2-factors and independent sets on claw-free graphs

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Abstract

In this paper, we show that if G is an l-connected claw-free graph with minimum degree at least three and $l \in \{2, 3\}$, then for any maximum independent set S, there exists a 2-factor in which each cycle contains at least $l - 1$ vertices in S.

1 Introduction

In this paper, we consider finite graphs. If no ambiguity can arise, we denote simply the order $|G|$ of G by n, the minimum degree $\delta(G)$ by δ and the independence number $\alpha(G)$ by α. All notation and terminology not explained in this paper is given in [1, 4].

A *2-factor* of a graph G is a spanning 2-regular subgraph of G. Choudum and Paulraj [3] and Egawa and Ota [5] independently showed that every claw-free graph with $\delta \geq 4$ has a 2-factor. For the upper bound of the number of cycles in 2-factors, Broersma, Paulusma and Yoshimoto [2] proved that a claw-free graph with $\delta \geq 4$

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has a 2-factor with at most $\max\left\{\dfrac{n-3}{\delta-1}, 1\right\}$ cycles. This upper bound is almost best possible. (See [17].) Faudree et al. [8] studied a pair of a maximum independent set and a 2-factor of a claw-free graph G which together dominate G and showed that if G is a claw-free graph with $\delta \geq \dfrac{2n}{\alpha} - 2$ and $n \geq \dfrac{3\alpha^3}{2}$, then for any maximum independent set S, G has a 2-factor with α cycles such that each cycle contains exactly one vertex in S. The following problems were posed in their article.

Conjecture A ([8]). *Let G be a claw-free graph.*

1. *If $\delta \geq \dfrac{n}{\alpha} \geq 5$, then there exist a maximum independent set S and a 2-factor with α cycles such that each cycle contains exactly one vertex of S.*

2. *If $\delta \geq \alpha + 1$, then for any maximum independent set S, there exists a 2-factor with α cycles such that each cycle contains exactly one vertex in S.*

In this paper, we study 2-factors that just divide a given maximum independent set S, i.e., we require that every cycle contains at least one vertex of S, and so the number of cycles in a 2-factor can be smaller than α. The original question was posed by Kaiser when the third author gave a lecture at University of West Bohemia. However, in general still we need the condition $\delta \geq n/\alpha$ because for any positive integer δ with $\dfrac{n}{\alpha} - \dfrac{1}{2\delta} < \delta < \dfrac{n}{\alpha}$, there exists an infinite family of line graphs with minimum degree δ every 2-factor of which contains more than α cycles (see [8], [17]). However 2-connectivity decreases the lower bound of minimum degrees. Our main result of this paper is the following.

Theorem 1. *If G is an l-connected claw-free graph with $\delta \geq 3$ and $l \in \{2, 3\}$, then for any maximum independent set S, G has a 2-factor such that each cycle contains at least $l - 1$ vertices in S.*

We will show this in Section 2. Since a 3-connected claw-free graph has a 2-factor in which each cycle contains at least two vertices in a given maximum independent set by Theorem 1, the number of the cycles in the 2-factor is at most $\dfrac{\alpha}{2}$. It is well known that the independence number of a claw-free graph is at most $\dfrac{2n}{\delta + 2}$ (for instance, see [8]), and so we obtain the following.

2

Corollary 2. *A 3-connected claw-free graph has a 2-factor with at most $\dfrac{\alpha}{2} \leq \dfrac{n}{\delta + 2}$ cycles.*

This improve the upper bound $\dfrac{n-3}{\delta - 1}$ given by Broersma et al. in [2] if the connectivity is at least three and also the upper bound $\dfrac{2n}{15}$ given by Jackson and Yoshimoto in [12] if $\delta \geq 14$.

Finally we give some additional definitions and notation. A subgraph D is *dominating* a graph G if $G - V(D)$ is edgeless. The degree of a vertex u in G is denoted by $d_G(u)$ and we denote the set of all the vertices of degree at least k in G by $V_{\geq k}(G)$, and $V_k(G) = \{d_G(u) = k \mid u \in G\}$. The *edge-degree* of an edge xy is defined as $d_G(x) + d_G(y) - 2$. An edge subset E_0 is called *independent* if no pair of edges in E_0 are adjacent. We denote the subgraph induced by the vertex set of a subgraph B in G by $G[B]$. A graph G with at least $k + 1$ edges is *essentially k-edge-connected* if for any edge set E_0 of at most $k - 1$ edges, $G \setminus E_0$ contains at most one component with edges.

