} be the subgraph of G induced by $\widehat{B} \cup C[u_0, v_0]$. It follows from conditions (B4) and (B5) that G_{m-k+1} is 2-connected. Let C_{m-k+1} be the outer cycle of G_{m-k+1}. By Lemma 7, note that (G_{m-k+1}, C_{m-k+1}) is a circuit graph with $G_{m-k+1} \not\cong K_3$. Then $2G_{m-k+1}$ has a spanning connected

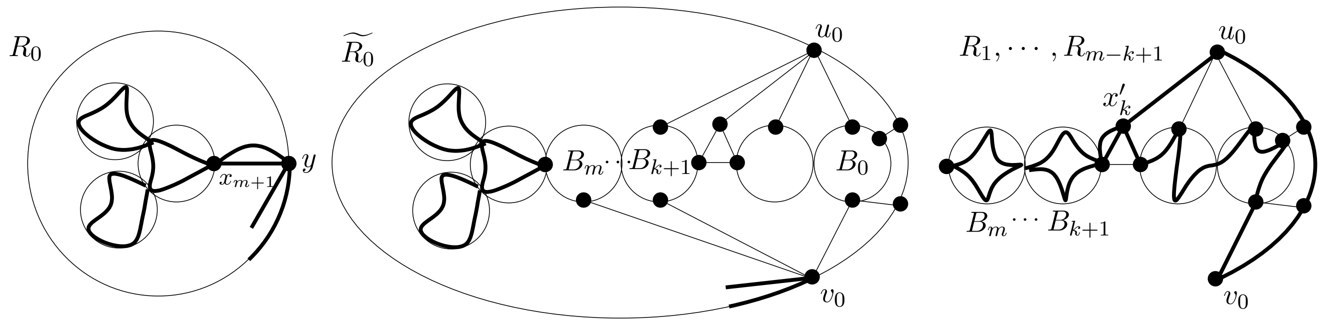

Figure 11: The graphs $\widetilde{R_0}, R_1, \cdots, R_{m-k+1}$ for Case 2.1 with case (III) and $t_e = 2$.

even subgraph R_{m-k+1} satisfying conditions (R1)–(R3). See Figure 11.

Let

$$R = \begin{cases} \widetilde{R_0} \cup \bigcup_{i=1}^{m-k+1} R_i \cup \{f, f'\} & \text{if } t_e = 2, \\ \widetilde{R_0} \cup \bigcup_{i=1}^{m-k+1} R_i & \text{otherwise.} \end{cases}$$

Note that $|E(R)| = |E(\widetilde{R_0})| + \sum_{i=1}^{m-k+1} |E(R_i)| + t_e$. Notice also that by the choice of k, we have $V(R_i \cap R_{i+1}) \neq \emptyset$ for $0 \leq i \leq m-k-1$. Moreover, if follows from the choice of f and f' that R is a spanning connected even subgraph satisfying conditions (R2) and (R3).

Note that $|V(G_0) \cap V(G_1)| \leq 1$, and $|V(G_i) \cap V(G_{i+1})| = 1$ for $1 \leq i \leq m-k-1$. In addition, since $V(G_{m-k}) \cap V(G_{m-k+1}) = \emptyset$, we have

$$\sum_{i=0}^{m-k+1} |V(G_i)| \leq |V(G)| + m - k + 1.$$

Thus, we obtain

$$\begin{aligned} |E(R)| &= |E(\widetilde{R_0})| + \sum_{i=1}^{m-k+1} |E(R_i)| + t_e \\ &\leq \sum_{i=0}^{m-k+1} \frac{4|V(G_i)| - 4}{3} \\ &= \frac{4\Big(\sum_{i=0}^{m-k+1} |V(G_i)| - (m-k+1)\Big) - 4}{3} \\ &\leq \frac{4|V(G)| - 4}{3}, \end{aligned}$$

and hence R also satisfies condition (R1). $\square$

Case 2.2. Both u_0 and v_0 have odd degree in $\widetilde{R_0}$.

Let

$$r = \begin{cases} k - 1 & \text{if } k \geq 1 \text{ and } B_k \text{ is } K_2, \\ k & \text{otherwise.} \end{cases}$$

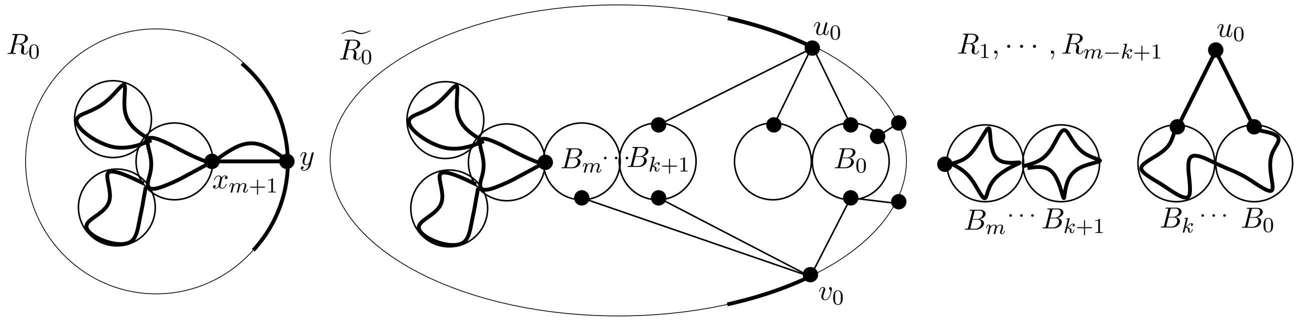

Figure 12: The graphs $\widetilde{R_0}, R_1, \cdots, R_{m-k+1}$ for Case 2.2 with (I) and b).

