Spanning k-forests with large components in $K_{1,k+1}$-free graphs

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Abstract

For an integer k with $k \geq 2$, a *k-tree* (resp. a *k-forest*) is a tree (resp. forest) with maximum degree at most k. In this paper, we show that for any integer k with $k \geq 3$, any connected $K_{1,k+1}$-free graph has a spanning k-tree or a spanning k-forest with only large components.

1 Introduction

In this paper, we only consider finite undirected graphs without loops or multiple edges. For standard graph-theoretic terminology not explained in this paper, we refer to [8] for the reader. For a graph G, let $\delta(G)$ and $\alpha(G)$ be the minimum degree and the independence number of G, respectively. For a positive integer k, if $\alpha(G) \geq k$, then let

$$\sigma_k(G) = \min\left\{ \sum_{x \in X} d_G(x) \colon X \text{ is an independent set of } G \text{ with } |X| = k \right\};$$

otherwise let $\sigma_k(G) = +\infty$.

To find a Hamiltonian cycle or a Hamiltonian path in a given graph is one of the most important problems in Graph Theory. Dirac [9] showed that any graph G of order

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n with $n \geq 3$ contains a Hamiltonian cycle if $\delta(G) \geq n/2$. Ore [22] improved this result as follows: any graph G of order $n \geq 3$ contains a Hamiltonian cycle if $\sigma_2(G) \geq n$. As a direct corollary of Ore's result, any graph G of order n with $\sigma_2(G) \geq n-1$ contains a Hamiltonian path.

In addition to hamiltonicity, several researchers have studied some structures with more relaxed properties. One of the huge targets along this line is a cycle or a path having large length. Actually, the result above by Ore was extended by Bermond [4] and independently Linial [20] to this direction. As a corollary of their result, we obtain the following: any connected graph G of order n contains a path of order at least $\min\{\sigma_2(G)-1, n\}$. More results on long cycles and paths have appeared, for example, see [10, 12]. See also the good books [5, 24].

Instead of finding "one" path with large length, we sometimes consider several pairwise disjoint paths. A *path-factor* of a graph G is a spanning subgraph of G in which each component is a path. Akiyama, Avis and Era [1], and Kaneko [17] gave necessary and sufficient conditions for the existence of a path-factor in which the length of each path is at least 2, and at least 3, respectively.

On the other hand, as a relaxed property of hamiltonicity, we consider a spanning tree with bounded maximum degree. For an integer k with $k \geq 2$, a *k-tree* is a tree with maximum degree at most k. When $k = 2$, a spanning 2-tree is equivalent to a Hamiltonian path. Therefore, a spanning k-tree has indeed a relaxed property of Hamiltonicity, and hence it has attracted much attention. Win [25] showed that any connected graph G of order n contains a spanning k-tree if $\sigma_k(G) \geq n-1$. Furthermore, several degree-type sufficient conditions for the existence of a spanning k-tree has been known, see [14, 18, 23]. In particular, Caro, Krasikov and Roditty [7] proved the following result, which shows the existence of a k-tree with large order. (In addition, Aung and Kyaw [3] later extended it.)

Theorem 1 (Caro, Krasikov and Roditty [7]) *Any connected graph G of order n contains a k-tree T with*

$$|T| \geq \min\{k\delta(G)+1, n\}.$$

We have surveyed relaxed properties of hamiltonicity in general connected graphs, and such properties are often considered in $K_{1,3}$-free graphs. For a positive integer r, recall that a graph having no $K_{1,r}$ as an induced subgraph is said to be *$K_{1,r}$-free*. Since $K_{1,3}$ is called a *claw*, we sometimes call a $K_{1,3}$-free graph a *claw-free graph*. Matthews and Sumner [21] proved the following.

Theorem 2 (Matthews and Sumner [21]) *Every connected claw-free graph G on n vertices contains a path of order at least* $\min\{2\delta(G)+2, n\}$.

Also several researchers have studied hamiltonicity or the existence of long cycles or long paths in $K_{1,r}$-free graphs [6, 13, 15, 19]. For more detail, see the survey [11] on claw-free graphs. Similarly to the situation on general graphs, Ando, Egawa, Kaneko, Kawarabayashi and Matsuda [2] showed a result on the existence of a path-factor with only large components.

Theorem 3 (Ando et al. [2]) *Let G be a connected $K_{1,3}$-free graph. Then G contains a path-factor such that each component has at least $\delta(G)+1$ vertices.*

Let k be an integer with $k \geq 2$. In this paper, we consider a spanning forest with maximum degree at most k, which is called a *spanning k-forest.* Note that a path-factor of a graph G is a spanning 2-forest. Before mentioning our main theorem on spanning k-forests, we introduce the following well-known proposition. For example, we obtain Proposition 4 combining Lemma 2.2 (ii) and Theorem 3.1 (i) in [16].

Proposition 4 *For an integer k with $k \geq 1$, every connected $K_{1,k+1}$-free graph contains a spanning $(k+1)$-tree.*

Moreover, it is known that this condition is best possible, that is, there exist infinitely many connected $K_{1,k+1}$-free graphs containing no spanning k-tree. Therefore, instead of a spanning k-tree, in a connected $K_{1,k+1}$-free graph, we are interested in a spanning k-forest with only large components, as in Theorem 3. This is the main purpose of this paper and we show the following two results.

Theorem 5 *Let G be a connected $K_{1,4}$-free graph. Then G contains a spanning 3-tree or a spanning 3-forest F such that for any component C of F, we have*

$$|V(C)| \geq \begin{cases} \sigma_3(G)+1 & \text{if } \Delta(C)=3, \\ \sigma_2(G)+1 & \text{if } \Delta(C)=2. \end{cases}$$

Theorem 6 *Let k be a positive integer with $k \geq 4$ and let G be a connected $K_{1,k+1}$-free graph. Then G contains a spanning k-tree or a spanning k-forest F such that for any component C of F, we have $\Delta(C) \geq k-1$ and*

$$|V(C)| \geq \begin{cases} \sigma_{2k-3}(G)-1 & \text{if } \Delta(C)=k, \\ \sigma_{k-1}(G)+1 & \text{if } \Delta(C)=k-1. \end{cases}$$

Note that the lower bounds on the orders of each component of a spanning k-forest obtained in Theorems 5 and 6 are almost best possible. We will show that in Section 2. After that, we will prove Theorems 5 and 6 in Section 3.

Here we give some terminology. For a graph G, we write $\omega(G)$ for the number of components in G. Let $V_k(G) = \{x \in V(G) : d_G(x) = k\}$ and $V_{\geq k}(G) = \{x \in V(G) : d_G(x) \geq k\}$. For $x, y \in V(G)$, a path connecting x and y in G is called an *x, y-path.* We define the *distance* of two vertices x and y in G, denoted by $d_G(x, y)$, as the length of a shortest x, y-path in G; if no such path exists, we set $d_G(x, y) = \infty$.

Let T be a rooted tree with root r. For a vertex v in $V(T)\backslash\{r\}$, the unique neighbor of v on the r, v-path in T is called the *parent* of v (denoted by v^-). A vertex in $N_T(v)\backslash\{v^-\}$ is a *child.* Sometimes we simply write v^{-2} for $(v^-)^-$ if v is not a neighbor of r in T. For $S \subseteq V(T)$, we define $S^+ = \bigcup_{x \in S}(N_T(x)\backslash\{x^-\})$. The set $(S^+)^+$ is simply denoted by S^{+2}.