2   Proof of Theorem 1

Let G_0 be an l-connected claw-free graph with $\delta \geq 3$ and $l \in \{2, 3\}$ and S_0 be any maximum independent set of G_0. We look for a 2-factor in G_0 in which each cycle contains at least $l - 1$ vertices in S_0.

We use Ryjáček closure of a claw-free graph G which is defined as follows: for each vertex x of G, $N_G(x)$ induces a subgraph $G[N_G(x)]$ with at most two components; otherwise there is an induced claw. If $G[N_G(x)]$ has two components, both of them must be cliques. In the case that $G[N_G(x)]$ is connected, we add edges joining all pairs of nonadjacent vertices in $N_G(x)$. The *closure $cl(G)$* of G is a graph obtained by recursively repeating this operation, as long as this is possible. Ryjáček [15] showed that the closure $cl(G)$ is uniquely determined and G is hamiltonian if and only if $cl(G)$ is hamiltonian.

Ryjáček, Saito and Schelp [16, Theorem 4] proved that for any mutually vertex-disjoint cycles $D_1, \ldots, D_p$ in $cl(G)$, a claw-free graph G has mutually vertex-disjoint

cycles $C_1, \ldots C_q$ with $p \geq q$ such that $\bigcup_{i=1}^{p} V(D_i) \subseteq \bigcup_{j=1}^{q} V(C_j)$. By modifying the proof, easily we can improve this result as follows:

Lemma 3. *If G is a claw-free graph and $D_1, \ldots, D_p$ are mutually vertex-disjoint cycles in $cl(G)$, then G has mutually vertex-disjoint cycles $C_1, \ldots C_q$ with $p \geq q$ such that for each D_i, there exists C_j such that $V(D_i) \subseteq V(C_j)$.*

If $cl(G_0)$ has a 2-factor $\bigcup_{i=1}^{p} D_i$ in which each cycle D_i contains at least $l - 1$ vertices in S_0, then by the above lemma, G_0 has vertex-disjoint cycles $C_1, \ldots, C_q$ such that for each D_i, there exists C_j such that $V(D_i) \subseteq V(C_j)$. Since

$$|C_j \cap S_0| \geq |D_i \cap S_0| \geq l - 1 \text{ and } \bigcup_{i=1}^{p} V(D_i) = V(cl(G_0)) = V(G_0),$$

$\bigcup_{j=1}^{q} C_j$ is a 2-factor of G_0 with the required properties. We rephrase moreover the above statement using the following result.

Lemma B (Ryjáček [15]). *For any claw-free graph G, there exists a triangle-free graph H such that $L(H) = cl(G)$.*

Let H_0 be a triangle-free graph such that $L(H_0) = cl(G_0)$. By the above facts, for Theorem 1, it is sufficient to show that:

$$L(H_0) \text{ has a 2-factor in which each cycle contains} \tag{1}$$
$$at \text{ least } l - 1 \text{ vertices in } S_0.$$

Let H be a graph and $\mathcal{D}$ a set of mutually edge-disjoint closed trails and stars in H. If every star has at least three edges and every edge in $E(G) \setminus \bigcup_{D \in \mathcal{D}} E(D)$ is incident to a closed trail in $\mathcal{D}$, then $\mathcal{D}$ is called a *system that dominates* H. Gould and Hynds [9] showed that the line graph $L(H)$ has a 2-factor with c cycles if and only if there exists a system that dominates H with c elements. Hence, we look for a system that dominates H_0 such that the corresponding 2-factor of $L(H_0)$ satisfies (1).

The set in H_0 corresponding to S_0 is an edge set. We denote the edge set also by S_0. Notice that S_0 is independent in G_0, but S_0 is not always independent in $L(H_0) = cl(G_0)$, i.e.,

$$S_0 \text{ is possibly not independent in } H_0.$$

4

In either case, the following claim implies (1) immediately.

Claim 1. *There exist edge-disjoint subgraphs $F_1, \ldots, F_p$ in H_0 such that*

1. *$\bigcup_{i=1}^{p} E(F_i) = E(H_0)$,*

2. *$L(F_i)$ is hamiltonian for all $i \leq p$ and*

3. *F_i contains at least $l - 1$ edges in S_0 for all $i \leq p$.*

Proof of Claim 1.

