# Spanning trees in 3-connected $K_{3,t}$-minor-free graphs*

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**Abstract**

In this paper, we show that for any even integer $t \geq 4$, every 3-connected graph with no $K_{3,t}$-minor has a spanning tree whose maximum degree is at most $t-1$. This result is a common generalization of the result by Barnette [1] and the one by Chen, Egawa, Kawarabayashi, Mohar and Ota [4].

Keywords: $K_{3,t}$-minor-free graphs, Spanning trees, Graphs on a surface

## 1 Introduction

In 1956, Tutte [20] proved that every 4-connected planar graph has a Hamilton cycle, and Thomassen [18] generalized this result; every 4-connected planar graph is Hamilton-connected. It is known that there exist infinitely many 3-connected planar graphs without a Hamilton cycle, even a Hamilton path. Therefore, we sometimes consider a relaxation of a Hamilton property in a 3-connected planar graph. For example, the property "having long cycles" is one of such relaxations. Chen and Yu

*A preliminary description of this work appeared previously in [16].

†Research partly supported by JSPS, Grant-in-Aid for Scientific Research (B)

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[6] showed that any 3-connected planar graph has a cycle of length at least $|G|^{\log_3 2}$, which was conjectured by Moon and Moser [14].

Notice that any planar graph has no $K_{3,3}$-minor. In this sense, a generalization of a result on a planar graph is considered. Chen, Sheppardson, Yu and Zang [5] improved the above result by Chen and Yu; for an integer $t \geq 3$, any 3-connected graph with no $K_{3,t}$-minor has a cycle of length at least $|G|^{r(t)}$, where $r(t) := \log_{(8t^{t+1})} 2$.

In this paper, we concentrate on the other relaxation of a Hamilton property. For an integer $t \geq 2$, a spanning tree $T$ of a graph is called a *$t$-tree* if the maximum degree of $T$ is at most $t$. Note that a 2-tree is equivalent to a Hamilton path, so actually a $t$-tree is a relaxation of a Hamilton property. Barnette [1] proved the following result on a 3-tree.

**Theorem 1 (Barnette [1])** *Every 3-connected planar graph has a 3-tree.*

We consider a generalization of Theorem 1 for the direction of graphs with no $K_{3,t}$-minor. Chen, Egawa, Kawarabayashi, Mohar and Ota [4] showed that any 3-connected graph with no $K_{3,t}$-minor has a $(t+1)$-tree. However, this bound is not sharp and in this paper, we show the following result. In Theorem 2, we deal with only the case where $t$ is even, but when $t$ is odd, we obtain Corollary 3 as a direct corollary. (Note that $K_{3,t}$-minor-free graphs are also $K_{3,t+1}$-minor-free.)

**Theorem 2** *Let $t \geq 4$ be an even integer and let $G$ be a 3-connected graph. If $G$ has no $K_{3,t}$-minor, then $G$ has a $(t-1)$-tree.*

**Corollary 3** *Let $t \geq 3$ be an integer and let $G$ be a 3-connected graph. If $G$ has no $K_{3,t}$-minor, then $G$ has a $t$-tree.*

In [4], Chen et al. provided a result that will enable us to show that Theorem 2 and Corollary 3 are best possible, see Section 3.

It is well known that any 3-connected graph with no $K_{3,3}$-minor except for $K_5$ is planar. Therefore, Corollary 3 for the case $t = 3$ corresponds to Theorem 1. Note that we use a very different method from Barnette's for the proof of Theorem 2, and hence it also gives another proof of Theorem 1.

On the other hand, Barnette [2] improved Theorem 1 to graphs on other surfaces; every 3-connected graph on the projective plane, the torus, and the Klein bottle has a 3-tree. Recently, it is considered for 3-connected graphs on a surface with higher genus. Ellingham [7] showed the following result, which was first asked by Brunet, Ellingham, Gao, Metzlar and Richter [3]; any 3-connected graph on a surface of Euler characteristic $\chi < 0$ has a $\lceil \frac{10-2\chi}{3} \rceil$-tree. However the upper bound of the maximum degree is not best possible. Later, Sanders and Zhao [17] gave a sharp result for a graph on a surface of Euler characteristic $\chi \leq -36$; any 3-connected graph on a surface of Euler characteristic $\chi \leq -36$ has an $\lceil \frac{8-2\chi}{3} \rceil$-tree. Note that the complete bipartite graph $K_{3,6-2\chi}$ attains this upper bound of the maximum degree of a spanning tree. In this paper, we also improve this result for any 3-connected graph on a surface of Euler characteristic $\chi \leq 0$.

**Theorem 4** *Let $G$ be a 3-connected graph on a surface of Euler characteristic $\chi \leq 0$. Then $G$ has an $\lceil \frac{8-2\chi}{3} \rceil$-tree.*

Note that several other results on the existence of a degree bounded spanning tree of graphs on a surface are known, for example, see [8, 12, 15, 19, 21].

Before giving the proofs of our main results, we show some lemmas in Section 2, and the best possibility of Theorem 2 and Corollary 3 in Section 3, respectively. The proof of Theorem 2 is divided into two parts. The first is to consider a minor minimal 3-connected graph having no $(t-1)$-tree, and the second is to find a $K_{3,t}$-minor in such a graph when $t$ is even. Actually, we show the following theorem in Section 4 as the first part.

**Theorem 5** *Let $k \geq 3$ be an integer. Then every 3-connected graph having no $k$-tree contains a 3-connected bipartite minor $R$ with bipartition $A$ and $B$ such that $|B| = (k-1)|A| + 2$, where $A := \{x \in V(R) : d_R(x) \geq k+1\}$ and $B := \{x \in V(R) : d_R(x) = 3\}$.*

Note that the graph $R$ in Theorem 5 has no $k$-tree by the conditions "$R$ is bipartite" and "$|B| = (k-1)|A| + 2$". (See Proposition 13 in Section 2.) In this sense, the graph $R$ in Theorem 5 is a minor minimal 3-connected graph having no $k$-tree. This result can play an important role when we consider the existence of a $k$-tree. In fact we obtain Theorem 4 as a corollary of Theorem 5, which we shall show in Section 6. As the second part, in Section 5, we find a $K_{3,t}$-minor in the graph $R$ in Theorem 5 for $k = t-1$ when $t$ is an even integer that is at least four.

