Book embedding of toroidal bipartite graphs

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Abstract

Endo [5] proved that every toroidal graph has a book embedding with at most seven pages. In this paper, we prove that every toroidal bipartite graph has a book embedding with at most five pages. In order to do so, we prove that every bipartite torus quadrangulation Q with n vertices admits two disjoint essential simple closed curves cutting the torus into two annuli so that each of the two annuli contains a spanning connected subgraph of Q with exactly n edges.

Keywords: torus, bipartite graph, book embedding, quadrangulation

1 Introduction

A *book embedding* of a graph G is to put the vertices along the *spine* (a segment) and each edge of G on a single *page* (a half-plane with the spine as its boundary) so that no two edges intersect transversely in the same page. We say that a graph G is *k-page embeddable* if G has a book embedding with at most k pages. The *pagenumber* (or sometimes called *stack number* or *book thickness*) of a graph G is the minimum of k such that G is k-page embeddable. This notion was first introduced by Bernhart and Kainen [1]. Since a book embedding is much concerned with theoretical computer science including VLSI design [3, 15], we are interested in bounding the pagenumber. Actually, we can find a number of researches giving upper bounds of the pagenumber for some graph classes, for example, complete bipartite graphs [6, 13] and k-trees [4, 8, 17]. Several algorithms to find an embedding of a given graph into a book with a few pages were also presented [11, 16].

On the other hand, the pagenumber has widely been studied from the aspect of graphs on surfaces. In fact, a graph G is 1-page embeddable if and only if G is outerplanar, and a graph G is 2-page embeddable if and only if G is a subgraph of a hamiltonian planar graph [1]. For a graph of genus g, Heath and Istrail [10] proved that its pagenumber is $O(g)$, and later, Melitz [12] improved this result to $O(\sqrt{g})$. Note that there exists a graph of genus g with the pagenumber $\Theta(\sqrt{g})$, see [10].

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Now we focus on graphs on a fixed surface. Bernhart and Kainen [1] first conjectured that the pagenumber of planar graphs could be large enough. However, this conjecture was disproved by Buss and Shor [2], who proved that every planar graph is 9-page embeddable. Later this upper bound was improved to seven by Heath [9], and finally, Yannakakis [18] obtained the sharp upper bound: every planar graph has the pagenumber at most four. He also gave a planar graph which cannot be embedded into the book with 3-pages [18].

As next to the spherical case, we are interested in *toroidal* graphs, i.e., graphs embeddable in the torus. Although the algorithm given in [10] only guaranteed 13-page embedding of a toroidal graph, Heath and Istrail conjectured that any toroidal graph has the pagenumber at most seven. Endo gave the positive answer to this conjecture [5].

In this paper, focusing bipartite graphs on surfaces, we consider their pagenumbers. Since bipartite graphs on surfaces can have fewer edges than general graphs, we can expect that a less pagenumber suffices. Actually, every bipartite planar graph is known to be 2-page embeddable [7], although there exists a general planar graph whose pagenumber is four. Extending this, we shall prove the following.

Theorem 1 *Every toroidal bipartite graph has a book embedding with at most five pages.*

A *surface* means a connected compact 2-dimensional manifold possibly with boundary. A simple closed curve l on a surface F^2 is said to be *essential* if l does not bound a 2-cell on F^2, and otherwise, it is *trivial*. Let G be a *map* on a surface, that is, a fixed embedding of a graph on the surface. Let $V(G)$ and $E(G)$ denote the sets of the vertices and the edges of G, respectively. We say that a cycle C of a graph G on a surface is *essential* if C can be regarded as an essential simple closed curve topologically.

An *even embedding* on a surface is a map on the surface with each face bounded by a cycle of even length. In particular, a *quadrangulation* is an even embedding with each face quadrilateral. It is easy to see that every even embedding on the sphere is bipartite, but this does not hold on any other non-spherical surface.

In order to prove Theorem 1, we shall prove that every bipartite torus quadrangulation Q with n vertices admits two disjoint essential simple closed curves γ and γ' cutting the torus into two annuli so that every vertex of Q is visited by either of γ and γ', and each of the two annuli contains a connected spanning subgraph with exactly n vertices. The proof is put in Section 2. Using the decomposition of the edges of Q, we can easily construct a 5-page embedding of Q. The algorithm to embed Q into a book is put in Section 3.