Because $L(H_0)$ is l-connected and has minimum degree at least three, H_0 is essentially l-edge-connected and the minimum edge-degree of H_0 is at least three. Since $l \in \{2, 3\}$, we can use the following lemma.

Lemma C (Yoshimoto [17]). *If H is an essentially 2-edge-connected graph with minimum edge-degree at least three, then there exists a system $\mathcal{D}$ that dominates H such that every vertex in $V_{\geq 3}(H - V_1(H))$ is in a closed trail in $\mathcal{D}$.*

Let $\mathcal{D}_0 = \{B_1, \ldots, B_p\}$ be such a system that dominates H_0 such that the number p of the elements in the system is smallest.

Suppose there exists a star B_i in $\mathcal{D}_0$. If the center of B_i is in $V_{\geq 3}(H_0 - V_1(H_0))$, then there exists a closed trail $B_j \in \mathcal{D}_0$ that passes through the center. As every edge in B_i is dominated by B_j, we can remove B_i from $\mathcal{D}_0$. This contradicts the minimality of the cardinality p of $\mathcal{D}_0$, and hence the center of B_i is in $V_2(H_0 - V_1(H_0))$. Since the minimum edge-degree is at least three,

$$a \text{ star } B_i \in \mathcal{D}_0 \text{ contains at least two pendant edges of } H_0. \tag{2}$$

Furthermore,

$$\text{if } H_0 \text{ is essentially 3-edge-connected, then there is no star in } \mathcal{D}_0. \tag{3}$$

We construct desired subgraphs $F_1, \ldots, F_p$ from $\mathcal{D}_0$ by distributing edges which are not used in $\bigcup_{i=1}^{p} B_i$. For a star $B_i \in \mathcal{D}_0$, let B_i' be the star that consists of all
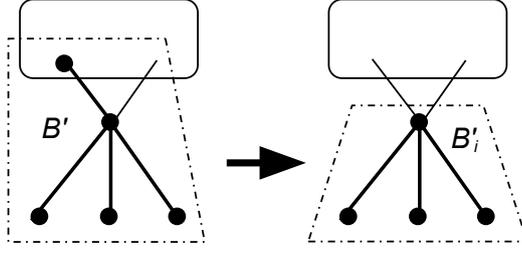
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Figure 1:

the pendant edges of H_0 in B_i. See Figure 1. By (2), B_i' contains at least two edges. For a closed trail $B_i \in \mathcal{D}_0$, we let $B_i' = B_i$. Let $\mathcal{D}_0' = \{B_1', \dots, B_p'\}$.

For each $B_i' \in \mathcal{D}_0'$, let

$$S_i = E(H_0[B_i']) \cap S_0 \text{ and } \widetilde{S} = S_0 \setminus \bigcup_{i=1}^{p} S_i$$

and

$$T_i \text{ be a maximum independent edge set of } H_0[B_i'] - V(S_i),$$

where $H_0[B_i']$ and $V(S_i)$ mean the subgraph of H_0 induced by $V(H_0)$ and $\{x, y \in V(G) \mid xy \in S_i\}$, respectively. Notice that there is a natural bijection between $V(G_0)$ and $E(H_0)$. Since the vertex subset in G_0 corresponding to $\bigcup_{i=1}^{p}(S_i \cup T_i)$ is independent,

$$\text{for any edge } f \in \bigcup_{i=1}^{p} T_i, \text{ there exists an edge } e \in \widetilde{S} \text{ adjacent to } f; \qquad (4)$$

otherwise the vertex subset in G_0 corresponding to $S_0 \cup \{f\}$ is an independent set of G_0 which is larger than S_0.