Let let G_{m-k+1} be the subgraph of G induced by

$$\begin{cases} B_0 & \text{a) if } r = 0, \\ \widehat{B} \cup \{u_0\} & \text{b) if } r \neq 0,\ \widehat{B} \text{ is connected, and } u_0 \text{ has a neighbor in } B_k - x_k, \\ \widehat{B} \cup \{v_0\} & \text{c) if } r \neq 0,\ \widehat{B} \text{ is connected, and } u_0 \text{ does not have a neighbor in } B_k - x_k, \\ \widehat{B} \cup \{u_0, v_0\} & \text{d) otherwise.} \end{cases}$$

See Figure 12. Note that it follows from conditions (B4) and (B5) that G_{m-k+1} is 2-connected, unless $r = 0$, and $B_0 \cong K_2$ or $B_0 \cong K_1$. Notice also that $G_{m-k+1} \cong K_i$ for $i = 1, 2, 3$ only when Case a) occurs.

If $G_{m-k+1} \cong K_1$ or $G_{m-k+1} \cong K_2$, then $2G_{m-k+1}$ has a spanning connected even subgraph R_{m-k+1} with $|E(R_{m-k+1})| = 0$ or $|E(R_{m-k+1})| = 2$, respectively. If $G_{m-k+1} \cong K_3$, then it follows from Proposition 6 that $2G_{m-k+1}$ has a spanning connected even subgraph R_{m-k+1} with $|E(R_{m-k+1})| = 3$. If G_{m-k+1} is isomorphic to neither K_1, K_2 nor K_3, then it follows from the minimality of (G, C) that $2G_{m-k+1}$ has a spanning connected even subgraph R_{m-k+1} satisfying conditions (R1)–(R3). In either case, $2G_{m-k+1}$ has a spanning connected even subgraph R_{m-k+1} with

$$|E(R_{m-k+1})| \leq \begin{cases} \frac{4|V(G_{m-k+1})|-2}{3} & \text{if Case a) occurs,} \\ \frac{4|V(G_{m-k+1})|-4}{3} & \text{otherwise.} \end{cases}$$

We first deal with Case a). Suppose first that $B_0 \cap B_1 = \emptyset$. In this case, B_0 has at least three neighbors in $C[u_0, v_0]$, and hence it follows from Claim 2 that $|V(C[u_0, v_0])| \geq 5$. Let h and h' be the two edges of $2G$ such that both h and h' correspond to an edge connecting u_0 and B_0.

Let

$$R = \begin{cases} \widetilde{R_0} \cup \bigcup_{i=1}^{m-k+1} R_i \cup C[u_0, v_0] \cup \{f, f', h, h'\} & \text{if } t_e = 2, \\ \widetilde{R_0} \cup \bigcup_{i=1}^{m-k+1} R_i \cup C[u_0, v_0] \cup \{h, h'\} & \text{otherwise.} \end{cases}$$

It follows from the construction that R is a spanning connected even subgraph of $2G$ such that R satisfies conditions (R2) and (R3). Now we shall check that R also satisfies condition (R1). Note that

$$\begin{aligned} |V(G)| &\geq |V(G_0)| + |V(C[u_0, v_0])| - 1 + \sum_{i=1}^{m-k} \big(|V(G_i)| - 1\big) + |V(G_{m-k+1})| \\ &= \sum_{i=0}^{m-k+1} |V(G_i)| + |V(C[u_0, v_0])| - (m - k + 1). \end{aligned}$$

Then we obtain

$$\begin{aligned}
|E(R)| &= |E(\widetilde{R_0})| + \sum_{i=1}^{m-k+1} |E(R_i)| + |E(C[u_0, v_0])| + t_e + 2 \\
&\leq \sum_{i=0}^{m-k} \frac{4|V(G_i)| - 4}{3} + \frac{4|V(G_{m-k+1})| - 2}{3} + |V(C[u_0, v_0])| - 1 + 2 \\
&= \frac{4\Big(\sum_{i=0}^{m-k+1} |V(G_i)| + |V(C[u_0, v_0])| - (m-k+1)\Big) - 4}{3} - \frac{|V(C[u_0, v_0])| - 5}{3} \\
&\leq \frac{4|V(G)| - 4}{3},
\end{aligned}$$

and hence R also satisfies condition (R1).

Therefore, we may assume that $B_0 \cap B_1 \neq \emptyset$. In this case, without using two edges h and h', the subgraph R_{m-k+1} is automatically connected to others. Instead of it, we only know that B_0 has at least two neighbors in $C[u_0, v_0]$, and hence it follows from Claim 2 that $|V(C[u_0, v_0])| \geq 3$.

