Hamilton cycles in 4-connected troidal triangulations

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1 Introduction

In 1956, Tutte [12] proved that every 4-connected planar graph has a hamilton cycle, and Thomassen [10] generalized this result; every 4-connected planar graph is hamilton-connected. Later, Thomas and Yu [7] improved Tutte's result: every 4-connected graph on the projective plane has a hamilton cycle. For graphs on the torus, Grünbaum [5] and independently Nash-Williams [6] conjectured the following:

Conjecture 1 (Grünbaum [5] and Nash-Williams [6]) *Every 4-connected graph on the torus has a hamilton cycle.*

Although Conjecture 1 is still open, there are some partial solutions to it. Altshuler [1] showed that every 6-connected toroidal graphs are Hamiltonian. (Note that such graph has to be a triangulation of the torus.) Brunet and Richter [3] proved that every 5-connected triangulation of the torus has a hamilton cycle, and Thomas and Yu [8] improved this to 5-connected graphs (not necessarily triangulation) on the torus. Dean and Ota [4], and Thomas, Yu and Zang [9] showed that every 4-connected graph on the torus has a 2-factor and a hamilton path, respectively.

In this paper, we solve the triangulation case of Conjecture 1.

Theorem 2 *Every 4-connected triangulation of the torus has a hamilton cycle.*

Remark that hamiltonicity of graphs on other surfaces has been also studied. Brunet, Nakamoto and Negami [2] showed that every 5-connected triangulation of the Klein bottle has a hamilton cycle. Thomassen [11] conjectured that for every closed surface $\mathbb{F}^2$, there exists an integer $r = r(\mathbb{F}^2)$ such that every 5-connected graph on $\mathbb{F}^2$ with representativity at least r has a hamilton cycle. Note that the representativity of a graph G on a non-spherical surface is the minimum number k such that every non-contractible simple closed

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curve intersects G at least k times. Yu [13] verified the triangulation case of Thomassen's conjecture. Generally, Thomassen's conjecture is still open.

For a graph G and a vertex x, we denote the set of neighbors of x by $N_G(x)$. When there is no fear of confusion, we write $N(x)$ for $N_G(x)$. For $X \subset V(G)$, let $N(X) = N_G(X) = \bigcup_{x \in X} N(x)$. A *block* in a graph is a maximal 2-connected subgraph. (We here regard K_2 as a 2-connected graph.) For a graph G, a pair $(K, \overline{K})$ of subgraphs of G is called *separation of* G if $V(G) = V(K) \cup V(\overline{K})$ and each edge of G are contained in exactly one of K and $\overline{K}$. A separation $(K, \overline{K})$ of G is a k*-separation* if $|K|, |\overline{K}| \geq k+1$ and $\left|V(K) \cap V(\overline{K})\right| = k$. Note that G is k-connected if and only if G has no r-separation for each $r < k$.

For a path P and two vertices $x, y \in V(P)$, xPy denotes the subpath of P between x and y. Let T be a subgraph of a graph G. A T*-bridge* of G is either an edge of $G - E(T)$ with both ends on T or a subgraph of G induced by the edges in a component of $G - V(T)$ and all edges from that component to T. A T-bridge which is an edge is called *trivial*; otherwise it is *non-trivial*. For a T-bridge B of G, the vertices in $B \cap T$ are the *attachments* of B (on T).

2 Lemmas used in the proof of Theorem 2

2.1 Known results concerning Tutte paths

Lemma 3 (Thomassen [10]) *Let G be a 2-connected plane graph and let D be the outer face cycle of G. Let $u, v \in V(D)$ and let $f \in E(D)$. Then G has a path T connecting u and v through f such that any T-bridge has at most three attachments and any T-bridge containing an edge of D has at most two attachments.*

Lemma 4 (Thomas and Yu, Lemma (2.4) in [7]) *Let G be a connected plane graph and let D be the boundary walk of the outer face of G. Let $u, v, x, y \in V(D)$ such that $x, y \notin V(P)$, where P is a subpath of D connecting u and v. Then $G - \{x, y\}$ has a path T connecting u and v such that any $(T \cup \{x, y\})$-bridge has at most three attachments, and any $(T \cup \{x, y\})$-bridge containing an edge of P has at most two attachments.*

Lemma 5 (Thomas and Yu, Lemma (2.7) in [7]) *Let G be a 2-connected plane graph and let D be the outer face cycle of G. Let $e_1, e_2, e_3 \in E(D)$. Then G has a cycle T containing e_1, e_2 and e_3 such that any T-bridge has at most three attachments, and any T-bridge containing an edge in D has at most two attachments.*

Lemma 6 (Thomas and Yu, Lemma (3.2) in [7]) *Let G be a 2-connected plane graph with outer cycle D and another facial cycle D', and let $e \in E(D)$. Then G contains a cycle T with $e \in E(T)$ such that any T-bridge has at most three attachments, any T-bridge* *containing an edge in $D \cup D'$ has at most two attachments, and no T-bridge contains edges in both D and D'.*

2.2 Lemmas concerning a 4-separation

In this section, we consider a 4-separation in a 4-connected triangulation. For a contractible cycle S of a graph G on the torus, let $\mathrm{int}(S)$ be the subgraph of G induced by all vertices in the inside of S.

Lemma 7 *Let G be a 4-connected triangulation of the torus. Suppose that G has a contractible cycle S of length four such that int(S) has a vertex. Let G' be the graph obtained from G by contracting int(S) into one vertex. Then G' is also a 4-connected triangulation of the torus with the same representativity of G. Moreover, if G' has a hamilton cycle, then G also has.*

Proof of Lemma 7. Assume that $\mathrm{int}(S)$ has at least two vertices. Let u_1, u_2, u_3, u_4 be vertices in S such that u_i's appear in S in the clockwise order. Let G' be the graph obtained from G by contracting $\mathrm{int}(S)$ to one vertex, say v. It is easy to see that G' is also a 4-connected triangulation of the torus. Since all non-contractible curve hitting G' at v can be moved homotopically so that it hits u_i for some i instead of v, G' has the same representativity as G.

Suppose that G' has a hamilton cycle T. We will show that G also has a hamilton cycle. Let M be the graph induced by $\mathrm{int}(S) \cup \{u_1, u_2, u_3, u_4\}$. Note that M is a 2-connected plane graph. We shall show that for any pair u_i and u_j with $1 \leq i < j \leq 4$, $M - \{u_k, u_h\}$ has a hamilton path T_M connecting u_i and u_j, where $\{u_k, u_h\} = V(S) - \{u_i, u_j\}$. If we can show that, replacing the vertex v in T with T_M appropriately, we obtain a hamilton cycle of G.

Suppose that $j = i + 1$ or $j = i - 1$, (let $u_5 = u_1$,) say $i = 1$ and $j = 2$ by symmetry. Then by Lemma 4, (letting $P = u_1u_2$,) M has a path T_M connecting u_1 and u_2 such that every $(T_M \cup \{u_3, u_4\})$-bridge has at most three attachments. However, if there exists a non-trivial $(T_M \cup \{u_3, u_4\})$-bridge B, then the attachments of B form a cut set in G of order at most three, a contradiction. Then $V(T_M) = V(M) - \{u_3, u_4\}$, and hence T_M is a desired path.

So we may assume that $j = i + 2$ or $j = i - 2$, say $i = 1$ and $j = 3$. Note that there exists a path P connecting u_1 and u_3 in $M - \{u_2, u_4\}$. (Otherwise $\{u_1, u_2, u_4\}$ or $\{u_2, u_3, u_4\}$ is a cut set of order three in G, a contradiction.) Let $v_1, v_2, \ldots, v_{r-1}$ be all the cut vertices of $M - \{u_2, u_4\}$ on P which separates u_1 from u_3, and let $v_0 = u_1$ and let $v_r = u_3$. We may assume that v_i's appear on P in this order. There exist blocks $M_1, M_2, \ldots, M_r$ in $M - \{u_2, u_4\}$ such that $v_{i-1}, v_i \in V(M_i)$ for any $1 \leq i \leq r$ and $M_1 \cup M_2 \cup \ldots \cup M_r = M - \{u_2, u_4\}$. For $1 \leq i \leq r$, let D_i be the outer face boundary of M_i. Since each M_i is a 2-connected plane graph or $M_i \simeq K_2$, it follows from Lemma

3 that there exists a path T_i connecting v_{i-1} and v_i in M_i such that any T_i-bridge has at most three attachments, and any T_i-bridge containing an edge of D_i has at most two attachments. Suppose that there exists a non-trivial T_i-bridge B in M_i. If B contains no edges of D_i, then the attachments of B form a cut set of order at most three in G, a contradiction. On the other hand, if B contains an edge of D, then the attachments of B together with u_2 or u_4 form a cut set in G, a contradiction again. Thus, there exists no non-trivial T_i-bridge in M_i, and hence T_i is a hamilton path in M_i. Therefore, letting $T_M = T_1 \cup T_2 \cup \ldots \cup T_r$, T_M is a hamilton path in $M - \{u_2, u_4\}$ connecting u_1 and u_3. This completes the proof of Lemma 7. $\Box$

2.3 Lemmas concerning a 5-separation

Next we consider a 5-separation of a 4-connected triangulation of the torus. In the following two lemmas, we consider the index of u as modulo 5.

Lemma 8 *Let G be a 4-connected triangulation of the torus. Suppose that G has no contractible cycle S' of length four such that int(S') has at least two vertices. Let $S = u_1u_2u_3u_4u_5u_1$ be a contractible cycle in G of length five and suppose that S has no inner chord. Let G' be the graph obtained from G by contracting int(S) to one vertex, say v, and suppose that G' has a hamilton cycle T. Then G also has a hamilton cycle unless T passes v from u_i to u_{i+1} for some i and there exists a vertex x in G of degree four and $V(S) - \{u_i, u_{i+1}\} \subset N_G(x)$.*

Proof of Lemma 8. Suppose first that T passes v from u_i to u_{i+2} for some i, say by symmetry $i = 1$. Let M be the graph induced by $\text{int}(S) \cup V(S)$. Since S has no inner chord, there exists a path P connecting u_1 and u_3 in $M - \{u_2, u_4, u_5\}$. Let $v_1, v_2, \ldots, v_{r-1}$ be all the cut vertices of $M - \{u_2, u_4, u_5\}$ on P which separates u_1 from u_3, and let $v_0 = u_1$ and let $v_r = u_3$. We may assume that v_i's appear on P in this order. There exist blocks $M_1, M_2, \ldots, M_r$ in $M - \{u_2, u_4, u_5\}$ such that $v_{i-1}, v_i \in V(M_i)$ for any $1 \leq i \leq r$ and $M_1 \cup M_2 \cup \ldots \cup M_r = M - \{u_2, u_4, u_5\}$. For $1 \leq i \leq r$, let D_i be the outer face boundary of M_i. Note that M_i is a 2-connected plane graph or M_i is an edge.