2 Edge-disjoint outer-annulus subgraphs

In this section, we will use various facts on torus quadrangulations Q, which were progressed in [14] in order to find a Hamiltonian cycle in the *radial graph* of Q, where the *radial graph* $R(G)$ of a map G is obtained from G by putting a new vertex into each face of G and joining it to all vertices lying on the corresponding boundary cycle, and deleting all edges in G.

An *orientation* of a graph G is an assignment of a direction to each edge of G. Let $\vec{G}$ denote the graph with an orientation and distinguish it from the undirected graph G. For a vertex v of $\vec{G}$, the *outdegree* of v is the number of directed edges outgoing from v

and denoted by $\mathrm{od}(v)$, and the *indegree* of v is that of incoming edges to v and denoted by $\mathrm{id}(v)$. We say that $\vec{G}$ is a *k-orientation* or *k-oriented* if each vertex of $\vec{G}$ has outdegree exactly k. For a directed edge $e = xy$ from x to y, the vertex x is called the *origin* of e and y the *terminus*.

The following was proved in [14].

Proposition 2 (Nakamoto and Ozeki [14]) *Every torus quadrangulation admits a 2-orientation.* ■

Moreover, by the 2-orientation of a bipartite quadrangulation Q, the following was proved in [14]. Let G be a map on a surface F^2. A *vertex-face curve* for G is a simple closed curve on F^2 passing through each vertex of G exactly once and each face of G exactly once, but crossing no edges. Clearly, a vertex-face curve for a quadrangulation Q corresponds to a Hamiltonian cycle of the radial graph of Q.

Theorem 3 (Nakamoto and Ozeki [14]) *Every bipartite quadrangulation admits an essential vertex-face curve.* ■

Let Q be a bipartite torus quadrangulation, and then by Theorem 3, Q has an essential vertex-face curve l. Cut Q along l to obtain a map on the annulus, denoted Q', each of whose vertices lies on the both boundary components. Such a map is called an *outer-annulus* map. Note that each vertex of Q is doubled in Q' to be a pair of vertices lying on the two distinct boundary components, but no edge of Q is doubled in Q' since l intersects only the endpoints of each edge in Q. An edge e of Q' is said to be *essential* if the two endpoints of e lie on distinct boundary components, and e is *trivial* otherwise.

Considering a 2-orientation $\vec{Q}$ of Q and an essential vertex-face curve l for Q, we can naturally obtain an outer-annulus map $\vec{A}$ with the orientation. Suppose that $\vec{A}$ has a vertex v of outdegree exactly 1. Let e_1 be an incoming edge to v and let e_2 be the outgoing edge from v. Since v lies on the boundary of the annulus, we can define the *right-turn* and the *left-turn* of the directed path $e_1 \cup e_2$ at a middle vertex v. (See Figure 1.) We say that v is *right-turned* (resp., *left-turned*) if the directed path $e_1 \cup e_2$ turns right (resp., left) at v for all edges e_1 incoming to v.

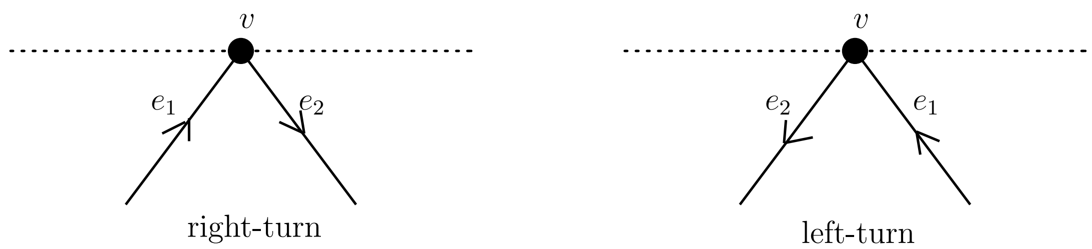

Figure 1: Right-turn and left-turn

Combining Lemmas 10 and 11 in [14], we can get the following. Since we always consider a bipartite graph Q, we let $V(Q) = B \cup W$ be the bipartition, where B and W are referred as *black* and *white* vertices, respectively.

