A toughness condition for a spanning tree with bounded total excesses

Kenta Ozeki*

National Institute of Informatics,
2-1-2 Hitotsubashi, Chiyoda-ku,
Tokyo 101-8430, Japan
and
JST, ERATO, Kawarabayashi Large Graph Project
e-mail: ozeki@nii.ac.jp

Abstract

Let k be an integer with $k \geq 2$. In terms of the toughness of a graph, Win gave a sufficient condition for the existence of a spanning k-tree, that is, a spanning tree in which the maximum degree is at most k. For a spanning tree T of a graph G, we define the *total excess* $\mathrm{te}(T,k)$ *of* T *from* k as $\mathrm{te}(T,k) := \sum_{v \in V(T)} \max\{d_T(v) - k, 0\}$, where $V(T)$ is the vertex set of T and $d_T(v)$ is the degree of a vertex v in T. Enomoto, Ohnishi and Ota extended Win's result by giving a condition that guarantees the existence of a spanning tree T with a bounded total excess. Here we further extend the result as follows. Let $\omega(G)$ be the number of components of a graph G. Let $k_1, k_2, \ldots, k_p$ be p integers with $k_1 \geq k_2 \geq \ldots \geq k_p \geq 2$, let $t_1, t_2, \ldots, t_p$ be p nonnegative integers, and let G be a connected graph. If G satisfies that $\omega(G-S) \leq (k_i - 2)|S| + 2 + t_i$ for every $1 \leq i \leq p$ and every $S \subset V(G)$, then G has a spanning tree T such that $\mathrm{te}(T, k_i) \leq t_i$ for every $1 \leq i \leq p$.

Keywords: Spanning k-trees, Toughness, Total excess

1 Introduction

The concept of toughness was first introduced by Chvátal [3] and has been extensively studied because of its relationship to the existence of a Hamiltonian cycle and a Hamiltonian path. In addition to those, it is also closely related to the existence of *a spanning k-tree,* which is a spanning tree in which the maximum degree is at most k. Note that a spanning 2-tree is a Hamiltonian path. In this paper, we focus on

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the relationship between the toughness and the existence of a spanning k-tree. For other topics on toughness, we refer the readers to a survey [1].

For a graph G, let $V(G)$ and $E(G)$ be the vertex set and the edge set of G, respectively, and let $\omega(G)$ be the number of components of G. A graph G is said to be *t-tough*, if $t \cdot \omega(G-S) \le |S|$ for every $S \subset V(G)$ with $\omega(G-S) \ge 2$. The *toughness* of a graph G, denoted by $\tau(G)$, is the maximum value of t for which G is t-tough, if G is not a complete graph; when G is a complete graph, let $\tau(G) := +\infty$. In other words, if G is not complete, then

$$\tau(G) = \min\left\{\frac{|S|}{\omega(G-S)} : S \subset V(G) \text{ and } \omega(G-S) \ge 2\right\}.$$

If a graph G has a Hamiltonian cycle, then G has to be 1-tough. So, having a high (at least 1) toughness is a necessary condition to have a Hamiltonian cycle. Chvátal [3] conjectured that the converse also holds, that is, "every graph with high toughness (not depending on the order of a graph) is Hamiltonian". This conjecture is still open, and Bauer, Broersma and Veldman [2] showed that at least $\frac{9}{4}$-tough is necessary even if this conjecture is true.

The following is a necessary condition for the existence of a spanning k-tree, which is similar to the case of a Hamiltonian cycle.

Proposition 1 *Let k be an integer with $k \ge 2$, and let G be a graph having a spanning k-tree. Then for every $S \subset V(G)$, $\omega(G-S) \le (k-1)|S|+1$.*

By Proposition 1, we see that any graph with spanning k-tree has toughness at least $\frac{1}{2(k-1)}$. Then it is natural to ask whether high toughness guarantees the existence of a spanning k-tree or not. As mentioned above, this problem for a Hamiltonian cycle is still open, but for a spanning k-tree, Win [8] proved the following theorem.

Theorem 2 (Win [8]) *Let k be an integer with $k \ge 2$, and let G be a connected graph. If $\omega(G-S) \le (k-2)|S|+2$ for every $S \subset V(G)$, then G has a spanning k-tree.*

For the case $k \ge 3$, Theorem 2 implies that every connected graph with toughness at least $\frac{1}{k-2}$ has a spanning k-tree. Hence high toughness guarantees the existence of a spanning k-tree. However for the case $k = 2$, that is, for a Hamiltonian path, Theorem 2 cannot give a sufficient condition on toughness.