Suppose that for some i, M_i has a 2-separation $(K, \overline{K})$ such that $v_{i-1}v_i \notin K - \overline{K}$. Note that u_4 and u_5 is neighbors of $K - \overline{K}$ since G is 4-connected. Therefore, for all $1 \leq j \leq r$ M_j does not have such a 2-separation, except for the case where $j = i$ and $(K, \overline{K})$. Take such a 2-separation $(K, \overline{K})$ so that $|K|$ is as small as possible. Since G is a plane graph, $K \subset F$ for all 2-separation $(F, \overline{F})$ such that both u_4 and u_5 are neighbors of $F - \overline{F}$. Let e be an edge in $K \cap D_i$. If there exists no such a 2-separation $(K, \overline{K})$, then we take an edge e arbitrary from D_i for some i.

Then it follows from Lemma 3 that for all $1 \leq j \leq r$ there exists a path T_j connecting v_{j-1} and v_j in M_j such that any T_j-bridge has at most three attachments, and any T_j-

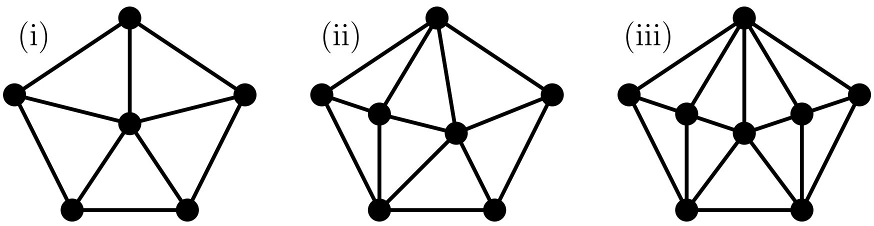

Figure 1:

bridge containing an edge of D_j has at most two attachments. In particular, when $j = i$, we can take such a path T_j so that it passes through e.

Suppose that there exists a non-trivial T_j-bridge B in M_j. If B contains no edges of D_j, then the attachments of B form a cut set of order at most three in G, a contradiction. Thus, we may assume that B contains an edge of D_j, and hence B has at most two attachments. Since G is 4-connected, $|N_G(B) \cap \{u_2, u_4, u_5\}| \geq 2$, so $\{u_4, u_5\} \subset N_G(B)$. However, by the choice of $(K, \overline{K})$, we obtain that $K \subset B$, which contradicts that T_M passes through $e_i \in E(K)$. Thus, there exists no non-trivial T_j-bridge in M_j, and hence T_j is a hamilton path in M_j. Letting $T_M = T_1 \cup T_2 \cup \ldots \cup T_r$, T_M is a hamilton path in $M - \{u_2, u_4, u_5\}$ connecting u_1 and u_3.

Then we may assume that T passes v from u_i to u_{i+1} for some i, say by symmetry $i = 1$. If $M' := M - \{u_3, u_4, u_5\}$ has a 1-separation $(F, \overline{F})$, (by symmetry, we may assume that $u_1, u_2 \in \overline{F}$,) then u_3, u_4, u_5 are neighbors of F since G is 4-connected. (Notice that u_1, u_2 are not neighbors of F since otherwise M has an inner chord.) Then $F \cap \overline{F} \cup \{u_3, u_4, u_5\}$ is a cut set of order four in G, and hence $F - \overline{F}$ consists of only one vertex, say x. In particular, x has the degree four and $\{u_3, u_4, u_5\} \subset N_G(x)$, and statement holds.

Hence we may assume that M' is 2-connected. Let D' be the outer face cycle of M'. Since G is a plane graph, there exist no three 2-separations $(K_1, \overline{K_1}), (K_2, \overline{K_2})$ and $(K_3, \overline{K_3})$ in M' such that $u_1, u_2 \in \overline{K_i} - K_i$ and $\left(K_i - \overline{K_i}\right) \cap \left(K_j - \overline{K_j}\right) = \emptyset$ for all $i, j = 1, 2, 3$. Let $(K_1, \overline{K_1})$ and $(K_2, \overline{K_2})$ be 2-separations such that $u_1, u_2 \in \overline{K_i} - K_i$ and $\left(K_i - \overline{K_i}\right) \cap \left(K_j - \overline{K_j}\right) = \emptyset$ for $i, j = 1, 2$ (if exist). Take such 2-separations so that $|K_1|$ and $|K_2|$ is as small as possible. Let e_1 and e_2 be edges in K_1 and K_2 contained in D', respectively. By Lemma 5, there exists a cycle T_M in M' with $u_1u_2, e_1, e_2 \in E(T_M)$ such that any T_M-bridge has at most three attachments, and any T_M-bridge containing an edge of D' has at most two attachments. Suppose that there exists a non-trivial T_M-bridge B in M'. If B contains no edges of D', then the attachments of B form a cut set of order at most three in G, a contradiction. Thus, we may assume that B contains an edge of D'. By the choice of K_1 and K_2, $K_1 \subset B$ or $K_2 \subset B$. However, B contains neither e_1 nor e_2, since T_R passes e_1 and e_2, a contradiction. $\Box$

Lemma 9 *Let G be a 4-connected triangulation of the torus. Suppose that G has no* *contractible cycle S' of length four such that int(S') has at least two vertices. Suppose that G has a contractible cycle S of length five such that int(S) has at least four vertices and does not have a chord. Then one of the graphs obtained from G by replacing $S \cup$int(S) with graphs in Figure 1, say G', is also a 4-connected triangulation of the torus with the same representativity of G. Moreover, if G' has a hamilton cycle, then G also has.*

Proof of Lemma 9. Let $S = u_1u_2u_3u_4u_5u_1$. First consider the graph G' obtained from G by contracting $\mathrm{int}(S)$ to one vertex. Suppose that G has a hamilton cycle, but G' does not have. Then by Lemma 8, there exists a vertex x of degree four in $\mathrm{int}(S)$ such that $V(S) - \{u_i, u_{i+1}\} \subset N_G(x)$, say $i = 4$.

Next, let $S' = u_1xu_3u_4u_5$ and consider the graph G'' obtained from G by contracting $\mathrm{int}(S')$ to one vertex. Suppose that G has a hamilton cycle, but G' does not have. Then by Lemma 8 and by the same reason as above, there exists a vertex y of degree four in $\mathrm{int}(S')$ such that y has three neighbors in S'. If $xy \in E(G)$, then $u_1u_2u_3z$ is a contractible cyce of order four, where $\{z\} = N_G(y) - \{u_1, x, u_3\}$, contradicting the assumption. Hence $xy \notin E(G)$, so we may assume that $N_G(y) \cap S' = \{u_1, u_4, u_5\}$.

Let $S'' = u_1xu_3yu_5$ and consider the graph G''' obtained from G by contracting $\mathrm{int}(S'')$ to one vertex. Suppose that G''' has a hamilton cycle, but G does not have. Then by Lemma 8, there exists a vertex z of degree four in $\mathrm{int}(S'')$ such that z has three neighbors in S''. However, similarly as above, $xz, yz \notin E(G)$, a contradiction.

In a same way as in the proof of Lemma 7, we can show that each of G', G'' and G'' are 4-connected triangulations of the torus with the same representativity of G. □

2.4 Lemmas concerning a 6-separation

In this section, we prove the following three lemmas, using Lemmas 3–5,

Lemma 10 *Let G be a 2-connected plane graph and let C be the boundary cycle of the outer face of G. Suppose that the length of C is exactly six, say $u_1u_2u_3u_4u_5u_6u_1$. Suppose that $G - \{u_1, u_4\}$ is 2-connected. Then $G - \{u_1, u_4\}$ has a cycle T containing the two edges u_2u_3 and u_5u_6 such that any $(T \cup \{u_1, u_4\})$-bridge has at most three attachments.*

Lemma 11 *Let G be a 2-connected plane graph and let C be the boundary cycle of the outer face of G. Suppose that the length of C is exactly six, say $u_1u_2u_3u_4u_5u_6u_1$. Suppose that $G - \{u_1, u_2, u_4\}$ is 2-connected. Then $G - \{u_1, u_2, u_4\}$ has a path T such that T connects u_5 and u_6, T passes u_3, and any $(T \cup \{u_1, u_2, u_4\})$-bridge has at most three attachments.*

Lemma 12 *Let G be a 2-connected plane graph and let C be the boundary cycle of the outer face of G. Suppose that the length of C is exactly six, say $u_1u_2u_3u_4u_5u_6u_1$. Suppose that $G - \{u_1, u_2\}$ is 2-connected, and $G - \{u_1, u_2, u_3, u_6\}$ is connected. Then $G - \{u_1, u_2\}$*

has two disjoint paths T_1 and T_2 such that T_1 connects u_3 and u_4, T_2 connects u_5 and u_6, and any $(T_1 \cup T_2 \cup \{u_1, u_2\})$-bridge has at most three attachments.

Proof of Lemma 10. Let $G' = G - \{u_1, u_4\}$. By Lemma 3, there exists a path Q in G' connecting u_2 and u_3 and passing the edge u_5u_6 such that any Q-bridge of G' has at most three attachments, and any Q-bridge containing an edge in D' has at most two attachments, where D' be the boundary cycle of G'. Let T be the cycle obtained from Q by adding the edge u_2u_3.

Let B be a non-trivial $(T \cup \{u_1, u_4\})$-bridge of G. Let $B' = B - \{u_1, u_4\}$. Note that B' is a T-bridge of G'. If B' contains no edge in D', then neither u_1 nor u_4 are attachments of B in G, and hence B also has at most three attachments in G. So suppose that B' contains an edge in D'. Then u_1 or u_4 is an attachment of B in G. Since G is a plane graph and T contains u_2, u_3, u_5 and u_6, exactly one of u_1 and u_4 is an attachment of B in G, by symmetry, say u_1. Since B' contains an edge of D', B' has at most two attachments in G'. Hence in G, B has at most three attachments in G, one of them is u_1 and the others are attachments in G'. □

Proof of Lemma 11. Let $G' = G - \{u_1, u_2, u_4\}$ and let D' be the outer face cycle of G'. Suppose that G' has a 2-separation $(K, \overline{K})$ such that $u_5, u_6 \notin K - \overline{K}$ and both u_1 and u_2 are neighbors of $K - \overline{K}$. Take such a 2-separation so that $|K|$ is as small as possible. Since G is a plane graph, $K \subset F$ for all 2-separation $(F, \overline{F})$ such that both u_1 and u_2 are neighbors of $F - \overline{F}$. Let e_1 be an edge in $K \cap D'$ which is not contained in $K \cap \overline{K}$. If there exists no such a 2-separation $(K_1, \overline{K_1})$, then we take an edge e_1 arbitrary. Let e_2 be an edge of D' incident with u_3.

By Lemma 5, G' has a path T connecting u_5 and u_6 through e_1 and e_2 such that any T-bridge of G' has at most three attachments, and any T-bridge of G' containing an edge in D' has at most two attachments.