# 2 Lemmas

In this section, we show several lemmas which are used in the proofs of our main theorems. We divide this section into three subsections depending on the main topic of each lemma, that is, 3-connectedness, a $k$-tree, and a $K_{3,t}$-minor, respectively.

## 2.1 Lemmas concerning 3-connectedness

First, we show some lemmas concerning 3-connectedness. Before showing them, we will define some terminology.

Let $G$ be a 3-connected graph. For an edge $e \in E(G)$, let $G/e$ be the graph obtained from $G$ by contracting $e$. An edge $e$ is called *contractible* if $G/e$ is also 3-

connected. An edge which is not contractible is *non-contractible*. For $A, B \subset V(G)$ with $A \cap B = \emptyset$, we denote the number of edges between $A$ and $B$ by $e_G(A, B)$.

The first lemma is a well-known result on contractible edges.

**Lemma 6 (Halin [10])** *Let $G$ be a 3-connected graph other than $K_4$. Then every vertex of degree three is incident with a contractible edge.*

A 3-connected graph $G$ is called *minimally 3-connected* if the graph obtained by deleting any edge from $G$ is not 3-connected. For a minimally 3-connected graph, Halin showed the following result. Let $V_3(G) := \{x \in V(G) : d_G(x) = 3\}$.

**Lemma 7 (Halin [10, 11])** *Let $G$ be a minimally 3-connected graph. Then the following statements hold.*

(i) $V_3(G) \neq \emptyset$.

(ii) *Any edge connecting two vertices in $V(G) - V_3(G)$ is contractible.*

(iii) *The graph obtained by contracting any edge connecting two vertices in $V(G) - V_3(G)$ is also minimally 3-connected.*

(iv) *Every cycle of $G$ contains at least two vertices of $V_3(G)$.*

As an immediate consequence of Lemma 7 (iv), we can see that $G - V_3(G)$ is a forest. (A similar property for minimally $k$-connected graphs was proved by Mader [13].) Moreover, contraction of any edge in $G - V_3(G)$ does not produce a new vertex of degree three. So, applying Lemma 7 (iii) repeatedly, we obtain the following fact.

**Lemma 8** *Let $G$ be a minimally 3-connected graph, and let $P$ be any connected subgraph of $G - V_3(G)$. Then, $G/P$, the graph obtained from $G$ by contracting $P$ into a single vertex, is also a minimally 3-connected graph.*

The remaining four lemmas in this subsection give the reduction methods for a 3-connected graph. We will use those results in Section 5, when we find a $K_{3,t}$-minor. The first one is well-known and easy to prove, and hence we omit the proof.

**Lemma 9** *Let $G$ be a graph other than $K_4$ and $x \in V(G)$ with $d_G(x) = 3$, say $N_G(x) = \{y_1, y_2, y_3\}$. Suppose that $y_1y_2, y_2y_3, y_3y_1 \in E(G)$. Then $G$ is 3-connected if and only if $G - x$ is 3-connected.*

**Lemma 10** *Let $G$ be a 3-connected graph and let $S$ be a cut set of $G$ with $|S| = 3$. Let $C$ be a component of $G - S$ and let $G'$ be the graph obtained from $G$ by deleting $C$ and adding edges so that $S$ forms a clique. Then $G'$ is also 3-connected.*

**Proof.** Suppose not, that is, there exists a cut set $S'$ of order at most two in $G'$. Since $S$ forms a clique in $G'$, all vertices in $S - S'$ are contained in the same component of $G' - S'$, say $C_1'$, and let $C_2' := G' - \big(S' \cup V(C_1')\big)$. This implies that $S'$ also separates $G$ into $C_1' \cup C$ and $C_2'$, which contradicts 3-connectedness of $G$. $\square$

**Lemma 11** *Let $G$ be a 3-connected graph, $S_1$ be a cut set of $G$ with $|S_1| = 3$, and $C_1$ be a component of $G - S_1$. Suppose that $S_1$ and $C_1$ are chosen so that $C_1$ is minimal, namely, there is no cut set $S'$ of $G$ and component $C'$ of $G - S'$ such that $S' \neq S_1$, $|S'| = 3$, and $C' \subset C_1$. Then there is a contractible edge of $G$ between $S_1$ and $C_1$.*

**Proof.** Assume that there exists a non-contractible edge $uv$ with $u \in S_1$ and $v \in V(C_1)$. Let $\{x_1, x_2\} := S_1 - \{u\}$ and let $C_2 := G - \big(S_1 \cup V(C_1)\big)$. Since $uv$ is not contractible, there exists a cut set $S_1'$ of $G$ with $|S_1'| = 3$ and $u, v \in S_1'$, and let $\{w\} := S_1' - \{u, v\}$. Let $C_1'$ be a component of $G - S_1'$ and let $C_2' := G - \big(S_1' \cup V(C_1')\big)$.

Suppose that $S_1 \cap V(C_1') = \emptyset$. If $V(C_1) \cap V(C_1') = \emptyset$, then $N_G(v) \cap V(C_1') \subset \big(S_1 \cup V(C_1)\big) \cap V(C_1') = \emptyset$, and this implies that $S_1' - \{v\}$ is also a cut set of $G$, a contradiction. Therefore we have $V(C_1) \cap V(C_1') \neq \emptyset$. Then, $S_1' = \{u, v, w\}$ is a cut set of $G$ such that $C_1 \cap C_1'$ is a component of $G - S_1'$ with $|V(C_1 \cap C_1')| < |V(C_1)|$, contradicting the choice of $S_1$. Thus, $S_1 \cap V(C_1') \neq \emptyset$. By the same way, we obtain $S_1 \cap V(C_2') \neq \emptyset$. Therefore, we may assume that $x_i \in S_1 \cap V(C_i')$ for $i = 1, 2$.