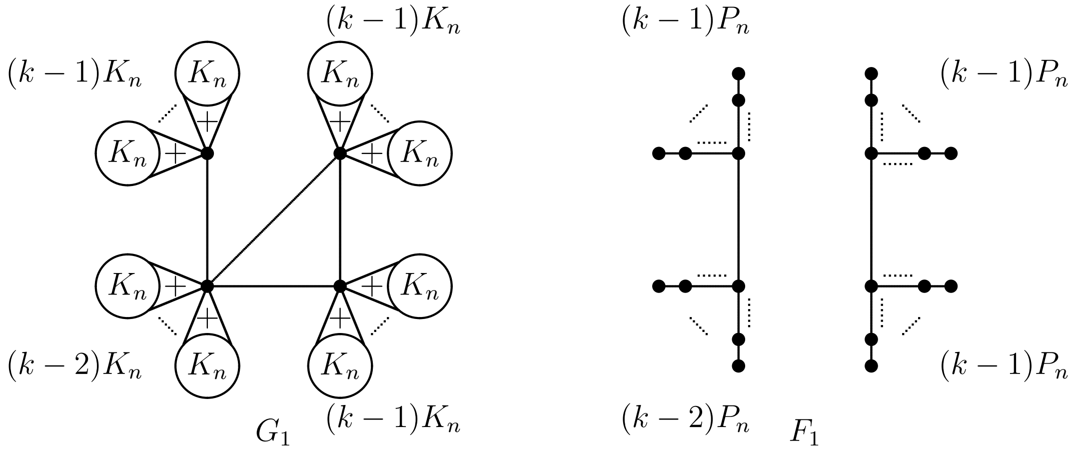

Figure 1: The graph G_1 and a spanning k-forest F_1

2 Theorems 5 and 6 are almost best possible

Throughout this section, let k be an integer with $k \geq 3$. Let G_1 be the graph as in the left side of Figure 1. Note that G_1 consists of $4k-5$ copies of the complete graph K_n, together with four additional vertices drawn in the center. Notice also that G_1 does not contain a spanning k-tree, G_1 is $K_{1,k+1}$-free and $\sigma_{2k-3}(G_1) = (2k-3)n$. Let F_1 be a spanning k-forest in G_1 such that the order of a smallest component of F_1 is as large as possible. (See the right side of Figure 1.) Each of the components in F_1 contains a vertex of degree k and the order of the smallest component in F_1 is $(2k-3)n+2 = \sigma_{2k-3}(G_1)+2$. So, the conditions "$\sigma_3(G)+1$" and "$\sigma_{2k-3}(G)-1$" in Theorems 5 and 6 are best possible, except for the constant term in Theorem 5.

Let G_2 be the graph as in the left side of Figure 2. Note that G_2 consists of $3k-3$ copies of the complete graph K_n, together with three additional vertices drawn in the center. Notice also that G_2 does not contain a spanning k-tree, G_2 is $K_{1,k+1}$-free and $\sigma_{k-1}(G_2) = (k-1)n$. Let F_2 be a spanning k-forest in G_2 such that (1) the order of a smallest component of F_2 is as large as possible, and (2) $\omega(F_2)$ is as small as possible, subject to (1). (See the right side of Figure 2.) Then F_2 has exactly two components, and let C be a smaller component of F_2. Note that C contains a vertex of degree $k-1$, and $|V(C)| = (k-1)n+1 = \sigma_{k-1}(G_1)+1$. Therefore, the condition "$\sigma_{k-1}(G)+1$" in Theorems 5 and 6 is best possible.

3 Proof of Theorems 5 and 6

In this section, we prove Theorems 5 and 6 at the same time. We now briefly sketch the outline of the proof. First we take a spanning k-forest F with certain conditions (the conditions (F1)–(F4) and (F5) in some cases). After showing two basic claims, we divide the proof into five parts, depending on the maximum degree of a component C of F and the value of k. We use the almost same method to all cases; we take appropriate number of leaves in C and consider the degree sum over those leaves in C, which shows

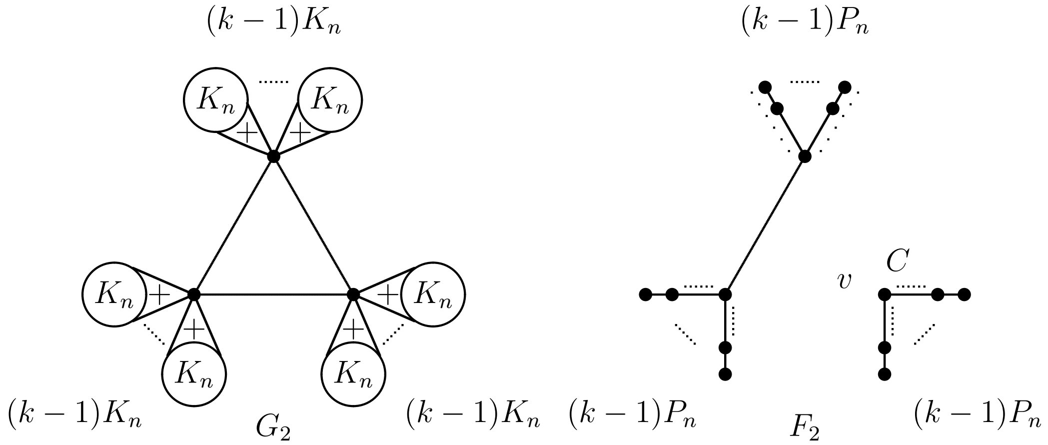

Figure 2: The graph G_2 and a spanning k-forest F_2

that the order of C is large enough. When we consider the degree sum, we will use a standard crossing argument, which appeared in some papers [3, 7, 18, 25] for general graphs. However, in this paper, we deal with $K_{1,k+1}$-free graphs, and hence we have to modify the argument. Indeed, this gives us some new difficulty in the proof, especially in Subcase 2.1. (In Subcase 2.1, we have to consider some set of vertices, called $\tilde{Y}_i^h$, with distance 3 from the set of leave over which we consider the degree sum, while some vertices with distance 2 are usually enough in standard crossing argument.)

Proof of Theorems 5 and 6.

Let k be an integer with $k \geq 3$, and G be a connected $K_{1,k+1}$-free graph. We take a spanning k-forest F so that F satisfying the following conditions:

(F1) $\omega(F)$ is as small as possible,

(F2) $|V_1(F)|$ is as small as possible, subject to (F1),

(F3) $|V_k(F)|$ is as small as possible, subject to (F2), and

(F4) $|V_{k-1}(F)|$ is as small as possible, subject to (F3).

If $\omega(F) = 1$, then F is a spanning k-tree of G, and hence both Theorems 5 and 6 hold. Thus, we may assume $\omega(F) \geq 2$. First we prove the following claims.

Claim 1 *Let C be a component of F. Then for each $v \in V(C)$ with $d_C(v) \leq k-2$, we have $N_G(v) \subseteq V(C)$.*

Proof of Claim 1.

Let C be a component of F. Suppose that there exists a vertex v in $V(C)$ such that $d_C(v) \leq k-2$ and a vertex u in $N_G(v) \backslash V(C)$. Let C' be the component of F such that $u \in V(C')$. If $d_F(u) \leq k-1$, then $F \cup \{uv\}$ is a spanning k-forest of G with $\omega(F \cup \{uv\}) < \omega(F)$, contradicting the condition (F1). Hence we may assume that $d_F(u) = k$.

If $\{v\} \cup N_F(u)$ is an independent set, then G contains an induced $K_{1,k+1}$ with center u, a contradiction. Therefore, there exists a vertex a in $N_F(u)$ such that $va \in E(G)$, or there exist two vertices a and b in $N_F(u)$ such that $ab \in E(G)$. If the former case occurs, then let $F_1 = (F \backslash \{ua\}) \cup \{vu, va\}$ and F_1 is a spanning k-forest of G with $\omega(F_1) < \omega(F)$, contradicting the condition (F1). Thus we may assume that the latter case occurs. If $d_F(a) \leq k-1$, then let $F_2 = (F \backslash \{ub\}) \cup \{uv, ab\}$ and F_2 is a spanning k-forest of G with $\omega(F_2) < \omega(F)$, contradicting the choice (F1). Hence $d_F(a) = k$. By symmetry, we obtain $d_F(b) = k$. Let $F_3 = (F \backslash \{au, bu\}) \cup \{uv, ab\}$. Then F_3 is a spanning k-forest of G with $\omega(F_3) = \omega(F)$. Moreover, we have $d_{F_3}(v) = d_F(v) + 1 \leq k-1$, $d_{F_3}(u) = k-1 \geq 2$, $d_{F_3}(a) = d_F(a) = k$ and $d_{F_3}(b) = d_F(b) = k$. This contradicts the condition (F2) when $d_F(v) = 1$; otherwise contradicts the condition (F3). □

Claim 2 *Let C be a component of F. If $\Delta(C) \geq 3$, then for each two leaves u and v of C, we have $uv \notin E(G)$.*

Proof of Claim 2.
Let P be the unique u,v-path in C. Since $\Delta(C) \geq 3$, there exists a vertex w in P such that $d_C(w) \geq 3$. Let $w' \in N_P(w)$. If $uv \in E(G)$, then let $F_1 = (F \backslash \{ww'\}) \cup \{uv\}$ and F_1 is a spanning k-forest of G with $\omega(F_1) = \omega(F)$ and $|V_1(F_1)| < |V_1(F)|$, contradicting the condition (F2). □

Since $\omega(F) \geq 2$ and G is connected, it follows from Claim 1 that any component C of F contains a vertex of degree at least $k-1$ in C. Let $\mathcal{C}$ be the set of components of F, and define

$$\begin{aligned} \mathcal{C}_1 &= \{C \in \mathcal{C} : \Delta(C) = k-1\}, \\ \text{and} \qquad \mathcal{C}_2 &= \mathcal{C} \backslash \mathcal{C}_1. \end{aligned}$$

Note that for each C in $\mathcal{C}_2$, we have $\Delta(C) = k$. Let C be a component of F. We will show that C has large order desired in Theorem 5 or 6. We divide the rest of the proof into two parts according as $C \in \mathcal{C}_1$ or $C \in \mathcal{C}_2$.