Let B be a non-trivial $(T \cup \{u_1, u_2, u_4\})$-bridge of G. Let $B' = B - \{u_1, u_2, u_4\}$. Note that B' is a T-bridge of G'. If B' contains no edge in D', then none of u_1, u_2 and u_4 are attachments of B in G, and hence B also has at most three attachments in G. So suppose that B' contains an edge in D'. Then B has at most two attachments in G. If at most one of u_1, u_2 and u_4 is an attachment of B in G, then B has at most three attachments in G, one of them is u_i for some $i = 1, 2, 4$ and the other two are attachments in G'. So, suppose that B' has two attachments in G' and at least two of u_1, u_2 and u_4 are attachments of B in G. However, this contradicts that G is a plane graph and T contains e_1 and e_2. This completes the proof of Lemma 11. □

Proof of Lemma 12. Let $G' = G - \{u_1, u_2, u_5, u_6\}$ and let D' be the outer face cycle of G'. Suppose that G' has a 2-separation $(K_1, \overline{K_1})$ such that $u_3, u_4 \notin K_1 - \overline{K_1}$ and both u_1 and u_2 are neighbors of $K_1 - \overline{K_1}$. Take such a 2-separation so that $|K_1|$ is as small as possible. Let e_1 be an edge in $K_1 \cap D'$. If there exists no such a 2-separation $(K_1, \overline{K_1})$, then we take an edge e_1 arbitrary from D'.

Similarly, if G' has a 2-separation $(K_2, \overline{K_2})$ such that $u_3, u_4 \notin K_2 - \overline{K_2}$ and both u_1 and u_6 are neighbors of $K_2 - \overline{K_2}$, then taking such a 2-separation so that $|K_2|$ is as small as possible, we let e_2 be an edge in $K_2 \cap D'$. If there exists no such a 2-separation $(K_2, \overline{K_2})$, then we take an edge e_2 arbitrary from D'.

By Lemma 5, G' has a path T_1 connecting u_3 and u_4 through e_1 and e_2 such that any T_1-bridge of G' has at most three attachments, and any T_1-bridge of G' containing an edge in D' has at most two attachments.

Let B_1 be a non-trivial $(T_1 \cup \{u_1, u_2, u_5, u_6\})$-bridge of G. Let $B_1' = B_1 - \{u_1, u_2, u_5, u_6\}$. Note that B_1' is a T_1-bridge of G'. If B_1' contains no edge in D', then none of u_1, u_2, u_5 and u_6 are attachments of B_1 in G, and hence B_1 also has at most three attachments in G. So suppose that B_1' contains an edge in D'. Then B_1' has at most two attachments in G'. If at most one of u_1, u_2, u_5 and u_6 is an attachment of B_1 in G, then B_1 has at most three attachments in G, one of them is u_i for some $i = 1, 2, 5, 6$ and the other two are attachments in G'.

So, suppose that B_1' has two attachments in G' and at least two of u_1, u_2, u_5 and u_6 are attachments of B_1 in G. Since G is a plane graph and T_1 contains e_1 and e_2, u_5 and u_6 are attachments of B_1 in G, and there exist no $(T_1 \cup \{u_1, u_2, u_5, u_6\})$-bridge B of G with $B \neq B_1$ such that B has at least four attachments in G.

Let v_1 and v_2 be attachments of B_1' in G'. Let M be the block of the subgraph of G induced by B_1 containing both u_5 and u_6. Since both u_5 and u_6 are neighbors of B_1' in G, M has at least three vertices, and hence M is 2-connected. Let D_M be the outer face cycle of M. If M has a 2-separation $(F, \overline{F})$ such that $u_5, u_6 \in V(\overline{F} - F)$ and both v_1 and v_2 are neighbors of $F - \overline{F}$. Take such a 2-separation so that $|F|$ is as small as possible. Let f be an edge in $F \cap D_M$. If there exists no such a 2-separation $(F, \overline{F})$, then we take an edge f in D_M arbitrary. By Lemma 3, there exists a path T_2 in M connecting u_5 and u_6 through f such that any T_2-bridge of M has at most three attachments. and any T_2-bridge containing an edge in D_M has at most two attachments.

Let B_M be a non-trivial $(T_2 \cup \{v_1, v_2\})$-bridge of G. Note that B_M is a $(T_1 \cup T_2)$-bridge of G, and B_M' is a T_2-bridge of M, where $B_M' = B_M - \{v_1, v_2\}$. If B_M' contains no edge in D_M, then neither v_1 nor v_2 are attachments of B_M in G, and hence B_M has also at most three attachments in G. If B_M' has only one attachment in M, then B_M has at most three attachments in G, one of them is an attachment in M, and the other two is v_1 and v_2. So we may suppose that B_M' contains an edge in D' and B_M' has exactly two attachments in M. Since G is a plane graph and T_1 passes f, at least one of v_1 and v_2 is not an attachment of B_M in G. Then B_M also has at most three attachments.

In either case, any $\big(T_1 \cup T_2 \cup \{u_1, u_2\}\big)$-bridge has at most three attachments, and hence T_1 and T_2 are desired paths in Lemma 11. $\square$

2.5 Lemmas concerning a l-separation with $l \leq 8$

Lemma 13 *Let G be a 2-connected plane graph and let D be the boundary cycle of the outer face of G. Suppose that each face of G is a triangle except for D and D has no chord. Suppose that the length of D is at most eight. Let $D = u_1u_2u_3 \dots u_lu_1$. Let $S \subset \{u_l\}$. Then the followings hold:*

(I-i) *Let $r = \lfloor \frac{l-2}{2} \rfloor$. Then $G - S$ has r disjoint paths $T_1, T_2, \dots, T_r$ such that T_1 connects u_1 and u_3, T_i connects u_{2i} and u_{2i+1} for each $2 \leq i \leq r-1$, T_r connects u_{2r} and u_{2r+2}, and any $\left(\bigcup_{i=1}^{r} T_i \cup S \cup \{u_2, u_{2r+1}\}\right)$-bridge of G has at most three attachments.*

(I-ii) *Let $r = \lfloor \frac{l-1}{2} \rfloor$. Then $G - S$ has r disjoint paths $T_1, T_2, \dots, T_r$ such that T_1 connects u_1 and u_3, T_i connects u_{2i} and u_{2i+1} for each $2 \leq i \leq r$, and any $\left(\bigcup_{i=1}^{r} T_i \cup S \cup \{u_2\}\right)$-bridge of G has at most three attachments.*

(II-i) *Let $r = \lfloor \frac{l-1}{2} \rfloor$. Then $G - S$ has r disjoint paths $T_1, T_2, \dots, T_r$ such that T_i connects u_{2i-1} and u_{2i} for each $1 \leq i \leq r-1$, T_r connects u_{2r-1} and u_{2r+1}, and any $\left(\bigcup_{i=1}^{r} T_i \cup S \cup \{u_{2r}\}\right)$-bridge of G has at most three attachments.*

(II-ii) *Let $r = \lfloor \frac{l}{2} \rfloor$. Then G has r disjoint paths $T_1, T_2, \dots, T_r$ such that T_i connects u_{2i-1} and u_{2i} for each $1 \leq i \leq r$, and any $\left(\bigcup_{i=1}^{r} T_i \cup S\right)$-bridge of G has at most three attachments.*

(III-i) *Let $r = \lfloor \frac{l-2}{2} \rfloor$. Then G has r disjoint paths $T_1, T_2, \dots, T_r$ such that T_i connects u_{2i} and u_{2i+1} for each $1 \leq i \leq r-1$, T_r connects u_{2r} and u_{2r+2}, and any $\left(\bigcup_{i=1}^{r} T_i \cup S \cup \{u_1, u_{2r+1}\}\right)$-bridge of G has at most three attachments.*

(III-ii) *Let $r = \lfloor \frac{l-1}{2} \rfloor$. Then G has r disjoint paths $T_1, T_2, \dots, T_r$ such that T_i connects u_{2i} and u_{2i+1} for each $1 \leq i \leq r$, and any $\left(\bigcup_{i=1}^{r} T_i \cup S \cup \{u_1\}\right)$-bridge of G has at most three attachments.*

Proof of Lemma 13.
TBA

2.6 A new lemma concerning a Tutte path

In the proof of Theorem 2, we need the following result, which is an improvement of Lemma 6 for the special case.

Lemma 14 *Let G be a 2-connected plane graph with outer cycle D and another facial cycle D', and let $u, v \in E(D)$. Suppose that every face of G except for D and D' is a triangle. Then G contains a path T connecting u and v such that any T-bridge has at most three attachments, any T-bridge containing an edge in $D \cup D'$ has at most two attachments, and no T-bridge contains edges in both D and D'.*

Proof of Lemma 14.

TBA

2.7 Other lemmas used in the proof

Lemma 15 *Let G be a graph on the torus, and let C be a shortest non-contractible cycle of G. Suppose that there exists a path P in G connecting two vertices of C, say u and v, such that each cycles $uCv \cup P$ and $vCu \cup P$ are non-contractible, where uCv and vCu be the two subpaths of C connecting u and v. Then $|V(C)| \leq 2|V(P)| - 2$.*

For a near triangulation, it is easy to prove the following lemma (see, for example, [3]). We implicitly use Lemma 16 in the proof of Theorem 2 many times.

Lemma 16 *Let G be a 2-connected plane graph and let D be the outer face cycle of G. Suppose that all faces of G except for D is a triangle. Let $(K, \overline{K})$ be a 1- or 2-separation of G. Then $G' = G - \big(K - \overline{K}\big)$ is also 2-connected. Moreover, for each 2-separation $(F, \overline{F})$ of G', $(F \cup K, \overline{F})$ or $(F, \overline{F} \cup K)$ is a 2-separation of G.*

3 Proof of Theorem 2

In this section, we will prove Theorem 2. We divide this section into six subsections.

3.1 Settings and main claim of the proof of Theorem 2

Let G be a 4-connected triangulation of the torus. For the case where the representativity of G is at most four, we can cut the torus and obtain a spanning plane subgraph of G satisfying some connectivity conditions. By using some results on Tutte paths, we can find a hamilton cycle in such plane graphs. (We omit the detail of the proof here.)

By Lemma 7, we may assume that for each contractible cycle S of length four, $\text{int}(S)$ consists of only one vertex. Since otherwise, we can contract $\text{int}(S)$ into one vertex, and it is enough to find a hamilton cycle in the obtained graph. If G has two adjacent vertices of degree four, then since G is a triangulation, the neighbors of the two vertices consists of four vertices which form a contractible cycle. By Lemma 9, we may similarly assume that for each contractible cycle S of length five, $\text{int}(S)$ consists of at most three vertices. Then we may also assume the following claim. Let $V_4(G)$ be the set of vertices in G of degree four.

Claim 1 *For each contractible cycle S of length four, $\text{int}(S)$ consists of one vertex in $V_4(G)$. Moreover, $V_4(G)$ is an independent set.*

Claim 2 *For each contractible cycle S of length five, $\text{int}(S)$ consists of at most three vertices.*

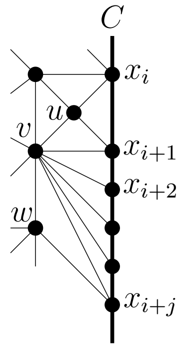

Figure 2: A bad configuration.