If $w \notin V(C_2) \cap S_1'$, then $V(C_2) \cap S_1' = \emptyset$ (because $|S_1'| = 3$, $u \in S_1' \cap S_1$ and $v \in V(C_1) \cap S_1'$), and $V(C_2) \cap V(C_i') = \emptyset$ for $i = 1, 2$ (because otherwise $\{u, x_i\}$ is a cut set of $G$), and hence $V(C_2) = \emptyset$, a contradiction. Hence $w \in S_1' \cap V(C_2)$.

If $V(C_1) \cap V(C_i') \neq \emptyset$ for some $i = 1, 2$, then $\{u, v, x_i\}$ is a cut set of $G$ such that $C_1 \cap C_i'$ is the component of $G - \{u, v, x_i\}$ with $|V(C_1 \cap C_i')| < |V(C_1)|$, contradicting the choice of $S_1$. This implies that $V(C_1) \cap V(C_i') = \emptyset$ for any $i = 1, 2$, and hence $V(C_1) = \{v\}$. Since $d_G(v) = 3$, it follows from Lemma 6 that there exists a contractible edge $e$ incident with $v$, which connects $S_1$ and $C_1$. $\square$

As an immediate corollary of Lemma 11, we obtain the following lemma.

**Lemma 12** *Let $G$ be a 3-connected graph and let $S$ be a cut set of $G$ with $|S| = 3$. Let $C$ be a component of $G - S$. Then there exists a cut set $S_1 \subset S \cup V(C)$ of $G$ with $|S_1| = 3$ and a contractible edge between $S_1$ and the component of $G - S_1$ contained in $C$.*

## 2.2 Lemmas concerning a $k$-tree

In this subsection, we introduce two results on the existence of a degree bounded spanning tree. Let $\omega(G)$ be the number of components of a graph $G$.

**Proposition 13** *Let $G$ be a graph having a $k$-tree. Then for any $S \subset V(G)$, $\omega(G - S) \leq (k - 1)|S| + 1$.*

**Proof.** Let $T$ be a $k$-tree of $G$. Then, we can easily see that $\omega(G-S) \leq \omega(T-S) \leq (k-1)|S| + 1$. $\square$

We also use the following theorem, which gives a criterion for a graph to have a spanning tree such that the vertices in a specified independent set have bounded degree.

**Theorem 14 (Frank and Gyárfás [9])** *Let $G$ be a connected graph and let $X \subset V(G)$ be an independent set. Then $G$ has a spanning tree $T$ such that $d_T(x) \leq k$ for all $x \in X$ if and only if for any $S \subset X$, $\omega(G - S) \leq (k - 1)|S| + 1$.*

## 2.3 Lemmas concerning a $K_{3,t}$-minor

In this subsection, we show the following two results on a $K_{3,t}$-minor. They can play important roles when we will find a $K_{3,t}$-minor in the proofs of our main theorems. Let $G$ be a graph and let $t \geq 3$ be an integer. For three vertices $x_1, x_2, x_3 \in V(G)$, a *$K_{3,t}$-minor in $G$ with base $x_1, x_2$ and $x_3$* is a $K_{3,t}$-minor such that these three vertices are contained in the distinct sets which correspond to the smaller partite set of $K_{3,t}$.

**Lemma 15** *Let $G$ be a 3-connected graph, and let $x, z \in V(G)$. Let $t \geq 3$ be an integer and let $y_1, \ldots, y_t \in V(G) - \{x, z\}$ with $d_G(y_j) = 3$ and $x, z \in N_G(y_j)$ for all $1 \leq j \leq t$. Suppose that $G$ has a $K_{3,t}$-minor such that $\{x\}$, $\{z\}$ and $V(G) - \{x, z, y_1, \ldots, y_t\}$ correspond to the smaller partite set of $K_{3,t}$ and $y_1$, $y_2$, $\ldots$, $y_t$ correspond to the others. Then for any $x_1, x_2 \in V(G) - \{x\}$, $G$ has a $K_{3,l}$-minor with base $x, x_1$ and $x_2$, where $l = \lceil \frac{t}{2} \rceil$.*

**Proof.** Let $x_1, x_2 \in V(G) - \{x\}$. If $z \in \{x_1, x_2\}$, say $z = x_1$, then contracting $\big(V(G) - \{x, z, y_1, \ldots, y_t\}\big) \cup \{x_2\}$ to one vertex, we can find a $K_{3,t}$-minor or a $K_{3,t-1}$-minor with base $x, x_1$ and $x_2$. Since $t-1 \geq l$, it contains a desired $K_{3,l}$-minor. Hence we may assume that $z \neq x_1$ and $z \neq x_2$.

Since $G$ is 3-connected, there exist two internally disjoint paths $P_1$ and $P_2$ connecting $\{x_1, x_2\}$ and $z$ in $G - x$. We partition $V(G) - \{x, z\}$ into two sets $X_1$ and $X_2$ so that both $X_1$ and $X_2$ induces a connected graph in $G$ and $V(P_i) - \{z\} \subset X_i$ for $i = 1, 2$. Let $Y_i := \{y_j : 1 \leq j \leq t \text{ and } y_j \in X_i\}$ for $i = 1, 2$. Note that $Y_1 \cap Y_2 = \emptyset$. Thus, we obtain that $|Y_1| \geq \lceil \frac{t}{2} \rceil$ or $|Y_2| \geq \lceil \frac{t}{2} \rceil$, say $|Y_1| \geq \lceil \frac{t}{2} \rceil \geq 2$. Note that $X_1 - Y_1$ is nonempty and connected, because $d_G(y_j) = 3$ and $\{x, z\} \subset N_G(y_j)$ for all $1 \leq j \leq t$. Then when we contract $X_2 \cup \{z\}$ to one vertex, all vertices in $Y_1$ remains, and hence there exists a $K_{3,l}$-minor with base $x, x_1$ and $x_2$, where $l = \lceil \frac{t}{2} \rceil$. (Note that $\{x\}$, $X_1 - Y_1$ and $X_2 \cup \{z\}$ correspond to the smaller partite set of $K_{3,l}$, and all vertices of $Y_1$ correspond to the others.) $\square$