Case 1 $C \in \mathcal{C}_1$.

By Claim 1 and the fact that $\omega(F) \geq 2$, we can choose a vertex v in $V_{k-1}(C)$ such that $N_{G \backslash C}(v) \neq \emptyset$. Let $C_1, \cdots, C_{k-1}$ be the components of $C \backslash \{v\}$. For $1 \leq i \leq k-1$, let v_i be the vertex in $V(C_i) \cap N_C(v)$ and u_i be a leaf of C with $u_i \in V(C_i)$.

We also divide the rest of the proof of Case 1 into two subcases.

Subcase 1.1 $k \geq 4$.

For $1 \leq i \leq k-1$, consider C_i as a rooted tree with root u_i. Let $U = \{u_1, \cdots, u_{k-1}\}$. Note that $|U| = k-1$ and U is an independent set by Claim 2. Now we estimate the order of C and the degree sum over all vertices in U. First, we prove the following claims.

Claim 3 (1) *For $1 \leq i \leq k-1$, we have $v_i \notin N_G(U \backslash \{u_i\})$.*

(2) *For* $1 \le i \le k-1$ *and each* $x \in V(C_i) \cap N_G(U \backslash \{u_i\})$*, we have* $d_{C_i}(x) \ge k-2$*.*

Proof of Claim 3.
(1) Assume that there exists an edge $v_i u_j$ in $E(G)$ with $i \neq j$. Let $F_1 = (F \backslash \{vv_i\}) \cup \{v_i u_j\}$. Then F_1 is a spanning k-forest of G with $\omega(F_1) = \omega(F)$ and $|V_1(F_1)| < |V_1(F)|$, contradicting the condition (F2).

(2) Suppose that there exists a vertex x in $V(C_i) \cap N_G(U \backslash \{u_i\})$ such that $d_{C_i}(x) < k-2$. Let u_j be a vertex in $U \backslash \{u_i\}$ such that $xu_j \in E(G)$, and let $F_2 = (F \backslash \{v_i v\}) \cup \{xu_j\}$. Note that F_2 is a spanning k-forest of G. If $d_F(v_i) \ge 3$, then F_2 satisfies $\omega(F_2) = \omega(F)$ and $|V_1(F_2)| < |V_1(F)|$, contradicting the condition (F2). On the other hand, if $d_F(v_i) = 2$, then F_2 satisfies $\omega(F_2) = \omega(F)$, $|V_1(F_2)| = |V_1(F)|$, $|V_k(F_2)| = |V_k(F)|$ and $|V_{k-1}(F_2)| < |V_{k-1}(F)|$, contradicting the condition (F4). □

Claim 4 *For* $1 \le i \le k-1$ *and each* $x \in V(C_i) \cap N_G(U \backslash \{u_i\})$*, we have* $d_{C_i}(x) \ge k-1$*.*

Proof of Claim 4.
Let i be an integer with $1 \le i \le k-1$ and $x \in V(C_i) \cap N_G(U \backslash \{u_i\})$. By Claim 3 (2), we obtain $d_{C_i}(x) \ge k-2$. Hence we may assume $d_{C_i}(x) = k-2$. Let $u_j \in U \backslash \{u_i\}$ such that $xu_j \in E(G)$, and let $F_1 = (F \backslash \{vv_i\}) \cup \{xu_j\}$. Then F_1 is a spanning k-forest of G. Note that F_1 satisfies $\omega(F_1) = \omega(F)$, $d_{F_1}(x) = k-1$, $d_{F_1}(v) = k-2$, $d_{F_1}(u_j) = 2$ and $d_{F_1}(v_i) \ge 1$. If $d_F(v_i) \ge 3$, then F_1 satisfies $\omega(F_1) = \omega(F)$ and $|V_1(F_1)| < |V_1(F)|$, contradicting the condition (F2). Otherwise, that is, if $d_F(v_i) = 2$, then F_1 satisfies $\omega(F_1) = \omega(F)$, $|V_1(F_1)| = |V_1(F)|$, $|V_k(F_1)| = |V_k(F)|$ and $|V_{k-1}(F_1)| = |V_{k-1}(F)|$. Therefore, F_1 satisfies the properties (F1), (F2), (F3) and (F4), and hence F_1 also has to satisfy Claim 1. It follows from the choice of v that $V(G \backslash C) \cap N_G(v) \neq \emptyset$, a contradiction. □

For $1 \le i \le k-1$, let $X_i = V(C_i) \cap N_G(U \backslash \{u_i\})$. We prove the following claims.

Claim 5 *For each integer* i *with* $1 \le i \le k-1$*, the following statements hold.*

(1) $|{X_i}^+| \ge (k-2)|X_i|$.

(2) $X_i^+ \cap N_G(u_i) = \emptyset$.

Proof of Claim 5.
(1) By Claims 3 (1) and 4, we obviously obtain $|{X_i}^+| \ge (k-2)|X_i|$.

(2) Suppose that there exist two vertices a in $V(C_i) \cap N_G(u_i)$ and b in $V(C_i) \cap N_G(u_j)$ with $j \neq i$ and $a^- = b$. Let $F_1 = (F \backslash \{vv_i, ab\}) \cup \{au_i, bu_j\}$. Then F_1 is a spanning k-forest of G with $\omega(F_1) = \omega(F)$ and $|V_1(F_1)| < |V_1(F)|$, contradicting the condition (F2). □

For $1 \leq i \leq k-1$, we obviously obtain

$$X_i^+ \cup (V(C_i) \cap N_G(u_i)) \cup \{u_i\} \subseteq V(C_i).$$

Therefore, it follows from Claims 5 (1) and (2) that

$$\begin{aligned}
|V(C_i)| &\geq |X_i^+| + |V(C_i) \cap N_G(u_i)| + |\{u_i\}| \\
&\geq (k-2)|X_i| + |V(C_i) \cap N_G(u_i)| + 1 \\
&\geq (k-2) \max_{j \neq i} |V(C_i) \cap N_G(u_j)| + |V(C_i) \cap N_G(u_i)| + 1 \\
&\geq \sum_{j=1}^{k-1} |V(C_i) \cap N_G(u_j)| + 1.
\end{aligned}$$

Clearly $\sum_{x \in U} |\{v\} \cap N_G(x)| \leq |U| = k-1$, and it follows from Claim 1 that $V(G \backslash C) \cap N_G(U) = \emptyset$. Therefore, we obtain

$$\begin{aligned}
|V(C)| &\geq \sum_{i=1}^{k-1} |V(C_i)| + |\{v\}| \\
&\geq \sum_{i=1}^{k-1} \Big(\sum_{j=1}^{k-1} |V(C_i) \cap N_G(u_j)| + 1 \Big) + 1 \\
&\geq \sum_{i=1}^{k-1} \Big(\sum_{j=1}^{k-1} |V(C_i) \cap N_G(u_j)| \Big) + \sum_{j=1}^{k-1} |\{v\} \cap N_G(u_j)| + 1 \\
&\geq \sum_{j=1}^{k-1} |V(C) \cap N_G(u_j)| + 1 \\
&\geq \sigma_{k-1}(G) + 1. \qquad \square
\end{aligned}$$

Subcase 1.2 $k = 3$.