When we consider toughness conditions, we focus on only the "ratio" between the order of a vertex set S and the number of components of $G-S$. Therefore, we have the following question; What happens when we change the constant part of the condition of Theorem 2?

As one of the solutions, Enomoto, Ohnishi and Ota [6] extended Theorem 2 in terms of *total excess*. For an integer k and a spanning tree T of a graph G, we define the *total excess* $\mathrm{te}(T,k)$ *of T from k* as

$$\mathrm{te}(T,k) := \sum_{v\in V(T)} \max\{d_T(v)-k, 0\},$$

where $d_T(v)$ is the degree of v in T. Note that a spanning tree T satisfies $\mathrm{te}(T,k)=0$ if and only if T is a spanning k-tree.

Theorem 3 (Enomoto, Ohnishi and Ota [6]) *Let k and t be integers with $k \geq 2$ and $t \geq 0$, and let G be a connected graph. If $\omega(G-S) \leq (k-2)|S|+2+t$ for every $S \subset V(G)$, then G has a spanning tree T with $\mathrm{te}(T,k) \leq t$.*

In this paper, we show the following result. That means that if a given graph satisfies the condition of Theorem 3 for some pairs (k,t) at the same time, then the graph has a spanning tree satisfying the total excess properties for all pairs.

Theorem 4 *Let $k_1, k_2, \ldots, k_p$ be p integers with $k_1 \geq k_2 \geq \ldots \geq k_p \geq 2$, and let $t_1, t_2, \ldots, t_p$ be p nonnegative integers. Let G be a connected graph. If $\omega(G-S) \leq (k_i-2)|S|+2+t_i$ for every $1 \leq i \leq p$ and for every $S \subset V(G)$, then G has a spanning tree T with $\mathrm{te}(T,k_i) \leq t_i$ for every $1 \leq i \leq p$.*

After giving the relationship among Theorems 2, 3 and 4 in Section 2, we will consider a concept that extends a spanning k-tree and show a slightly stronger result (Theorem 8) than Theorem 4 in Section 3. We will prove it in Section 4.

2 An application of Theorem 4

Fix a connected graph G and take the minimum integer k which satisfies "$\omega(G-S) \leq (k-1)|S|+2$ for every $S \subset V(G)$". Assume that $k \geq 2$. By the minimality of k, we have "$\omega(G-S) > (k-2)|S|+2$ for some $S \subset V(G)$". This means that G satisfies the assumption of Theorem 2 for $k+1$, but not for k. Then by using Theorem 2 for $k+1$, we have that

$$G \text{ has a spanning } (k+1)\text{-tree } T.$$

On the other hand, since "$\omega(G-S) \leq (k-1)|S|+2$ for every $S \subset V(G)$", there exists an integer t satisfying "$\omega(G-S) \leq (k-2)|S|+2+t$ for every $S \subset V(G)$". Then by Theorem 3, we also have that

$$G \text{ has a spanning tree } T' \text{ with } \mathrm{te}(T',k) \leq t.$$

But, the upper bound on the maximum degree of T' that we get from Theorem 3 is $k+t$, which is directly obtained from the definition of total excess. So between T and T', we cannot determine which spanning tree has a stronger property. However, setting $k_1 = k+1$, $t_1 = 0$, $k_2 = k$ and $t_2 = t$, Theorem 4 implies that

$$G \text{ has a spanning } (k+1)\text{-tree } T'' \text{ with } \mathrm{te}(T'',k) \leq t.$$

Therefore, using Theorem 4, we can find a spanning tree having the properties both of T and T'.

3 Total excess from f

In this section, we consider a concept that extends a spanning k-tree. When we consider a spanning k-tree, we set the bound k on the degree to all vertices uniformly. Let G be a graph and let f be a mapping from $V(G)$ to the integers. A spanning tree T of G is called a *spanning f-tree* if the degree of x in T is bounded by $f(x)$ for every $x \in V(T)$. Clearly, when f is a constant mapping taking value k, a spanning f-tree is a spanning k-tree.