**Lemma 16** *Let $G$ be a graph and let $y \in V(G)$ with $d_G(y) = 3$. Let $\{x_1, x_2, x_3\} = N_G(y)$. Let $G' := G + \{x_1x_2, x_2x_3, x_3x_1\}$. If $G'$ has a $K_{3,t}$-minor for some $t \geq 4$, then $G$ also has.*

**Proof.** Suppose that $G'$ has a $K_{3,t}$-minor. Let $X_1, X_2, X_3$ be the sets of vertices of $G'$ whose contractions form the smaller partite set of $K_{3,t}$, and let $Y_1, Y_2, \ldots, Y_t$ be the others. We take such sets so that the union of them is as small as possible.

Suppose that $y \in \bigcup_{i=1}^{3} X_i \cup \bigcup_{j=1}^{t} Y_j$. If $y \in X_i$ for some $1 \leq i \leq 3$, then $|X_i| \geq 2$ (because $t \geq 4$ and $d_G(y) = 3$), however $X_i - \{y\}$ can play the same role as $X_i$ because $\{x_1, x_2, x_3\}$ is a clique in $G'$, which contradicts the minimality of $X_i$. Thus $y \in Y_j$ for some $1 \leq j \leq t$. If $|Y_j| \geq 2$, then we can remove $y$ from $Y_j$, again contradicting the choice of $Y_j$. These imply that $\{y\} = Y_j$ for some $1 \leq j \leq t$ if $y \in \bigcup_{i=1}^{3} X_i \cup \bigcup_{j=1}^{t} Y_j$.

Assume that $G$ has no $K_{3,t}$-minor. Then at least one edge in $\{x_1x_2, x_2x_3, x_3x_1\}$, say $x_1x_2$, is contained in $X_i$ or $Y_j$, or connects $X_i$ and $Y_j$, for some $1 \leq i \leq 3$ and $1 \leq j \leq t$. Suppose first that $x_1x_2$ is contained in $X_1$. In this case, $N_G(y) \cap X_2 = \emptyset$ or $N_G(y) \cap X_3 = \emptyset$, and hence $y \notin \bigcup_{i=1}^{3} X_i \cup \bigcup_{j-1}^{t} Y_j$. Thus, by adding $y$ into $X_1$, we also obtain a $K_{3,t}$-minor in $G$. When $x_1x_2$ is contained in $Y_j$ for some $1 \leq j \leq t$, we can find a $K_{3,t}$-minor by the same way.

Therefore we may assume that no two vertices in $x_1, x_2, x_3$ are contained in the same set. This implies that $x_1x_2$ connects $X_i$ and $Y_j$, say $X_1$ and $Y_1$, and $x_1 \in X_1$. Then by adding $y$ into $X_1$ if $x_3 \in \bigcup_{j=2}^{t} Y_j$; otherwise into $Y_1$, we can find a $K_{3,t}$-minor in $G$. $\square$

# 3 Best possibility

In this section, we show the best possibility of Theorem 2 and Corollary 3. We use the following proposition shown in [4].

**Proposition 17 ([4, Proposition 7])** *Let $t \geq 3$ be an odd integer. Then there exist infinitely many 3-connected graphs $G$ having no $K_{3,t}$-minor and containing a subset $S \subset V(G)$ with $\omega(G - S) \geq (t-1)(|S| - 2)$.*

In particular, by using the construction from [4], we can construct 3-connected graphs $G$ in Proposition 17 with one extra property that $|S|$ is sufficiently large with respect to $t$. This means that $S$ can be chosen so that $(t-1)(|S|-2) > (t-2)|S|+1$, and hence by Proposition 13, $G$ has no $(t-1)$-tree. Hence for every odd integer $t$

with $t \geq 3$, there exist infinitely many 3-connected graphs having no $K_{3,t}$-minor and no $(t-1)$-tree. Then Corollary 3 is best possible.

For an even integer $t$ with $t \geq 4$, letting $t' = t-1$, we have found infinitely many 3-connected graphs $G$ having no $K_{3,t'}$-minor and no $(t'-1)$-tree. Then such $G$ have no $K_{3,t}$-minor and no $(t-2)$-tree. Hence Theorem 2 is also best possible.

# 4 Minor minimal 3-connected graphs having no $k$-tree

In this section, we show Theorem 5. As mentioned before, Theorem 5 gives the properties of a minor minimal 3-connected graph having no $k$-tree.

**Proof of Theorem 5.**

Suppose that Theorem 5 does not hold, and let $G$ be a minimum counter example, that is, $|E(G)|$ is minimum among all 3-connected graphs having no $k$-tree and no desired bipartite minor. Since $K_4$ has a 2-tree, $G$ is not isomorphic to $K_4$. Note that $G$ is minimally 3-connected by the minimality of $|E(G)|$. We will show that $G$ itself satisfies the desired conditions of $R$ in Theorem 5. Again by the minimality of $|E(G)|$, any graph obtained from $G$ by contracting some edges which is 3-connected has a $k$-tree. (Note that the minor relation satisfies the transitive law.)

Let $B := V_3(G) = \{x \in V(G) : d_G(x) = 3\}$. By Lemma 7 (i), $B \neq \emptyset$. The following claim was essentially shown by Sanders and Zhao in [17], but for self-containedness, we will prove it.