Let

$$R = \begin{cases} \widetilde{R_0} \cup \bigcup_{i=1}^{m-k+1} R_i \cup C[u_0, v_0] \cup \{f, f'\} & \text{if } t_e = 2, \\ \widetilde{R_0} \cup \bigcup_{i=1}^{m-k+1} R_i \cup C[u_0, v_0] & \text{otherwise.} \end{cases}$$

It follows from the construction that R is a spanning connected even subgraph of $2G$ such that R satisfies conditions (R2) and (R3). Note that

$$\begin{aligned}
|V(G)| &\geq |V(G_0)| + |V(C[u_0, v_0])| - 1 + \sum_{i=1}^{m-k+1} (|V(G_i)| - 1) \\
&= \sum_{i=0}^{m-k+1} |V(G_i)| + |V(C[u_0, v_0])| - (m-k+2).
\end{aligned}$$

Then we obtain

$$\begin{aligned}
|E(R)| &= |E(\widetilde{R_0})| + \sum_{i=1}^{m-k+1} |E(R_i)| + |E(C[u_0, v_0])| + t_e \\
&\leq \sum_{i=0}^{m-k} \frac{4|V(G_i)| - 4}{3} + \frac{4|V(G_{m-k+1})| - 2}{3} + |V(C[u_0, v_0])| - 1 \\
&= \frac{4\Big(\sum_{i=0}^{m-k+1} |V(G_i)| + |V(C[u_0, v_0])| - (m-k+2)\Big) - 4}{3} - \frac{|V(C[u_0, v_0])| - 3}{3} \\
&\leq \frac{4|V(G)| - 4}{3},
\end{aligned}$$

and hence R also satisfies condition (R1). This discussion completes Case a).

Now we consider the remaining three cases; Case b), c) and d). Let

$$R = \begin{cases} \widetilde{R_0} \cup \bigcup_{i=1}^{m-k+1} R_i \cup C[u_0, v_0] \cup \{f, f'\} & \text{if } t_e = 2, \\ \widetilde{R_0} \cup \bigcup_{i=1}^{m-k+1} R_i \cup C[u_0, v_0] & \text{otherwise.} \end{cases}$$

By the construction, R is a spanning connected even subgraph of $2G$ such that R satisfies conditions (R2) and (R3). (Notice that R_{m-k+1} shares u_0 and/or v_0 with others.) Now we shall check that R also satisfies condition (R1). Note that $|V(C[u_0,v_0])| \geq 3$ by Claim 2.

Suppose first that Case b) or c) occurs. Then

$$\begin{aligned}|V(G)| &\geq |V(G_0)| + |V(C[u_0,v_0])| - 1 + \sum_{i=1}^{m-k+1} (|V(G_i)| - 1) \\ &= \sum_{i=0}^{m-k+1} |V(G_i)| + |V(C[u_0,v_0])| - (m-k+2).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}|E(R)| &= |E(\widetilde{R_0})| + \sum_{i=1}^{m-k+1} |E(R_i)| + |E(C[u_0,v_0])| + t_e \\ &\leq \sum_{i=0}^{m-k} \frac{4|V(G_i)| - 4}{3} + \frac{4|V(G_{m-k+1})| - 4}{3} + |V(C[u_0,v_0])| - 1 \\ &= \frac{4\Big(\sum_{i=0}^{m-k+1} |V(G_i)| + |V(C[u_0,v_0])| - (m-k+2)\Big) - 4}{3} - \frac{|V(C[u_0,v_0])| - 1}{3} \\ &\leq \frac{4|V(G)| - 4}{3},\end{aligned}$$

and hence R also satisfies condition (R1).

Finally, suppose that Case d) occurs. In this case,

$$\begin{aligned}|V(G)| &= |V(G_0)| + |V(C[u_0,v_0])| - 1 + \sum_{i=1}^{m-k} (|V(G_i)| - 1) + |V(G_{m-k+1})| - 2 \\ &= \sum_{i=0}^{m-k+1} |V(G_i)| + |V(C[u_0,v_0])| - (m-k+3).\end{aligned}$$

Note that $\widehat{B}$ is not connected, and hence B_0 has at least three neighbors in $C[u_0,v_0]$. Hence it follows from Claim 2 that $|V(C[u_0,v_0])| \geq 5$. Thus, we obtain

$$\begin{aligned}|E(R)| &= |E(\widetilde{R_0})| + \sum_{i=1}^{m-k+1} |E(R_i)| + |E(C[u_0,v_0])| + t_e \\ &\leq \sum_{i=0}^{m-k+1} \frac{4|V(G_i)| - 4}{3} + |V(C[u_0,v_0])| - 1 \\ &= \frac{4\Big(\sum_{i=0}^{m-k+1} |V(G_i)| + |V(C[u_0,v_0])| - (m-k+3)\Big) - 4}{3} - \frac{|V(C[u_0,v_0])| - 5}{3} \\ &\leq \frac{4|V(G)| - 4}{3},\end{aligned}$$

and hence R also satisfies condition (R1).

This discussion completes the proof of Theorem 5. $\square$

At the end of this section, we explain that we can find a spanning connected even subgraph R of $2G$ such that R satisfies condition (R1)–(R3) in $O(n^2)$-time, where n is the number of vertices of a graph G. In all part of the proof, we used the fact that (G, C) is a minimum counterexample, all of which means, in the algorithmic side, that we consider a decomposition of a given graph into some smaller (circuit) graphs that are pairwise edge-disjoint. First the total iteration number is $O(m)$, where m is the number of edges, since at each step, at least one edge (that is not an edge added later) is deleted. In the process, we sometimes add some new edges, but at each step, we add constant number of edges, and hence, in total, we deal with only $O(m)$ new edges. The important point is that all edges which were used in the spanning connected even subgraph in the smaller graph are surely used in the obtained spanning connected even subgraph, and all other edges in the smaller graph are never used in it. Since it takes $O(m)$-time to find a maximal extended chain of blocks, (this can be done by the similar way to find a block decomposition) and there are $O(m)$ iterations, our algorithm takes $O(m^2)$-time. Since the input graphs are planar, we have $O(m^2) = O(n^2)$.