Similarly to the case of a spanning k-tree (Proposition 1), it is known that the existence of a spanning f-tree implies a toughness type condition. Moreover the sufficient condition in Theorem 2 was extended to the case of a spanning f-tree by Ellingham, Nam and Voss.

Proposition 5 *Let G be a graph and let f be a mapping from $V(G)$ to the integers. If G has a spanning f-tree, then $\omega(G-S) \leq \sum_{v\in S} \big(f(v)-1\big)+1$ for every $S \subset V(G)$.*

Theorem 6 (Ellingham, Nam and Voss [4]) *Let G be a connected graph and let f be a mapping from $V(G)$ to the positive integers. If $\omega(G-S) \leq \sum_{v\in S} \big(f(v) - 2\big) + 2$ for every $S \subset V(G)$, then G has a spanning f-tree.*

Similarly to the total excess from k, Ohnishi and Ota [7] also defined the total excess from a mapping f, and they extended Theorem 6 as follows. For a mapping f from $V(G)$ to the integers and for a spanning tree T, let

$$\mathrm{te}(T, f) := \sum_{v\in V(T)} \max\{d_T(v) - f(v), 0\}.$$

(In [7], they proved Theorem 7 only for a mapping f from $V(G)$ to the nonnegative integers. However, their proof also can work for the case where f takes negative integers.)

Theorem 7 (Ohnishi and Ota [7]) *Let G be a connected graph and let f be a mapping from $V(G)$ to the integers. If $\omega(G-S) \leq \sum_{v\in S}(f(v)-2)+2+t$ for every $S \subset V(G)$, then G has a spanning tree T with $\mathrm{te}(T, f) \leq t$.*

With motivation same as improving Theorem 3 to Theorem 4, we also obtain the following result. For two mappings f_1 and f_2 from $V(G)$ to the integers, we write $f_1 \geq f_2$ if $f_1(v) \geq f_2(v)$ for every $v \in V(G)$. (Note that the relation "$\geq$" is a partial order on the set of mappings from $V(G)$ to the integers.)

Theorem 8 *Let G be a connected graph and let $f_1 \geq f_2 \geq \cdots \geq f_p$ be mappings from $V(G)$ to the integers. For $1 \leq i \leq p$, let t_i be nonnegative integers. If $\omega(G-S) \leq \sum_{v \in S} \big(f_i(v) - 2\big) + 2 + t_i$ for every $1 \leq i \leq p$ and for every $S \subset V(G)$, then G has a spanning tree T with $\mathrm{te}(T, f_i) \leq t_i$ for every $1 \leq i \leq p$.*

Note that the condition "$f_1 \geq f_2 \geq \cdots \geq f_p$" is necessary, in the sense that Theorem 8 does not hold for non-comparable mappings. In order to show that, we consider the following graph G and two mappings f_1 and f_2; Let G be the graph in Figure 1. We define $f_1(x) = 5$, $f_1(y) = 2$, $f_2(x) = 2$ and $f_2(y) = 5$, and for $z \neq x, y$, let $f_1(z)$ and $f_2(z)$ be a sufficiently large integer. Let $t_1 = t_2 = 1$. Then G satisfies that $\omega(G - S) \leq \sum_{v \in S} \big(f_i(v) - 2\big) + 2 + t_i$ for every $1 \leq i \leq 2$ and for every $S \subset V(G)$. Although, for $i \in \{1, 2\}$, G has a spanning tree T_i with $\mathrm{te}(T_i, f_i) \leq t_i$, G has no spanning tree T with $\mathrm{te}(T, f_1) \leq t_1$ and $\mathrm{te}(T, f_2) \leq t_2$.

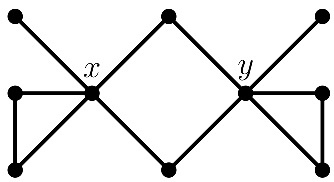

Figure 1: The graph G.

As mentioned before, when f_i is a constant mapping taking value k_i for each i, Theorem 8 is equivalent to Theorem 4. Thus, Theorem 8 is stronger than Theorem 4. So in Section 4, we will prove Theorem 8 instead of Theorem 4.

4 Proof of Theorem 8

For the proof of Theorem 8, we need the following proposition. We can find the proof of it in some papers, for example in [8]. But for self-containedness, we show it here.