**Claim 1 ([17, Lemma 3.2])** *$B \cup \{x \in V(G) : 4 \leq d_G(x) \leq k\}$ is independent.*

*Proof.* Assume that there exist two vertices $y_1, y_2 \in B \cup \{x \in V(G) : 4 \leq d_G(x) \leq k\}$ such that $y_1y_2 \in E(G)$.

Suppose first that $y_1$ is incident with a contractible edge, say $y_1u$ (possibly $u = y_2$). By the minimality of $G$, $G/y_1u$ has a $k$-tree $T'$. Let $T''$ be the subgraph of $G$ which has the same edge set as $T'$. When there exist more than one edges corresponding to one edge of $T'$, we choose one of them arbitrarily. Notice that $d_{T''}(y_1) \leq d_G(y_1) - 1 \leq k-1$. If $d_{T''}(u) \leq k-1$, then $T'' + uy_1$ is also a $k$-tree of $G$. Hence we may assume that $d_{T''}(u) = k$. In this case, $d_{T''}(y_1) = 0$, and hence $T' + y_1y_2$ is a $k$-tree of $G$.

Therefore we may assume that any edge incident with $y_1$ is not contractible. By symmetry, any edge incident with $y_2$ is not contractible. In particular, $y_1y_2$ is not contractible, and hence $y_1 \in V_3$ or $y_2 \in V_3$ by Lemma 7 (ii), which contradicts Lemma 6. □

Suppose that there exists an edge in $G - B$. Let $P := x_0x_1 \cdots x_l$ ($l \geq 1$) be a maximal path in $G - B$. Since $G - B$ is a forest by Lemma 7 (iv), $x_0$ and $x_l$ are leaves of $G-B$, and $x_ix_j \notin E(G)$ for any $0 \le i < j \le l$ with $j \ne i+1$. By Lemma 8, $G/P$ is 3-connected. Let $x$ be the vertex in $G/P$ obtained by contracting $P$. Then by the minimality of $G$, we can find a $k$-tree $T'$ of $G/P$. We consider the subgraph $T''$ of $G$ consisting of the same edges as $T'$. When there exist more than one edges corresponding to one edge of $T'$, we choose one of them arbitrarily. Note that $T''$ is a spanning forest of $G$ consisting of $l+1$ components each of which contains one of $x_0, x_1, \cdots, x_l$. If $d_{T''}(x_i) \le k-2$ for all $0 \le i \le l$, then $T'' \cup P$ is a $k$-tree of $G$. Therefore suppose that $d_{T''}(x_i) \ge k-1$ for some $0 \le i \le l$.

Suppose that there exists an index $j \ne i$ such that $d_{T''}(x_j) \ge 1$. Then since $\sum_{r=0}^{l} d_{T''}(x_r) = d_{T'}(x) \le k$, we have $d_{T''}(x_i) = k-1$, $d_{T''}(x_j) = 1$, and $d_{T''}(x_r) = 0$ for any $0 \le r \le l$ with $r \ne i, j$. By the symmetry, we may assume that $j > i$. Since both $x_0$ and $x_l$ has degree one in $G-B$, we can take vertices $u \in N_G(x_0) \cap B$ and $v \in N_G(x_l) \cap B$. (Possibly $u = v$.) We consider the graph $T$ obtained from $T''$ by adding the paths $x_iPx_j$, $ux_0Px_{i-1}$ (if $i \ne 0$), and $x_{j+1}Px_lv$ (if $j \ne l$). By the condition of $T''$, $T$ is a spanning tree of $G$ and $d_T(y) \le k$ for any $y \in V(G)$. When $d_{T''}(x_j) = 0$ for any $j \ne i$, we consider the graph $T$ obtained from $T''$ by adding the paths $ux_0Px_{i-1}$ (if $i \ne 0$), and $x_{i+1}Px_lv$ (if $i \ne l$). Again $T$ is a spanning tree of $G$ and $d_T(y) \le k$ for any $y \in V(G)$. In either cases, this contradicts that $G$ has no $k$-tree.

Thus, there exists no edge in $G-B$, and hence $A := V(G) - B$ is independent. Then Claim 1 and this imply that $G$ is a bipartite graph with bipartition $A$ and $B$ and $d_G(x) \ge k+1$ for all $x \in A$.

If $G$ has a spanning tree $T$ such that $d_T(x) \le k$ for all $x \in A$, then $T$ is a $k$-tree of $G$, because $d_T(y) \le d_G(y) = 3$ for all $y \in B$, a contradiction. Hence it follows from Theorem 14 that there exists $\widetilde{A} \subset A$ such that $\omega(G-\widetilde{A}) \ge (k-1)|\widetilde{A}| + 2$.

Suppose that $A - \widetilde{A} \ne \emptyset$ or $\omega(G-\widetilde{A}) \ge (k-1)|\widetilde{A}|+3$. In the former case, we choose $y \in B$ so that $y$ is not an isolated vertex in $G-\widetilde{A}$. In the latter case, we choose $y \in B$ arbitrarily. By Lemma 6, there exists a contractible edge incident with $y$, say $yx$ and let $G'$ be the graph obtained from $G$ by contracting $yx$. We name the new vertex of $G'$ as $v$. Let $A' := \big(A - \{x\}\big) \cup \{v\}$, and $B' := B - \{y\}$. Let $\widetilde{A}' := \big(\widetilde{A} - \{x\}\big) \cup \{v\}$ if $x \in \widetilde{A}$; otherwise let $\widetilde{A}' := \widetilde{A}$. By the definition of $G'$, note that $\omega(G' - \widetilde{A}') \ge \omega(G - \widetilde{A}) - 1$. In particular, in the former case, $\omega(G' - \widetilde{A}') \ge \omega(G - \widetilde{A})$ since $y$ is not an isolated vertex in $G - \widetilde{A}$. In either case, we obtain that $\omega(G' - \widetilde{A}') \ge (k-1)|\widetilde{A}'| + 2$. By Proposition 13, $G'$ does not have a $k$-tree, which contradicts the minimality of $G$. Thus, $\widetilde{A} = A$ and $\omega(G - \widetilde{A}) = |B| = (k-1)|A| + 2$. $\square$

# 5 Proof of Theorem 2

First we will show the following result. In the last part of this section, we shall prove Theorem 2 by using it.