4 Proof of Lemma 11

Let y be the vertex of G_0 obtained by contracting $\bigcup_{i=0}^{m-1} V(B_i) \cup I_H(B_m) \cup C[u_0, v_0]$. We consider two cases depending on the types of $B_0 \cdots B_m$.

Type 1.1, 1.2, 1.4 or 2.1.

Suppose first that $B_0, \dots B_m$ is of Type 1.1, 1.2, or 1.4. Let B_{m+1} be the block of H as defined in Section 2.2. Let D be the graph consisting of the components B of H such that B contains B_i for some $0 \le i \le m+1$. Let $D' = D - \bigcup_{i=0}^{m-1} B_i - I_H(B_m)$.

When $B_0, \dots B_m$ is of Type 2.1, then let D' be the component as defined in Section 2.2. Let D be the graph consisting of D' and the components of H containing B_i for some $0 \le i \le m$.

In either cases, D is a subgraph of H such that D satisfies conditions (D1) and (D2). Note that D' is connected by conditions (B1)–(B3). We prove the following two claims.

Claim 5 *The subgraph D' has at least two end blocks.*

Claim 6 *For each end block B of D', at least one of u_0 and v_0 is not a neighbor of $I_H(B)$, unless $B_0, \dots B_m$ is of Type 1.4 and $B = B_{m+1}$, or $B_0, \dots B_m$ is of Type 2.1 and there exists an end block B_{m+1} of D' such that both u_0 and v_0 has a neighbor in $I_H(B_{m+1})$ and for each end block B of D' with $B \neq B_{m+1}$, at least one of u_0 and v_0 is not a neighbor of $I_H(B)$.*

Proof of Claims 5 and 6. We prove these claims considering three cases, depending on the types of $B_0, \dots B_m$.

Type 1.1 or 1.2.

For these two types, the abstract tree of D has a vertex of degree at least three such that it corresponds to x_{m+1} (Type 1.1) or B_{m+1} (Type 1.2). Therefore, D has at least three end blocks, and hence D' has at least two end blocks. Thus, Claim 5 holds. Claim 6 is obvious by the planarity of G, Lemma 10 (ii) and condition (B5).

Type 1.4.

Recall that by the definition of Type 1.4, there exists a block B_{m+2} with $B_{m+2} \neq B_m$ such that $|V(B_{m+1} \cap B_{m+2})| = 1$ and $B_{m+1} - x_{m+2}$ has a neighbor in $C[v_0, u_0] - \{u_0, v_0\}$, where $\{x_{m+2}\} =$

$V(B_{m+2} \cap B_{m+1})$. Then D' contains the two blocks B_{m+1} and B_{m+2}, and hence Claim 5 holds. Since $B_{m+1} - x_{m+2}$ has a neighbor in $C[v_0, u_0] - \{u_0, v_0\}$, Claim 6 follows from the planarity of G.

Type 2.1.

In this type, Claim 5 trivially holds. So, we only show Claim 6. Suppose that there exist two end blocks B_{m+1} and B'_{m+1} of D' such that both u_0 and v_0 has a neighbor in $I_H(B_{m+1})$ and in $I_H(B'_{m+1})$. Then by the planarity of G, at least one of the sequences $B_0 \cdots B_m B_{m+1}$ and $B_0 \cdots B_m B'_{m+1}$ satisfies conditions (B6) and (B7). However, since both sequence trivially satisfies conditions (B1)–(B5), we have a contradiction to the maximality of m. This discussion implies that at least one of u_0 and v_0 is not a neighbor of $I_H(B)$ for each end block B of D', or there exists an end block B_{m+1} of D' such that both u_0 and v_0 has a neighbor in $I_H(B_{m+1})$ and at least one of u_0 and v_0 is not a neighbor of $I_H(B)$ for each end block B of D' with $B \neq B_{m+1}$.

This discussion completes the proofs of Claims 5 and 6. $\square$

It follows from Claim 6 and the definition of a circuit graph that D' has a neighbor that is neither u_0 nor v_0, and it must be contained in $C[v_0, u_0]$ by Lemma 10 (ii). Let u' and v' be the neighbors of D' in $C[v_0, u_0] - \{u_0, v_0\}$ such that u' and v' are closest to v_0 and to u_0 in $C[v_0, u_0]$, respectively. Let F be the subgraph of G induced by $D' \cup C[u', v']$. We have the following claim.

Claim 7 *The graph F is 2-connected.*

Proof. Suppose not, that is, there exists a cut vertex in F.

Since there exists a cycle in F passing through $C[u', v']$ and inside of D', it is clear that there exists an end block B of F such that B contains no vertex of C and has no neighbor in $C[u', v']$. Note that B is also an end block of H (and $B \neq B_{m+1}$ when $B_0, \ldots B_m$ is of Types 1.4 and 2.1 and the exceptional case occurs). Therefore, since (G, C) is a circuit graph, the block B has at least two neighbors in C. By Claim 6, at least one of u_0 and v_0 is not a neighbor of B, and hence B has a neighbor w in C with $w \neq u_0, v_0$.