Proposition 9 *Let T be a tree and let $S \subset V(T)$ with $S \neq \emptyset$. Then $\omega(T - S) \geq \sum_{v \in S} \big(d_T(v) - 2\big) + 2$.*

Proof of Proposition 9. We prove this proposition by induction on $|S|$. When $|S| = 1$, say $S = \{v\}$, we obtain that $\omega(T - S) = d_T(v) = \big(d_T(v) - 2\big) + 2$. So we may assume that $|S| \geq 2$.

Regard T as a directed tree from an arbitrary vertex, and take the farthest vertex v of S from the root. By the choice of v, only one component of $T - v$ can have a vertex in S.

Let $S' := S - \{v\}$. By induction hypothesis, $\omega(T - S') \geq \sum_{v \in S'} \big(d_T(v) - 2\big) + 2$. Let C be the component of $T - S'$ containing v. Note that $d_C(v) \geq d_T(v) - 1$ by the choice of v. (If v is adjacent to no vertices in S, then $d_C(v) = d_T(v)$; Otherwise

$d_C(v) = d_T(v) - 1.$). Since C is divided into $d_C(v)$ components when we remove the vertex v from C, we obtain

$$\begin{aligned}\omega(T-S) &= \omega(T-S') - 1 + d_C(v) \\ &\geq \sum_{v\in S'} \big(d_T(v)-2\big) + 2 - 1 + d_T(v) - 1 \\ &= \sum_{v\in S} \big(d_T(v)-2\big) + 2. \qquad \square\end{aligned}$$

For a graph T and for $S \subset V(T)$, we define *an S-bridge of T* as either (i) an edge between two vertices of S, or (ii) a subgraph of T induced by the edges in a component of $T - S$ together with all edges between that component and S. For a mapping f from $V(G)$ to the integers and an integer k, let $f + k$ be the mapping that takes the value $f(v) + k$ for every $v \in V(G)$. For a graph T, let $V_f(T) := \{v \in V(T) : d_T(v) = f(v)\}$ and $V_{\geq f}(T) := \{v \in V(T) : d_T(v) \geq f(v)\}$.

In order to prove Theorem 8, we use a similar method to the one used in [5] by Ellingham and Zha. In fact, the statements of Claims 2 and 3 can be found in the proof of Theorem 2.1 in [5]. However, they deal with only one pair (k, t) of integers (in particular, only for the case $t = 0$). It is not easy to deal with many pairs (k_i, t_i) at the same time. Indeed, we have to deal with spanning trees used in the proof of Theorem 8 more carefully. For example, in the proof of Theorem 8, we first set conditions (1)–(p) for a spanning tree T, considering the order on $f_1, f_2, \ldots, f_p$. This is a key idea.

Proof of Theorem 8. Let G be a connected graph and let $f_1 \geq f_2 \geq \cdots \geq f_p$ be mappings from $V(G)$ to the integers, which satisfy the assumptions of Theorem 8.

We take a spanning tree T of G such that

(1) $\max\{\mathrm{te}(T, f_1) - t_1, 0\}$ is as small as possible,

(2) $\max\{\mathrm{te}(T, f_2) - t_2, 0\}$ is as small as possible, subject to (1),
$\cdots$

(p) $\max\{\mathrm{te}(T, f_p) - t_p, 0\}$ is as small as possible, subject to (1) $\cdots$ ($p - 1$).

If T satisfies $\mathrm{te}(T, f_i) \leq t_i$ for every $1 \leq i \leq p$, then there is nothing to prove. Hence we may assume that there exists an integer i with $\mathrm{te}(T, f_i) > t_i$, that is, $\max\{\mathrm{te}(T, f_i) - t_i, 0\} > 0$. We take such an integer i as small as possible. By the minimality of i, we obtain $\mathrm{te}(T, f_j) \leq t_j$ for every $1 \leq j \leq i - 1$, and hence $\max\{\mathrm{te}(T, f_j) - t_j, 0\} = 0$.

For $S \subset V(G)$, we define $\mathcal{T}(S)$ as the set of spanning trees T' of G such that for every S-bridge C' of T', there exists an S-bridge C of T with $V(C') = V(C)$ and $\mathrm{te}(C', f_j) \leq \mathrm{te}(C, f_j)$ for every $1 \leq j \leq i$. Note that $\mathcal{T}(S) \subset \mathcal{T}(S')$ for $S' \subset S$.