**Theorem 18** *Let $t \geq 4$ be an even integer. Let $G$ be a 3-connected graph and let $B \subset V(G)$ be an independent set of $G$. Let $A := V(G) - B$. Suppose that $d_G(y) = 3$ and $N_G(y)$ is a clique for all $y \in B$. If $|B| = (t-2)(|A|-2)+2$, then $G$ has a $K_{3,t}$-minor.*

**Proof.** We will show this theorem by induction on $|A|$. When $|A| = 3$, we can easily find a $K_{3,t}$-minor, where three vertices of $A$ correspond to the smaller partite set of $K_{3,t}$ and $(t-2)(|A|-2)+2 = t$ vertices of $B$ correspond to the others. Hence we may assume that $|A| \geq 4$.

Assume that $G$ has no $K_{3,t}$-minor. Let $H := G - B$. By Lemma 9, $H$ is also 3-connected. The following two claims follow from the non-existence of a $K_{3,t}$-minor in $G$.

**Claim 2** *For any $x_1, x_2 \in A$, $|N_G(x_1) \cap N_G(x_2) \cap B| \leq t-1$.*

*Proof.* Suppose that there exist $x_1, x_2 \in A$ such that $|N_G(x_1) \cap N_G(x_2) \cap B| \geq t$. Since $G$ is 3-connected, $G' := G - \{x_1, x_2\}$ is connected. Note that $d_{G'}(y) = 1$ for any $y \in N_G(x_1) \cap N_G(x_2) \cap B$, and hence $G' - \big(N_G(x_1) \cap N_G(x_2) \cap B\big)$ is also connected. Then by contracting $G' - \big(N_G(x_1) \cap N_G(x_2) \cap B\big)$ to one vertex, we obtain a $K_{3,t}$-minor, where $x_1, x_2$ and the contracted vertices correspond to the smaller partite set of $K_{3,t}$, and $t$ vertices in $N_G(x_1) \cap N_G(x_2) \cap B$ correspond to the others. $\square$

**Claim 3** *For any pair of distinct triples $x_1^i, x_2^i, x_3^i \in A$ $(i = 1, 2)$, we obtain that $\sum_{i=1}^{2} \left| \bigcap_{j=1}^{3} N_G(x_j^i) \cap B \right| \leq t-1$.*

*Proof.* Suppose that there exists a pair of distinct triples $x_1^i, x_2^i, x_3^i \in A$ $(i = 1, 2)$ such that $\sum_{i=1}^{2} \left| \bigcap_{j=1}^{3} N_G(x_j^i) \cap B \right| \geq t$. Since $H$ is 3-connected, there exist three pairwise disjoint paths $P_1, P_2, P_3$ connecting $\{x_1^1, x_2^1, x_3^1\}$ and $\{x_1^2, x_2^2, x_3^2\}$. Then by contracting each path $P_1$, $P_2$ and $P_3$ to one vertex, respectively, we obtain a $K_{3,t}$-minor, where three contracted vertices correspond to the smaller partite set of $K_{3,t}$ and $t$ vertices in $\bigcup_{i=1}^{2} \big( \bigcap_{j=1}^{3} N_G(x_j^i) \cap B \big)$ correspond to the others. $\square$

Suppose first that $H$ is isomorphic to $K_4$ and let $\{x_1, x_2, x_3, x_4\} = V(H) = A$. By Claim 3, $|B_i| + |B_j| \leq t-1$ for any $1 \leq i < j \leq 4$, where $B_i := \bigcap_{j \neq i} N_G(x_j) \cap B$ for $1 \leq i \leq 4$. This implies that the cardinality of at least three of $B_i$'s is at most

$\lfloor \frac{t-1}{2} \rfloor = \frac{t-2}{2}$, say $|B_i| \leq \frac{t-2}{2}$ for $1 \leq i \leq 3$. Then

$$\begin{aligned} |B| &= |B_1| + |B_2| + \big(|B_3| + |B_4|\big) \\ &\leq 2 \cdot \frac{t-2}{2} + (t-1) \\ &= 2t-3, \end{aligned}$$

however this contradicts the condition "$|B| = (t-2)(|A|-2) + 2 = 2t-2$". Thus, $H$ is not isomorphic to $K_4$, so $|A| \geq 5$.

**Claim 4** *For any $x_1x_2 \in E(H)$ with $|N_G(x_1) \cap N_G(x_2) \cap B| \leq t-2$, $x_1x_2$ is not a contractible edge in $H$.*

*Proof.* Suppose that $x_1x_2$ is a contractible edge in $H$. Let $\widetilde{H}$ be the graph obtained from $H$ by contracting $x_1x_2$, $\widetilde{A} := V(\widetilde{H})$, and $\widetilde{B} := B - \big(N_G(x_1) \cap N_G(x_2) \cap B\big)$. Note that $|\widetilde{A}| = |A| - 1$, and

$$\begin{aligned} |\widetilde{B}| &= |B| - |N_G(x_1) \cap N_G(x_2) \cap B| \\ &\geq (t-2)(|A|-2) + 2 - (t-2) \\ &= (t-2)(|\widetilde{A}|-2) + 2. \end{aligned}$$

Let $\widetilde{G}$ be the graph on $\widetilde{A} \cup \widetilde{B}$ with $E(\widetilde{G}) := E(\widetilde{H}) \cup \{xy : x \in \widetilde{A}, y \in \widetilde{B} \text{ and } xy \in E(G)\}$. By the definition, $\widetilde{G} - \widetilde{B} = \widetilde{H}$, and hence $\widetilde{G}$ is also 3-connected, by Lemma 9. If $|\widetilde{B}| > (t-2)(|\widetilde{A}|-2) + 2$, then we can remove appropriate number of vertices of $\widetilde{B}$ so that the equality holds. (Note that this operation does not effect the other conditions of Theorem 18 by Lemma 9.) Then by the induction hypothesis, $\widetilde{G}$ has a $K_{3,t}$-minor, and hence $G$ also has, a contradiction. □