In this case, we have $\Delta(C) = k - 1 = 2$, which implies that C is a path. Let u_1 and u_2 be the end-vertices of C. We give an orientation to C from u_1 to u_2. If $(V(C) \cap N_G(u_1))^- \cap (V(C) \cap N_G(u_2)) \neq \emptyset$ or $u_1u_2 \in E(G)$, then $G[V(C)]$ contains a Hamiltonian cycle. Hence for each vertex x in $V(C)$, there exists a Hamiltonian path in $G[V(C)]$ such that x is an end-vertex of it. Therefore, it follows from Claim 1 that $N_G(C) \subseteq V(C)$, contradicting the connectedness of G and the fact $\omega(F) \geq 2$. This implies that $(V(C) \cap N_G(u_1))^- \cap (V(C) \cap N_G(u_2)) = \emptyset$ and $u_1u_2 \notin E(G)$. Note that $u_2 \notin (V(C) \cap N_G(u_1))^- \cup (V(C) \cap N_C(u_2))$. Hence we obtain

$$\begin{aligned}
|V(C)| &\geq |(V(C) \cap N_G(u_1))^-| + |V(C) \cap N_G(u_2)| + 1 \\
&\geq d_C(u_1) + d_C(u_2) + 1.
\end{aligned}$$

On the other hand, it follows from Claim 1 that $V(G \backslash C) \cap N_G(u_i) = \emptyset$ for $i = 1, 2$. Hence we conclude that $|V(C)| \geq d_C(u_1) + d_C(u_2) + 1 \geq \sigma_2(G) + 1$. $\square$

Case 2 $C \in \mathcal{C}_2$.

In order to complete the proof of Case 2, we first observe that $|V_{\geq k-1}(C)| \geq 2$. To show this, suppose $|V_{\geq k-1}(C)| \leq 1$. Thus $|V_k(C)| = 1$, say $v \in V_k(C)$. Let $N_C(v) = \{v_1, v_2, \cdots, v_k\}$. Since $|V_{\geq k-1}(C)| = 1$ and $\omega(F) \geq 2$, it follows from Claim 1 that $V(G\backslash C) \cap N_G(v) \neq \emptyset$. Let $v' \in V(G\backslash C) \cap N_G(v)$ and C' be the component of F containing v'. It follows from Claim 1 that $d_F(v') \geq k-1$. Since G is $K_{1,k+1}$-free, the set $\{v', v_1, v_2, \cdots, v_k\}$ is not an independent set. If $\{v_1, v_2, \cdots, v_k\}$ is not an independent set, then without loss of generality, we may assume that $v_1v_2 \in E(G)$. Since $|V_{\geq k-1}(C)| = 1$, we have $d_C(v_1) \leq k-2$ and $d_C(v_2) \leq k-2$. Hence F_1 is a spanning k-forest of G, where $F_1 = (F\backslash\{vv_1\}) \cup \{v_1v_2\}$. If v_2 is a leaf of C, then F_1 satisfies $\omega(F_1) = \omega(F)$ and $|V_1(F_1)| < |V_1(F)|$, contradicting the condition (F2). On the other hand, if v_2 is not a leaf of C, then F_1 satisfies $\omega(F_1) = \omega(F)$, $|V_1(F_1)| = |V_1(F)|$ and $|V_k(F_1)| < |V_k(F)|$, contradicting the condition (F3). These conclude that $\{v_1, v_2, \cdots, v_k\}$ is an independent set. Hence $v'v_i \in E(G)$ for some integer i with $1 \leq i \leq k$, say $i = 1$ by symmetry. Thus, we have $V(G\backslash C) \cap N_G(v_1) \neq \emptyset$. However it follows from Claim 1 that $d_F(v_1) \geq k-1$, a contradiction.

Therefore, we obtain $|V_{\geq k-1}(C)| \geq 2$, say $v, v' \in V_{\geq k-1}(C)$ with $v \neq v'$. By the assumption of Case 2 and the symmetry, we may assume that $v \in V_k(C)$.

Subcase 2.1 $k \geq 5$.

Now we reconstruct the component C of F and choose two vertices v and v' so that

(F5) the distance in C between v and v' is as small as possible, subject to (F4).

Let C_{l+1} be the component of $C\backslash\{v, v'\}$ such that $v, v' \in N_C(C_{l+1})$. Note that if $vv' \in E(C)$, then $C_{l+1} = \emptyset$. Let $C_1, C_2, \cdots, C_l$ be the components of $C\backslash\{v, v'\}$ such that $C_i \neq C_{l+1}$ for $1 \leq i \leq l$. Let $\mathcal{C}_v = \{C_i : V(C_i) \cap N_C(v) \neq \emptyset\}\backslash\{C_{l+1}\}$. Note that if $d_C(v') = k$, then $l = 2k-2$; otherwise $l = 2k-3$. For $1 \leq i \leq 2k-3$, let u_i be a leaf of C with $u_i \in V(C_i)$. Let $U = \{u_1, \cdots, u_{2k-3}\}$ and $U_v = \{u_i : C_i \in \mathcal{C}_v\}$. If $v \in N_C(C_i)$, then let v_i be a vetex in $C_i \cap N_C(v)$; otherwise let v_i be a vetex in $C_i \cap N_C(v')$. If $C_{l+1} \neq \emptyset$, then let ${v'}_{l+1}$ be a vertex in $V(C_{l+1}) \cap N_C(v')$ and P be the path in C_{l+1} connecting v_{l+1} and v'_{l+1}. For $1 \leq i \leq 2k-3$, we consider C_i as a rooted tree with the root u_i, and we consider C_{l+1} as a rooted tree with root v_{l+1}. If $l = 2k-2$, then we also consider C_{2k-2} as a rooted tree with root v_{2k-2}.

We obtain the following claim. Since we can prove (1)–(5) similarly to the ones in Claim 3, we omit the proof of those statements.

Claim 6 (1) *For $1 \leq i \leq 2k-3$, we have $v_i \notin N_G(U\backslash\{u_i\})$.*

(2) *If $l = 2k-2$, then $v_{2k-2} \notin N_G(U)$.*

(3) *For $1 \leq i \leq 2k-3$ and $x \in V(C_i) \cap N_G(U\backslash\{u_i\})$, we have $d_C(x) \geq k-2$.*

(4) *If $l = 2k-2$, then for $x \in V(C_{2k-2}) \cap N_G(U)$, we have $d_C(x) \geq k-2$.*

(5) *For $x \in V(C_{l+1}) \cap N_G(U)$, we have $d_C(x) \geq k-2$.*

(6) $N_G(U) \cap V(P) = \emptyset$.

Proof of Claim 6 (6).
Suppose that there exists a vertex x in P and a vertex u_i in U such that $xu_i \in E(G)$. It follows from Claim 6 (5) that $d_C(x) \geq k-2$, and from the condition (F5) that $d_C(x) \leq k-2$. Thus, we have $d_C(x) = k-2$.

Suppose first $u_i \in U_v$. Let $F_1 = (F \backslash \{vv_i\}) \cup \{xu_i\}$. Then F_1 is a spanning k-forest of G with $\omega(F_1) = \omega(F)$, $|V_1(F_1)| \leq |V_1(F)|$ and $|V_k(F_1)| < |V_k(F)|$, contradicting the condition (F2) or (F3). Hence $u_i \in U \backslash U_v$. In this case, let $F_2 = (F \backslash \{v'v_i\}) \cup \{xu_i\}$. Then F_2 is a spanning k-forest of G with $\omega(F_2) = \omega(F)$ and $|V_1(F_2)| \leq |V_1(F)|$. If $d_C(v') = k$, then $|V_k(F_2)| < |V_k(F)|$, contradicting the condition (F2) or (F3). On the other hand, if $d_C(v') = k-1$, then $|V_k(F_2)| = |V_k(F)|$, $|V_{k-1}(F_2)| \leq |V_{k-1}(F)|$, $d_{F_2}(x) = k-1$ and $d_{F_2}(v,x) < d_F(v,v')$, contradicting at least one of the conditions (F2), (F4) and (F5). $\Box$

For $1 \leq i \leq 2k-3$, let $X_i = V(C_i) \cap N_G(U \backslash \{u_i\})$, and $X_{l+1} = V(C_{l+1}) \cap N_G(U)$. In addition, if $l = 2k-2$, then let $X_{2k-2} = V(C_{2k-2}) \cap N_G(U)$. Similarly to the proof of Claim 5, we can prove the following claim using Claim 6, except for (2).