Let $T' \in \mathcal{T}(S)$. By the condition of T', we see that $d_{T'}(v) = d_T(v)$ for each $v \in S$. Therefore, for every $1 \le j \le i$,

$$\begin{aligned} & \mathrm{te}(T, f_j) - \mathrm{te}(T', f_j) \\ = & \sum_{C\text{: an } S\text{-bridge of } T} \mathrm{te}(C, f_j) \; - \sum_{C'\text{: an } S\text{-bridge of } T'} \mathrm{te}(C', f_j) \\ \ge & \; 0. \end{aligned}$$

Thus, T' satisfies $\mathrm{te}(T', f_j) \le \mathrm{te}(T, f_j)$ for every $1 \le j \le i$, and hence T' also satisfies the conditions (1)–($i-1$). On the other hand, by the choice of T, we have $\mathrm{te}(T', f_i) \ge \mathrm{te}(T, f_i)$, since otherwise $\max\{\mathrm{te}(T', f_i) - t_i, 0\}$ is smaller than that for T, contradicting (i) in the choice of T. These imply the following claim.

Claim 1 *For every $T' \in \mathcal{T}(S)$, T' also satisfies $\mathrm{te}(T', f_j) \le \mathrm{te}(T, f_j)$ for every $1 \le j \le i$. In particular, $\mathrm{te}(T', f_i) = \mathrm{te}(T, f_i) > t_i$.*

Let $A_0 := V_{\ge f_i+1}(T)$. For $l \ge 1$, let

$$A_l := A_0 \cup \{v \in V(G) : d_{T'}(v) = f_i(v) \text{ for every } T' \in \mathcal{T}(A_{l-1})\}.$$

Since $\mathcal{T}(S) \subset \mathcal{T}(S')$ for $S' \subset S$, we obtain $A_0 \subset A_1 \subset A_2 \subset \cdots$. The following two claims are crusial for the proof.

Claim 2 *For every edge xy in G such that x and y are contained in distinct components of $T - A_0$, we have $x \in A_1$ or $y \in A_1$.*

Proof. Assume that there exists an edge $xy \in E(G)$ such that $x \notin A_1$, $y \notin A_1$ and x and y are contained in distinct components of $T - A_0$. Then $x \notin A_1$ implies that G has a spanning tree $T_x \in \mathcal{T}(A_0)$ such that $d_{T_x}(x) < f_i(x)$. By symmetry, G has a spanning tree $T_y \in \mathcal{T}(A_0)$ such that $d_{T_y}(y) < f_i(y)$. By changing an A_0-bridge of T_x containing y with the corresponding A_0-bridge of T_y, we can find a spanning tree T_{xy} of G such that $d_{T_{xy}}(x) < f_i(x)$ and $d_{T_{xy}}(y) < f_i(y)$. Since $T_x, T_y \in \mathcal{T}(A_0)$, for every A_0-bridge C_{xy} of T_{xy}, there exists an A_0-bridge C of T with $V(C_{xy}) = V(C)$ and $\mathrm{te}(C_{xy}, f_j) \le \mathrm{te}(C, f_j)$ for every $1 \le j \le i$. This implies that $T_{xy} \in \mathcal{T}(A_0)$.

Let z be a vertex in A_0 lying on the unique path in T_{xy} between x and y, and let zw be an edge incident with z lying on the above path. (Possibly, $w = x$ or $w = y$.) Since x and y are contained in distinct components of $T - A_0$, we can find such a vertex z. Then $T_z = (T_{xy} - zw) \cup xy$ is a spanning tree of G with $d_{T_z}(x) \le d_{T_{xy}}(x) + 1 \le f_i(x)$, $d_{T_z}(y) \le d_{T_{xy}}(y) + 1 \le f_i(y)$, and $d_{T_z}(z) = d_{T_{xy}}(z) - 1$. Since $d_{T_z}(x) \le f_i(x) \le f_j(x)$ and $d_{T_z}(y) \le f_i(y) \le f_j(y)$ for every $1 \le j \le i$, we obtain $\max\{d_{T_z}(x) - f_j(x), 0\} = 0$ and $\max\{d_{T_z}(y) - f_j(y), 0\} = 0$. Therefore, for every $1 \le j \le i$,

$$\begin{aligned}
&\mathrm{te}(T_{xy}, f_j) - \mathrm{te}(T_z, f_j) \\
&= \sum_{v \in \{x,y,z,w\}} \max\{d_{T_{xy}}(v) - f_j(v), 0\} - \sum_{v \in \{x,y,z,w\}} \max\{d_{T_z}(v) - f_j(v), 0\} \\
&= \sum_{v \in \{z,w\}} \Big(\max\{d_{T_{xy}}(v) - f_j(v), 0\} - \max\{d_{T_z}(v) - f_j(v), 0\} \Big) \\
&\ge 0.
\end{aligned}$$