**Claim 5** *$H$ is 4-connected.*

*Proof.* Suppose that there exists a cut set $S$ of $H$ with $|S| = 3$. Let $H_i$ for $i = 1, 2$ be two components of $H - S$. By Lemma 12, for each $i = 1, 2$, there exists a cut set $S_i \subset S \cup V(H_i)$ of $H$ with $|S_i| = 3$ and a contractible edge $u_iv_i$ with $u_i \in S_i$ and $v_i \in V(C_i)$, where $C_i$ is a component of $H - S_i$ contained in $H_i$. Let $H_i'$ be the graph induced by $S_i \cup C_i$ in $H$ with additional edges which make $S_i$ a clique. By Lemma 10, $H_i'$ is 3-connected. By Claims 2 and 4, $|N_G(u_i) \cap N_G(v_i) \cap B| = t-1$.

Let $A_i' := V(H_i')$ and $B_i' := \{y \in B : N_G(y) \subset A_i'\}$ for $i = 1, 2$. Let $G_i'$ be the graph on $A_i' \cup B_i'$ with $E(G_i') := E(H_i') \cup \{xy : x \in A_i', y \in B_i' \text{ and } xy \in E(G)\}$. Note that $G_i' - B_i' = H_i'$ and hence $G_i'$ is 3-connected by Lemma 9. Thus, $G_i' - \{u_i, v_i\}$ is connected, therefore $G_i' - \{u_i, v_i\} - \big(N_G(u_i) \cap N_G(v_i) \cap B\big)$ is also connected, because for any $y \in N_G(u_i) \cap N_G(v_i) \cap B$, the degree of $y$ in $G_i' - \{u_i, v_i\}$ is one. Then by contracting $V(G_i') - \{u_i, v_i\} - \big(N_G(u_i) \cap N_G(v_i) \cap B\big)$ to one vertex, we find a $K_{3,t-1}$-minor such that $u_i, v_i$ and the contracted vertex correspond to the smaller partite set of $K_{3,t-1}$ and the vertices of $N_G(u_i) \cap N_G(v_i) \cap B$ correspond to the others. Then by Lemma 15 for $x = u_i$, $\{x_1, x_2\} = S_i - \{u_i\}$, $G'_i$ has a $K_{3,l}$-minor with base vertices in $S_i$, where $l = \lceil \frac{t-1}{2} \rceil = \frac{t}{2}$. Since $S$ separates $G'_1 - S_1$ and $G'_2 - S_2$ in $G$, the sets of vertices in $G'_1$ and $G'_2$ corresponding to the larger partite set of $K_{3,l}$ do not intersect.

Since $H$ is 3-connected, there exist three disjoint paths $P_1, P_2, P_3$ in $H$ connecting $S_1$ and $S_2$. Since $S_i$ is a cut set of $H$ separating $A'_i - S_i$ and $H'_{3-i}$, we have $V(P_j) \cap (A'_i - S_i) = \emptyset$ for $i = 1, 2$ and for $j = 1, 2, 3$. Note that $P_j$ is also a path of $G$. Thus, by contracting each $P_j$ to one vertex, we can combine two $K_{3,l}$-minors of $G'_1$ and $G'_2$ to obtain a $K_{3,t}$-minor in $G$.

Therefore, $H$ has no cut set of order three, and hence $H$ is 4-connected. (Note that $H$ is not isomorphic to $K_4$.) This completes the proof of Claim 5. $\square$

By Claims 2, 4 and 5, we obtain $|N_G(x_1) \cap N_G(x_2) \cap B| = t-1$ for any $x_1x_2 \in E(G)$ with $x_1, x_2 \in A$. Let $x \in A$. Then $e_G(N_H(x), N_G(x) \cap B) = (t-1)|N_H(x)|$ and $e_G(N_G(x) \cap B, N_H(x)) = 2|N_G(x) \cap B|$ because $d_G(y) = 3$ for any $y \in N_G(x) \cap B$, and hence $(t-1)|N_H(x)| = 2|N_G(x) \cap B|$. This equality and the fact "$t$ is even" imply that $|N_G(x) \cap B|$ is a multiple of $t-1$. Also by Claim 5, we have $|N_H(x)| \geq 4$, and hence $|N_G(x) \cap B| \geq 2(t-1)$.

Let $A_1 := \{x \in A : |N_G(x) \cap B| = 2(t-1)\}$. Then for each $x \in A_1$, $|N_H(x)| = 4$. Suppose that $|A_1| \leq 5$. Then

$$\begin{aligned} e_G(A, B) &\geq 2(t-1)|A_1| + 3(t-1)(|A| - |A_1|) \\ &= 3(t-1)|A| - (t-1)|A_1| \\ &\geq 3(t-1)|A| - 5(t-1). \end{aligned}$$

On the other hand,

$$\begin{aligned} e_G(A, B) &= 3|B| \\ &= 3(t-2)(|A| - 2) + 6 \\ &= 3(t-1)|A| - 6(t-1) - 3|A| + 12 \\ &< 3(t-1)|A| - 5(t-1) \end{aligned}$$

since $|A| \geq 5$ and $t \geq 4$, a contradiction. Hence $|A_1| \geq 6$.

Let $x^1 \in A_1$ and let $X^1 := N_G(x^1) \cap A$. Since $|X^1 \cup \{x\}| = 5$, we can choose a vertex $x^2 \in A_1 - (X^1 \cup \{x^1\})$. Let $X^2 := N_G(x^2) \cap A$. Note that $X^i$ is a cut set of $H$ separating $x^i$ and $V(H) - \big(\{x^i\} \cup X^i\big)$ for $i = 1, 2$.