Claim 7 (1) *For $1 \leq i \leq l+1$, we have $|{X_i}^+| \geq (k-3)|X_i|$.*

(2) *For $1 \leq i \leq l+1$, we have $X_i^+ \cap X_i = \emptyset$.*

(3) *For $1 \leq i \leq 2k-3$, we have $X_i^+ \cap N_G(u_i) = \emptyset$.*

Proof of Claim 7 (2).
Suppose that there exist two vertices a and b in X_i such that $ab \in E(C_i)$. Suppose $a, b \in N_G(u_j)$ for some integer j with $1 \leq j \leq 2k-3$ and $j \neq i$. If $u_j \in U_v$, then let $F_1 = (F \backslash \{vv_j, ab\}) \cup \{au_j, bu_j\}$ and F_1 is a spanning k-forest of G with $\omega(F_1) = \omega(F)$, $1 \leq d_{F_1}(v_j) = d_F(v_j) - 1$ and $d_{F_1}(u_j) = 3 \leq k-2$. Furthermore we obtain $d_{F_1}(v) = d_F(v) - 1$. This implies that if $d_F(v_j) \geq 3$, then $\omega(F_1) = \omega(F)$ and $|V_1(F_1)| < |V_1(F)|$; otherwise, $\omega(F_1) = \omega(F)$, $|V_1(F_1)| = |V_1(F)|$ and $|V_k(F_1)| < |V_k(F)|$. In either case, that contradicts the condition (F2) or (F3). We can similarly get a contradiction even for the case that $u_j \in U \backslash U_v$.

Hence we may assume that $a \in N_G(u_j)$ and $b \in N_G(u_h)$ for some integers j and h with $1 \leq j, h \leq 2k-3$, $j \neq i$, $h \neq i$ and $j \neq h$. Suppose $i \neq l+1$. If $v \in N_F(C_i)$, then let $F_2 = (F \backslash \{vv_i, ab\}) \cup \{au_j, bu_h\}$ and F_2 is a spanning k-forest of G such that $\omega(F_2) = \omega(F)$ and $|V_1(F_2)| < |V_1(F)|$, contradicting the condition (F2). On the other hand, if $v' \in N_F(C_i)$, then let $F_3 = (F \backslash \{v'v_i, ab\}) \cup \{au_j, bu_h\}$ and F_3 is a spanning k-forest of G such that $\omega(F_3) = \omega(F)$ and $|V_1(F_3)| < |V_1(F)|$, contradicting the condition (F2). In either case, we have a contradiction.

Therefore, $i = l+1$. Without loss of generality, we may assume $d_C(v,a) < d_C(v,b)$. It follows from Claim 6 (6) that $a, b \notin V(P)$. Suppose $u_j \in U_v$, and let $F_4 = (F \backslash \{vv_{l+1}, ab\}) \cup \{au_j, bu_h\}$. Then F_4 is a spanning k-forest of G which satisfies $\omega(F_4) = \omega(F)$ and $|V_1(F_4)| < |V_1(F)|$, contradicting the condition (F2). Thus, we may assume

$u_j \in U\backslash U_v$. In this case, let $F_5 = (F\backslash\{v'v'_{l+1}, ab\}) \cup \{au_j, bu_h\}$. Then F_5 is a spanning k-forest of G such that $\omega(F_5) = \omega(F)$ and $|V_1(F_5)| < |V_1(F)|$, contradicting the condition (F2). $\square$

For a positive integer h, we define $Y_i^h = \{x \in X_i : |N_G(x) \cap (U\backslash\{u_i\})| = h\}$ for $1 \leq i \leq 2k-3$, and $Y_{l+1}^h = \{x \in X_{l+1} : |N_G(x) \cap U| = h\}$. If $l = 2k-2$, then we also define $Y_{2k-2}^h = \{x \in X_{2k-2} : |N_G(x) \cap U| = h\}$.

Claim 8 *For $h \geq k$ and $1 \leq i \leq l+1$, we have $Y_i^h = \emptyset$.*

Proof of Claim 8.
Suppose $Y_i^h \neq \emptyset$ for some integers h and i with $h \geq k$ and $1 \leq i \leq 2k-3$, say $y \in Y_i^h$. Without loss of generality, we may assume that $\{u_1, \cdots, u_h\} = N_G(y) \cap (U\backslash\{u_i\})$. It follows from Claim 2 that $\{u_1, \cdots, u_h\}$ is an independent set. If $h \geq k+1$, then $G[\{y, u_1, \cdots, u_h\}]$ contains an induced $K_{1,k+1}$, a contradiction. Therefore, we have $h = k$. Since y is not a leaf of C, there exists a vertex z with $z^- = y$. It follows from Claim 7 (2) that for each integer h' with $1 \leq h' \leq h$, we have $zu_{h'} \notin E(G)$. Hence $G[\{y, z, u_1, \cdots, u_h\}] \simeq K_{1,k+1}$, a contradiction again. When $i = l+1$ or $i = 2k-2$ (if exists), we can similarly show $Y_i^h = \emptyset$ for each integer h with $h \geq k$. $\square$

For $h \geq 1$ and $1 \leq i \leq l+1$, we define

$$\tilde{Y}_i^h \ = \ \{x \in V(C_i) : x^{-2} \in Y_i^h \text{ and } x^- z \in E(G) \text{ for some } z \in N_C(x^{-2})\}.$$

Claim 9 *For $2 \leq h \leq k-1$ and $1 \leq i \leq l+1$, we have $|\tilde{Y}_i^h| \geq (h-2)|Y_i^h|$.*

Proof of Claim 9.
If $h = 2$, then it is clear that $|\tilde{Y}_i^h| \geq (h-2)|Y_i^h|$. Hence we may assume that $3 \leq h \leq k-1$. Suppose $1 \leq i \leq 2k-3$. We shall show that for $y \in Y_i^h$, we can find at least $h-2$ vertices x in $\tilde{Y}_i^h$ with $x^{-2} = y$. Since C is a tree, this immediately proves Claim 9.

Let y be a vertex in Y_i^h and let $\{u_1, \cdots, u_h\} = (U\backslash\{u_i\}) \cap N_G(y)$. It follows from Claim 2 that $\{u_1, \cdots, u_h\}$ is an independent set. By Claim 6 (3), we have $d_{C_i}(y) \geq k-2$. Let $Z = \{z_1, z_2, \cdots\} = N_{C_i}(y)$. Without loss of generality we can assume that $z_1 = y^-$. Let

$$\tilde{Z} = \{z_i \in Z\backslash\{z_1\} : \ z_i z' \in E(G) \text{ for some } z' \in Z\}.$$

We now prove $|\tilde{Z}| \geq h-2$. Suppose $|\tilde{Z}| \leq h-3$. It follows from the choice of $\tilde{Z}$ that $Z\backslash\tilde{Z}$ is an independent set. Furthermore it follows from Claim 7 (2) that $\{u_1, \cdots, u_h\} \cup (Z\backslash\tilde{Z})$ is an independent set with $\left|\{u_1, \cdots, u_h\} \cup (Z\backslash\tilde{Z})\right| = h + d_{C_i}(y) - |\tilde{Z}| \geq h + (k-2) - (h-3) \geq k+1$, contradicting the $K_{1,k+1}$-free condition.

Thus, we obtain $|\tilde{Z}| \geq h-2$. Suppose that there exists a vertex z in $\tilde{Z}$ such that z is a leaf of C_i. Let $z' \in N_{C_i}(y)$ with $zz' \in E(G)$. If $z \neq v_i$, then let $F_1 = (F\backslash\{z'y\}) \cup \{zz'\}$; otherwise, let $F_1 = (F\backslash\{zy, z'y\}) \cup \{zz', yu_1\}$. In either case, F_1 is a spanning k-forest of G such that $\omega(F_1) = \omega(F)$ and $|V_1(F_1)| < |V_1(F)|$, contradicting the condition (F2). Therefore, z is not a leaf of C_i for each $z \in \tilde{Z}$. This discussion implies that for each

$z \in \tilde{Z}$, we can take a vertex x in $N_{C_i}(z)\backslash\{z^-\}$. Since such a vertex is contained in $\tilde{Y}_i^h$, there exist at least $h-2$ vertices x in $\tilde{Y}_i^h$ with $x^{-2} = y$.