By the choice of u' and v', we have $w \notin V\big(C[v_0, u'] - \{u'\}\big) \cup V\big(C[v', u_0] - \{v'\}\big)$. It follows from Lemma 10 (ii) that $w \notin V(C[u_0, v_0]) - \{u_0, v_0\}$. Hence $w \in V(C[u', v'])$, but this contradicts that B is an end block of F. This discussion completes the proof of Claim 7. $\square$

If (G_0, C_0) is a circuit graph, then we are done. So, suppose that (G_0, C_0) is not a circuit graph. Since G_0 is 2-connected, there exists a cut set S of order two such that $G_0 - S$ has a component containing no vertices of C_0. Recall that y is the vertex of G_0 obtained by contracting $\bigcup_{i=0}^{m-1} B_i \cup I_H(B_m) \cup C[u_0, v_0]$. Since (G, C) is a circuit graph, we have $y \in S$. Let z be the vertex in S with $z \neq y$.

Note that F is an induced subgraph of $G_0 - y$. Thus, z is contained in F and z separates some vertices in F from C. However, it follows from Claim 7 that F is 2-connected, and hence $z = u' = v'$. By the choice of u' and v', we have $F - z = D'$ and D' has no neighbor in $C[v_0, u_0] - \{v_0, z, u_0\}$. It follows from Lemma 10 (ii) that D' has no neighbor in $C[u_0, v_0] - \{v_0, u_0\}$. Hence, D' has only at most three neighbors, which are u_0, v_0 and z.

It follows from Lemma 10 (ii) and the planarity of G that D' has at most two end blocks. (Since otherwise, at least one of the end blocks can have only one neighbor in C, contradicting that (G, C) is a circuit graph.) On the other hand, it follows from Claim 5 that D' has at least two end blocks, and hence D' has exactly two end blocks. Moreover, all of u_0, z and v_0 are neighbors of D' and hence conditions (D3) and (D4) hold.

Let $G'_0 = G_0 - D'$. Let x be a vertex in $G'_0 - V(C_0)$. Note that x is also a vertex in $G - V(C)$. Since (G, C) is a circuit graph, there exist three pairwise internally disjoint paths P_1, P_2 and P_3 in

G from x to C. Since D consists of some components of H, the paths P_1, P_2, and P_3 exist even in $G - D$. Let w_i be the end vertex of P_i in C for $i \in \{1, 2, 3\}$. It follows from Lemma 10 (ii) that $w_1, w_2, w_3 \notin V(C[u_0, v_0]) - \{u_0, v_0\}$. By the existence of D' and z and the planarity of G, at most one of the vertices w_1, w_2 and w_3 belongs to $\{u_0, v_0\}$. These discussions imply that P_1, P_2 and P_3 are still paths in G'_0 from x to C_0 such that P_1, P_2 and P_3 are pairwise disjoint except for x. This holds for every vertex x in $G'_0 - V(C_0)$, and hence (G'_0, C_0) is also a circuit graph. Therefore (D5) also holds.

This discussion completes the proof of Lemma 11 for a maximal extended chain of blocks of Type 1.1, 1.2, 1.4 or 2.1. □

Type 1.3 or 2.2.

Let x be a vertex in $G_0 - V(C_0)$. Note that x is also a vertex in $G - V(C)$. Since (G, C) is a circuit graph, there exist three paths P_1, P_2 and P_3 in G from x to C such that P_1, P_2 and P_3 are pairwise disjoint except for x. By the conditions of Types 1.3 and 2.2 and by Lemma 10 (ii), two of the paths P_1, P_2 and P_3, say P_1 and P_2 by symmetry, use no vertex in $\bigcup_{i=0}^{m-1} B_i \cup I_H(B_m) \cup C[u_0, v_0]$. Then we can find three paths P_1, P_2 and P'_3 in G_0 from x to C_0 such that P_1, P_2 and P'_3 are pairwise disjoint except for x, where P'_3 is the path in G_0 from x to y such that P'_3 corresponds to P_3, or $P'_3 = P_3$ if P_3 does not use a vertex in $I_H(B_m) \cup \{u_0, v_0\}$. This holds for every vertex x in $G_0 - V(C_0)$, and hence (G_0, C_0) is also a circuit graph. This completes the proof for Type 1.3 or 2.2, and the proof of Lemma 11. □

5 Sharpness

In this section, we give several examples showing sharpness of our results.

Let H be a graph embedded in a closed surface. For each face of H, we put a vertex v in its interior and join v with the vertices on its boundary. The resulting graph G is the *face subdivision* of H. Obviously, the representativity of G is at least that of H when H is embedded in a non-spherical closed surface.

The following proposition shows that Theorems 2 and 3 are "essentially" tight.