Since $H$ is 4-connected, there exist four pairwise disjoint paths $P_1, P_2, P_3, P_4$ connecting $X^1$ and $X^2$. (Possibly some of them might consist of only one vertex.) Note that $x^i \notin V(P_j)$ for any $i = 1, 2$ and any $1 \leq j \leq 4$. Let $z^i_j$ be an end vertex of $P_j$ in $X^i$ for $i = 1, 2$ and $1 \leq j \leq 4$. Let $Y^i_{jk} := N_G(z^i_j) \cap N_G(z^i_k) \cap N_G(x^i) \cap B$ and let $Y_{jk} := Y^1_{jk} \cup Y^2_{jk}$ for $i = 1, 2$ and $1 \leq j < k \leq 4$. By Claim 3, we obtain that $|Y_{jk}| \leq t-1$ for any $1 \leq j < k \leq 4$.

Suppose that there exists $Y_{jk}$, say $Y_{12}$, such that $Y_{12} = \emptyset$. Since $|Y_{12} \cup Y_{13} \cup Y_{14}| = \sum_{i=1}^{2} |N_G(x^i) \cap N_G(z_1^i) \cap B| = 2(t-1)$ and $|Y_{jk}| \leq t-1$, we obtain $|Y_{13}| = |Y_{14}| = t-1$. Then $|Y_{13}^1| \geq \frac{t}{2}$ or $|Y_{13}^2| \geq \frac{t}{2}$, because $t$ is an even integer. By symmetry, $|Y_{14}^1| \geq \frac{t}{2}$ or $|Y_{14}^2| \geq \frac{t}{2}$. In either case, $|Y_{13}^i| + |Y_{14}^{i'}| \geq t$ for some $i, i' = 1, 2$, which contradicts Claim 3. Therefore $Y_{jk} \neq \emptyset$ for any $1 \leq j < k \leq 4$.

Note that $\sum_{1 \leq j < k \leq 3} |Y_{jk}| = \sum_{i=1}^{2} \big(|N_G(x^i) \cap B| - |N_G(x^i) \cap N_G(z_4^i) \cap B|\big) = 2\big(2(t-1) - (t-1)\big) = 2(t-1)$. We may assume that $|Y_{12}| \geq |Y_{13}| \geq |Y_{23}|$. Then we have $|Y_{12} \cup Y_{13}| \geq \frac{4(t-1)}{3} \geq t$. Since $Y_{23} \neq \emptyset$, $P_2 \cup P_3 \cup Y_{23}$ is connected. Then by contracting $P_1$, $P_2 \cup P_3 \cup Y_{23}$ and $x^1 z_4^1 \cup P_4 \cup z_4^2 x^2$ to one vertex, respectively, we obtain a $K_{3,t}$-minor, where three contracted vertices correspond to the smaller partite set of $K_{3,t}$ and $t$ vertices in $Y_{12} \cup Y_{13}$ correspond to the others. $\Box$

**Proof of Theorem 2.**

Let $t \geq 4$ be an even integer and suppose that $G$ has no $(t-1)$-tree. Then by Theorem 5 for $k = t-1$, $G$ contains a 3-connected bipartite minor $R$ with bipartition $A$ and $B$ such that $d_R(y) = 3$ for all $y \in B$ and $|B| = (t-2)|A| + 2$. Let $\widetilde{R}$ be the graph obtained from $R$ by adding edges $x_1x_2, x_2x_3, x_1x_3$ for all $y \in B$, where $\{x_1, x_2, x_3\} = N_R(y)$, and by removing $2(t-2)$ vertices of $B$. Note that $\widetilde{R}$ is also 3-connected by Lemma 9. Let $\widetilde{B}$ be the set of remaining vertices of $B$ in $\widetilde{R}$. Then $|\widetilde{B}| = |B| - 2(t-2) = (t-2)|A| + 2 - 2(t-2) = (t-2)(|A| - 2) + 2$. Thus, it follows from Theorem 18 that $\widetilde{R}$ has a $K_{3,t}$-minor. By Lemma 16, $R$ also has a $K_{3,t}$-minor, and hence $G$ has, a contradiction. $\Box$

# 6 Proof of Theorem 4

In order to prove Theorem 4, we will use the following lemma. Since it directly follows from Euler's formula, we omit the proof.

**Lemma 19** *Let $G$ be a bipartite graph of order at least 3 on a surface of Euler characteristic $\chi$. Then $|E(G)| \leq 2|V(G)| - 2\chi$.*

**Proof of Theorem 4.**

Let $k := \lceil \frac{8-2\chi}{3} \rceil$ and suppose that $G$ is a 3-connected graph on a surface of Euler characteristic $\chi \leq 0$ having no $k$-tree. Since $\chi \leq 0$, we have $k \geq 3$. By Theorem 5, $G$ contains a 3-connected bipartite minor $R$ with bipartition $A$ and $B$ such that $d_R(y) = 3$ for all $y \in B$ and $|B| \geq (k-1)|A| + 2$. Note that $|A| \geq 3$. Then it follows from Lemma 19 and the fact $d_R(y) = 3$ for all $y \in B$ that

$$3|B| = e_R(A, B) \leq 2|V(R)| - 2\chi = 2|A| + 2|B| - 2\chi,$$

and hence

$$2|A| - 2\chi \geq |B| \geq (k-1)|A| + 2.$$

This implies that

$$\begin{aligned} -2\chi - 2 &\geq (k-3)|A| \\ &\geq \left(\left\lceil \frac{8-2\chi}{3} \right\rceil - 3\right) \cdot 3 \\ &\geq -2\chi - 1, \end{aligned}$$

a contradiction. □

## Acknowledgements

The authors would like to thank anonymous referees for carefully reading the previous manuscript and helpful suggestions. In particular, the authors are grateful to one of the referees for pointing out some mistakes of it.

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