We can similarly prove the case where $i = l+1$ and $i = 2k-2$ if $l = 2k-2$. $\Box$

Claim 10 *The following statements hold.*

(1) *For $2 \leq h < h' \leq k-1$ and $1 \leq i \leq l+1$, we have $\tilde{Y}_i^h \cap \tilde{Y}_i^{h'} = \emptyset$.*

(2) *For $2 \leq h \leq k-1$ and $1 \leq i \leq l+1$, we have $\tilde{Y}_i^h \cap X_i^+ = \emptyset$.*

(3) *For $2 \leq h \leq k-1$ and $1 \leq i \leq 2k-3$, we have $\tilde{Y}_i^h \cap (V(C_i) \cap N_G(u_i)) = \emptyset$.*

Proof of Claim 10.
(1) Since C is a tree, it follows from the choice of $\tilde{Y}_i^h$ that $\tilde{Y}_i^h \cap \tilde{Y}_i^{h'} = \emptyset$.

(2) By the choice of $\tilde{Y}_i^h$, we have $\tilde{Y}_i^h \subseteq X_i^{+2}$. It follows from Claim 7 (2) that $X_i \cap X_i^+ = \emptyset$. Hence $\tilde{Y}_i^h \cap X_i^+ = \emptyset$.

(3) Suppose $\tilde{Y}_i^h \cap (V(C_i) \cap N_G(u_i)) \neq \emptyset$, say $x \in \tilde{Y}_i^h \cap (V(C_i) \cap N_G(u_i))$. It follows from the definition of $\tilde{Y}_i^h$ that $x^{-2} \in X_i$ and there exists a vertex z in $N_{C_i}(x^{-2})$ such that $x^- z \in E(G)$. Assume $x^{-2} \in N_G(u_j)$ with $j \neq i$. Let $F_1 = (F \backslash \{vv_i, xx^-, x^{-2}z\}) \cup \{u_j x^{-2}, u_i x, x^- z\}$ when $C_i \in \mathcal{C}_v$, or $F_1 = (F \backslash \{v'v_i, xx^-, x^{-2}z\}) \cup \{u_j x^{-2}, u_i x, x^- z\}$ when $C_i \notin \mathcal{C}_v$. In either case, F_1 is a spanning k-forest of G such that $\omega(F_1) = \omega(F)$ and $|V_1(F_1)| < |V_1(F)|$, contradicting the condition (F2). $\Box$

By the choice of X_i and Y_i^h and Claim 8, for $1 \leq i \leq l+1$, we have $\sum_{h=1}^{k-1} |Y_i^h| = |X_i|$. Then it follows from Claim 9 that for $1 \leq i \leq 2k-3$, we have

$$\begin{aligned}
\sum_{j \neq i} |V(C_i) \cap N_G(u_j)| &= \sum_{h=1}^{k-1} h|Y_i^h| \\
&\leq 2|X_i| + \sum_{h=2}^{k-1} (h-2)|Y_i^h| \\
&\leq 2|X_i| + \sum_{h=2}^{k-1} |\tilde{Y}_i^h|.
\end{aligned}$$

On the other hand,

$$V(C_i) \supseteq {X_i}^+ \cup \Big(\bigcup_{h=2}^{k-1} \tilde{Y}_i^h \Big) \cup \big(V(C_i) \cap N_G(u_i)\big) \cup \{u_i\},$$

and it follows from Claims 7 (3) and 10 (1)–(3) that ${X_i}^+$, $\tilde{Y}_i^2$, $\tilde{Y}_i^3$, ... , $\tilde{Y}_i^{k-1}$, $V(C_i) \cap N_G(u_i)$, and $\{u_i\}$ are pairwise disjoint. It follows from Claim 7 (1) and the assumption

$k \geq 5$ that

$$\begin{aligned}
|V(C_i)| &\geq |X_i^+| + \sum_{h=2}^{k-1} |\tilde{Y}_i^h| + |V(C_i) \cap N_G(u_i)| + 1 \\
&\geq (k-3)|X_i| + \sum_{h=2}^{k-1} |\tilde{Y}_i^h| + |V(C_i) \cap N_G(u_i)| + 1 \\
&\geq 2|X_i| + \sum_{h=2}^{k-1} |\tilde{Y}_i^h| + |V(C_i) \cap N_G(u_i)| + 1 \\
&\geq \sum_{j=1}^{2k-3} |V(C_i) \cap N_G(u_j)| + 1.
\end{aligned}$$

For C_{l+1}, we have

$$V(C_{l+1}) \supseteq X_{l+1} \cup X_{l+1}^+ \cup \Big(\bigcup_{h=2}^{k-1} \tilde{Y}_{l+1}^h \Big),$$

$$\text{and} \quad \sum_{j=1}^{2k-3} |V(C_{l+1}) \cap N_G(u_j)| \leq |X_{l+1}| + \sum_{h=2}^{k-1} (h-1)|Y_{l+1}^h|.$$

Hence it follows from Claims 7 (1), 9 and 10 (1) and (2) that

$$\begin{aligned}
|V(C_{l+1})| &\geq |X_{l+1}| + |X_{l+1}^+| + \sum_{h=2}^{k-1} |\tilde{Y}_{l+1}^h| \\
&\geq (k-2)|X_{l+1}| + \sum_{h=2}^{k-1} |\tilde{Y}_{l+1}^h| \\
&\geq 2|X_{l+1}| + \sum_{h=2}^{k-1} |\tilde{Y}_{l+1}^h| \\
&\geq |X_{l+1}| + \sum_{h=2}^{k-1} (h-1)|Y_{l+1}^h| \\
&\geq \sum_{j=1}^{2k-3} |V(C_{l+1}) \cap N_G(u_j)|.
\end{aligned}$$

If $l = 2k-2$, then the similar argument implies that

$$\begin{aligned}
|V(C_{2k-2})| &\geq |X_{2k-2}{}^+| + \sum_{h=2}^{k-1} |\tilde{Y}_{2k-2}^h| \\
&\geq |X_{2k-2}| + \sum_{h=2}^{k-1} (h-1)|Y_{2k-2}^h| \\
&\geq \sum_{j=1}^{2k-3} |V(C_{2k-2}) \cap N_G(u_j)|.
\end{aligned}$$

Note that $|N_G(v) \cap U| \leq k$, since otherwise we can find an induced $K_{1,k+1}$ with center v, a contradiction. Similarly, we have $|N_G(v') \cap U| \leq k$. By the above argument and the fact $N_{G \backslash C}(U) = \emptyset$, if $l = 2k-2$, then

$$\begin{aligned}
|V(C)| &= \sum_{i=1}^{l+1} |V(C_i)| + |\{v, v'\}| \\
&\geq \sum_{i=1}^{2k-3} \Big(\sum_{j=1}^{2k-3} |V(C_i) \cap N_G(u_j)| + 1 \Big) + \sum_{j=1}^{2k-3} |V(C_{2k-2}) \cap N_G(u_j)| \\
&\quad + \sum_{j=1}^{2k-3} |V(C_{l+1}) \cap N_G(u_j)| + 2 \\
&= \sum_{j=1}^{2k-3} \sum_{i=1}^{l+1} |V(C_i) \cap N_G(u_j)| + 2k - 1 \\
&\geq \sum_{j=1}^{2k-3} \sum_{i=1}^{l+1} |V(C_i) \cap N_G(u_j)| + \Big(\sum_{j=1}^{2k-3} |\{v, v'\} \cap N_G(u_j)| - 2k \Big) + 2k - 1 \\
&= \sum_{j=1}^{2k-3} |V(C) \cap N_G(u_j)| - 1 \\
&\geq \sigma_{2k-3}(G) - 1.
\end{aligned}$$

When $l = 2k-3$, we similarly obtain

$$|V(C)| \geq \sigma_{2k-3}(G) - 1. \qquad \square$$

Subcase 2.2 $k = 4$.

Similarly to Subcase 2.1, we define v, v', $C_1, \cdots C_{l+1}$, u_i, v_i, U and U_v. Moreover, we reconstruct C so that

(F5) the distance in C between v and v' is as small as possible, subject to (F4).

Note that $d_C(v) = 4$, and $d_C(v') = 3$ or $d_C(v') = 4$. Hence $l = 5$ (if $d_C(v') = 3$) or $l = 6$ (if $d_C(v') = 4$). We consider C_i as a rooted tree with root u_i for $1 \leq i \leq 5$ and C_{l+1} with root v_{l+1}. If $l = 6$, we also regard C_6 as a rooted tree with the root v_6. Let $X_i = V(C_i) \cap N_G(U \backslash \{u_i\})$ for $1 \leq i \leq 5$, and $X_{l+1} = V(C_{l+1}) \cap N_G(U)$. If $l = 6$, then also let $X_6 = V(C_6) \cap N_G(U)$. Similarly to the proofs of Claims 6 (1),(2) and 7 (3), we can show the following claim, except for (2).