(The first equality follows from the fact that "$d_{T_{xy}}(v) = d_{T_z}(v)$ for every $v \in V(T_{xy}) - \{x, y, z, w\}$", and the second follows from the fact that "$0 \le \max\{d_{T_{xy}}(x) - f_j(x), 0\} \le \max\{d_{T_z}(x) - f_j(x), 0\} = 0$ and $0 \le \max\{d_{T_{xy}}(y) - f_j(y), 0\} \le \max\{d_{T_z}(y) - f_j(y), 0\} = 0$". The last inequality follows from the fact that "$d_{T_{xy}}(z) > d_{T_z}(z)$ and $d_{T_{xy}}(w) \ge d_{T_z}(w)$".)

Therefore, by Claim 1 and by the fact that $T_{xy} \in \mathcal{T}(A_0)$, $\mathrm{te}(T_z, f_j) \le \mathrm{te}(T_{xy}, f_j) \le \mathrm{te}(T, f_j)$ for every $1 \le j \le i$. This implies that T_z also satisfies the conditions (1)–$(i-1)$. Since $d_{T_z}(z) = d_{T_{xy}}(z) - 1$, $d_{T_{xy}}(z) \ge f_i(z) + 1$ and $d_{T_{xy}}(w) \ge d_{T_z}(w)$, we have

$$\begin{aligned}
&\mathrm{te}(T_{xy}, f_i) - \mathrm{te}(T_z, f_i) \\
&= \sum_{v \in \{z,w\}} \Big(\max\{d_{T_{xy}}(v) - f_i(v), 0\} - \max\{d_{T_z}(v) - f_i(v), 0\} \Big) \\
&\ge d_{T_{xy}}(z) - f_i(z) - \big(d_{T_z}(z) - f_i(z)\big) \\
&> 0.
\end{aligned}$$

Thus, by Claim 1, $\mathrm{te}(T_z, f_i) < \mathrm{te}(T_{xy}, f_i) = \mathrm{te}(T, f_i)$. Since $\mathrm{te}(T, f_i) > t_i$, we obtain that $\max\{\mathrm{te}(T_z, f_i) - t_i, 0\} < \max\{\mathrm{te}(T, f_i) - t_i, 0\}$, which contradicts (i) in the choice of T. $\square$

Claim 3 *For every $l \ge 1$, and every edge xy in G such that x and y are contained in distinct components of $T - A_l$, we have $x \in A_{l+1}$ or $y \in A_{l+1}$.*

Proof. Suppose that there exists an edge xy in G such that $x \notin A_{l+1}$, $y \notin A_{l+1}$ and x and y are contained in distinct components of $T - A_l$. We may assume that l is as small as possible. Then $x \notin A_{l+1}$ and $y \notin A_{l+1}$ imply that G has spanning trees $T_x, T_y \in \mathcal{T}(A_l)$ such that $d_{T_x}(x) < f_i(x)$ and $d_{T_y}(y) < f_i(y)$, respectively. In the same way as in the proof of Claim 2, we take the spanning tree T_{xy}.

Let z be a vertex in A_l lying on the unique path in T_{xy} between x and y and let zw be an edge incident with z lying on the above path. By the minimality of l, $z \in A_l - A_{l-1}$, and x, y, z and w are contained in the same A_{l-1}-bridge of T. Let $T_z = (T_{xy} - zw) \cup xy$. By the definition of T_z, we have $d_{T_z}(x) \le d_{T_{xy}}(x) + 1 \le f_i(x) \le f_j(x)$ and $d_{T_z}(y) \le d_{T_{xy}}(y) + 1 \le f_i(y) \le f_j(y)$ for every $1 \le j \le i$.