Proposition 12 *For each closed surface $\mathbb{F}^2$ with Euler characteristic χ, there exist infinitely many triangulations G on $\mathbb{F}^2$ such that every spanning closed walk of G has length at least $\frac{4}{3}(|V(G)| - \chi)$. Moreover, if $\chi \leq 1$, then such a graph G can be chosen so that the representativity of G is arbitrarily large.*

Proof. Let T be a triangulation of $\mathbb{F}^2$ with $|V(T)| = t$, and let G be the face subdivision of T. Then, $|V(G) - V(T)| = 2(t - \chi)$ and $|V(G)| = 3t - 2\chi$. Let W be a spanning closed walk of G. Since $V(G) - V(T)$ is independent in G, we have

$$\text{Length}(W) \;\geq\; 2|V(G) - V(T)| \;=\; 4(t - \chi) \;=\; \frac{4}{3}(|V(G)| - \chi),$$

where Length(W) denotes the length of W. In order to make the representativity of G large, we take a triangulation T with large representativity. □

Unfortunately, this proposition does not show the sharpness of Theorem 2. In fact, we expect that $\frac{4}{3}(n - 2)$ (for $n \geq 8$) will be the sharp bound for the planar case.

However, the following proposition shows that Theorem 5 is best possible.

Proposition 13 *There exist infinitely many circuit graphs G such that every spanning closed walk of G has length at least $\frac{4}{3}(|V(G)|-1)$.*

Proof. Let T be a triangulation of the sphere with $|V(T)| = t$, and let G be obtained from the face subdivision of T by deleting one vertex of T. Then, $|V(G) - V(T)| = 2t - 4$ and $|V(G)| = 3t - 5$. Let W be a spanning closed walk of G. Since $V(G) - V(T)$ is independent in G, we have

$$\mathrm{Length}(W) \;\geq\; 2|V(G) - V(T)| \;=\; 2(2t-4) \;=\; \frac{4}{3}(|V(G)|-1),$$

as desired. □

Finally, we note that we have not found any example showing that the assumption on the representativity in Theorem 3 is necessary. It might be true that every 3-connected graph G embedded in a closed surface with Euler characteristic χ has a spanning closed walk with at most $\frac{4}{3}(|V(G)|-\chi)$ edges.

Acknowledgments

We are gracteful to the referees for the several useful comments.

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Appendix: Proof of Theorem 3

Using the result in [24], Kawarabayashi, Nakamoto and Ota [16] proved the following result. (See Theorem 3.4 in [16], and also Theorems 3.1 and 3.2. The subgraph we can find by Lemma 14 is called a *starlike I3CP* graph in [16].) Note that the second part of Lemma 14 is not stated in [16] explicitly, and hence we show it in this paper.

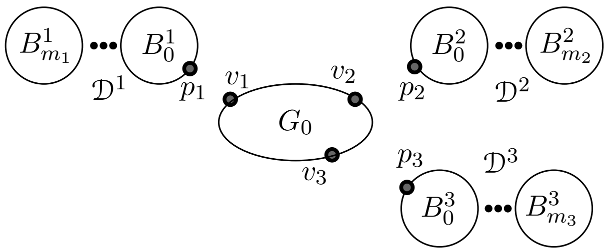

Figure 13: A circuit graph (G_0, C_0) and three chains of blocks $\mathcal{D}^1, \mathcal{D}^2, \mathcal{D}^3$ desired in Lemma 14.

Lemma 14 *For any closed surface $\mathbb{F}^2$ with Euler characteristic χ, there exists a positive integer $r = r(\mathbb{F}^2)$ such that if G is a 3-connected graph on $\mathbb{F}^2$ with representativity at least r, then G has a spanning subgraph obtained from a circuit graph (G_0, C_0) and t chains of blocks $\mathcal{D}^1, \mathcal{D}^2 \cdots, \mathcal{D}^t$, say $\mathcal{D}^i = B^i_0 B^i_1 \cdots B^i_{m_i}$ for $1 \le i \le t$, with $t \le -2\chi + 2$ by identifying a vertex v_i in C_0 and a vertex p_i of $I_{\mathcal{D}^i}(B^i_0)$ for each $1 \le i \le t$. Moreover, we can take such a spanning subgraph so that for $1 \le i \le t$, there exists a vertex u_i in C_0 such that u_i is a neighbor of $I_{\mathcal{D}^i}(B^i_j)$ in G for $0 \le j \le m_i$ with $I_{\mathcal{D}^i}(B^i_j) \ne \emptyset$.*

To prove the second part, we need a few terminology that were defined in [16].

First we consider the case where $\mathbb{F}^2$ is an orientable surface. Then the genus of $\mathbb{F}^2$ is g, where $g = (2-\chi)/2$. Let $\Gamma(\mathbb{F}^2) = \{a_1, \ldots, a_g, b_1 \ldots, b_g\}$ be a set of simple closed curves on $\mathbb{F}^2$ such that (i) $\{a_1, \ldots, a_g\}$ is a set of g pairwise non-homotopic disjoint essential simple closed curves, (ii) $\{b_1, \ldots, b_g\}$ is a set of g pairwise disjoint simple closed curves such that each b_i crosses a_i exactly once and never crosses a_j for $1 \le j \le g$ with $j \ne i$. Let G be a 3-connected graph on a closed surface $\mathbb{F}^2$. Suppose that G has two sets of pairwise disjoint cycles $\{A_1, A'_1, \ldots, A_g, A'_g\}$ and $\{B_1, B'_1, \ldots, B_g, B'_g\}$ such that (i) A_i and A'_i are homotopic to $a_i \in \Gamma(\mathbb{F}^2)$, (ii) B_j and B'_j are homotopic to $b_j \in \Gamma(\mathbb{F}^2)$, and (iii) each of A_i and A'_i is disjoint from B_j and B'_j if $j \ne i$.