Let $C = x_1 \dots x_m x_1$ be a non-contractible cycle of G. A vertex u in $V_4(G) - V(C)$ is called *good for* C if u has three consecutive neighbors in C. Let $V_{good}(C)$ be the set of good vertices for C. Take a non-contractible cycle C such that

(C1) C is as short as possible,

(C2) and $3|V_4(G) \cap V(C)| + 2|V_{good}(C)|$ is as large as possible, subject to (C1).

A *bad configuration for* C consists of one vertex in $V_4(G) - V(C)$, say u, two vertices in $G - V(C)$, say v and w, and at least four consecutive vertices in C, say $x_i, x_{i+1}, \dots, x_{i+j}$ ($j \geq 3$), together with the edges $x_i u, x_{i+1} u, uv, vx_{i+1}, vx_{i+2}, \dots, vx_{i+j}, vw, wx_{i+j}$, see Figure 2. Moreover, the same configuration for the reverse orientation of C is also a bad configuration for C.

By the choice of C, we have the following claim. Note that we use condition (C2) here only.

Claim 3 *There are no bad configurations for* C.

Proof. Suppose that there is a bad configuration for C, and let $u, v, w, x_i, \dots, x_{i+j}$ as in Figure 2. We may assume that $i = 1$. If $j \geq 4$, then the cycle $x_1 uvx_{1+j} \dots x_m x_1$ is shorter than C, contradicting the condition (C1). Thus, we have that $j = 3$.

Suppose that there exists a good vertex y for C such that $x_1, x_2, x_3 \in N_G(y)$. Note that y is adjacent with C from the other side of u, v, w. Let $C' = x_1 y x_3 x_4 \dots x_m x_1$. Note that $|V(C')| = |V(C)|$, and hence C' also satisfies the condition (C1). Since $y \in V_4(G)$, we have that $|V_4(G) \cap V(C')| = |V_4(G) \cap V(C)| + 1$. By the existence of u, v, w, $|V_{good}(C')| = |V_{good}(C)| - 1$. Hence $3|V_4(G) \cap V(C')| + 2|V_{good}(C')| = 3|V_4(G) \cap V(C)| + 2|V_{good}(C)| + 1$, contradicting the condition (C2). Therefore, there exists no good vertex y for C such that $x_1, x_2, x_3 \in N_G(y)$. Similarly, we can show that there exists no good vertex y for C such that $x_3, x_4, x_5 \in N_G(y)$. By Claim 1, there exists no good vertex y for C such that $x_2, x_3, x_4 \in N_G(y)$.

Let $C'' = x_1x_2vx_4\dots x_mx_1$. Note that $|V(C'')| = |V(C)|$, and hence C' also satisfies the condition (C1). Notice also that $u \in V_{good}(C'')$. Then it follows from the above facts that $|V_{good}(C'')| \geq |V_{good}(C)| + 1$. If $x_3 \notin V_4(G)$, then $3|V_4(G) \cap V(C'')| + 2|V_{good}(C'')| \geq 3|V_4(G) \cap V(C)| + 2|V_{good}(C)| + 2$, contradicting the condition (C2).

So, $x_3 \in V_4(G)$ and $|V_4(G) \cap V(C'')| = |V_4(G) \cap V(C)| - 1$. In this case, $x_3 \in V_{good}(C'')$ and hence $|V_{good}(C'')| \geq |V_{good}(C)| + 2$. Then

$$\begin{aligned} 3|V_4(G) \cap V(C'')| + 2|V_{good}(C'')| &\geq 3\big(|V_4(G) \cap V(C)| - 1\big) + 2\big(|V_{good}(C)| + 2\big) \\ &= 3|V_4(G) \cap V(C)| + 2|V_{good}(C)| + 1 \end{aligned}$$

contradicting the condition (C2). □

The main idea of the proof is the following claim.

Claim 4 *There exist a partition $\{U_0, U_{K_0}\} \cup \mathscr{U}$ of $V(G) - V(C)$, and a mapping φ from $\{U_{K_0}\} \cup \mathscr{U}$ to $E(C)$, and two vertices $u_1, u_2 \in U_0$ satisfying the followings.*

(U1) *Let H_0 be the subgraph of G induced by U_0. Then H_0 is a 2-connected graph on the cylinder. Moreover, for all 2-separation $(K, \overline{K})$, $K \cap \overline{K} \subset V(D_0) \cap V(D'_0)$ or K or $\overline{K}$ (by symmetry say K) can be bounded by a disc on the cylinder and $u_1 \in K$, and for all 3-separation $(K, \overline{K})$, two of the vertices of $K \cap \overline{K}$ are contained in D_0 or D'_0, where D_0 and D'_0 are the boundary cycles of H_0.*

(U2) *There exist two paths Q_0 and Q'_0 in the subgraph of G induced by $U_{K_0} \cup \{x_i, x_{i+1}, u_1, u_2\}$ such that $u_1, u_2 \in V(D_0)$ or $u_1, u_2 \in V(D'_0)$, Q_0 connects x_i and u_1, Q'_0 connects x_{i+1} and u_2, and $V(Q_0) \cup V(Q'_0) = U_{K_0} \cup \{x_i, x_{i+1}, u_1, u_2\}$, where $x_ix_{i+1} = \varphi(U_{K_0})$. (Note that we allow the case where $U_{K_0} = \emptyset$.)*

(U3) *For each $U \in \mathscr{U}$, the subgraph of G induced by $U \cup \{x_i, x_{i+1}\}$ has a hamilton path connecting x_i and x_{i+1}, where $x_ix_{i+1} = \varphi(U)$.*

(U4) *φ is an injection.*

Before showing Claim 4, here we prove Theorem 2 assuming Claim 4.

Suppose that there are a partition $\{U_0, U_{K_0}\} \cup \mathscr{U}$ of $V(G) - V(C)$, and a mapping φ from $\{U_{K_0}\} \cup \mathscr{U}$ to $E(C)$ satisfying conditions (U1)–(U4).

Let H_0 be the subgraph of G induced by U_0, and let D_0 and D'_0 be the boundary cycles of H_0. Note that $u_1, u_2 \in V(D_0)$ or $u_1, u_2 \in V(D'_0)$. By Lemma 14, H_0 contains a path T_0 connecting u_1 and u_2 such that any T_0-bridge has at most three attachments, any T_0-bridge containing an edge in $D_0 \cup D'_0$ has at most two attachments, and no T_0-bridge contains edges in both D_0 and D'_0.

Suppose that there exists a non-trivial T_0-bridge B of H_0. By symmetry, we may assume that B contains no edge in D'_0. Note that the attachments of B forms a cut set of H_0. If B contains an edge in D_0, then B has at most two attachments, but this contradicts that the condition on 2-separations of H_0 in (U1) and the fact that T_0 connects u_1 and u_2. So, B contains no edge in D_0. Hence by the condition of T_0, B has exactly three attachments. and by the condition on H_0, two of them are contained in D_0 or D'_0. However, this implies that B contains an edge in $D_0 \cup D'_0$, and hence B has at most two attachments, a contradiction. Therefore, there exists a non-trivial T_0-bridge B of H_0, and hence T_0 is a hamilton cycle of H_0.

Let $U \in \mathscr{U}$ and let $x_i x_{i+1} = \varphi(U)$. By condition (U3), the subgraph of G induced by $U \cup \{x_i, x_{i+1}\}$ has a hamilton path Q_U connecting x_i and x_{i+1}.

Let

$$T = \Big(C - \{x_i x_{i+1} \in \varphi(U) : U \in \mathscr{U}\}\Big) \cup \bigcup_{U \in \mathscr{U}} Q_U.$$

Note that $\{U_0, U_{K_0}\} \cup \mathscr{U}$ is a partition of $V(G) - V(C)$ and Q_U is a path passing all vertices in U and connecting x_i and x_{i+1}. Then it follows from condition (U4) that T is a hamilton cycle in $G - U_0 - U_{K_0}$.

Let $x_i x_{i+1} = \varphi(U_{K_0})$. By condition (U2), the subgraph of G induced by $U_{K_0} \cup \{x_i, x_{i+1}, u_1, u_2\}$ has two paths Q_0 and Q'_0 such that Q_0 connects x_i and u_1, Q'_0 connects x_{i+1} and u_2, and $V(Q_0) \cup V(Q'_0) = U_{K_0} \cup \{x_i, x_{i+1}, u_1, u_2\}$.

Then

$$\big(T_0 - u_1 u_2\big) \cup \big(T - \varphi(U_{K_0})\big) \cup Q_0 \cup Q'_0$$

is a hamilton cycle of G. This completes the proof of Theorem 2. □

3.2 Preliminary for the proof of Claim 4

It only remains to show Claim 4. We will show it in the rest of this paper. Since C is 2-sided, we can distinguish the right side and the left side of C. For simplicity, we use the cutting method (along C), which is first considered by Thomassen [11].

From G, we obtain the graph G' on the cylinder in the following way. First add a cycle $C' = x'_1 \dots x'_m x'_1$. If an edge yx_i is incident with C on its left side in G, we delete yx_i and add instead yx'_i. Finally attach a disc to C and another disc to C' to get the sphere. Let G' be the graph obtained by cutting as above.

Note that we assumed that the representativity of G is at least 5. Here we further assume that the representativity of G is at least 7, For the case where the representativity of G is 5 or 6, we can use the similar argument to the case where the representativity at least 7.

Suppose that G' has a path P connecting C and C'. Then P corresponds to a path as in Lemma 15, and hence P has at least 5 vertices, since otherwise $|V(C)| \leq 6$, which contradicts that we assumed that G has representativity at least 7. Then each path in G' connecting C and C' has at least 5 vertices.

Let $H = G' - V(C) - V(C')$ (note that $H = G - V(C)$). Since C has no chord and G is a triangulation, H is a connected graph on the cylinder. Let D and D' be the two boundary walks of H. We may assume that each vertex in D has a neighbor in C, and each vertex in D' has a neighbor in C'. Since each path in G' connecting C and C' has at least 5 vertices, we have the following claim.

Claim 5 *Each path in H connecting D and D' has at least three vertices.*

For a 1- or 2-separation $(K, \overline{K})$ of H, it follows from Claim 5 that exactly one of the followings holds; $N(K-\overline{K}) \cap V(C) = \emptyset$, $N(K-\overline{K}) \cap V(C') = \emptyset$, $N(\overline{K}-K) \cap V(C) = \emptyset$ and $N(\overline{K}-K) \cap V(C') = \emptyset$. By symmetry, in the rest of this paper, we may assume that all 1- or 2-separation $(K, \overline{K})$ of H satisfies that $N(K - \overline{K}) \cap V(C) = \emptyset$ or $N(K - \overline{K}) \cap V(C') = \emptyset$. By this notation, $\overline{K} - K$ has neighbors in both C and C' since G is a triangulation. By the 4-connectedness of G, $N(K - \overline{K}) \cap V(C) \neq \emptyset$ if $N(K - \overline{K}) \cap V(C') = \emptyset$, and $N(K - \overline{K}) \cap V(C') \neq \emptyset$ if $N(K - \overline{K}) \cap V(C) = \emptyset$.