Lemma 4 (Nakamoto and Ozeki [14]) *Let Q be a bipartite torus quadrangulation. Then Q has a 2-orientation $\vec{Q}$ and an essential vertex-face curve l satisfying the following: Cutting $\vec{Q}$ along l, we obtain a connected outer-annulus map $\vec{A}$ such that*

(4-1) each vertex of $\vec{A}$ has outdegree exactly 1,

(4-2) $\vec{A}$ has exactly one essential directed cycle $\vec{C}$ each of whose edges is essential,

(4-3) each black (resp., white) vertex of $\vec{A}$ with incoming edges is right-turned (resp., left-turned).

Let $\vec{C} = b_1w_1 \dots b_kw_k$ be the directed cycle in (4-2), called a *zigzag cycle*, where $k \geq 2$, and each b_i is a black vertex and each w_i is white. See Figure 2 (1), where the rectangle shows an annulus by identifying the top and the bottom. Then, since each edge of $\vec{C}$ is essential, the edges of $\vec{C}$ cut the annulus into $2k$ regions, called *V-regions*. So a V-region is bounded by two edges $b_iw_i, b_{i+1}w_i$ or b_iw_{i-1}, b_iw_i for some i (the indices are taken modulo k). In the former, w_i is called the *top* of the V-region, and a segment between b_i and b_{i+1} is called the *bottom*. For two vertices a, b on the bottom, let $[a, b]$ be the *interval* of the bottom between a and b. Let $(a, b) = [a, b] - \{a, b\}$, and we define $[a, b)$ and $(a, b]$ by the same manner. See Figure 2 (2).

The following is a main result in this section, which modifies Lemma 4 to decompose a given bipartite torus quadrangulation Q into two spanning outer-annulus maps.

Lemma 5 *Let Q be a bipartite torus quadrangulation. Then Q admits a 2-orientation $\vec{Q}$ satisfying the following: $\vec{Q}$ can be separated into two connected spanning outer-annulus maps $\vec{Q}_1$ and $\vec{Q}_2$ by cutting $\vec{Q}$ along two disjoint essential simple closed curves γ_1 and γ_2 so that for $i = 1, 2$,*

(5-1) each vertex of $\vec{Q}_i$ has outdegree exactly 1,

(5-2) $\vec{Q}_i$ has exactly one essential directed cycle $\vec{C}_i$ each of whose edges is essential,

(5-3) each black (resp., white) vertex of $\vec{Q}_i$ with incoming edges is right-turned (resp., left-turned).

Proof. Let $\vec{Q}$ be a 2-orientation satisfying Lemma 4. Modifying it, we shall construct another 2-orientation of Q satisfying the lemma.

Let γ be an essential vertex-face curve for $\vec{Q}$ cutting open $\vec{Q}$ into an outer-annulus map $\vec{A}$. By Lemma 4, let $\vec{C} = b_1w_1 \dots b_kw_k$ be the zigzag cycle of $\vec{A}$ for some integer $k \geq 2$, where each b_i is a black vertex and each w_i is a white vertex. Note that each vertex of $\vec{Q}$ is double in $\vec{A}$. Then all b_i's lie on the same boundary component of the annulus, denoted γ_b, and all w_i's lie on the other boundary component, denoted γ_w. Note that $b_1, w_1, b_2, w_2, \dots, b_k, w_k$ correspond to distinct vertices of $\vec{Q}$ lying on γ, but they are not necessarily consecutive on γ. We give a direction to γ along $\vec{C}$, and let $\vec{\gamma}$ denote γ with the direction.