Let $\mathcal{A}_i$ denote the annulus bounded by A_i and A'_i. We suppose further that G has pairwise disjoint paths $C_1, C'_1, \ldots, C_{g-1}, C'_{g-1}$ such that (iv) each of C_i and C'_i connects A'_i and A'_{i+1}, and runs across both of the annuli $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$, namely, C_i and C'_i intersect every closed curve in $\mathcal{A}_i$ (resp., $\mathcal{A}_{i+1}$) homotopic to a_i (resp., a_{i+1}), and (v) C_i and C'_i together with a segment of A'_i and a segment of A'_{i+1} bound a strip (a thin 2-cell region) that intersects none of $A_1, A'_1, \ldots, A_g, A'_g, B_1, B'_1, \ldots, B_g$ and B'_g except for A_i, A'_i, A_{i+1} and A'_{i+1}.

We say that G *satisfies Cutting Condition* if G has the above $4g$ cycles and $2g-2$ paths. Note that the argument in Lemma 14 works well even if a closed curve b_1 is 1-sided. In this case, we have to take a single cycle separating a thin Möbius band including b_1 and disjoint from all A_j, A'_j, B_j and B'_j for $2 \le j \le g$, instead of the annulus bounded by B_1 and B'_1. In the proof of Lemma 14, we only deal with the case where $\mathbb{F}^2$ is orientable, since the non-orientable case is similar to it.

Proof of Lemma 14.

The first part of Lemma 14 follows from the proof in [16], but to show the second part, we give an overview of the proof in [16].

By a result in [24], there exists a positive integer $r = r(\mathbb{F}^2)$ such that if G is a 3-connected graph on $\mathbb{F}^2$ with representativity at least r, then G satisfies Cutting Condition.

Consider an annulus $\mathcal{A}_k$ bounded by two cycles A_k and A'_k of G for $1 \leq k \leq g$. By choosing A_k and A'_k so that $\mathcal{A}_k$ contains as few faces as possible, we can show that $\mathcal{A}_k$ has no inner vertex. Let G' be the graph obtained from G by removing all inner edges of $\mathcal{A}_k$ together with the region $\mathcal{A}_k$ for all k with $1 \leq k \leq g$. Then G' is a spanning plane subgraph of G that is embedded in the sphere with $2g$ boundary components $A_1, A'_1, \ldots, A_g, A'_g$. Then we will cut G' by $B_1, B'_1, \ldots, B_g, B'_g, C_1, C'_1, \ldots, C_{g-1}$ and C'_{g-1}.

Let $\widetilde{\mathcal{B}_k}$ be the annulus bounded by B_k and B'_k. Take a subpath Q_1 of B_k and a subpath Q_2 of B'_k both joining A_k and A'_k in G_0. Consider the strip $\mathcal{B}_k$ contained in $\widetilde{\mathcal{B}_k} - (\widetilde{\mathcal{B}_k} \cap \mathcal{A}_k)$ and bounded by $A_k \cup A'_k \cup Q_1 \cup Q_2$. Let x, y, x' and y' be the four corners of the strip $\mathcal{B}_k$ such that $x \in V(A_k) \cap V(Q_1)$, $y \in V(A_k) \cap V(Q_2)$, $x' \in V(A'_k) \cap V(Q_1)$, and $y' \in V(A'_k) \cap V(Q_2)$. Let H be the plane graph consisting of the vertices and the edges contained in $\mathcal{B}_k$. Then there exists four paths P_1, P_2, Q_1 and Q_2 such that $P_1 \subset A_k$, $P_2 \subset A'_k$, $Q_1 \subset B_k$, $Q_2 \subset B'_k$ and $P_1 \cup P_2 \cup Q_1 \cup Q_2$ bounds H. Choosing B_j and B'_j so that the number of faces in $\mathcal{B}_j$ is as small as possible, we can add the edges xy and $x'y'$ through inside of H, and the region $\mathcal{B}'_j$ bounded by $Q_1 \cup Q_2 \cup \{xy, x'y'\}$ contains no inner vertex. Let D and D' be the plane subgraph of $H \cup \{xy, x'y'\}$ bounded by $P_1 \cup \{xy\}$ and by $P_2 \cup \{x'y'\}$, respectively. Then both $D - y$ and $D - y'$ is a chain of blocks, which may consists of only one vertex. Now remove all inner edges of $\mathcal{B}'_k$ together with all faces in $\mathcal{B}'_k$, all edges connecting y and $D - y$, and all edges connecting y' and $D' - y$.

After doing the same operation as above for all pairs of C_k and C'_k, in total, we get a circuit graph (G_0, C_0) and $4g - 2$ chains of blocks $\mathcal{D}^1, \cdots, \mathcal{D}^{4g-2}$ by identifying a vertex v_i in C_0 and a vertex p_i of $\mathcal{D}^i$ for $1 \leq i \leq 4g - 2$. (Notice that each $\mathcal{D}^i$ corresponds to $D - y$ or $D' - y'$ for some pair B_k and B'_k or some pair C_k and C'_k as above. Identifying the two vertices v_i and p_i, we obtain the corresponding vertex x.) If $\mathcal{D}^i$ consists of only one vertex, (that is, if $D - y$ or $D' - y$ consists of only x,) then we can delete it. Thus, we obtain t chains of blocks with $t \leq 4g - 2 = -2\chi + 2$, and hence the first part holds.