We call a 1- or 2-separation $(K, \overline{K})$ of H *maximal* if there exists no 1- or 2-separation $(F, \overline{F})$ of H $\big($we also assume that $N(F - \overline{F}) \cap V(C) = \emptyset$ or $N(F - \overline{F}) \cap V(C') = \emptyset\big)$ such that $K \subset F$ and $F \neq K$. Let

$$\begin{aligned}
\mathscr{B}_1 &= \big\{X_K = V\big(K - \overline{K}\big) : \\
&\qquad (K, \overline{K}) \text{ is a maximal 1-separation of } H \text{ and } X_K \text{ has no neighbors in } C'\big\}, \\
\mathscr{B}_1' &= \big\{X_K = V\big(K - \overline{K}\big) : \\
&\qquad (K, \overline{K}) \text{ is a maximal 1-separation of } H \text{ and } X_K \text{ has no neighbors in } C\big\}, \\
\mathscr{B}_2 &= \big\{X_K = V\big(K - \overline{K}\big) : \\
&\qquad (K, \overline{K}) \text{ is a maximal 2-separation of } H \text{ and } X_K \text{ has no neighbors in } C'\big\}, \\
\mathscr{B}_2' &= \big\{X_K = V\big(K - \overline{K}\big) : \\
&\qquad (K, \overline{K}) \text{ is a maximal 2-separation of } H \text{ and } X_K \text{ has no neighbors in } C\big\}, \\
\mathscr{B} &= \mathscr{B}_1 \cup \mathscr{B}_1' \cup \mathscr{B}_2 \cup \mathscr{B}_2'.
\end{aligned}$$

In the rest of this paper, for each $X_K \in \mathscr{B}_1 \cup \mathscr{B}_1' \cup \mathscr{B}_2 \cup \mathscr{B}_2'$, we naturally define $(K, \overline{K})$ as the 1- or 2-separation of H with $X_K = K - \overline{K}$. Moreover, let y_K be the unique vertex in $K \cap \overline{K}$ when $X_K \in \mathscr{B}_1 \cup \mathscr{B}_1'$, and let $\{y_K, z_K\} = K \cap \overline{K}$ when $X_K \in \mathscr{B}_2 \cup \mathscr{B}_2'$.

Let $X_K \in \mathscr{B}$. Note that by Lemma 16, $H - (K - \overline{K})$ is also a graph on the cylinder with two boundary cycles contained in D and D' and has fewer 1- or 2-separations. Let $H_1 = H - \bigcup_{X_K \in \mathscr{B}} X_K$, and let D_1 and D_1' be two boundary cycles of H_1. We may assume that all vertices in D_1 has a neighbor in C, and all vertices in D_1' has a neighbor in C'. By Lemma 16, H_1 is 3-connected. Moreover, since G is 4-connected, for all 3-separation $(F, \overline{F})$, we have that two of the vertices in $F \cap \overline{F}$ are contained in D_1 or D_1'. Actually, the vertex set of H_1 is a candidate for U_0 in Claim 4. Since each path in H connecting D and D' has at least four vertices, we have the following claim.

Claim 6 *Each path in H_1 connecting D_1 and D_1' has at least two vertices.*

We have the following claims.

Claim 7 *Let $X_K \in \mathscr{B}_1 \cup \mathscr{B}_1'$. Then there exists an integer i such that x_i, x_{i+1}, x_{i+2} (if $X_K \in \mathscr{B}_1$,) or x_i', x_{i+1}', x_{i+2}' (if $X_K \in \mathscr{B}_1'$) are the neighbor of X_K in G'. Moreover, X_K consists of only one vertex in $V_4(G')$.*

Claim 8 *Let $X_K \in \mathscr{B}_2 \cup \mathscr{B}_2'$. Then there exist two integers i and j such that (changing the names of y_K and z_K if necessary) $y_K x_i, z_K x_j \in E(G')$ and the cycle $y_K x_i C x_j z_K y_K$ (when $X_K \in \mathscr{B}_1$) or $y_K x_i' C' x_j' z_K y_K$ (when $X_K \in \mathscr{B}_1'$) bounds a disk containing all vertices of X_K. Moreover, $i < j \leq i + 3$.*

Proof of Claim 7. Let $X_K \in \mathscr{B}_1 \cup \mathscr{B}_1'$. We may assume that $X_K \in \mathscr{B}_1$. Since G is a triangulation, y_K has two neighbors in C, say x_i and x_j, such that the cycle $y_K x_i C x_j y_K$ bounds a disk containing all vertices of X_K. If $j \leq i + 1$, then y_K, x_i (and x_{i+1}) form a 2- or 3-cut of G separating X_K, a contradiction. On the other hand, if $j \geq i + 3$, then the cycle $x_1, \dots, x_i, y_K, x_j, \dots, x_m, x_1$ is a non-contractible cycle which is shorter than C, contradicting condition (C1). Then $j = i + 2$, and hence $y_K, x_i, x_{i+1}, x_{i+2} y_K$ is a contractible cycle of length four. It follows from Claim 1 that X_K consists of only one vertex in $V_4(G)$. $\square$

Proof of Claim 8. Let $X_K \in \mathscr{B}_2 \cup \mathscr{B}_2'$. Here we assume that $X_K \in \mathscr{B}_2$, but we can prove the case where $X_K \in \mathscr{B}_2'$ in a same way. Since G is a triangulation, changing the names of y_K and z_K if necessary, y_K and z_K have neighbors x_i and x_j in G', respectively, such that $y_K x_i C x_j z_K y_K$ bounds a disk containing all vertices of X_K. We may assume that $j \geq i$. If $i = j$, then y_K, z_K and x_i form a 3-cut of G, a contradiction. So we may assume that $j > i$. On the other hand, if $j \geq i + 4$, then the cycle $x_1, \dots, x_i, y_K, z_K x_j, \dots, x_m x_1$ is a non-contractible cycle which is shorter than C, contradicting condition (C1). $\square$

Let $X_K \in \mathscr{B}_1 \cup \mathscr{B}_1'$. Then by Claim 7, there exists an integer i such that x_i, x_{i+1}, x_{i+2} (if $X_K \in \mathscr{B}_1$,) or x_i', x_{i+1}', x_{i+2}' (if $X_K \in \mathscr{B}_1'$) are the neighbor of X_K in G'. In this case, let $\alpha(K) = i$ and $\beta(K) = i + 2$. Let $X_K \in \mathscr{B}_2 \cup \mathscr{B}_2'$. In this case, let $\alpha(K)$ and $\beta(K)$ be the integers i and j as in Claim 8, respectively.

We devide $\mathscr{B}_2$ into three sets as follows:

$$\begin{aligned}
\mathscr{B}_{2,1} &= \{X_K \in \mathscr{B}_2 : \beta(K) = \alpha(K) + 1\}. \\
\mathscr{B}_{2,2} &= \{X_K \in \mathscr{B}_2 - \mathscr{B}_{2,1} : \beta(K) = \alpha(K) + 2\}. \\
\mathscr{B}_{2,3} &= \mathscr{B}_2 - \big(\mathscr{B}_{2,1} \cup \mathscr{B}_{2,2}\big).
\end{aligned}$$

By Claim 8, for each $X_K \in \mathscr{B}_{2,3}$, $\beta(K) = \alpha(K) + 3$. Similarly, we partition $\mathscr{B}_2'$ into $\mathscr{B}_{2,1}', \mathscr{B}_{2,2}'$ and $\mathscr{B}_{2,3}'$.

3.3 Finding the set U_{K_0}

In this subsection, we will find a vertex set U_{K_0} as desired in condition (U2).

Case 1. m is odd.

Case 1.1. $\mathscr{B}_{2,1} \cup \mathscr{B}'_{2,1} \neq \emptyset$.

Let $X_{K_0} \in \mathscr{B}_{2,1} \cup \mathscr{B}'_{2,1}$. We may assume that $X_{K_0} \in \mathscr{B}_{2,1}$, $\alpha(K_0) = m$ and $\beta(K_0) = 1$. Note that $y_{K_0}, z_{K_0}, x_m, x_1$ form a 4-cut in G, by Claim 1, X_{K_0} consists of only one vertex, say u_0. In this case, letting $U_{K_0} = X_{K_0} = \{u_0\}$, $u_1 = y_{K_0}$, $u_2 = z_{K_0}$, and $\varphi(U_{K_0}) = x_m x_1$, we can find two paths $Q_0 = x_m u_1$ and $Q'_0 = x_1 u_0 u_2$ in the subgraph of G' induced by $U_{K_0} \cup \{x_m, x_1, u_1, u_2\}$. In this case, let $m' = m$.

Case 1.2. $\mathscr{B}_{2,1} = \mathscr{B}'_{2,1} = \emptyset$ and $\mathscr{B}_{2,3} \cup \mathscr{B}'_{2,3} \neq \emptyset$.

Let $X_{K_0} \in \mathscr{B}_{2,3} \cup \mathscr{B}'_{2,3}$. We may assume that $X_{K_0} \in \mathscr{B}'_{2,3}$, $\alpha(K_0) = m-2$ and $\beta(K_0) = 1$. Let R be the subgraph of G' induced by $X_{K_0} \cup \{x_{m-2}, x_{m-1}, x_m, x_1, y_{K_0}, z_{K_0}\}$, and let $R' = R - \{x_{m-2}, x_1\}$. Note that R is 2-connected. If R' is not 2-connected, then $x_{m-2}x_1 \in E(G')$ or there exists a cut vertex u in R' with $ux_{m-2}, ux_1 \in E(G')$, since G is a triangulation. However, both contradicts condition (C1) of C. Hence R' is also 2-connected. Then by Lemma 10, $R - \{x_{m-2}, x_1\}$ has a cycle T_0 containing the two edges $x_{m-1}x_m$ and $y_{K_0}z_{K_0}$ such that any $(T \cup \{x_{m-2}, x_1\})$-bridge has at most three attachments. If there exists a non-trivial $(T \cup \{x_{m-2}, x_1\})$-bridge B, then the attachments of B forms a cut set of order at most three, contradicting that G is 4-connected. So, $V(T) = V(R) - \{x_{m-2}, x_1\} = X_{K_0} \cup \{x_{m-1}, x_m, y_{K_0}, z_{K_0}\}$. Let $U_{K_0} = X_{K_0}$, $u_1 = y_{K_0}$, $u_2 = z_{K_0}$, and $\varphi(U_{K_0}) = x_{m-1}x_m$. Let Q_0 and Q'_0 be the two paths of $T - \{x_{m-1}x_m, u_1u_2\}$ such that Q_0 connects x_{m-1} and u_1, Q'_0 connects x_m and u_2. Then they are desired ones in condition (U2). In this case, let $m' = m-1$.

Case 1.3. $\mathscr{B}_{2,1} = \mathscr{B}'_{2,1} = \mathscr{B}_{2,3} = \mathscr{B}'_{2,3} = \emptyset$.