We shall prove that $\vec{Q}$ has a directed essential cycle $\vec{D}$ such that D is homotopic to γ and consists only of trivial edges of A, where D is the undirected cycle corresponding

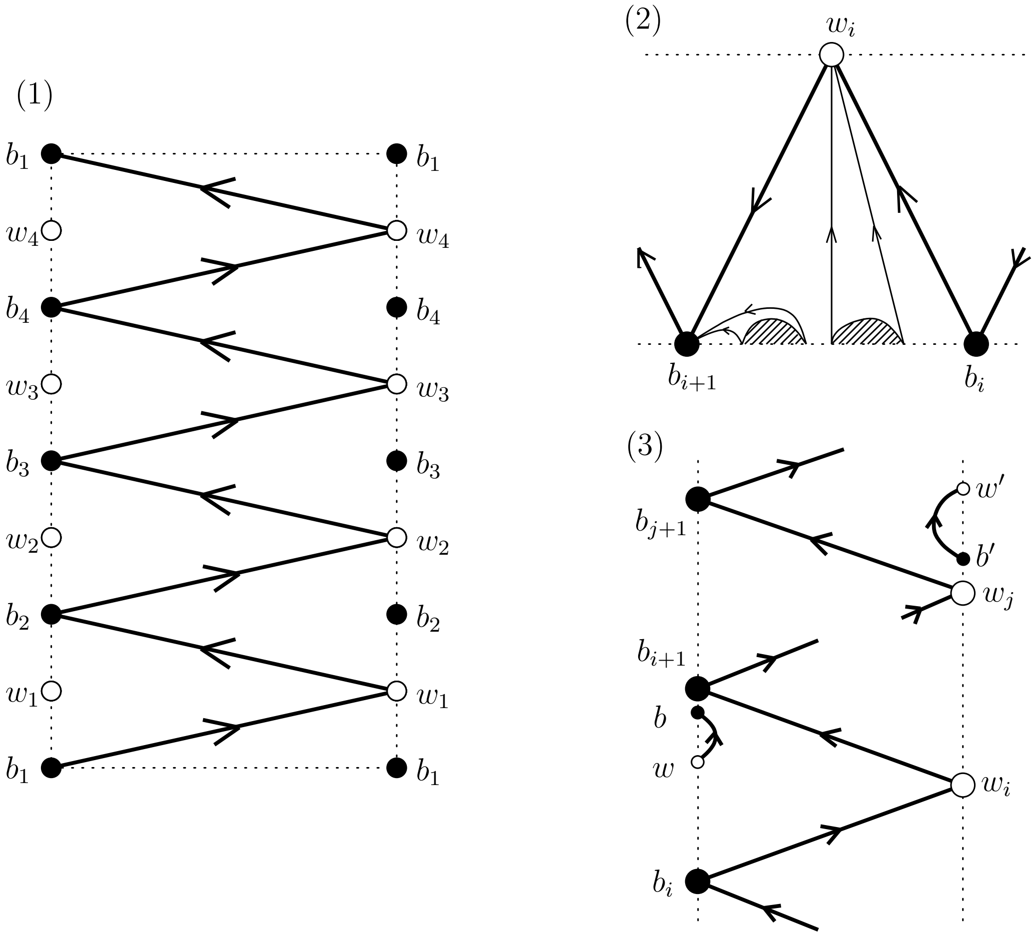

Figure 2: Zigzag cycle $\vec{C}$, V-region and directed cycle $\vec{D}$

to $\vec{D}$. Observe that γ_b contains at least one white vertex, say w, since γ contains white vertices in Q. Let Δ_i be the V-region of $\vec{A}$ bounded by two edges b_iw_i, w_ib_{i+1}, and we may suppose that w is contained in the interval (b_i, b_{i+1}). By (4-1), w has an outgoing edge e_w in $\vec{A}$, and let b be the terminus of e_w. Observe that b must be contained in $[b_i, b_{i+1}]$. However we have $b \neq b_i$, for otherwise (i.e., $b = b_i$), then b_i were not right-turned, a contradiction. Moreover, b must be contained in $(w, b_{i+1}]$. (For otherwise, i.e., if b is contained in (b_i, w), then the right-turn at b proceeds to the interior of the 2-cell region of $\vec{A}$ bounded by the trivial edge wb, and the left-turn at the next vertex, say $\tilde{w}$, proceeds to the interior of the smaller region bounded the trivial edge $b\tilde{w}$. Since every vertex in $\vec{A}$ has outdegree 1, we can choose an outgoing edge at every vertex. However, this argument does not continue since $\vec{A}$ is finite, a contradiction.) Hence we can find a trivial edge $\vec{wb}$ such that b_i, w, b and b_{i+1} appear in $\vec{\gamma}$ in this order. See Figure 2(3).