Then the following holds:

For every A_{l-1}-bridge C_z of T_z, there exists an A_{l-1}-bridge C of T with $V(C_z) = V(C)$ and $\mathrm{te}(C_z, f_j) \leq \mathrm{te}(C, f_j)$ for every $1 \leq j \leq i$.

This is trivial for an A_{l-1}-bridge C_z of T_z with $x, y, z, w \notin V(C_z)$, since C_z is same as some A_{l-1}-bridge of T_{xy} and $T_{xy} \in \mathcal{T}(A_{l-1})$. For an A_{l-1}-bridge C_z of T_z with $x, y, z, w \in V(C_z)$, there is a unique A_{l-1}-bridge C_{xy} of T_{xy} such that $V(C_z) = V(C_{xy})$. Then for every $1 \leq j \leq i$,

$$
\begin{aligned}
&\mathrm{te}(C_{xy}, f_j) - \mathrm{te}(C_z, f_j) \\
&= \sum_{v \in \{x,y,z,w\}} \max\{d_{C_{xy}}(v) - f_j(v), 0\} - \sum_{v \in \{x,y,z,w\}} \max\{d_{C_z}(v) - f_j(v), 0\} \\
&= \sum_{v \in \{z,w\}} \Big(\max\{d_{T_{xy}}(v) - f_j(v), 0\} - \max\{d_{T_z}(v) - f_j(v), 0\} \Big) \\
&\geq 0.
\end{aligned}
$$

(The first equality follows from the fact "$d_{C_z}(v) = d_{C_{xy}}(v)$ for every $v \in V(C_{xy}) - \{x, y, z, w\}$", and the second follows from the fact "$d_{C_z}(x) \leq f_j(x)$ and $d_{C_z}(y) \leq f_j(y)$". The last inequality follows from the fact "$d_{C_z}(z) < d_{C_{xy}}(z)$ and $d_{C_z}(w) \leq d_{C_{xy}}(w)$".)

Therefore $T_z \in \mathcal{T}(A_{l-1})$. However, $z \in A_l$ and $d_{T_z}(z) = d_T(z) - 1 = f_i(z) - 1$, which contradicts the definition of A_l. $\square$

Since $A_0 \subset A_1 \subset \cdots \subset V_{\geq f_i}(T)$, there exists an integer l such that $A_l = A_{l+1}$. By Claim 2 or 3, we have $\omega(T - A_l) = \omega(G - A_l)$. Since $V_{\geq f_i+1}(T) = A_0 \subset A_l \subset V_{\geq f_i}(T)$, we have $\mathrm{te}(T, f_i) = \sum_{v \in A_l} \big(d_T(v) - f_i(v)\big)$. It follows from Proposition 9 and these equalities that

$$
\begin{aligned}
\omega(G - A_l) &= \omega(T - A_l) \\
&\geq \sum_{v \in A_l} \big(d_T(v) - 2\big) + 2 \\
&= \sum_{v \in A_l} \big(f_i(v) - 2\big) + \sum_{v \in A_l} \big(d_T(v) - f_i(v)\big) + 2 \\
&= \sum_{v \in A_l} \big(f_i(v) - 2\big) + \mathrm{te}(T, f_i) + 2 \\
&> \sum_{v \in A_l} \big(f_i(v) - 2\big) + t_i + 2.
\end{aligned}
$$

This contradicts the assumption in Theorem 8 and completes the proof of Theorem 8. $\square$

5 Concluding remarks

In this paper, we have considered a toughness type condition for the existence of a spanning tree with bounded total excess. In particular, we proved Theorem 8, which states that when a given graph satisfies the toughness type condition of Theorem 3 for some pairs (k, t) at the same time, then the graph has a spanning tree satisfying the total excess properties for all pairs. Using it, we can find a spanning tree with bounded maximum degree and bounded total excess, for example, see Section 2.

On the other hand, we do not know whether the coefficient "$k - 2$" of $|S|$ of the condition in Theorem 2 is best possible. It seems interesting to consider the improvement of Theorem 2. (However, if we can increase the coefficient "$k - 2$" to "$k - 2 + \varepsilon$" for some $\varepsilon > 0$ and $k = 2$, then it gives the positive answer to Chvátal's conjecture. So this problem seems difficult.) If we succeed to increase the coefficient of $|S|$ in Theorem 2, then we might be also able to improve Theorems 3 and 4.

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