Since m is odd and each $X_K \in \mathscr{B}'_1 \cup \mathscr{B}'_{2,2}$ has exactly two neighbors in C', there exists an edge in C', by symmetry say $x'_m x'_1$, such that for each $X_K \in \mathscr{B}'_1 \cup \mathscr{B}'_{2,2}$, at least one of x'_m and x'_1 is not a neighbor of X_K.

Case 1.3.1. There exists $X_{K_0} \in \mathscr{B}_1$ such that $\alpha(K_0) = m-1$ or $\alpha(K_0) = m$.

We may assume that $\alpha(K_0) = m$. Note that X_{K_0} consists of only one vertex, say u_0. Let $u_2 = y_{K_0}$. Since G is a triangulation, it follows from Claim 7 that x_m has a neighbor u_1 in D_0 with $u_1 \neq u_2$. Let $U_{K_0} = X_{K_0}$, $\varphi(U_{K_0}) = x_m x_1$, $Q_0 = x_m u_1$, and $Q'_0 = x_1 u_0 u_2$. Then they are desired ones in condition (U2) in Claim 4. Let $m' = m$.

Case 1.3.2. There exists $X_{K_0} \in \mathscr{B}_{2,2}$ such that $\alpha(K_0) = m-1$ or $\alpha(K_0) = m$.

We may assume that $\alpha(K_0) = m$. By Claim 2, X_{K_0} consists of at most three vertices. Let $U_{K_0} = X_{K_0}$ and $\varphi(U_{K_0}) = x_m x_1$. In all cases, we can find two paths Q_0 and Q'_0 in the subgraph of G induced by $U_{K_0} \cup \{x_m, x_1, y_{K_0}, z_{K_0}\}$ as desired in condition (U2) in Claim

4. Let $m' = m$.

Case 1.3.3. Otherwise.

Since G is a triangulation, there exist two edges u_1x_m and u_2x_1 such that $u_1, u_2 \in V(G') - \big(V(C) \cup V(C')\big)$ and $u_1 \neq u_2$. Let $U_{K_0} = \emptyset$, $\varphi(U_{K_0}) = x_mx_1$, $Q_0 = x_mu_1$, and $Q_0' = x_1u_2$, which are desired in condition (U2) in Claim 4. Let $m' = m$.

Case 2. m is even.

Case 2.1. $\mathscr{B}_1 \cup \mathscr{B}_1' \neq \emptyset$.

Let $X_{K_0} \in \mathscr{B}_1 \cup \mathscr{B}_1'$. We may assume that $X_{K_0} \in \mathscr{B}_1$, and x_{m-1}, x_m and x_1 are neighbors of X_{K_0}. By Claim 7, X_{K_0} consists of only one vertex, say u_0. Let $u_1 = y_{K_0}$. Since G is a triangulation, it follows from Claim 7 that x_1 has a neighbor u_2 in D_0 with $u_2 \neq u_1$. Let $U_{K_0} = X_{K_0}$, $\varphi(U_{K_0}) = x_mx_1$, $Q_0 = x_mu_0u_1$, and $Q_0' = x_1u_2$. Then they are desired ones in condition (U2) in Claim 4. Let $m' = m$.

Case 2.2. $\mathscr{B}_1 = \mathscr{B}_1' = \emptyset$ and $\mathscr{B}_{2,3} \cup \mathscr{B}_{2,3}' \neq \emptyset$

Let $X_{K_0} \in \mathscr{B}_{2,3} \cup \mathscr{B}_{2,3}'$. We may assume that $X_{K_0} \in \mathscr{B}_{2,3}$, $\alpha(K_0) = m-1$, and $\beta(K_0) = 2$. Let R be the subgraph of G' induced by $X_{K_0} \cup \{x_{m-1}, x_m, x_1, x_2, y_{K_0}, z_{K_0}\}$, and let $R' = R - \{x_{m-1}, x_2\}$. In a similar way to Case 1.2, it follows from Lemma 10 that $R - \{x_{m-1}, x_2\}$ has a cycle T_0 containing the two edges x_mx_1 and $y_{K_0}z_{K_0}$ such that $V(T_0) = V(R) - \{x_{m-1}, x_2\} = X_{K_0} \cup \{x_m, x_1, y_{K_0}, z_{K_0}\}$. Let $U_{K_0} = X_{K_0}$, $u_1 = y_{K_0}$, $u_2 = z_{K_0}$, and $\varphi(U_{K_0}) = x_mx_1$. Let Q_0 and Q_0' be the two paths of $T - \{x_mx_1, u_1u_2\}$ such that Q_0 connects x_m and u_1, Q_0' connects x_1 and u_2. Then they are desired ones in condition (U2). In this case, let $m' = m$.

Case 2.3. $\mathscr{B}_1 = \mathscr{B}_1' = \mathscr{B}_{2,3} = \mathscr{B}_{2,3}' = \emptyset$ and $\mathscr{B}_{2,2} \cup \mathscr{B}_{2,2}' \neq \emptyset$.

Let $X_{K_0} \in \mathscr{B}_{2,2} \cup \mathscr{B}_{2,2}'$. We may assume that $X_{K_0} \in \mathscr{B}_{2,2}$, $\alpha(K_0) = m-1$ and $\beta(K_0) = 1$. By Claim 2, X_{K_0} consists of at most three vertices. Let $U_{K_0} = X_{K_0}$, $u_1 = y_{K_0}$, $u_2 = z_{K_0}$ and $\varphi(U_{K_0}) = x_mx_1$. In all cases, we can find two paths Q_0 and Q_0' in the subgraph of G induced by $U_{K_0} \cup \{x_{m-1}, x_m, u_1, u_2\}$ as desired in condition (U2) in Claim 4. Let $m' = m$.

Case 2.4. $\mathscr{B}_1 = \mathscr{B}_1' = \mathscr{B}_{2,2} = \mathscr{B}_{2,2}' = \mathscr{B}_{2,3} = \mathscr{B}_{2,3}' = \emptyset$ and there exist two $X_{K_0}, X_{K_0'} \in \mathscr{B}_{2,1}$ such that $\alpha(K_0) = \beta(K_0')$.

Note that each of X_{K_0} and $X_{K_0'}$ consists of only one vertex, say u_0 and u_0', respectively. In this case, we may assume that $\alpha(K_0) = \beta(K_0') = m-1$, that is, $u_0x_{m-1}, u_0x_m, u_0'x_m, u_0'x_1 \in E(G)$. Let $U_{K_0} = X_{K_0} \cup X_{K_0'}$, $u_1 = y_{K_0}$, $u_2 = y_{K_0'}$, $\varphi(U_{K_0}) = x_mx_1$, $Q_0 = x_mu_0y_1$ and $Q_0' = x_1u_0'y_2$. Then they are desired ones in condition (U2). Let $m' = m$.

Case 2.5. Otherwise.

By the assumption, "for each $X_K \in \mathscr{B}'_{2,1}$, $\alpha(K) \neq m$" or "for each $X_K \in \mathscr{B}'_{2,1}$, $\alpha(K) \neq 1$". By symmetry, we may assume that the former holds.

Suppose first that there exists $X_{K_0} \in \mathscr{B}'_{2,1}$ such that $\alpha(K_0) = m$. Note that X_{K_0} consists of only one vertex, say u_0. In this case, letting $U_{K_0} = K_0 - \overline{K_0} = \{u_0\}$, $u_1 = y_{K_0}$, $u_2 = z_{K_0}$, and $\varphi(U_{K_0}) = x_m x_1$, we can find two paths $Q_0 = x_m u_1$ and $Q'_0 = x_1 u_0 u_2$ in the subgraph of G' induced by $U_{K_0} \cup \{x_m, x_1, u_1, u_2\}$.

Suppose next that for each $X_K \in \mathscr{B}_{2,1}$, $\alpha(K) \neq m$. Since G is a triangulation, it follows from condition (C1) that there exist two edges $x_m u_1$ and $x_1 u_2$ such that $u_1, u_2 \in V(G') - \big(V(C) \cup V(C')\big)$ and $u_1 \neq u_2$. Let $U_{K_0} = \emptyset$, $\varphi(U_{K_0}) = x_m x_1$, $Q_0 = x_m u_1$, and $Q'_0 = x_1 u_2$, which are desired in condition (U2) in Claim 4.

In either case, let $m' = m$.

3.4 Preliminary for finding the family $\mathscr{U}$

Let

$$E_L = \begin{cases} \{x_i x_{i+1} : 1 \leq i \leq m'-1 \text{ and } i \text{ is odd}\} \cup \{x_m x_1\} & \text{if Case 2.1 occurs,} \\ \{x_i x_{i+1} : 1 \leq i \leq m'-1 \text{ and } i \text{ is odd}\} & \text{otherwise,} \end{cases}$$

$$\text{and} \quad E_R = \{x'_i x'_{i+1} : 1 \leq i \leq m'-1 \text{ and } i \text{ is even}\}.$$

Note that except for Case 1.2, the edges in E_L appear alternately in C, and the edges in E_R appear alternately in C'. We will partition $V(G) - V(C) - U_{K_0}$ into some sets satisfying conditions (U1), (U3) and (U4) so that each $U \in \mathscr{U}$ has no neighbors in C or in C', and moreover, $\varphi(U) \in E_L$ if U has no neighbors in C; otherwise $\varphi(U) \in E_R$. This guarantees condition (U4).

3.5 The set $U \in \mathscr{U}$ such that U has no neighbors in C

In this subsection, we will find some sets satisfying condition (U3) in Claim 4 from $\mathscr{B}_1 \cup \mathscr{B}_2$. Note that for $\mathscr{B}'_1 \cup \mathscr{B}'_2$, we will use the same method in the next subsection.

We divide $\mathscr{B}_1$ and $\mathscr{B}_2$ into the following subsets.

$$\begin{aligned}
\text{Let} \quad \mathscr{B}_1^A &= \big\{X_K \in \mathscr{B}_1 : x_{\alpha(K)} x_{\beta(K)} \in E_R \text{ and } K \neq K_0\big\}, \\
\mathscr{B}_1^B &= \mathscr{B}_1 - \mathscr{B}_1^A - \{X_{K_0}\}, \\
\mathscr{B}_{2,1}^A &= \big\{X_K \in \mathscr{B}_{2,1} : x_{\alpha(K)} x_{\beta(K)} \in E_R \text{ and } K \neq K_0\big\}, \\
\mathscr{B}_{2,1}^B &= \mathscr{B}_{2,1} - \mathscr{B}_{2,1}^A - \{X_{K_0}\}, \\
\mathscr{B}_{2,2}^A &= \big\{X_K \in \mathscr{B}_{2,2} : x_{\alpha(K)} x_{\alpha(K)+1} \in E_R \text{ and } K \neq K_0\big\}, \\
\mathscr{B}_{2,2}^B &= \mathscr{B}_{2,2} - \mathscr{B}_{2,2}^A - \{X_{K_0}\}, \\
\mathscr{B}_{2,3}^A &= \big\{X_K \in \mathscr{B}_{2,3} : x_{\alpha(K)} x_{\alpha(K)+1}, x_{\alpha(K)+2} x_{\alpha(K)+3} \in E_R \text{ and } K \neq K_0\big\}, \\
\text{and} \quad \mathscr{B}_{2,3}^B &= \mathscr{B}_{2,3} - \mathscr{B}_{2,3}^A - \{X_{K_0}\}.
\end{aligned}$$

We shall construct the family $\mathscr{W}$ of vertex sets, the set of paths $\mathscr{Q}$, and the mapping ψ from $\mathscr{W}$ to E_R in an order along with C from x_1. In each step, we update the graph H_1 and the boundary D_1 if necessary, preserving the following property; each path in H_1 connecting D_1 and D_1' has at least three vertices. Moreover we will define $\mathscr{W}$, $\mathscr{Q}$ and ψ so that they satisfy condition (U3) in Clam 4, that is, for each $W_K \in \mathscr{W}$, the subraph of G induced by $W_K \cup \{x_i, x_{i+1}\}$ has a hamilton path $Q_K \in \mathscr{Q}$ connecting x_i and x_{i+1}, where $x_i x_{i+1} = \psi(W_K)$.