Let's consider the black vertex b' on γ_w such that $b' = b$ in Q, which may be supposed to be contained in the bottom (w_j, w_{j+1}) of a V-region with top b_{j+1} for some j. By (4-1), b' has an outgoing edge $e_{b'}$ in $\vec{A}$, and let w' is the terminus of $e_{b'}$. By completely the same argument, we can conclude that $b'w'$ is a trivial edge as shown in Figure 2(3). In this way, we can successively take trivial directed edges, and so these sequence of edges corresponds to a directed walk in $\vec{Q}$ starting at the vertex w. Since $\vec{Q}$ is finite, we can take a repeated vertex, say x, which appears in the walk at first time. Then the directed cycle starting and ending at x is a required one, and we let $\vec{D} = p_1 q_1 \cdots p_k q_k$ be the directed cycle with $x = p_1$, where for each i, p_i and q_i are black and white vertices.

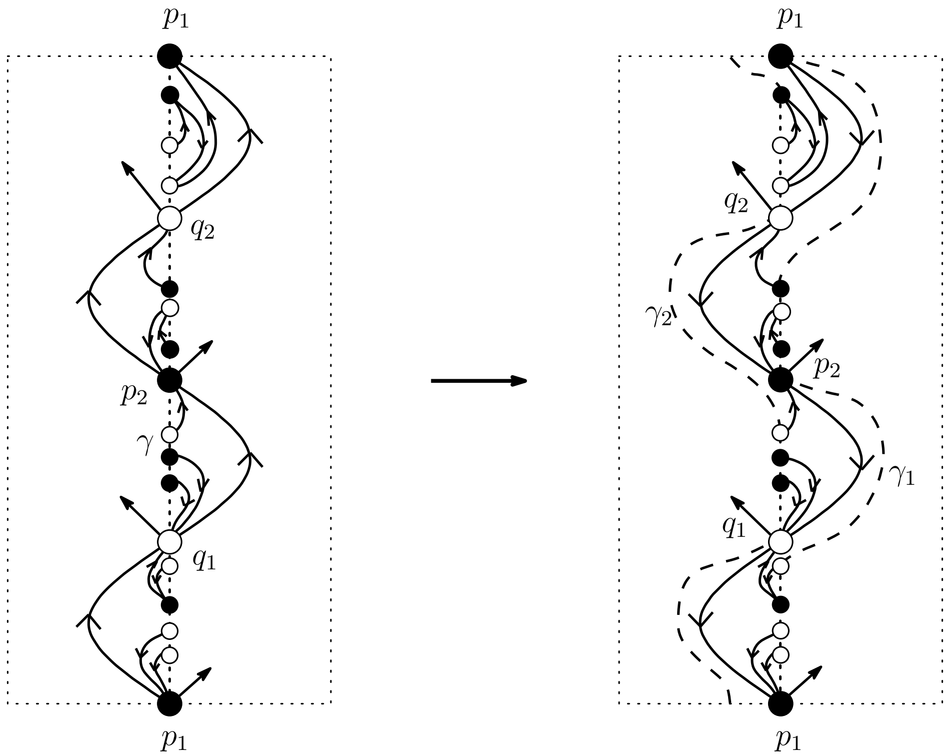

Figure 3: Reversal of the direction of $\vec{C}$ splits γ into γ_1 and γ_2

Now we reverse the direction of $\vec{D}$ in $\vec{Q}$ to obtain another 2-orientation of Q denoted $\overleftarrow{Q}$, where $\overleftarrow{D} = q_k p_k \cdots q_1 p_1$ is the inverse orientation of $\vec{D}$. Modify γ in the resulting 2-orientation $\overleftarrow{Q}$ so that (5-1) and (5-3) are satisfied. Then γ splits into two essential simple closed curves, say γ_1 and γ_2, so that γ_1 passes through all black vertices $p_1, \ldots, p_k$, and γ_2 through all white vertices $q_1, \ldots, q_k$, as shown in Figure 3. Since γ_1 and γ_2 are disjoint on the torus and every vertex of Q lies on either γ_1 or γ_2, each of the annuli bounded by γ_1 and γ_2 has a spanning subgraph of $\overleftarrow{Q}$. If we let A be the annulus containing $\vec{C}$, then $\vec{C}$ is still a zigzag cycle in the map in A. Moreover, $\overleftarrow{D} = p_1 q_1 \cdots p_k q_k$ is a zigzag cycle of the map in the other annulus. Then both of the maps on A and A' satisfies (5-2), since an annulus contains at most one zigzag cycle. ■