Claim 11 *The following statements hold.*

(1) *For* $1 \leq i \leq 5$, *we have* $X_i^+ \cap N_G(u_i) = \emptyset$.

(2) *If* $l = 6$, *then* $X_6 \cap {X_6}^+ = \emptyset$.

(3) *For* $1 \leq i \leq 5$, *then* $v_i \notin N_G(U \backslash \{u_i\})$.

(4) *If* $l = 6$*, then* $v_6 \notin N_G(U)$.

Proof of Claim 11 (2).
Since $l = 6$, we have $d_C(v') = 4$. Suppose that there exist two vertices x and y in X_6 with $y = x^-$. If there exists a vertex u_j in $N_G(x) \cap N_G(y) \cap U$, then let $F_1 = (F \backslash \{xy, vv_j\}) \cup \{xu_j, yu_j\}$ (when $vv_j \in E(C)$) or $F_1 = (F \backslash \{xy, v'v_j\}) \cup \{xu_j, yu_j\}$ (when $v'v_j \in E(C)$). Then F_1 is a spanning 4-forest of G. Moreover, if $d_C(v_j) \geq 3$, then F_1 satisfies $\omega(F_1) = \omega(F)$ and $|V_1(F_1)| < |V_1(F)|$; otherwise $\omega(F_1) = \omega(F)$, $|V_1(F_1)| = |V_1(F)|$ and $|V_4(F_1)| < |V_4(F)|$. In either case, that contradicts the condition (F2) or (F3). Therefore, there exist a vertex u_j in $N_G(x) \cap U$ and a vertex u_h in $N_G(y) \cap U$ such that $j \neq h$. Let $F_2 = (F \backslash \{xy, vv_j\}) \cap \{xu_j, yu_h\}$ (when $vv_j \in E(C)$) or $F_2 = (F \backslash \{xy, v'v_j\}) \cap \{xu_j, yu_h\}$ (when $v'v_j \in E(C)$). Then F_2 is a spanning 4-forest of G. In either case, we can lead a contradiction by the argument similar to the above. □

Claim 12 (1) *For* $1 \leq i \leq 5$ *and* $x \in X_i$*, we have* $|N_G(x) \cap U \backslash \{u_i\}| \leq 2$.

(2) *If* $l = 6$*, then for each* $x \in X_6$*, we have* $|N_G(x) \cap U| \leq 2$.

(3) *For* $x \in X_{l+1}$*, we have* $|N_G(x) \cap U| \leq 2$.

Proof of Claim 12.
(1) Suppose that there exists a vertex x in X_i such that $|N_G(x) \cap U \backslash \{u_i\}| \geq 3$, say $u_j, u_h, u_p \in N_G(x) \cap U \backslash \{u_i\}$. Since u_j, u_h and u_p are leaves of C, it follows from Claim 2 that $u_j u_h, u_j u_p, u_h u_p \notin E(G)$ and x is not a leaf of C_i. Hence x has the parent x^- and $N_C(x) \backslash \{x^-\} \neq \emptyset$, say $z \in N_C(x) \backslash \{x^-\}$. We can choose x^- and z so that z or x^- is contained in Q, where Q is the unique x, v_i-path in C_i. Suppose that there exists an edge connecting $\{u_j, u_h, u_p\}$ and $\{z, x^-\}$. Assume $u_j z \in E(G)$ and $u_j, u_h \in U_v$. If z is contained in Q, then let $F_1 = (F \backslash \{vv_j, zx\}) \cup \{u_j z, u_h x\}$; otherwise, let $F_1 = (F \backslash \{vv_h, zx\}) \cup \{u_j z, u_h x\}$. In either case, F_1 is a spanning 4-forest of G such that $\omega(F_1) = \omega(F)$ and $|V_1(F_1)| < |V_1(F)|$, contradicting the condition (F2). Therefore, we have $u_j z \notin E(G)$ if $u_j, u_h \in U_v$. The similar argument implies that $u_j z \notin E(G)$ even if $u_j \notin U_v$ or $u_h \notin U_v$. Hence, we have $u_j z \notin E(G)$. By the same argument, we also see $u_h z, u_p z, u_j x^-, u_h x^-, u_p x^- \notin E(G)$.

Therefore, if $zx^- \notin E(G)$, then $G[\{x, z, x^-, u_j, u_h, u_p\}]$ is an induced $K_{1,5}$, a contradiction. Hence $zx^- \in E(G)$. Let $F_2 = (F \backslash \{xz, xx^-, vv_j\}) \cup \{zx^-, xu_j, xu_h\}$ (when $vv_j \in E(C)$) or $F_2 = (F \backslash \{xz, xx^-, v'v_j\}) \cup \{zx^-, xu_j, xu_h\}$ (when $v'v_j \in E(C)$). Then F_2 is a spanning 4-forest of G with $\omega(F_2) = \omega(F)$ and $|V_1(F_2)| < |V_1(F)|$, contradicting the condition (F2).

We can show (2) and (3) by the argument similar to (1). □

Claim 13 (1) *For* $1 \leq i \leq 5$*, we have* $|X_i^+| \geq \sum_{j \neq i} |V(C_i) \cap N_G(u_j)|$.

(2) *If* $l = 6$*, then* $|X_6^+| \geq \sum_{u_i \in U} |V(C_6) \cap N_G(u_i)|$.

Proof of Claim 13.
(1) Let i be an integer with $1 \leq i \leq 5$. We first show that each vertex x in X_i has at least one child, and at least two children if $|N_G(x) \cap U \backslash \{u_i\}| = 2$. Let $x \in X_i$. Then it follows from Claims 2 and 11(3) that x is not a leaf of C_i, and hence there exists at least one child of x. Now we assume $|N_G(x) \cap U \backslash \{u_i\}| = 2$. Let u_j and u_h two vertices in $N_G(x) \cap U \backslash \{u_i\}$, Q be the unique u_j, x-path in C, and Q' be the unique u_h, x-path in C. Note that both Q and Q' contains at least one of v and v'. If neither Q nor Q' contains v, then both contains v', and hence $d_C(v') = 4$. Thus, at least one of Q and Q', say Q, contains a vertex of degree exactly four in C. Let e be an edge in Q incident to a vertex of degree exactly four in C.

Suppose that $d_C(x) = 2$. Let $F_1 = (F \backslash \{e\}) \cup \{xu_j\}$. Then F_1 is a spanning 4-forest of G. Since $d_C(x) = 2$, we have $d_{F_1}(x) < 4$. Note that F_1 satisfies $\omega(F_1) = \omega(F)$, $|V_1(F_1)| \leq |V_1(F)|$ and $|V_4(F_1)| < |V_4(F)|$, contradicting the condition (F2) or (F3). Thus $d_C(x) \geq 3$. It follows from Claim 11 (3) that $x \neq v_i$. Hence x has at least two children in $V(C_i)$.

This and Claim 12 (1) imply that $|X_i^+| \geq 2|X_i| \geq \sum_{j \neq i} |V(C_i) \cap N_G(u_j)|$, which completes the proof of (1).

Similarly to (1), we can conclude the proof of (2). □

If there exists a vertex x in C_{l+1} such that $d_F(x) \geq 3$, then this contradicts the choice (F5). Hence if $C_{l+1} \neq \emptyset$, then C_{l+1} is a path.

Claim 14 *The following statements hold.*

(1) $V(C_{l+1}) \cap N_G(U_v) = \emptyset$.

(2) $v'_{l+1} \notin N_G(U)$.

(3) *If* $x \in V(C_{l+1}) \cap N_G(U \backslash U_v)$, *then there exists a vertex* z *in* $N_{C_{l+1}}(x) \backslash \{x^-\}$ *and* $z \notin N_G(U \backslash U_v)$.

Proof of Claim 14.
(1) Suppose that there exists a vertex x in $V(C_{l+1}) \cap N_G(U_v)$. Let $u_j \in U_v$ such that $xu_j \in E(G)$. Since C_{l+1} is a path, we have $d_F(x) = 2$. Let $F_1 = (F \backslash \{vv_j\}) \cup \{xu_j\}$. Then F_1 is a spanning 4-forest of G such that $\omega(F_1) = \omega(F)$, $|V_1(F_1)| \leq |V_1(F)|$ and $|V_4(F_1)| < |V_4(F)|$, contradicting the condition (F2) or (F3).