Let first $\mathscr{W} = \mathscr{Q} = \emptyset$ and

$$\mathfrak{J} = \{j : j = \alpha(K) \text{ for some } X_K \in \mathscr{B}_1 \cup \mathscr{B}_2\}.$$

In each step, we take the minimum integer $j \in \mathfrak{J}$. Let $X_K \in \mathscr{B}_1 \cup \mathscr{B}_2$ with $j = \alpha(K)$. Let $x_i = x_{\alpha(K)}$ if $X_K \in \mathscr{B}_1^A \cup \mathscr{B}_{2,1}^A \cup \mathscr{B}_{2,2}^A \cup \mathscr{B}_{2,3}^A$; otherwise let $x_i = x_{\alpha(K)+1}$. Note that in all cases, $x_i x_{i+1} \in E_R$.

Case 1. $X_K \in \mathscr{B}_1 \cup \mathscr{B}_{2,1}^A$.

In this case, X_K consists of only one vertex, say u. Let $W_K = X_K - \{u\}$ and $\psi(W_K) = x_i x_{i+1}$, and let $Q_K = x_i u x_{i+1}$. We add W_K into $\mathscr{W}$, add Q_K into $\mathscr{Q}$, and delete j from $\mathfrak{J}$.

Case 2. $X_K \in \mathscr{B}_{2,3}^A$.

Recall that in this case, $x_{\alpha(K)} x_{\alpha(K)+1} \in E_R$. Let M be the subgraph of G induced by $X_K \cup \{y_K, z_K, x_{\alpha(K)}, x_{\alpha(K)+1}, x_{\alpha(K)+2}, x_{\beta(K)}\}$. and let $M' = M - \{y_K, z_K\}$. If M' is not 2-connected, there exists a cut vertex, say u, in M'. Since G is a triangulation, we have that $u x_{\alpha(K)}, u x_{\beta(K)} \in E(G')$, but the cycle $x_1 \ldots x_{\alpha(K)} u x_{\beta(K)} \ldots x_m x_1$ is shorter than C, a contradiction. Thus, M' is 2-connected. Similarly, we can show that $M' - \{x_{\alpha(K)}, x_{\beta(K)}\}$ is connected. Then by Lemma 12, M' has two paths Q_K and $Q_{K'}$ such that Q_K connects $x_{\alpha(K)}$ and $x_{\alpha(K)+1}$, $Q_{K'}$ connects $x_{\alpha(K)+2}$ and $x_{\beta(K)}$, and any $(Q_K \cup Q_{K'} \cup \{y_K, z_K\})$-bridge of M has at most three attachments. If there exists a non-trivial $(Q_K \cup Q_{K'} \cup \{y_K, z_K\})$-bridge of M, then the attachments of it forms a cut set of order at most three in G' and also in G, contradicting that G is 4-connected. Thus, $V(Q_K) \cup V(Q_{K'}) = X_K \cup \{x_{\alpha(K)}, x_{\alpha(K)+1}, x_{\alpha(K)+2}, x_{\beta(K)}\}$. Let $W_K = V(Q_K) - \{x_{\alpha(K)}, x_{\alpha(K)+1}\}$, $W_{K'} = V(Q_{K'}) - \{x_{\alpha(K)+2}, x_{\beta(K)+1}\}$, $\psi(W_K) = x_{\alpha(K)} x_{\alpha(K)+1}$, and $\psi(W_{K'}) = x_{\alpha(K)+2} x_{\beta(K)}$. We add W_K and $W_{K'}$ into $\mathscr{W}$, add Q_K and $Q_{K'}$ into $\mathscr{Q}$, and delete j from $\mathfrak{J}$.

Case 3. Otherwise.

When $X_K \in \mathscr{B}_{2,1}^B \cup \mathscr{B}_{2,2}^A$, we define the path P_K as follows: P_K is the subpath of D such that P_K starts from z_K, each vertex in P_K is a neighbor of $x_{\beta(K)}$, and the end vertex, say w_K, of P_K with $w_K \neq z_K$ satisfies $w_K x_{\beta(K)+1} \in E(G')$. Since G is a triangulation, there exists such a path P_K and a vertex w_K, See Figure 3. Note that P_K has no chord, since otherwise, the two end vertices of a chord and $x_{\beta(K)}$ form a 3-cut of G, contradicting that G is 4-connected. In this case, we call X_K *Type I*.

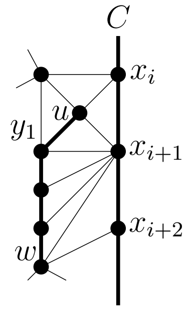

Figure 3: The path P_K.

For $X_K \in \mathscr{B}^B_{2,2} \cup \mathscr{B}^B_{2,3}$, let $w_K = z_K$ and let P_K be the path consisting of only w_K. We call X_K *Type II* and *Type III*, if $X_K \in \mathscr{B}^B_{2,2}$ and if $X_K \in \mathscr{B}^B_{2,3}$, respectively.

Subclaim 1 *The folowings hold.*

(i) *If X_K is of Type I and $X_K \in \mathscr{B}^B_{2,1}$, then there exists a path R_K such that $V(R_K) = X_K \cup \{x_{\beta(K)}, z_K\}$ and R_K connects $x_{\beta(K)}$ and z_K.*

(ii) *If X_K is of Type I and $X_K \in \mathscr{B}^A_{2,2}$, then there exist two paths Q^1_K and R_K such that $V(Q^1_K) \cup V(R_K) = X_K \cup \{x_{\alpha(K)}, x_{\alpha(K)+1}, x_{\beta(K)}, z_K\}$, Q^1_K connects $x_{\alpha(K)}$ and $x_{\alpha(K)+1}$, and R_K connects $x_{\beta(K)}$ and z_K.*

(iii) *If X_K is of Type II, then there exists a path R_K such that $V(R_K) = X_K \cup \{x_{\alpha(K)+1}, z_K\}$ and R_K connects $x_{\alpha(K)+1}$ and z_K.*

(iv) *If X_K is of Type III, then there exist two paths Q^1_K such that $V(Q^1_K) = X_K \cup \{x_{\alpha(K)+1}, x_{\alpha(K)+2}\}$, and Q^1_K connects $x_{\alpha(K)+1}$ and $x_{\alpha(K)+2}$.*

Proof of Subclaim 1. By Claims 1 and 2, statements (i)–(iii) are obvious. Statement (iv) follows Lemma 11. □

Let R_K (and Q^1_K) be the path(s) as in Subclaim 1 in (i)–(iv). When X_K is of Type I and $X_K \in \mathscr{B}^A_{2,2}$, let $W_K = V(Q^1_K) - \{x_{\alpha(K)}, x_{\alpha(K)+1}\}$ and let $\psi(W_K) = x_{\alpha(K)}x_{\alpha(K)+1}$. When X_K is of Type III, let $W_K = V(Q^1_K) - \{x_{\alpha(K)+1}, x_{\alpha(K)+2}\}$ and let $\psi(W_K) = x_{\alpha(K)+1}x_{\alpha(K)+2}$. In either cases, we add W_K into $\mathscr{W}$ and add Q_K into $\mathscr{Q}$.

Let $H'_1 = H_1 - V(P_K)$. We call a maximal 1- or 2-separation $(F, \overline{F})$ of H'_1 *malignant* if $u_1 \notin F$ and $F - \overline{F}$ has a neighbor in P_K. (Recall that we assumed that $F - \overline{F}$ has no neighbor in C or in C'. In this case, since $F - \overline{F}$ has a neighbor in P_K, $F - \overline{F}$ has no neighbor in C'.) Let $X_F = V(F - \overline{F})$. For a malignant 1- or 2-separation $(F, \overline{F})$ of H'_1, let u_F and v_F be neighbors of X_F in P_K which is closed to z_K and w_K, respectively.

Subclaim 2 *For two maximal* 1- *or* 2*-separations* $(F_1, \overline{F_1})$ *and* $(F_2, \overline{F_2})$ *of* H'_1, $u_{F_1}, v_{F_1}, u_{F_2}, v_{F_2}$ *appear in* P_K *in this order, or in the order* $u_{F_2}, v_{F_2}, u_{F_1}, v_{F_1}$. *(Possibly* $v_{F_1} = u_{F_2}$ *or* $v_{F_2} = u_{F_1}$ *in each case.)*

Proof of Subclaim 2. Subclaim 2 follows the facts that $(F_1, \overline{F_1})$ and $(F_2, \overline{F_2})$ are maximal and G' is a plane graph. □

We call a malignant 1- or 2-separation $(B_1, \overline{B_1})$ of H'_1 *bad for* K if X_{B_1} has a neighbor in C. When $(F, \overline{F})$ is not bad, we call it *good (for* K*)*. Since G' is a plane graph, there are no two bad 1- or 2-separation of H'_1 for K.

Subclaim 3 *For a good malignant* 1- *or* 2*-separation* $(F, \overline{F})$ *of* H'_1 *for* K, X_F *has at least two neighbors in* P_K. *Moreover, there exists a path* P_F *in* G' *connecting* u_F *and* v_F *with* $V(P_F) = X_F \cup V(u_F P_K v_F)$.

Proof of Subclaim 3. Let M be the subgraph of G' induced by $F \cup u_F P_K v_F \cup x_{\beta(K)}$. Then M is a plane graph with outer face boundary of length four or five. Then by Claim 1 or Lemma 11, we can find a desired path. □

Subclaim 3 guarantees that for a good malignant 1- or 2-separation $(F, \overline{F})$ of H'_1 for K, replacing the subpath $u_F P_K v_F$ of P_K with P_F, we can "insert" all vertices in X_F into the path P_K. Moreover, by Subclaim 2, we can insert all good malignant 1- or 2-separation for K independently. Let $\widetilde{P_K}$ be the path obtained from P_K by inserting all vertices in X_F for all good malignant 1- or 2-separation $(F, \overline{F})$ of H'_1 for K.