Let R be the bipartite graph with partite sets $\bigcup_{i=1}^{p} T_i$ and $\widetilde{S}$ obtained by joining all $f \in \bigcup_{i=1}^{p} T_i$ and $e \in \widetilde{S}$ which are adjacent in H_0. See Figure 2. By (4),

$$d_R(f) \geq 1 \text{ for all } f \in \bigcup_{i=1}^{p} T_i.$$

If there is $X \subseteq \bigcup_{i=1}^{p} T_i$ such that $|N_R(X)| < |X|$, then the vertex subset in G_0 corresponding to $(S_0 - N_R(X)) \cup X$ is independent and the order is greater than S_0. This contradicts the maximality of S_0, and so by Hall's theorem, R has a matching
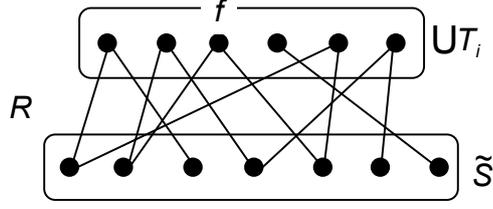
Figure 2:

M covering $\bigcup_{i=1}^{p} T_i$. Using M, we define a mapping $\gamma : \bigcup_{i=1}^{p} T_i \to \widetilde{S}$ as follows: for $f \in \bigcup_{i=1}^{p} T_i$, let $\gamma(f) := e_f$ where $f e_f \in M$.

The set S_0 can be partitioned into the following mutually disjoint subsets:

$$\bigcup_{i=1}^{p} S_i, \ \bigcup_{i=1}^{p} \gamma(T_i) \ \text{and} \ \widetilde{S} \setminus (\bigcup_{i=1}^{p} \gamma(T_i)).$$

Using this partition, we distribute edges in S_0 to $\mathcal{D}_0'$, i.e., we define a mapping $\varphi : S_0 \to \mathcal{D}_0'$ as follows:

1. For $e \in S_i$ $(1 \le i \le p)$, we define $\varphi(e) := B_i'$.

2. For $e \in \gamma(T_i)$ $(1 \le i \le p)$, we define $\varphi(e) := B_i'$.

3. For $e \in \widetilde{S} \setminus \bigcup_{i=1}^{p} \gamma(T_i)$, there exists $B_i' \in \mathcal{D}_0'$ such that e is incident to B_i' since $\mathcal{D}_0$ is a system that dominates H_0. We let $\varphi(e) := B_i'$ for arbitrary B_i' containing an end of e.

Because $S_i \cup \gamma(T_i) \subseteq \varphi^{-1}(B_i')$ and γ is a one-to-one correspondence,

$$|\varphi^{-1}(B_i')| \ge |S_i| + |\gamma(T_i)| = |S_i| + |T_i|$$

for any $B_i' \in \mathcal{D}_0'$. Therefore $\varphi^{-1}(B_i')$ contains at least $|S_i| + |T_i|$ edges in S_0. If B_i' is a star, then $S_i = E(B_i') \cap S_0$ and so $|S_i| + |T_i| = 1$. If B_i' is a closed trail, then $|S_i| + |T_i| \ge 2$ because H_0 is triangle-free. Therefore,

$$\begin{cases} \textit{if } B_i' \textit{ is a star,} & \varphi^{-1}(B_i') \textit{ contains one edge in } S_0 \\ \textit{if } B_i' \textit{ is a closed trail,} & \varphi^{-1}(B_i') \textit{ contains at least two edges in } S_0. \end{cases} \quad (5)$$

Now we divide our argument into two cases.

Case 1. $l = 3$, i.e., H_0 is essentially 3-edge-connected.

In this case, there is no star in $\mathcal{D}_0$ by (3), and so $\mathcal{D}_0' = \mathcal{D}_0$. Since $\mathcal{D}_0$ is a system that dominates H_0, every edge in $E(H_0) \setminus \bigcup_{i=1}^{p} E(B_i')$ is incident to some B_i' in $\mathcal{D}_0'$. We define a mapping ψ from

$$\widetilde{E} = E(H_0) \setminus (\bigcup_{i=1}^{p} E(B_i') \cup S_0)$$

to $\mathcal{D}_0'$ as follows: for all $e \in \widetilde{E}$, let $\psi(e) := B_i'$ for arbitrary B_i' containing an end of e.

Let

$$F_i = B_i' \cup \varphi^{-1}(B_i') \cup \psi^{-1}(B_i').$$

Obviously $F_1, \ldots, F_p$ are mutually edge-disjoint and $\bigcup_{i=1}^{p} E(F_i) = E(H_0)$. Since every edge in F_i is on the closed trail B_i or incident to B_i, F_i has a dominating closed trail. Harary and Nash-Williams [11] showed that $L(H)$ is hamiltonian if and only if H has a dominating closed trail or H is a star with at least three edges. Therefore $L(F_i)$ is hamiltonian. By (5), every F_i contains at least two edges in S_0, and hence $F_1, \ldots, F_p$ are desired subgraphs.