Now we will prove the second part. Suppose $t \geq 1$ and let $1 \leq i \leq t$. We may assume that $\mathcal{D}^i \neq \emptyset$ and $\mathcal{D}^i = D - y$ for D in the above argument. We use the same terminology as above. Let $\mathcal{D}^i = B^i_0 B^i_1 \cdots B^i_{m_i}$ and $x = p_i \in I_{\mathcal{D}^i}(B^i_0)$. Since G is 3-connected, for $0 \leq j \leq m_i$ with $I_{\mathcal{D}^i}(B^i_j) \neq \emptyset$, the set $I_{\mathcal{D}^i}(B^i_j)$ has a neighbor u^i_j in G with $u^i_j \notin V(D)$. Moreover, we may assume that $u^i_0 \neq p_i$ and $u^i_{m_i} \neq y$. By the construction, $u^i_0, u^i_1, \cdots$ appear in A'_k in this order. (Possibly, $u^i_j = u^i_{j+1}$ for some j.) If $u^i_0 \neq u^i_{m_i}$, then $A'_k \cup (\mathcal{D}^i - p_i)$ induces a 2-connected graph. So we can add $\mathcal{D}^i - p_i$ into A'_k and decrease the integer t. Thus, $u^i_0 = u^i_1 = \cdots = u^i_{m_i}$. Let $u_i = u^i_0$. Then u_i is a neighbor of $I_{\mathcal{D}^i}(B^i_j)$ in G for $0 \leq j \leq m_i$ with $I_{\mathcal{D}^i}(B^i_j) \neq \emptyset$. $\square$

Proof of Theorem 3. By Lemma 14, there exists an integer r such that if G is a 3-connected graph on $\mathbb{F}^2$ with representativity at least r, then G has a spanning subgraph as in Lemma 14. Since (G_0, C_0) is a circuit graph, it follows from Theorem 5 that $2G_0$ has a spanning even subgraph R_0 with

$$|E(R_0)| \leq \frac{4|V(G_0)| - 4}{3}. \tag{2}$$

Let $1 \leq i \leq t$ and let G_i be the subgraph induced by $\{u_i\} \cup \bigcup_{j=0}^{m_i} V(B^i_j) - \{p_i\}$ and let C_i be the outer boundary of G_i. Note that (G_i, C_i) is a circuit graph, unless u_i has degree one. The exceptional case happens only when (i) $m_i = 0$, or (ii) $m_i = 1$ and B^i_0 is an edge. Moreover, when (G_i, C_i) is not a circuit graph, then (G^*_i, C^*_i) is a circuit graph, where $G^*_i = G_i - u_i$ and $C^*_i = C_i - u_i$, unless G^*_i has at most two vertices.

If (G_i, C_i) is a circuit graph, then it follows from Theorem 5 or Proposition 6 that $2G_i$ has a spanning even subgraph R_i with $|E(R_i)| \le \frac{4|V(G_i)|-3}{3}$. On the other hand, suppose that (i) or (ii) holds. If (G_i^*, C_i^*) is a circuit graph, then it follows from Theorem 5 or Proposition 6 that $2G_i^*$ has a spanning even subgraph R_i^* with $|E(R_i^*)| \le \frac{4|V(G_i^*)|-3}{3}$. Let R_i be the subgraph of $2G_i$ obtained from R_i^* by adding the two edges in $2G_i$ incident with u_i. Then $|E(R_i)| \le \frac{4|V(G_i^*)|-3}{3} + 2 = \frac{4|V(G_i)|-1}{3}$. If G_i^* is isomorphic to K_2, then the two vertices of G_i^* and p_i (when (i) occurs) or u_i (when (ii) occurs) form a triangle. In these cases, the triangle, say R_i, is a spanning even subgraph of $2G_i$ with $|E(R_i)| = 3 = \frac{4|V(G_i)|-3}{3}$. Finally if G_i^* consists of only one vertex, then let R_i is the spanning even subgraph consisting of the two edges e and e' of $2G_i$ such that both e and e' correspond to an edge connecting the unique vertex in G_i^* and u_i. Then $|E(R_i)| = 2 = \frac{4|V(G_i)|-2}{3}$. In either case, we obtain a spanning even subgraph R_i of $2G_i$ with

$$|E(R_i)| \le \frac{4|V(G_i)|-1}{3}. \tag{3}$$

Let $R = \bigcup_{i=0}^{t} R_i$. Since G_i shares only one vertex with G_0 for $1 \le i \le t$, we have

$$|V(G)| = \sum_{i=0}^{t} |V(G_i)| - t.$$

Then R is a spanning even subgraph of $2G$. Moreover, by inequalities (2) and (3),

$$\begin{aligned}
|E(R)| &= \sum_{i=0}^{t} |E(R_i)| \\
&\le \frac{4|V(G_0)|-4}{3} + \sum_{i=1}^{t} \frac{4|V(G_i)|-1}{3} \\
&= \frac{4\sum_{i=0}^{t} |V(G_i)| - t - 4}{3} \\
&= \frac{4\big(|V(G)|+t\big) - t - 4}{3} \\
&= \frac{4|V(G)| + 3t - 4}{3} \\
&\le \frac{4|V(G)| + 3(-2\chi+2) - 4}{3} \\
&= \frac{4|V(G)| - 6\chi + 2}{3}.
\end{aligned}$$

By Proposition 4, this discussion completes the proof of Theorem 3. □