(2) It follows from Claim 14 (1) that $v'_{l+1} \notin N_G(U_v)$. Suppose that there exists a vertex u_j in $U \backslash U_v$ such that $v'_{l+1}u_j \in E(G)$. Since C_{l+1} is a path, we have $d_F(v'_{l+1}) = 2$. Let $F_2 = (F \backslash \{v'v'_{l+1}\}) \cap \{v'_{l+1}u_j\}$. Then F_2 is a spanning 4-forest of G such that $\omega(F_2) = \omega(F)$ and $|V_1(F_2)| < |V_1(F)|$, contradicting the condition (F2).

(3) Let $x \in V(C_{l+1}) \cap N_G(U \backslash U_v)$ and $u_j \in U \backslash U_v$ with $xu_j \in E(G)$. It follows from Claim 14 (2) that there exists a vertex z in $N_{C_{l+1}}(x) \backslash \{x^-\}$. Suppose that $u_j z \in E(G)$. Let $F_3 = (F \backslash \{xz, v'v'_{l+1}\}) \cup \{xu_j, zu_j\}$. Then F_3 is a spanning 4-forest of G. Note that

$d_{F_3}(u_j) = 3$. Since $z \neq v'_{l+1}$, we have $d_{F_3}(v, u_j) < d_F(v, v')$. Moreover, F_3 also satisfies $\omega(F_3) = \omega(F)$ and $|V_1(F_3)| = |V_1(F)|$. If $d_C(v') = 4$, then $|V_4(F_3)| < |V_4(F)|$, which contradicts the condition (F3). On the other hand, if $d_C(v') = 3$, then $|V_4(F_3)| = |V_4(F)|$ and $|V_3(F_3)| = |V_3(F)|$, which contradicts the condition (F5) since $d_{F_3}(v, u_j) < d_F(v, v')$. Therefore, $u_j z \notin E(G)$. Hence there exists an edge zu_h in G with $h \neq j$ and $u_h \in U \backslash U_v$. Let $F_4 = (F \backslash \{xz, v'v_h\}) \cup \{xu_j, zu_h\}$. Then F_4 is a spanning 4-forest of G with $\omega(F_4) = \omega(F)$ and $|V_1(F_4)| < |V_1(F)|$, contradicting the condition (F2). $\square$

It follows from Claims 11 (1), (3) and 13 (1) that for $1 \leq i \leq 5$, we obtain

$$\begin{aligned} |V(C_i)| &\geq |X_i^+| + |V(C_i) \cap N_G(u_i)| + 1 \\ &\geq \sum_{u_j \in U} |V(C_i) \cap N_G(u_j)| + 1. \end{aligned}$$

For C_{l+1}, since $|U \setminus U_v| \leq 2$, it follows from Claim 14 (1) that $|N_G(x) \cap U| \leq 2$ for each $x \in V(C_{l+1})$. Hence by Claims 14 (2) and (3), we have

$$|V(C_{l+1})| \geq \sum_{u_j \in U} |V(C_{l+1}) \cap N_G(u_i)|.$$

If $l = 6$, then it follows from Claims 11 (4) and 13 (2) that

$$|V(C_6)| \geq |X_6^+| + 1 \geq \sum_{u_j \in U} |V(C_6) \cap N_G(u_j)| + 1.$$

Note that $|N_G(v) \cap U| \leq 4$ and $|N_G(v') \cap U| \leq 4$, since otherwise G contains an induced $K_{1,5}$. Hence we obtain

$$\begin{aligned} |V(C)| &= \sum_{i=1}^{l+1} |V(C_i)| + |\{v, v'\}| \\ &\geq \sum_{i=1}^{l+1} \sum_{u_j \in U} |V(C_i) \cap N_G(u_j)| + 5 + 2 \\ &\geq \sum_{i=1}^{l+1} \sum_{u_j \in U} |V(C_i) \cap N_G(u_j)| + \Big(|N_G(v) \cap U| + |N_G(v') \cap U| - 8\Big) + 7 \\ &\geq \sigma_5(G) - 1. \qquad \square \end{aligned}$$

Subcase 2.3 $k = 3$.

Similarly to Subcase 1.1, we define v, C_i, u_i, v_i and U for $1 \leq i \leq 3$. Note that $d_C(v) = 3$. We consider C_i as a rooted tree with root u_i for $1 \leq i \leq 3$. For $1 \leq i \leq 3$, we define $X_i = V(C_i) \cap N_G(U \backslash \{u_i\})$.

Claim 15 *The following statements hold.*

(1) *For $1 \leq i \leq 3$, we have $v_i \notin N_G(U \backslash \{u_i\})$.*

(2) *For* $1 \le i \le 3$*, we have* $X_i^+ \cap (N_G(u_i)) = \emptyset$*.*

Since we can prove Claim 15 by the way similar to the proofs of Claims 3 (1) and 5 (2), we omit the proof of it.

Claim 16 *For* $1 \le i \le 3$ *and* $x \in X_i$*, we have* $|N_G(x) \cap (U \backslash \{u_i\})| = 1$*.*

Proof of Claim 16.
Let i be an integer with $1 \le i \le 3$. Suppose that there exists a vertex x in X_i such that $|N_G(x) \cap (U \backslash \{u_i\})| = 2$. Let $u_j, u_h \in N_G(x) \cap (U \backslash \{u_i\})$ with $\{i, j, h\} = \{1, 2, 3\}$. Since u_j and u_h are leaves of C, it follows from Claim 2 that $u_j u_h \notin E(G)$. It follows from Claims 2 and 15 (1) that x is not a leaf of C_i. Hence there exist two vertices z and x^- in C_i with $z^- = x$. We can choose such vertices z and x^- so that z or x^- are contained in the x, v_i-path Q on C_i. Assume that there exists an edge connecting $\{u_j, u_h\}$ and $\{z, x^-\}$. If $x^- u_j \in E(G)$, then let $F_1 = (F \backslash \{xx^-, vv_h\}) \cup \{x^- u_j, xu_j\}$ (if x^- is contained in Q) or $F_1 = (F \backslash \{xx^-, vv_h\}) \cup \{x^- u_j, xu_h\}$ (if x^- is not contained in Q) and F_1 is a spanning 3-forest of G with $\omega(F_1) = \omega(F)$ and $|V_1(F_1)| < |V_1(F)|$, contradicting the condition (F2). Therefore, $x^- u_j \notin E(G)$. Similarly we can show that there exists no edge connecting $\{u_j, u_h\}$ and $\{z, x^-\}$. If $zx^- \notin E(G)$, then $G[\{x, z, x^-, u_j, u_h\}]$ is an induced $K_{1,4}$, a contradiction. Hence $x^- z \in E(G)$. Let $F_2 = (F \backslash \{xz, xx^-, vv_j\}) \cup \{x^- z, u_j x, u_h x\}$. Then F_2 is a spanning 3-forest of G with $|V_1(F_2)| < |V_1(F)|$, contradicting the condition (F2). $\square$

For $1 \le i \le 3$, we obtain

$$X_i^+ \cup (V(C_i) \cap N_G(u_i)) \cup \{u_i\} \subseteq V(C_i).$$

It follows from Claim 16 that $|V(C_i) \cap N_G(u_j)| + |V(C_i) \cap N_G(u_h)| = |X_i|$ for $\{i, j, h\} = \{1, 2, 3\}$. Therefore, by Claim 15, we obtain

$$\begin{aligned} |V(C_i)| &\ge |X_i^+| + |V(C_i) \cap N_G(u_i)| + 1 \\ &\ge \sum_{j=1}^{3} |V(C_i) \cap N_G(u_j)| + 1. \end{aligned}$$

Obviously $|\{v\} \cap N_G(u_i)| \le 1$ for $1 \le i \le 3$. It follows from Claim 1 that $V(G \backslash C) \cap N_G(u_i) = \emptyset$ for $1 \le i \le 3$. Hence we obtain

$$\begin{aligned} |V(C)| &= \sum_{i=1}^{3} |V(C_i)| + 1 \\ &\ge \sum_{i=1}^{3} \Big(\sum_{j=1}^{3} |V(C_i) \cap N_G(u_j)| + 1 \Big) + 1 \\ &\ge \sum_{i=1}^{3} d_G(u_i) + 1 \\ &\ge \sigma_3(G) + 1. \qquad \square \end{aligned}$$

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