Case 2. $l = 2$, i.e., H_0 is essentially 2-edge-connected.

In this case, $\mathcal{D}_0'$ possibly includes stars with exactly two edges by (2) that do not induce cycles in $L(H_0)$. So, we distribute edges in $\widetilde{E} = E(H_0) \setminus \left(\bigcup_{i=1}^{p} E(B_i') \cup S_0 \right)$ to $\mathcal{D}_0'$ such that every star contains at least three edges.

Suppose B_i' is a star in $\mathcal{D}_0'$ with $|E(B_i')| = 2$. If $S_i = \emptyset$, then there is $f \in T_i$, and so $\gamma(f) \in \varphi^{-1}(B_i')$. Since $\gamma(f) \notin E(B_i')$,

$$B_i' \cup \varphi^{-1}(B_i') \text{ contains at least three edges.} \tag{6}$$

Therefore it is not necessary to distribute an edge in $\widetilde{E}$ to B_i' if $S_i = \emptyset$. However, in the case that $|E(B_i')| = 2$ and $S_i \neq \emptyset$, we have to distribute an edge in $\widetilde{E}$ to B_i'.

Let $\mathcal{T}$ be the set of all stars $B_i' \in \mathcal{D}_0'$ with $|E(B_i')| = 2$ and $S_i \neq \emptyset$. The following claim implies that there exist two edges in $\widetilde{E}$ incident to the center of $B_i' \in \mathcal{T}$.

Claim 2. *For all $B_i' \in \mathcal{T}$, there is no edge in $S_0 \setminus S_i$ incident to B_i'.*

Proof. Suppose that there exists an edge $e_1 \in S_0$ such that $e_1 \notin E(B_i')$ and e_1 is incident with the center of B_i'. Let $e_2 \in S_i$, let $e_3 \in E(B_i') - \{e_2\}$, and let e_4 be the edge incident with the center of B_i' other than e_1, e_2, e_3. In the graph $cl(G_0)$, four vertices e_1, e_2, e_3, e_4 form a clique. However $e_1 e_2 \notin E(G_0)$ because $e_1, e_2 \in S_0$ and S_0 is independent on G_0. Thus, the edge $e_1 e_2$ of $cl(G_0)$ is added through the closure operation. This implies that $N_{G_0}(e_2) \subseteq \{e_3, e_4\}$, which contradicts that $\delta \geq 3$. Thus, Claim 2 holds. $\square$

Let K be the bipartite graph with partite sets $\widetilde{E}$ and $\mathcal{T}$ obtained by joining all $f \in \widetilde{E}$ and $B_i' \in \mathcal{T}$ if f is incident to the center of B_i'. Since for any $B_i' \in \mathcal{T}$, the center of B_i' is in $V_2(H_0 - V_1(H_0))$, we have $d_K(B_i') = 2$ by Claim 2. Since $d_K(f) \leq 2$ for all $f \in \widetilde{E}$, the bipartite graph K has a matching M covering $\mathcal{T}$ by Hall's Theorem. Using M, we define $\psi : \widetilde{E} \to \mathcal{D}_0'$ as follows: let $f \in \widetilde{E}$. If f is used in M, we define $\psi(f) := B_i'$ where $f B_i' \in M$; otherwise let $\psi(f) := B_i'$ for arbitrary B_i' containing an end of f. Because the matching M covers $\mathcal{T}$, we have $\psi^{-1}(B_i') \neq \emptyset$ for any $B_i' \in \mathcal{T}$. Since $\psi^{-1}(B_i') \cap E(B_i') = \emptyset$, $B_i' \cup \psi^{-1}(B_i')$ contains at least three edges.

Let

$$F_i = B_i' \cup \varphi^{-1}(B_i') \cup \psi^{-1}(B_i')$$

for $1 \leq i \leq p$. Obviously $F_1, \ldots, F_p$ are mutually edge-disjoint and $\bigcup_{i=1}^{p} E(F_i) = E(H_0)$. Since every star F_i contains at least three edges, $L(F_i)$ is hamiltonian for all F_i by the theorem of Harary and Nash-Williams [11]. As F_i contains an edge in S_0 by (5), $F_1, \ldots, F_p$ are desired subgraphs. The proof of Claim 1 is completed.

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