Suppose first that H'_1 has no bad 1- or 2-separation for K.

When X_K is of Type I, let $Q^+_K = R_K \cup \widetilde{P_K} \cup w_K x_{\beta(K)+1}$. Let $W^+_K = V(Q^+_K) - \{x_{\beta(K)}, x_{\beta(K)+1}\}$ and $\psi(W^+_K) = x_{\beta(K)} x_{\beta(K)+1}$. When X_K is of Type II, let $Q^+_K = R_K \cup w_K x_{\beta(K)}$. Let $W^+_K = V(Q^+_K) - \{x_{\beta(K)-1}, x_{\beta(K)}\}$ and $\psi(W^+_K) = x_{\beta(K)-1} x_{\beta(K)}$. In either case, we add W^+_K into $\mathscr{W}$ and add Q^+_K into $\mathscr{Q}$, (do nothing when X_K is of Type III), update H_1 by deleting all vertices in $\widetilde{P_K}$, and delete all integer j' with $j \le j' \le \beta(K)$ (when X_K is of Type I) or all integer j' with $j \le j' \le \beta(K) - 1$ (when X_K is of Type II or III) from $\mathfrak{J}$, and go to next $j \in \mathfrak{J}$.

Suppose that H'_1 has a bad 1- or 2-separation $(B_1, \overline{B_1})$ for K. In this case, note that $u_{B_1} = v_{B_1}$. For simplicity, we call X_{B_1} Types I–III so that the type of X_{B_1} is same as the one of X_K.

Consider the following two types:

Type a. $(B_1, \overline{B_1})$ is a bad 1-separation.

Type b. $(B_1, \overline{B_1})$ is a bad 2-separation.

Let and z_{B_1} be the unique vertex in $B_1 \cap \overline{B_1}$ which is contained in D. When $(B_1, \overline{B_1})$ is a 2-separation, let y_{B_1} be the unique vertex $B_1 \cap \overline{B_1}$ which is not contained in D. Let

$\beta(B_1)$ be the maximum integer k such that $z_{B_1}x_k \in E(G')$. We also consider the following two types:
Type i. $x_{\beta(B_1)-1}x_{\beta(B_1)} \in E_R$.
Type ii. $x_{\beta(B_1)}x_{\beta(B_1)+1} \in E_R$.

Let M_1 be the induced subgraph of G' which is bounded by the contractible cycle $u_{B_1}x_{\beta(K)}\dots x_{\beta(B_1)}z_{B_1}u_{B_1}$ (when X_{B_1} is of Type a,) or $u_{B_1}x_{\beta(K)}\dots x_{\beta(B_1)}z_{B_1}y_{B_1}u_{B_1}$ (when X_{B_1} is of Type b). Note that M_1 satisfies the conditions in Lemma 13 with $u_1 = u_{B_1}$ and $u_l = y_{B_1}$ (if X_{B_1} is of Type a) or $u_l = z_{B_1}$ (if X_{B_1} is of Type b). Let $S = \{u_l\}$ and $S = \emptyset$ if X_{B_1} is of Type a and of Type b, respectively, and apply Lemma 13 for the graph M_1. (Notice that the number in Lemma 13 corresponds to the one in each Types. For example, if X_{B_1} is of Type I-a-i, apply Lemma 13 (I-i) with $S = \{u_l\}$, and if X_{B_1} is of Type II-b-ii, apply Lemma 13 (II-ii) with $S = \emptyset$, and so on.) Then in each case, M_1 has r disjoint paths $T_1, \dots T_r$ as in Lemma 13. Since G is 4-connected, $\bigcup_{k=1}^{r} V(T_k) - V(C) = V(M_1) - V(C)$ (when Type a) or $\bigcup_{k=1}^{r} V(T_k) - V(C) = V(M_1) - V(C) - \{y_{B_1}\}$ (when Type b). Let

$$
\begin{aligned}
Q^1_{B_1} &= \begin{cases} R_K \cup z_K\widetilde{P_K}u_{B_1} \cup T_1 & \text{Type I or II,} \\ T_1 & \text{Type III,} \end{cases} \\
Q^i_{B_1} &= T_i \qquad \text{for each } 2 \le i \le r-1, \\
\text{and} \qquad R_{B_1} &= T_r.
\end{aligned}
$$

Moreover, for each $1 \le i \le r-1$, letting $x_{\alpha(Q^i_{B_1})}$ and $x_{\alpha(Q^i_{B_1})+1}$ be the end vertices of $Q^i_{B_1}$, $W^i_{B_1} = V(Q^i_{B_1}) - \{x_{\alpha(Q^i_{B_1})}, x_{\alpha(Q^i_{B_1})+1}\}$, and $\psi(W^i_{B_1}) = x_{\alpha(Q^i_{B_1})}x_{\alpha(Q^i_{B_1})+1}$. We add $W^i_{B_1}$ to $\mathscr{W}$ and add $Q^i_{B_1}$ to $\mathscr{Q}$ for each $1 \le i \le r-1$.

When X_{B_1} is of Type i, let $w_{B_1} = z_{B_1}$ and let P_{B_1} be the path consisting of only w_{B_1}. When X_{B_1} is of Type ii, let P_{B_1} be the subpath of D such that P_{B_1} starts from z_{B_1}, each vertex in P_{B_1} is a neighbor of $x_{\beta(B_1)}$, and the end vertex, say w_{B_1}, of P_{B_1} with $w_{B_1} \neq z_{B_1}$ satisfies $w_{B_1}x_{\beta(B_1)+1} \in E(G')$. Similarly to the case for P_K, there exists such a path P_{B_1} and a vertex w_{B_1}, since G is a triangulation. Let $H''_1 = H'_1 - V(P_{B_1})$.

By the same way as the case for P_K, we can "insert" all vertices of X_F into P_{B_1} for all good malignant 1- or 2-separation $(F, \overline{F})$ of H''_1 for B_1. Let $\widetilde{P_{B_1}}$ be the path obtained by the insertion. If H''_1 has no bad 1- or 2-separation for B_1, then similarly as the case for X_K, we add some sets to $\mathscr{W}$ and go to next $j \in \mathscr{J}$. Actually, we consider the following operations:

When X_{B_1} is of Type i, let $Q^+_{B_1} = R_{B_1} \cup w_{B_1}x_{\beta(B_1)}$. Let $W^+_{B_1} = V(Q^+_{B_1}) - \{x_{\beta(B_1)-1}, x_{\beta(B_1)}\}$ and $\psi(W^+_{B_1}) = x_{\beta(B_1)-1}x_{\beta(B_1)}$. When X_{B_1} is of Type ii, let $Q^+_{B_1} = R_{B_1} \cup \widetilde{P_{B_1}} \cup w_{B_1}x_{\beta(B_1)+1}$ Let $W^+_{B_1} = V(Q^+_{B_1}) - \{x_{\beta(B_1)}, x_{\beta(B_1)+1}\}$ and $\psi(W^+_{B_1}) = x_{\beta(B_1)}x_{\beta(B_1)+1}$.

In either cases, we add $W^+_{B_1}$ into $\mathscr{W}$, add $Q^+_{B_1}$ into $\mathscr{Q}$, update H_1 by deleting all vertices in $\widetilde{P_{B_1}}$, and delete all integer j' with $j \le j' \le \beta(B_1) - 1$ (when X_{B_1} is of Type i) or all integer j' with $j \leq j' \leq \beta(B_1)$ (when X_{B_1} is of Type ii) from $\mathcal{J}$, and go to next $j \in \mathcal{J}$.

Suppose that H_1'' has a bad 1- or 2-separation $(B_2, \overline{B_2})$ for B_1. In this case, we call that X_{B_2} is of Type I if X_{B_1} is of Type ii, and X_{B_2} is of Type II if X_{B_1} is of Type i, respectively. In a same way as for B_1, we define the paths $Q^i_{B_2}$'s, R_{B_2}, the sets $W^i_{B_2}$'s, and $\psi(W^i_{B_2})$'s, add $W^i_{B_2}$ to $\mathscr{W}$, and add $Q^i_{B_2}$ to $\mathscr{Q}$.

Continue the above operation inductively, as long as there exists a bad 1- or 2-separation for the previous set. (Note that since we defined a malignant separation $(F, \overline{F})$ so that $u_1 \notin X_F$, the above operation has to stop.) When a bad 1- or 2-separation B_l exists and B_{l+1} does not exist, finally, we delete all integer j' with $j \leq j' \leq \beta(B_l) - 1$ (when X_{B_l} is of Type i) or all integer j' with $j \leq j' \leq \beta(B_l)$ (when X_{B_l} is of Type ii) from $\mathcal{J}$, and go to next $j \in \mathcal{J}$.

3.6 The set $U \in \mathscr{U}$ and completion of the proof of Claim 4

For $\mathscr{B}_1'$ and $\mathscr{B}_2'$, dividing them into the sets $\mathscr{B}_1'^A, \mathscr{B}_1'^B, \mathscr{B}_{2,1}'^A, \ldots$ and so on. In a same way as getting $\mathscr{W}$ and ψ in the case for $\mathscr{B}_1$ and $\mathscr{B}_2$, we obtain the family of vertex sets $\mathscr{W}'$, the set of paths $\mathscr{Q}'$, and a mapping ψ' from $\mathscr{W}'$ to E_L, preserving the property that H_1' is still a graph on the cylinder. (That means we never delete a vertex contained in both of two boundary cycles.) However, in this process, we have to consider the cycle C' with "reverse" orientation. For example, in each step, we consider the maximum integer $j \in \mathcal{J}'$, we define the path P_K as a subpath of D' such that P_K starts from y_K, each vertex in P_K is a neighbor of $x_{\alpha'(K)}$, and the end vertex, say w_K, of P_K with $w_K \neq y_K$ satisfies $w_K x_{\alpha'(K)-1} \in E(G')$.

Now suppose that we get the family of vertex sets $\mathscr{W}'$, the set of paths $\mathscr{Q}'$ and the mapping ψ'. Let $\mathscr{U} = \mathscr{W} \cup \mathscr{W}'$, $U_0 = V(G) - V(C) - U_{K_0} - \bigcup_{U \in U} U$, $\varphi(U) = \psi(U)$ if $U \in \mathscr{W}$, and $\varphi(U) = \psi'(U)$ if $U \in \mathscr{W}'$. By the construction, for each $U \in \mathscr{U}$, each $Q \in \mathscr{Q}$ which corresponds to U is a hamilton path of the subgraph of G induced by $U \cup \{x_i, x_{i+1}\}$ connecting x_i and x_{i+1}, where $x_i x_{i+1} = \varphi(U)$. So condition (U3) holds. Since we deleted all 1- or 2-separations from H_1, H_0 satisfies condition (U1). Since both ψ and ψ' are injections, $\psi(\mathscr{W}) \subset E_R$ and $\psi'(\mathscr{W}') \subset E_L$, we have that φ is an injection, so condition (U4) holds.

This completes the proof of Claim 4, and Theorem 2. $\square$

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