Lemma 6 *Let Q_j be the map in Lemma 5 for $j = 1, 2$, and let Δ be the graph contained in a V-region of $\vec{Q}_j$ bounded by two directed edges $b_i w_i, w_i b_{i+1}$. Then Δ is a directed tree such that*

(i) b_i has no incoming edge, and

(ii) there is a directed path to b_{i+1} from any vertex.

We say that the subtree of Δ toward w_i is *essential*, and that toward b_{i+1} but not w_i is *trivial*. See Figure 2(2).

Proof. By (5-3), the statement (i) clearly holds. Hence, by (5-1), from any vertex v other than b_i, we can extend a directed path toward w_i or b_{i+1}. Therefore, if (ii) does not hold, then we may suppose that Δ has a cycle in Q_1. Then, since Q_1 is connected and has a cycle $\vec{C}_1$, Q_1 has at least two cycles, and hence $|E(Q_1)| \geq |V(Q_1)| + 1$. On the other hand, Q_2 is a connected graph with at least one cycle, and hence $|E(Q_2)| \geq |V(Q_2)|$. Since $|V(Q_1)| = |V(Q_2)| = |V(Q)|$ and each edge of Q is contained in either Q_1 or Q_2, we have $|E(Q)| \geq 2|V(Q)| + 1$, contrary to Euler's formula for torus quadrangulations. ■

3 Constructing book embeddings

In this section, we shall construct a book embedding of a toroidal bipartite graph Q on the torus with at most five pages. Observe that every bipartite graph G on a surface can be extended to a quadrangulation Q on the surface by adding vertices and edges suitably. Moreover, if Q has a 5-page embedding, then we can get a 5-page embedding of G, which is obtained from the embedding of Q by removing the vertices and edges added. Hence we shall prove the following, which immediately gives a proof of Theorem 1.

Theorem 7 *Every bipartite torus quadrangulation is* 5*-page embeddable.*

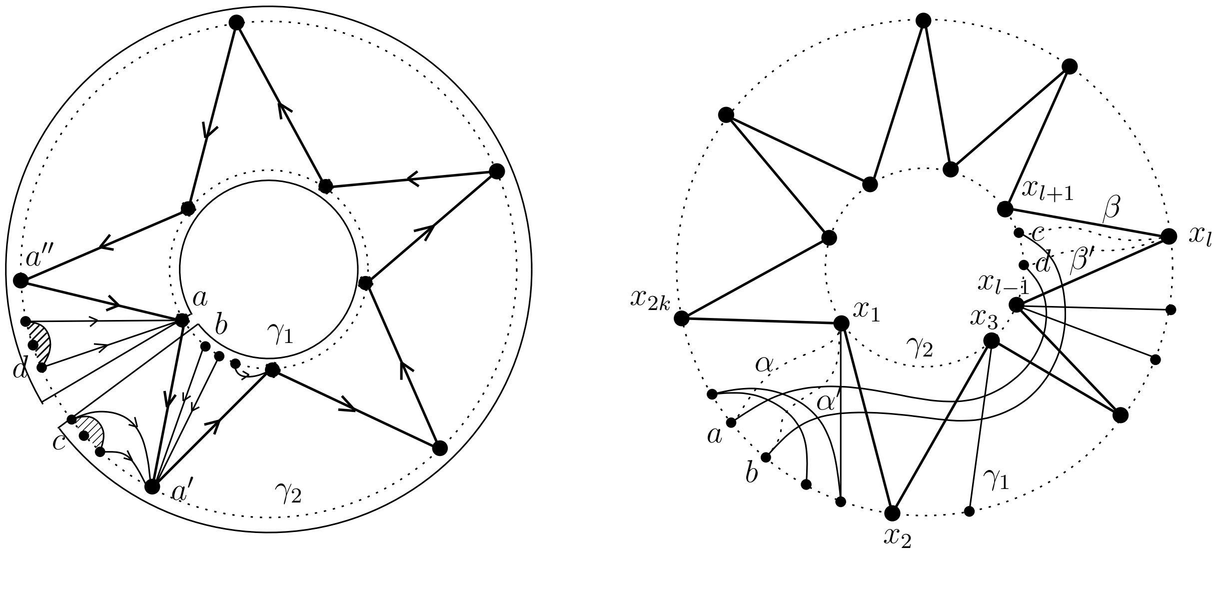

Figure 4: Structures of $\vec{Q}_1$ and $\vec{Q}_2$

Proof. Let Q be a bipartite torus quadrangulation. By Lemma 5, Q has a 2-orientation $\vec{Q}$ which can be decomposed into two spanning connected outer-annulus maps $\vec{Q}_1$ and $\vec{Q}_2$ with zigzag cycles $\vec{C}_1$ and $\vec{C}_2$, respectively. Let $\vec{\gamma}_1$ and $\vec{\gamma}_2$ be the disjoint essential simple closed curves cutting $\vec{Q}$ into $\vec{Q}_1$ and $\vec{Q}_2$, which are oriented along $\vec{C}_1$. (For the symbols used here, those without the arrows stand for the corresponding undirected ones.)

Consider a V-region of $\vec{Q}_1$ bounded by directed edges $a''a$ and aa', where $a''a$ and aa' are contained in $\vec{C}_1$, and a lies on $\vec{\gamma}_1$ and a', a'' on $\vec{\gamma}_2$. By Lemma 6, we can choose a vertex, say d, in the bottom on $\vec{Q}_1$ contained in an essential subtree which is the last vertex with respect to the linear ordering of vertices in the bottom on $\vec{\gamma}_2$ from a'' to a' (if the component does not exists, then we let $d = a''$), and the first vertex, say c, in the bottom contained in a trivial subtree (if the component does not exists, then we let $c = a'$). Let b be the vertex of V-region with top a' which is next to a in the bottom. See the left of Figure 4.

Now we shall construct a 5-page embedding of Q.

For the spine. Fix the vertices of Q in the spine of the book in the following order: First arrange all vertices on γ_1 in the order from a to b. Following them, we arrange all vertices on γ_2 in the order from c to d.

Page 1 and 2. We first embed the edges of $Q_1 - aa'$ in one page. This can naturally be done since the plane embedding of $Q_1 - aa'$ can be regarded as that in a rectangle with corners a, b, c, d, as shown in the left of Figure 4, and the vertices of $Q_1 - aa'$ on the boundary of the rectangle lie in the order of the vertices in the spine.

Next we embed a single edge aa' in the second page, as shown in Figure 5.

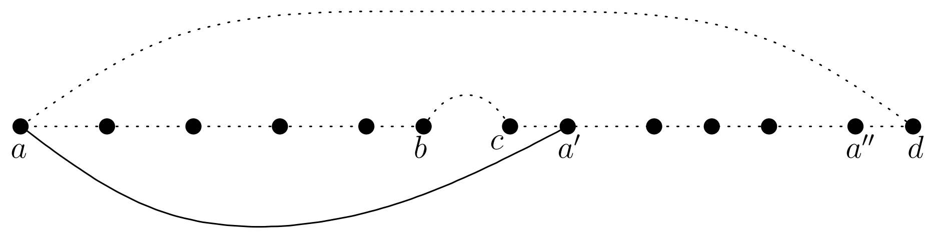

Figure 5: Embedding of Q_1 in page 1 and 2.

Page 3, 4 and 5. We embed the edges of Q_2 using at most three pages, as follows. Let $C_2 = x_1x_2 \cdots x_{2k}$ be the zigzag cycle of Q_2. Let Δ_i be the V-region with top x_i, for $i = 1, \ldots, 2k$. We may suppose that a, b are contained in the bottom of Δ_1, and that c, d are contained in the bottom of Δ_l for some even integer l, since a and b are consecutive on γ_1, and so are c and d on γ_2. So take disjoint simple curves $\gamma_{a,d}$ and $\gamma_{b,c}$ joining a and d, and b and c in Q_2, respectively, so that they cross $x_1x_2, x_2x_3, \ldots, x_{l-1}x_l$, and some essential edges contained in $\Delta_1, \ldots, \Delta_l$. We may suppose that $\gamma_{a,d}$ and $\gamma_{b,c}$ are taken to have a minimum number of crossing points with the edges of Q_2.

Take a segment α in Δ_1 joining x_1 and a, and a segment β in Δ_l joining x_l and c. Let R be the rectangle obtained from Q_2 by cutting along α and β, which does not contain b. Then all edges of Q_2 contained in R can be embedded in the third page, similarly to the first page. See the top of Figure 6.

Take a segment α' in Δ_1 joining x_1 and b, and a segment β' in Δ_l joining x_l and d. Let R' be the rectangle obtained from Q_2 by cutting along α' and β', which does not contain a. Then all edges in R' can be embedded in the forth page. See the top of Figure 6.

If the four segments α, α', β and β' can be taken without crossing edges, then Q is 4-page embeddable. Hence, assuming some edges cross these segments, we embed all of them in the fifth page. If an edge e_1 must cross α or α', then the two endpoints of e_1 are contained in the intervals $[a, x_{2k}]$ and $[b, x_2]$, respectively, in the bottom of Δ_1. All of such edges can be embedded in a rectangle R_1 bounded by $[a, x_{2k}]$, $\overline{x_{2k}x_2}$, $[b, x_2]$, $\overline{ab}$, where $\overline{pq}$ denotes a simple curve in Δ_1 joining two points p and q. Do the same for edges crossing β, β', and then those edges can be embedded in a rectangle R_2 bounded by $[c, x_{l+1}], \overline{x_{l+1}x_{l-1}}, [d, x_{l-1}], \overline{cd}$. Clearly, the edges of Q_2 contained in R_1 and R_2 can be embedded in the fifth page simultaneously, as shown in the bottom of Figure 6. Hence we are done. ■

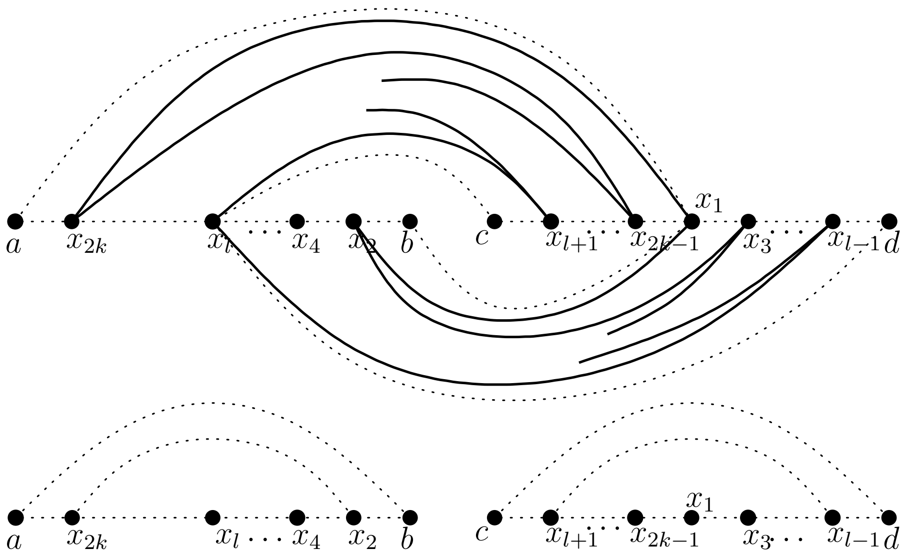

Figure 6: Embedding of Q_2 in page 3, 4 and 5.

In the proof of Theorem 7, we embed only a single edge of Q in the second page, and hence we will get the following.

THEOREM 8 *Let Q be a bipartite graph embeddable in the torus. Then Q has an edge e such that $Q - e$ is 4-page embeddable.* ■

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